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$\mathcal{D}$-modules on representations of Capelli type

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Let (G, V) be an irreducible multiplicity-free finite-dimensional representation of a connected reductive complex group G, as classified by V.G. Kac [17], and G' its derived subgroup. Denote by $\mathfrak{g}$ the Lie algebra of G, and $U(\mathfrak{g})$ its universal enveloping algebra. Assume that there exists a polynomial f generating the algebra of G'-invariant polynomials on V ($\mathbb{C}[V]^{G'} \simeq \mathbb{C}[f]$) and such that $f \notin \mathbb{C}[V]^G$. Such representations are said to be of Capelli type if the algebra of G-invariant differential operators is the image of the center of $U(\mathfrak{g})$ under the differential of the G-action. They fall into eight cases given by R. Howe and T. Umeda [14]: five infinite families and three "exceptional" examples.

We prove that the category of regular holonomic $\mathcal{D}_V$-modules invariant under the action of G' is equivalent to the category of graded modules of finite type over a suitable algebra $\mathcal{A}$, except for few special cases. Indeed the Levasseur's conjecture [28, Conjecture 5.17, p. 508] fails in these cases because of the disconnectedness of the stabilizers of some "smaller" orbits.

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1. Introduction

Let G be a complex connected reductive algebraic group, and let $G' = [G,G]$ be its derived subgroup. Denote by (G,ρ,V) or (G,V) a rational finite-dimensional linear representation of G ($\rho : G \longrightarrow GL(V,\mathbb{C})$) and $\mathbb{C}[V]$ the algebra of polynomials on V. The action of G on V extends to $\mathbb{C}[V]$. We will denote by $\mathbb{C}[V]^G \subset \mathbb{C}[V]$ the subalgebra of G-invariant polynomials on V. We assume that (G,V) is a multiplicity-free space, that is, the associated representation of G on $\mathbb{C}[V]$ decomposes without multiplicities. In other words, each irreducible representation of G occurs at most once in $\mathbb{C}[V]$ (see Definition 2). Note that the irreducible multiplicity-free actions have been classified by V.G. Kac [17]. For other classification and properties of multiplicity-free spaces, we refer to the work by C. Benson and G. Ratcliff [1], F. Knop [26], A. Leahy [27]. Assume furthermore that the multiplicity-free space (G,V) has a one-dimensional quotient, that is, there exists a polynomial f on V such that the subalgebra $\mathbb{C}[V]^{G'}$ of G'-invariant polynomials on V is isomorphic to the polynomial algebra with one variable f (i.e., $\mathbb{C}[V]^{G'} \simeq \mathbb{C}[f]$), and such that $f \notin \mathbb{C}[V]^G$ (see Definition 3). Then, it is known that: G acts on V with an open orbit, and in this case the representation (G,V) is called a prehomogeneous vector space (see M. Sato [43,44] or T. Kimura [25, Chap. 2]). Moreover, it is shown in [25, p. 39, Proposition 2.22] that: for such a reductive prehomogeneous vector space, there exists a constant coefficient differential operator Δ and a polynomial

$$b(s) = c(s+1)(s+\lambda_1+1)\cdots(s+\lambda_{d-1}+1) \in \mathbb{R}_d[s], \quad c>0, \tag{1}$$

called the Bernstein–Sato polynomial of f such that

$$\Delta f^{s+1} = b(s) f^s. \tag{2}$$

M. Kashiwara [19] has shown that the roots of this polynomial are rational, i.e., $\lambda_j \in \mathbb{Q}$ for $1 \le j \le d-1$.

As usual $\mathcal{D}_V$ is the sheaf of rings of differential operators on V with holomorphic coefficients. Let us now point out that the action of G on $\mathbb{C}[V]$ extends to $\Gamma(V,\mathcal{D}_V)^{\mathrm{pol}}$ the $\mathbb{C}$-algebra of differential operators on V with polynomial coefficients in $\mathbb{C}[V]$. This gives rise to a natural algebra: the Weyl algebra $\Gamma\left(V,\mathcal{D}_V\right)^G$ of polynomial coefficients G-invariant differential operators on V.

If G is a Lie group, denote by $\mathfrak{g}$ its Lie algebra and $U(\mathfrak{g})$ the associated universal enveloping algebra. A representation as above (G,V) is said to be of "Capelli type" if (G,V) is an irreducible multiplicity-free representation (MF for short) such that: the subalgebra of G-invariant global algebraic sections $\Gamma\left(V,\mathcal{D}_V\right)^G$ is the image of $Z\left(U\left(\mathfrak{g}\right)\right)$, the center of $U(\mathfrak{g})$, under the differential $\tau : \mathfrak{g} \longrightarrow \Gamma(V,\mathcal{D}_V)^{\mathrm{pol}}$ of the G-action, i.e.,

$$\tau\left(Z\left(U\left(\mathfrak{g}\right)\right)\right) = \Gamma\left(V,\mathcal{D}_V\right)^G$$

(see Definition 4). Note that these representations have been studied by R. Howe and T. Umeda in [15,46]: they fall into eight cases. There are five infinite families and three "exceptional" examples listed below.

	$(G,\ V)$	$\deg f$	$b(s)$
(1)	$(SO(n)\times\mathbb{C}^*,\ \mathbb{C}^n)$	2	$(s+1)(s+\frac{n}{2})$
(2)	$(GL(n),\ S^2\mathbb{C}^n)$	n	$\prod_{i=1}^n(s+\frac{i+1}{2})$
(3)	$(GL(n),\ \Lambda^2\mathbb{C}^n),\ n$ even	$\frac{n}{2}$	$\prod_{i=1}^n(s+2i-1)$
(4)	$(GL(n)\times SL(n),\ M_n(\mathbb{C}))$	n	$\prod_{i=1}^n(s+i)$
(5)	$(Sp(n)\times GL(2),\ (\mathbb{C}^{2n})^2)$	2	$(s+1)(s+2n)$
(6)	$(SO(7)\times\mathbb{C}^*,\ spin=\mathbb{C}^8)$	2	$(s+1)(s+4)$
(7)	$(G_2\times\mathbb{C}^*,\ \mathbb{C}^7)$	2	$(s+1)(s+\frac{7}{2})$
(8)	$(GL(4)\times Sp(2), M_4(\mathbb{C}))$	4	$(s+1)(s+2)(s+3)(s+4)$

If (G,V) is of Capelli type; in particular if (G,V) is MF, then V.G. Kac [17] asserts that V is decomposed into a finite union of G-orbits $(V_k)_{0\leqslant k\leqslant d}$. Let us denote by $\Lambda := \bigcup_{k=0}^{d}\overline{T^*_{V_k}V}\subset T^*V$ the lagrangian subvariety which is the union of the closure of conormal bundles to the G-orbits (see [38]).

Recall that a coherent $\mathcal{D}_V$-module $\mathcal{M}$ is said to be holonomic if its characteristic variety $\operatorname{char}(\mathcal{M})$ is lagrangian. Equivalently, the characteristic variety is of dimension equal to $\dim V$. The holonomic $\mathcal{D}_V$-module $\mathcal{M}$ is called regular if there exists a global good filtration $F\mathcal{M}$ on $\mathcal{M}$ such that the annihilator of $\mathrm{gr}^F\mathcal{M}$ (i.e., the ideal $\mathrm{ann}_{\mathbb{C}[T^*V]}\mathrm{gr}^F\mathcal{M}$) is a radical ideal in $\mathrm{gr}^F\mathcal{D}_V$ (see [20, Definition 5.2] or [24, Corollary 5.1.11]).

Denote by $\mathrm{Mod}^{\mathrm{rh}}_\Lambda(\mathcal{D}_V)$ the full category whose objects are holomorphic regular holonomic $\mathcal{D}_V$-modules $\mathcal{M}$, whose characteristic variety $\operatorname{char}(\mathcal{M})$ is contained in Λ, equivalently those which admit global good filtrations stable under the induced action of the Lie algebra $\mathfrak{g}$ of G on $\mathcal{M}$ (see Remark 13). The general problem consists in the description of the category $\mathrm{Mod}^{\mathrm{rh}}_\Lambda(\mathcal{D}_V)$.

The expected shape to the general solution of the family of problems is as follows. Let us first recall that G' is the derived subgroup of G. We denote by

$$\bar{\mathcal{A}} := \Gamma\left(V,\mathcal{D}_V\right)^{G'}\subset\Gamma(V,\mathcal{D}_V)^{\mathrm{pol}}$$

the $\mathbb{C}$-algebra formed by G'-invariant global algebraic sections of $\mathcal{D}_V$, i.e. the algebra of polynomial coefficients G'-invariant differential operators. This algebra is well understood (see [15,28]), in particular it contains θ the Euler vector field on V. Note that R. Howe and T. Umeda [15, Proposition (7.1), p. 578] have proved that when (G,V) is of Capelli type, the algebra $\Gamma\left(V,\mathcal{D}_V\right)^G$ of G-invariant operators is a polynomial algebra on a canonically defined set of generators (Theorem 9). These generators are precisely called the Capelli operators. Using these, T. Levasseur [28, Theorem 4.11, p. 491], H. Rubenthaler [40, p. 1346, Proposition 3.1, 1)] or [41, p. 24, Theorem 5.3.3] and Z. Yan [47, Theorem 1.9] gave a general description of algebra $\bar{\mathcal{A}}$. We should also mention the contribution by M. Muro, in the real case $(G, V) = (GL(n, \mathbb{R}), S^2(\mathbb{R}^n))$ in [31, Proposition 2.1, p. 356]. Finally, when $(G, V) = (GL(n, \mathbb{C}) \times SL(n, \mathbb{C}), M_n(\mathbb{C}))$, $(GL(2m, \mathbb{C}),\ \Lambda^2\mathbb{C}^{2m})$, the author obtained a concrete description with explicit relations in [33, Proposition 6, p. 120], [34, Proposition 5, pp. 637–638].

If $\mathcal{J} := \operatorname{ann}\mathbb{C}[V]^{G'} = \operatorname{ann}\mathbb{C}[f] \subset \bar{\mathcal{A}}$ denotes the two sided ideal annihilator of G'-invariant polynomials on V, we consider $\mathcal{A}$ the quotient algebra $\bar{\mathcal{A}}/\bar{\mathcal{J}}$, going modulo a suitable ideal $\bar{\mathcal{J}}$ of $\bar{\mathcal{A}}$ described in section 4: $\bar{\mathcal{J}}$ is the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ defined by specific relations (33), (34), (35), (36) of Proposition 11. Following the work by Benson–Ratcliff [1], Howe–Umeda [15], Knop [26] and Levasseur [28], we will deduce that the quotient algebra $\mathcal{A}$ is generated by the following three operators and relations (see Corollary 12): θ the Euler vector field on V, f the multiplication by the polynomial $f(x)$ of degree d, and the differential operator $\Delta := f\left(\dfrac{\partial}{\partial x}\right)$ as above satisfying the Bernstein–Sato equations:

$$\begin{aligned} \Delta f &= c(\frac{\theta}{d}+1)(\frac{\theta}{d}+\lambda_1+1)\cdots(\frac{\theta}{d}+\lambda_{d-1}+1), \\ f\Delta &= c\frac{\theta}{d}(\frac{\theta}{d}+\lambda_1)\cdots(\frac{\theta}{d}+\lambda_{d-1}), \quad c>0 \end{aligned} \tag{3}$$

and the relations

$$[\theta, f] = df, \qquad [\theta, \Delta] = -d\Delta. \tag{4}$$

Let $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ be the category whose objects are finitely generated left $\mathcal{A}$-modules T such that for each $s \in T$, the $\mathbb{C}$-vector space spanned by the set $\{\theta^n s\ /\ n \geq 1\}$ is finite dimensional. In other words, this category consists of all graded left $\mathcal{A}$-modules T of finite type for θ the Euler vector field on V.

The functor $\Psi : \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, defined by taking $\Psi(\mathcal{M})$ to be the set of all $\mathfrak{g}$-invariant θ-homogeneous global sections of $\mathcal{M}$, with quasi-inverse $\Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ defined by $\Phi(T) := \mathcal{D}_V \otimes_{\mathcal{A}} T$, give the equivalence of categories for the Capelli type representations, except in few cases described in Remark 1:

Theorem 25. *Let (G, V) be a representation of Capelli type with a one-dimensional quotient except for few special cases. Then the categories $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ are equivalent.*

We have proved this theorem in the following four infinite families (see [32–36]):

- $(G = Sp(n) \times GL(2),\ \left(\mathbb{C}^{2n}\right)^2)$
- $(G = GL(n),\ V = \Lambda^2\mathbb{C}^n)$, n even

- $(G = GL(n) \times SL(n),\ V = M_n(\mathbb{C}))$
- $(G = SO(n) \times \mathbb{C}^*,\ V = \mathbb{C}^n)$

Remark 1. Actually, Levasseur conjectured [28, Conjecture 5. 17, p. 508] this equivalence of categories for all the eight Capelli type representations, unfortunately this conjecture fails for the following "special cases":

$$(GL(n, \mathbb{C}), S^2\mathbb{C}^n) \quad \text{and for } n = 3,\ \ (SO(3) \times \mathbb{C}^*,\ \mathbb{C}^3). \tag{5}$$

Indeed, the proof of the conjecture is equivalent to the fact that any object in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its G'-invariant global sections (see Theorem 16). This argument fails in the above special cases because the "smaller" orbits here are not simply-connected (i.e. the stabilizers of these orbits are disconnected). More precisely:

- for $n = 3$, $(SO(3) \times \mathbb{C}^*,\ \mathbb{C}^3)$, the quadratic cone is not simply connected,[1]
- for $(GL(n, \mathbb{C}), S^2\mathbb{C}^n)$ the action of the general linear group on symmetric matrices, all the orbits (except the big one) are not simply-connected. The disconnectedness of the stabilizers of these orbits is an obstruction to the G'-invariant sections of the $\mathcal{D}_V$-module, as it can be seen below.

Counterexample. Consider $n = 2$, in this case the symmetric matrices $S^2\mathbb{C}^n$ coincides with the adjoint representation of $G' = SL(2, \mathbb{C})$. There is a simple G'-equivariant $\mathcal{D}$-module on the nilpotent cone on which the center of $SL(2, \mathbb{C})$ is acting through the sign. This $\mathcal{D}$-module does not admit any non-zero G'-invariant section as a quasi-coherent sheaf, and therefore is not generated by G'-invariant sections. For general $n > 2$, at least for n even, consider the orbit in $S^2\mathbb{C}^n$ corresponding to quadratic forms of rank $0 < r < n$, with r odd. Then there exists a simple $SL(n)$-equivariant $\mathcal{D}$-module $\mathcal{M}$ on that orbit such that the central element -1 in $SL(n)$ acts by -1 on global sections of $\mathcal{M}$. In particular, there are no $SL(n)$-invariant global sections.

It turns out that the equivalence between the categories $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ leads to a description of the "analytic" regular holonomic $\mathcal{D}_V$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ in terms of "algebraic homogeneous" $\mathcal{D}_V$-modules.

By the way, we should note that the problem of classifying holomorphic regular holonomic $\mathcal{D}$-modules or equivalently perverse sheaves on a complex manifold (thanks to the Riemann–Hilbert correspondence) has been treated by several authors. The first such result (around 1980) was Deligne's quiver description of perverse sheaves on an affine line with only one possible singularity at the origin [7], which under the Riemann–Hilbert correspondence is the case where $G = \mathbb{C}^\times$ acts on $V = \mathbb{C}$ by scalar multiplication. Deligne's description uses a characterization of constructible sheaves given in [8,9]. We should

[1] The case $n = 3$, $(SO(3) \times \mathbb{C}^*,\ \mathbb{C}^3)$ has been studied in [35, pp. 243–246].

also mention the contribution of L. Boutet de Monvel [2], who gave a classification of holomorphic regular holonomic $\mathcal{D}$-modules in one variable by using pairs of finite dimensional $\mathbb{C}$-vector spaces and certain linear maps. A. Galligo, M. Granger and P. Maisonobe [10] obtained using the Riemann–Hilbert correspondence, a classification of regular holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules with singularities along the hypersurface $x_1 \cdots x_n = 0$ by 2^n-tuples of $\mathbb{C}$-vector spaces with a set of linear maps. L. Narváez-Macarro [37] treated the case $y^2 = x^p$ using the method of Beilinson and Verdier and generalized this study to the case of reducible plane curves. R. MacPherson and K. Vilonen [29] treated the case with singularities along the curve $y^n = x^m$. T. Braden and M. Grinberg [4] studied perverse sheaves on complex $n \times n$-matrices, symmetric matrices and $2n \times 2n$-skew-symmetric matrices, each stratified by the rank. They gave an explicit description of the category of such perverse sheaves as the category of the representations of a quiver. In [33,34], the author classified regular holonomic $\mathcal{D}$-modules associated to the same stratification, and in [32,35,36] to other stratifications using $\mathcal{D}$-modules theoretical methods. This paper is organized as follows:

In Section 2, we recall notions on the so called representations of Capelli type. In section 3, we review some useful results: in particular the one's saying that: any coherent $\mathcal{D}_V$-module equipped with a global good filtration, invariant under the action of the Euler vector field θ, is generated by finitely many global sections of finite type for θ. Section 4 deals with the concrete description of $\overline{\mathcal{A}}$ the algebra of algebraic G'-invariant differential operators following Benson–Ratcliff [1], Howe–Umeda [15], Knop [26], and Levasseur results [28, Theorem 4.11, p. 491]. In section 5, we establish the main result, namely Theorem 25. This is done by means of the central Theorem 16 saying that: any object $\mathcal{M}$ in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ (except in special cases) is generated by finitely many global G'-invariant sections. This result leads to the equivalence of categories between the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$: the image by this equivalence of a regular holonomic $\mathcal{D}_V$-module being its set of θ-homogeneous global sections, which are invariant under the action of G'.

We refer the reader to [3,13,20–23] for notions on $\mathcal{D}$-modules theory.

2. Review on representations of Capelli type with a one-dimensional quotient

Let G be a connected reductive complex algebraic group. We denote by G' its derived subgroup.

Let $\rho : G \longrightarrow GL(V)$ be a finite dimensional representation of G, again denoted by (G, V). Recall that a polynomial $f \in \mathbb{C}[V]$ is called a relative invariant of (G, V) if there exists a rational character $\chi \in \mathcal{X}(G)$ such that $g \cdot f = \chi(g) f$ for all $g \in G$. One says (see [25, Chap. 2]) that the representation (G, V) is a (reductive) prehomogeneous vector space if G has an open dense orbit Ω in V. In that case, we denote the complement of the open dense orbit by $S := V \backslash \Omega$, it is called the singular set of (G, V). Then, it is known (see [25, p. 26, Theorem 2.9]) that, the one-codimensional irreducible components of S are of the form $\{f_i = 0\}$, $1 \leq i \leq r$, for some relative invariants f_i. The f_i are algebraically independent, and are called the basic or fundamental relative invariants of (G, V). Note that, any relative invariant can be (up to non-zero constant) written as $\prod_{i=1}^{r} f_i$. When the singular set S is an hypersurface, the prehomogeneous vector space (G, V) is said to be regular (see [25, p. 43, Theorem 2.28]).

2.1. Multiplicity-free representations

Let us denote by $\mathfrak{g}$ the Lie algebra of the connected reductive Lie group G, and by $\mathfrak{t}$ the Lie algebra of a maximal torus of G. Denote by B the set of dominant weights of $(\mathfrak{g}, \mathfrak{t})$. For a fix finite-dimensional representation (G, V) of the reductive group G, we recall that the action of G on V extends to the algebra of polynomials on V. Then, the rational G-module $\mathbb{C}[V]$ decomposes as

$$\mathbb{C}[V] \simeq \bigoplus_{\beta \in B} E(\beta)^{m(\beta)}, \tag{6}$$

where $E(\beta)$ is an irreducible $\mathfrak{g}$-module with highest weight $\beta \in B$ and $m(\beta) \in \mathbb{N} \cup \{\infty\}$. We recall that the finite-dimensional linear representation (G, V) is said to be multiplicity-free (MF for short) if its associated representation of G on $\mathbb{C}[V]$ decomposes without multiplicities. This means that each irreducible representation $E(\beta)$ of G occurs at most once in $\mathbb{C}[V]$. More precisely, we recall the following definition [28, Definition 4.1, p. 484]:

Definition 2. The representation (G, V) is called multiplicity-free if in (6): $m(\beta) \leq 1$ for all β. In this case

$$\mathbb{C}[V] = \bigoplus_{\beta \in B} V(\beta)^{m(\beta)}, \quad m(\beta) = 0, 1,$$

where $V(\beta)$ is isomorphic to $E(\beta)$.

Note that, a classification of MF representations can be found in [1,17,27], and a complete list of irreducible MF representations is given in [15, table, p. 612] or [28, appendix, p. 508].

2.1.1. Multiplicity-free spaces with a one-dimensional quotient

As above, G' is the derived subgroup of the complex Lie group G. We recall the following definition:

Definition 3. (See Levasseur [28].) A multiplicity-free-space (G, V) is said to have a one-dimensional quotient if there exists a non-constant polynomial $f_0 \in \mathbb{C}[V]$ such that $f_0 \notin \mathbb{C}[V]^G$, and such that $\mathbb{C}[V]^{G'} \simeq \mathbb{C}[f_0]$.

2.2. Representations of "Capelli type"

We continue with (G, V) the finite dimensional representation of the connected reductive Lie group G. We have denoted by $\mathfrak{g} = \mathrm{Lie}\,(G)$ the Lie algebra of G. We consider τ the differential of the G-action defined as follows:

$$\tau : \mathfrak{g} \longrightarrow \Gamma(V, \mathcal{D})^{\mathrm{pol}}, \tag{7}$$

where $\Gamma(V, \mathcal{D})^{\mathrm{pol}}$ is the algebra of global algebraic sections of $\mathcal{D}_V$, i.e. the algebra of polynomial coefficients differential operators. For any element ξ in $\mathfrak{g}$, the image $\tau(\xi)$ is a linear derivation on $\mathbb{C}\,[V]$ given by

$$\tau(\xi)(\phi)(v) = \frac{d}{dt}_{|t=0} \left(e^{t\xi} \cdot \phi\right)(v) = \frac{d}{dt}_{|t=0} \phi\left(e^{-t\xi} \cdot v\right), \tag{8}$$

for all $\phi \in \mathbb{C}[V]$, $v \in V$. This image is homogeneous of degree zero in the sense that $[\theta, \tau(\xi)] = 0$. Denote by $U\,(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. The map τ yields a homomorphism denoted again by τ, and defined by

$$\tau : U\,(\mathfrak{g}) \longrightarrow \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}. \tag{9}$$

Recall that the group G acts naturally on $\Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$: $\forall\, g \in G,\ \ \forall\, \phi \in \mathbb{C}[V],\ \ \forall\, P \in \Gamma\,(V, \mathcal{D}_V)^{\mathrm{pol}}$,

$$(g \cdot P)\,(\phi) = g \cdot P\left(g^{-1} \cdot \phi\right). \tag{10}$$

The differential of this action is given by $P \mapsto [\tau(\xi), P]$ for $\xi \in \mathfrak{g}$, $P \in \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$. Therefore, a subspace $I \subset \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$ is stable under G (resp. G') if and only if $[\tau(\mathfrak{g}), I] \subset I$ (resp. $[\tau(\mathfrak{g}'), I] \subset I$). Then, we know from [28] that the subalgebra of polynomial coefficients G-invariant differential operators

$$\Gamma(V, \mathcal{D}_V)^G = \left\{P \in \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}} :\ [\tau(\mathfrak{g}), P] = 0\right\} \tag{11}$$

is contained in the one's of G'-invariant differential operators

$$\bar{\mathcal{A}} := \Gamma(V, \mathcal{D}_V)^{G'} = \left\{P \in \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}} :\ [\tau(\mathfrak{g}'), P] = 0\right\}. \tag{12}$$

In particular, if $Z\,(U(\mathfrak{g})) = U\,(\mathfrak{g})^G$ is the center of $U\,(\mathfrak{g})$ then

$$\tau\,(Z\,(U(\mathfrak{g}))) \subset \Gamma\,(V, \mathcal{D}_V)^G. \tag{13}$$

Now, we give the following definition (see [28, Definition 5.1]):

Definition 4. We say that the representation (G, V) is of Capelli type if:

- (G, V) is irreducible and MF;
- $\tau(Z(U(\mathfrak{g}))) = \Gamma(V, \mathcal{D}_V)^G$.

Remark 5. In the list of irreducible MF representations (G, V) given by Howe and Umeda (see [15, table, p. 612] or [28, appendix, p. 508]), there are exactly eight of them which are of Capelli type with one-dimensional quotient (see Appendix A).

3. Coherent $\mathcal{D}$-modules generated by their θ-homogeneous global sections

We shall denote by $\mathcal{D}_V$ the sheaf of rings of differential operators on V with holomorphic coefficients. If x denotes a typical element of V, and $\partial := \dfrac{\partial}{\partial x}$ its dual in $\mathcal{D}_V$, let $\theta := \mathrm{Trace}(x\partial)$ be the Euler vector field on V.

Definition 6. Let $\mathcal{M}$ be a $\mathcal{D}_V$-module. A section u in $\mathcal{M}$ is said to be homogeneous if $\dim_{\mathbb{C}} \mathbb{C}[\theta]\, u < \infty$, i.e. the $\mathbb{C}$-vector space spanned by the set $\{\theta^n u \;/\; n \geq 1\}$ is finite dimensional. The section u is said to be homogeneous of degree $\lambda \in \mathbb{C}$, if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j u = 0$.

Let us recall the following result which will be used later (see [35, Theorem 1.3]):

Theorem 7. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module, equipped with a global good filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ stable under the action of θ. Then,*

i) $\mathcal{M}$ is generated over $\mathcal{D}_V$ by finitely many homogeneous global sections, i.e.,

$$\mathcal{M} = \mathcal{D}_V \{s_1, \cdots, s_k \;\in \Gamma(V, \mathcal{M}),\; \dim_{\mathbb{C}} \mathbb{C}[\theta]\, s_j < \infty,\; 0 \leqslant j \leqslant k\}.$$

ii) For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the vector space $\Gamma(V, \mathcal{M}_k) \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta - \lambda)^p\right]$ of homogeneous global sections in $\mathcal{M}_k$, of degree λ, is finite dimensional.

Remark 8. We will describe a holomorphic classification of regular holonomic $\mathcal{D}_V$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$, but Theorem 7 permits to reduce these objects to "algebraic homogeneous" $\mathcal{D}_V$-modules.

4. Algebraic invariant differential operators on a class of multiplicity-free spaces

As in the introduction, (G, V) is a finite-dimensional representation of a connected reductive Lie group G and $G' := [G, G]$ is the derived subgroup of G. Recall that the action of the group G extends to various algebras, namely $\mathbb{C}[V] = S(V^*)$ the algebra of polynomial functions on V, $\Gamma(V,\mathcal{D}_V)^{\mathrm{pol}}$ the algebra of differential operators with polynomial coefficients in $\mathbb{C}[V]$, and $\mathbb{C}[V^*]=S(V)$ identified with differential operators with constant coefficients. We thus obtain algebras of invariants: $\mathbb{C}[V]^G$, $S(V)^G$, and $\Gamma\left(V,\mathcal{D}_V\right)^G$.

If (G,V) is a prehomogeneous vector space, let $f_0,\cdots,f_m$ be its fundamental relative invariants and let $\chi_j\in\mathcal{X}(G)$, $0\le j\le m$, be their weight. There exist relative invariants $f_j^*(\partial)\in S(V)$ with weight χ_j^{-1}, $0\le j\le m$ (see [28, Section 3.1]). We set $\Delta_j:=f_j^*(\partial)$ for $j=0,\cdots,m$.

It is known that the algebra $\mathbb{C}[V]^{G'}$ of G'-invariant polynomials is a polynomial ring

$$\mathbb{C}[V]^{G'}=\mathbb{C}[f_0,\cdots,f_m], \tag{14}$$

and that

$$S(V)^{G'}=\mathbb{C}[\Delta_0,\cdots,\Delta_m] \tag{15}$$

(see [28, Lemma 4.2, (d) and formula (4.3), p. 487]).

Consider the following multiplication map

$$\begin{array}{rccc} m: & \mathbb{C}[V]\otimes S(V) & \longrightarrow & \Gamma(V,\mathcal{D}_V)^{\mathrm{pol}} \\ & \phi\otimes f & \longmapsto & \phi f(\partial). \end{array} \tag{16}$$

One knows from Howe–Umeda [15] that through this map the $(\mathbb{C}[V],G)$-module $\Gamma\left(V,\mathcal{D}_V\right)^{\mathrm{pol}}$ identifies with $\mathbb{C}[V]\otimes S(V)$:

$$\Gamma\left(V,\mathcal{D}_V\right)^{\mathrm{pol}}\simeq\mathbb{C}[V]\otimes S(V) \tag{17}$$

where the group G acts on $\Gamma\left(V,\mathcal{D}_V\right)^{\mathrm{pol}}$ as follows: $\forall\,\phi\in\mathbb{C}[V]$, $\forall\,P\in\Gamma\left(V,\mathcal{D}_V\right)^{\mathrm{pol}}$

$$\left(g\cdot P\right)(\phi)=g\cdot P\left(g^{-1}\cdot\phi\right). \tag{18}$$

First, we are interesting in the description of the algebras of G-invariant differential operators on a multiplicity-free space following the work by Benson–Ratcliff [1], Howe–Umeda [15], Knop [26] and Levasseur [28]. Actually, the isomorphism m is G-invariant, hence the algebra of G-invariant differential operators decomposes as a direct sum of one-dimensional irreducible G-modules $\mathbb{C}E_\gamma$:

$$\Gamma(V,\mathcal{D}_V)^G=\bigoplus_{\gamma\in\Gamma}\mathbb{C}E_\gamma \tag{19}$$

where Γ is the subsemigroup of B generated by certain linearly independent elements $\gamma_0,\cdots,\gamma_r\in B$ (see [15,28]). Let

$$E_\gamma\left(x,\partial_x\right):=m\left(E_\gamma\right)\qquad\in\quad\Gamma(V,\mathcal{D}_V)^G \tag{20}$$

be the operator corresponding to E_γ. The operators $E_\gamma\,(x, \partial_x)$ are called Capelli operators. Put

$$E_j := E_{\gamma_j}\,(x, \partial_x) \qquad 0 \leq j \leq r. \tag{21}$$

We know from [15, Proposition 7.1] that giving a multiplicity-free representation is equivalent to giving a commutative algebra of G-invariant differential operators:

$$(G, V) \quad \text{multiplicity-free} \quad \Longleftrightarrow \quad \Gamma(V, \mathcal{D}_V)^G \quad \text{commutative.} \tag{22}$$

In that case the algebra $\Gamma(V, \mathcal{D}_V)^G$ is generated by the Capelli operators E_j for $0 \leq j \leq r$ (see [15, Theorem 9.1] or [1, Corollary 7.4.4]):

Theorem 9 *(Howe–Umeda). For a fix multiplicity-free representation (G, V), the algebra*

$$\Gamma(V, \mathcal{D}_V)^G = \mathbb{C}\,[E_0, \cdots, E_r]$$

is a commutative polynomial ring.

From now on, we focus our attention in the subalgebras of G (resp. G')-invariant global algebraic sections of $\mathcal{D}_V$ on multiplicity-free representations with a one-dimensional quotient.

4.1. Invariant differential operators on multiplicity-free spaces with one-dimensional quotient

Recall that G' denotes the derived subgroup of G. Recall also that a multiplicity-free representation (G, V) is said to be with one-dimensional quotient if there exists a polynomial function $f \in \mathbb{C}[V]$ such that

$$\mathbb{C}[V]^{G'} = \mathbb{C}[f] \qquad \text{and} \qquad f \notin \mathbb{C}[V]^G. \tag{23}$$

In fact, the polynomial function f is a relative invariant of degree d of weight $\chi \in \mathcal{X}(G)$, and there exists an associated relative invariant differential operator $f^* := f(\partial) \in \mathbb{C}[V^*]$ of degree d with weight χ^{-1}. More precisely, set $\Delta := f^*(\partial)$. We know from Sato–Bernstein–Kashiwara (see [25, Proposition 2.22] and [19]) that there exists a polynomial $b(s) \in \mathbb{R}[s]$ of degree n called the Bernstein–Sato polynomial such that:

$$\begin{array}{rl} i) & b(s) = c\prod_{j=0}^{d-1}(s + \lambda_j + 1), \quad c > 0; \\ ii) & \Delta(f^{s+1}) = b(s)f^s; \\ iii) & \lambda_j + 1 \in \mathbb{Q}^{*+}, \quad 0 \leqslant j \leqslant d-1, \quad \lambda_0 = 0 \end{array} \tag{24}$$

where $\mathbb{Q}^{*+}$ is the set of non-zero positive rational numbers.

Set

$$f := f_0 \quad \text{and} \quad \Delta := \Delta_0 = f^*(\partial). \tag{25}$$

Following T. Levasseur [28, Section 4.2], recall that if (G, V) is a multiplicity-free representation of one-dimensional quotient then we have

$$\mathbb{C}[V]^{G'} = \mathbb{C}[f], \qquad S(V)^{G'} = \mathbb{C}[V^*]^{G'} = \mathbb{C}[\Delta] \qquad \text{and} \qquad E_0 = f\Delta. \tag{26}$$

Now, consider $\overline{\mathcal{A}} := \Gamma(V, \mathcal{D}_V)^{G'}$ the algebra of G'-invariant (polynomial coefficients) differential operators on V:

$$\overline{\mathcal{A}} \supset \Gamma(V, \mathcal{D}_V)^G \quad \text{and} \quad \mathcal{J} := \left\{ P \in \Gamma(V, \mathcal{D}_V)^G \,/\quad Pf^m = 0 \quad \text{for all } m \in \mathbb{N} \right\} \subset \overline{\mathcal{A}} \tag{27}$$

is the annihilator of the G'-invariant polynomial functions on V.

Recall that θ denotes the Euler vector field on V, $\theta \in \Gamma(V, \mathcal{D}_V)^G$. T. Levasseur [28, Lemma 4.10] proved that: for any G-invariant differential operator $P \in \Gamma(V, \mathcal{D}_V)^G$, there exists an associated Bernstein–Sato polynomial $b_P(s) \in \mathbb{C}[s]$ such that the operator $P - b_P(\theta)$ belongs to $\mathcal{J}$. In particular, one can find a polynomial $b_{E_j}(s)$ associated with each Capelli operator E_j, $0 \leqslant j \leqslant r$, such that if we consider Ω_j to be

$$\Omega_j := E_j - b_{E_j}(\theta) \in \mathcal{J} \quad \text{for } j = 0, \cdots, r, \tag{28}$$

then we obtain the following results [28, Theorem 4.11, (i), (v)]:

Theorem 10. *If (G, V) is a fix multiplicity-free representation with one-dimensional quotient, then*

$$\overline{\mathcal{A}} = \mathbb{C}\langle f, \Delta, \theta, \Omega_1, \cdots, \Omega_r \rangle, \tag{29}$$

$$\mathcal{J} = \Sigma_{j=1}^r \overline{\mathcal{A}} \Omega_j. \tag{30}$$

Note that, the operators f and Δ do not commute nor do not commute with the operators $\Omega_1, \cdots, \Omega_r$.

By the way, using these results, T. Levasseur [28, Theorem 4.15] gives a duality (of Howe type) correspondence between (multiplicity-free) representations (with a one-dimensional quotient) of G and lowest weight modules over the Lie algebra generated by f and Δ (which is infinite dimensional when the degree of f is ≥ 3). Actually, this duality recovers and extends results obtained by H. Rubenthaler when the representation (G, V) is of "commutative parabolic type" (see [39, Proposition 4.2] and also [11, Corollary 4.5.17]).

We should note that when (G, V) is irreducible, then

$$\Omega_r = 0, \quad \text{the two sided ideal} \quad \mathcal{J} = \Sigma_{j=0}^{r-1} \overline{\mathcal{A}} \Omega_j = \Sigma_{j=0}^{r-1} \Omega_j \overline{\mathcal{A}}, \quad \text{and} \tag{31}$$

$$\overline{\mathcal{A}} = \mathbb{C} \langle f, \Delta, \theta, \Omega_1, \cdots, \Omega_{r-1} \rangle . \tag{32}$$

In the case $(GL(n, \mathbb{R}), S^2(\mathbb{R}^n))$ of the real general linear group action on real symmetric matrices, M. Muro proved this formula in [31, Proposition 2.1, p. 356]. When $(G, V) = (GL(n, \mathbb{C}) \times SL(n, \mathbb{C}), M_n(\mathbb{C}))$, $(GL(2m, \mathbb{C}),\ \Lambda^2\mathbb{C}^{2m})$, this non-commutative algebra is obtained with explicit relations in [34, Proposition 5, pp. 637–638], [33, Proposition 6, p. 120]. Actually, the result (32) generalizes the one's of H. Rubenthanler (see [40, Proposition 3.1] or [41, Theorem 5.3.3]) obtained when (G, V) is an irreducible regular prehomogeneous representation of commutative parabolic type. We have the following proposition.

Proposition 11. *Let (G, V) be an irreducible multiplicity-free representation with a one-dimensional quotient. The following relations hold in the quotient algebra $\overline{\mathcal{A}}/\mathcal{J}$:*

$$[\theta, f] = df, \tag{33}$$

$$[\theta, \Delta] = -d\Delta, \tag{34}$$

$$f\Delta = c\frac{\theta}{d}(\frac{\theta}{d} + \lambda_1) \cdots (\frac{\theta}{d} + \lambda_{d-1}), \quad c > 0 \tag{35}$$

$$\Delta f = c(\frac{\theta}{d} + 1)(\frac{\theta}{d} + \lambda_1 + 1) \cdots (\frac{\theta}{d} + \lambda_{d-1} + 1), \tag{36}$$

$$f_j\Delta_j = c_j\frac{\theta}{d}(\frac{\theta}{d} + \lambda_1) \cdots (\frac{\theta}{d} + \lambda_{d-j-1}), \ \ c_j > 0, \quad 0 \le j \le r \tag{37}$$

where $\lambda_k \in \mathbb{Q}$ for $k = 0, \cdots, d-1$.

Proof. We should note that by [28, Remark 4.12, (2)], we have the homogeneity of degree d (resp. $-d$) of the polynomial f (resp. Δ), that is, the formula (33), (34).

Recall that $\Omega_j := E_j - b_{E_j}(\theta) \in \mathcal{J}$, for $j = 0, \cdots, r$, so we clearly have

$$E_j = b_{E_j}(\theta) \ \text{ in } \overline{\mathcal{A}}/\mathcal{J}. \tag{38}$$

Recall also that from [28, p. 490], we have $E_0 = f\Delta$ and $b_{E_0}(s) = b(s-1)$ where $b(s) = c(s+1)(s+\lambda_1+1)\cdots(s+\lambda_{d-1}+1)$ is the b-function of f. Then, using this last in (38), we get (35)

$$f\Delta = c\frac{\theta}{d}(\frac{\theta}{d} + \lambda_1) \cdots (\frac{\theta}{d} + \lambda_{d-1}) \quad \text{in} \quad \overline{\mathcal{A}}/\mathcal{J}.$$

Next, since $\Delta f^{s+1} = b(s)f^s$, that is, $(\Delta f)f^s = b(s)f^s$ we get the formula (36):

$$\Delta f = b(\theta) \mod \mathcal{J}.$$

More generally, we may take $E_j = f_j\Delta_j$ and using (38) we get

$$f_j\Delta_j = b_{E_j}(\theta) \text{ in } \overline{\mathcal{A}}/\mathcal{J}$$

with $b_{E_j}(s) = b_j(s-1) = c_j s(s+\lambda_1)\cdots(s+\lambda_{d-j-1})$, $c_j > 0$, $0 \leq j \leq r$, that is, the formula (37). $\square$

Let $\mathcal{K}$ be the ideal of $\overline{\mathcal{A}}/\mathcal{J}$ defined by the relations (33), (34), (35), (36) of Proposition 11. Then the preimage of $\mathcal{K}$ under the quotient map $\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\mathcal{J}$ is an ideal of $\overline{\mathcal{A}}$ containing properly $\mathcal{J}$. Let us denote by $\overline{\mathcal{J}}$ the preimage in $\overline{\mathcal{A}}$ of the ideal $\mathcal{K}$. Denote by $\mathcal{A}$ the quotient algebra of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$:

$$\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}. \tag{39}$$

We have the following corollary which is a particular case of T. Levasseur's result in [28, Theorem 3.9, p. 483] or H. Rubenthaler [40, Theorem 2.8, p. 1345], [41, Theorem 7.3.2, p. 37]:

Corollary 12. *The quotient algebra $\mathcal{A}$ is generated by f, θ, Δ satisfying the relations (33), (34), (35), (36):*

$$\begin{aligned}
[\theta, f] &= df,\\
[\theta, \Delta] &= -d\Delta,\\
f\Delta &= c\frac{\theta}{d}(\frac{\theta}{d}+\lambda_1)\cdots(\frac{\theta}{d}+\lambda_{d-1}),\\
\Delta f &= c(\frac{\theta}{d}+1)(\frac{\theta}{d}+\lambda_1+1)\cdots(\frac{\theta}{d}+\lambda_{d-1}+1).
\end{aligned}$$

5. $\mathcal{D}_V$-modules on representations of "Capelli type" with one-dimensional quotient generated by their invariant global sections

In this section, we continue with the representation (G, V) of the connected (reductive) Lie group G as in Section 4, and G' its derived subgroup. It is well known, in this case, that G (resp. G') acts on V with finitely many orbits $(V_k)_{0\leqslant k\leqslant d}$ (see [17]). Let $\Lambda \subset T^*V$ be the lagrangian subvariety which is the union of the closure of conormal bundles $T^*_{V_k}V$ (see Panyushev [38]). We recall that the action of G on V defines a morphism (see (7), (8)) $\tau : \mathfrak{g} \longrightarrow \Theta_V,\ \xi \mapsto \tau(\xi)$ from the Lie algebra $\mathfrak{g}$ of G to the subalgebra Θ_V of $\mathcal{D}_V$ consisting of vector fields on V, i.e. the tangent sheaf on V. So the lagrangian variety Λ is defined by the common zeros of the principal symbols of vector fields corresponding to infinitesimal generators of G.

Recall that a $\mathcal{D}_V$-module is said to be holonomic if it is coherent and its characteristic variety is lagrangian. Equivalently the characteristic variety is of dimension equal to $\dim V$. A holonomic $\mathcal{D}_V$-module $\mathcal{M}$ is regular if there exists a global good filtration $F\mathcal{M}$ on $\mathcal{M}$ such that the annihilator of $\mathrm{gr}^F\mathcal{M}$ (i.e., the ideal $\mathrm{ann}_{\mathbb{C}[T^*V]}\mathrm{gr}^F\mathcal{M}$) is a radical ideal in $\mathrm{gr}^F\mathcal{D}_V$ (see [20, Definition 5.2] or [24, Corollary 5.1.11]). As in the introduction, we denote by $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ the full category consisting of all holomorphic regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in Λ. Let $\mathcal{M}$ be an object in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. We know from Brylinski and Kashiwara [6, p. 389, (1.2.4)] that $\mathcal{M}$ has a global good filtration $(\mathcal{M}_j)_{j\in\mathbb{Z}}$ satisfying the following condition:

For a differential operator P of degree m ($P \in \Gamma(U, \mathcal{D}_V(m))$ where U is an open subset of V), if its principal symbol $\sigma_m(P)$ vanishes on the characteristic variety $\mathrm{char}(\mathcal{M})$, then we have

$$P\mathcal{M}_j \subset \mathcal{M}_{j+m-1} \quad \text{for any } j \in \mathbb{Z}. \tag{40}$$

In particular, if ξ is a vector field (corresponding to an infinitesimal generator of G) which describes the characteristic variety Λ, its principal symbol vanishes on $\Lambda \supset \mathrm{char}(\mathcal{M})$ (so vanishes on $\mathrm{char}(\mathcal{M})$). Then the relation (40) implies that

$$\xi\mathcal{M}_j \subset \mathcal{M}_{j+1-1}, \quad \text{that is} \tag{41}$$

$$\xi\mathcal{M}_j \subset \mathcal{M}_j \quad \text{for any } j \in \mathbb{Z}. \tag{42}$$

Then we have the following

Remark 13. The objects of the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ are holomorphic regular holonomic $\mathcal{D}_V$-modules equipped with global good filtrations which are preserved by the action of the Lie algebra $\mathfrak{g}$ of G.

We recall the following definition:

Definition 14. Let G be an algebraic group acting on a smooth variety V, and $\alpha : G \times V \longrightarrow V$ the group action morphism ($\alpha(g, v) = g \cdot v$ ($g \in G$, $v \in V$)). One says that the group G acts on a well filtered $\mathcal{D}_V$-module $\mathcal{M}$ if it preserves the good filtration on $\mathcal{M}$, and there exists an isomorphism of $\mathcal{D}_{G\times V}$-modules $u : \alpha^+(\mathcal{M}) \xrightarrow{\sim} \mathrm{pr}_V^+(\mathcal{M})$ satisfying the associativity condition coming from the group multiplication of G ($\mathrm{pr}_V : G \times V \longrightarrow V, (g, v) \longmapsto v$ is the projection onto V).

We specialize further to the case where (G, V) is of Capelli type, i.e., (G, V) is an irreducible multiplicity-free-space such that $\Gamma(V, \mathcal{D}_V)^G$ is equal to the image of the center of $U(\mathfrak{g})$ under the differential $\tau : \mathfrak{g} \longrightarrow \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$ of the G-action (see Definition 4). More precisely, assume that (G, V) is a representation of Capelli type with a one-dimensional quotient, i.e., there exists a non-constant polynomial f such that $f \notin \mathbb{C}[V]^G$, and such that $\mathbb{C}[V]^{G'} \simeq \mathbb{C}[f]$ (see Definition 3).

Let $\widetilde{G}$ be the universal covering group of the group G, and $\mathcal{M}$ be a holomorphic regular holonomic $\mathcal{D}_V$-module in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. From [35, Proposition 1.6], the infinitesimal action of G on $\mathcal{M}$ lifts to an action of $\widetilde{G}$ on $\mathcal{M}$ compatible with the action of G on V and $\mathcal{D}_V$. Therefore, we deduced the following remark:

Remark 15. The action of G on V extends to an action of the universal covering $\widetilde{G}$ on $\mathcal{D}_V$-modules $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. Specially the derived subgroup G' acts on $\mathcal{M}$. Note that this action is the one that derivates are given by $\mathcal{D}$-module action of corresponding vector fields as specified in Remark 13.

This section consists in the proof of the main general argument of the paper. We show, except in few special cases (see Remark 1, (5)), that any $\mathcal{D}_V$-module $\mathcal{M}$ in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its invariant global sections under the action of G'.

Theorem 16. *A $\mathcal{D}_V$-module $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its G'-invariant global sections, except in special cases.*[2]

Actually the proof proceeds in a number of steps described below.

5.1. Extension of sections and G'-invariance

For the proof of Theorem 16, we shall use an algebraic point of view. Since the concerning $\mathcal{D}_V$-modules are regular holonomic, it is equivalent to consider the algebraic case or the analytic one. We need the following two lemmas in the proof:

Lemma 17. ([45, Lemma 1, p. 247, $n°$ 55]) *Let V be an affine variety, f a regular function on V, and Ω the set of points $x \in V$ such that $f(x) \neq 0$. Let $\mathcal{F}$ be a coherent algebraic sheaf on V, and $s \in \Gamma(\Omega, \mathcal{F})$ a section of $\mathcal{F}$ on Ω. Then, for any large enough $N \in \mathbb{N}$, there exists a section s' of $\mathcal{F}$ on the whole V ($s' \in \Gamma(V, \mathcal{F})$), such that $s' = sf^N$ on Ω, i.e.,*

$$s'_{|\Omega} = sf^N. \tag{43}$$

Lemma 18. *Consider G' the complex algebraic group acting on the affine algebraic variety V, f a G'-invariant regular function on V $\left(f \in \mathbb{C}[V]^{G'}\right)$, Ω the complement in V of the hypersurface defined by $f = 0$, and $\mathcal{F}$ a G'-equivariant coherent algebraic sheaf on V. Then, for any G'-invariant section s of $\mathcal{F}$ on Ω ($s \in \Gamma(\Omega, \mathcal{F})^{G'}$) there exists some $N \gg 0$ such that sf^N extends to a G'-invariant global section.*

Proof. Recall that V is an affine algebraic variety, i.e. $V = \mathrm{Spec}A$, where $A := \mathbb{C}[V]$ is an affine algebra over $\mathbb{C}$ and $\Omega = \mathrm{Spec}A[\frac{1}{f}]$ with $A[\frac{1}{f}] = \mathbb{C}[V][\frac{1}{f}] = \mathbb{C}[\Omega]$.

[2] Remark 1, (5).

Since $\mathcal{F}$ is a coherent algebraic sheaf on V, then $\mathcal{F}$ is a finitely generated A-module. We consider the restriction of $\mathcal{F}$ on Ω:

$$\mathcal{F}[\Omega] := \mathcal{F} \bigotimes_A A[\frac{1}{f}]. \tag{44}$$

The previous lemma says that any section s of $\mathcal{F}$ on Ω ($s \in \Gamma(\Omega, \mathcal{F})$) extends to a global section m ($m \in \Gamma(V, \mathcal{F})$) such that

$$m_{|\Omega} = sf^p \quad \text{for} \;\; p \gg 0. \tag{45}$$

So, from (44) and (45), the section s can be written as

$$s = \frac{m}{f^r} \quad \text{for} \;\; r \gg 0. \tag{46}$$

Recall that the group G' acts on A and on $\mathcal{F}$. Then, for any $g \in G'$ acting on s, we have

$$g.s = g.\left(\frac{m}{f^r}\right) = \frac{g.m}{g.f^r}. \tag{47}$$

Since s is a G'-invariant section ($g.s = s$) and f is a G'-invariant regular function ($f = g.f$), then the previous equality becomes:

$$s = \frac{g.m}{f^r}. \tag{48}$$

Using (46) we get

$$\frac{m}{f^r} = \frac{g.m}{f^r} \quad \Longleftrightarrow \quad \frac{m - g.m}{f^r} = 0. \tag{49}$$

This means that there exists a large integer $N \gg 0$ such that

$$(m - g.m)f^N = 0 \quad \Longleftrightarrow \quad mf^N = (g.m)f^N. \tag{50}$$

Since f is G'-invariant ($f^N = g.f^N$), this last becomes

$$mf^N = (g.m)(g.f^N), \tag{51}$$

that is,

$$mf^N = g.(mf^N). \tag{52}$$

Thus mf^N is a G'-invariant global section extending s $\left(mf^N \in \Gamma(V, \mathcal{F})^G\right)$. $\square$

5.2. Simple holonomic $\mathcal{D}_V$-modules with support on the closure of the orbits

Recall that for each irreducible multiplicity free representation (G, V) with a one-dimensional quotient there exists a polynomial $f \in \mathbb{C}[V]$, called the relative invariant of (G, V), satisfying $g \cdot f = \chi(g) f$ for all $g \in G$, where $\chi \in \mathcal{X}(G)$ is a rational character. In fact, f is a G'-invariant homogeneous polynomial of degree d. The action of G on V has $d+1$ orbits which we denote by $(V_k)_{0 \leqslant k \leqslant d}$. We should note that the G-orbits closures $\left(\overline{V}_k\right)_{0 \leqslant k \leqslant d}$ in such an action are linearly ordered by inclusion as proved in [15, pp. 607–608, (13.1)–(13.6)]:

$$V = \overline{V}_d \supsetneq \overline{V}_{d-1} \supsetneq \cdots \supsetneq \overline{V}_0 = \{0\}.$$

Let us denote again by f the mapping $f : V \longrightarrow \mathbb{C}, x \longmapsto f(x)$. Here we introduce some subquotient modules of the inverse image by the mapping f of the $\mathcal{D}_{\mathbb{C}}$-module $\mathcal{O}_{\mathbb{C}}\left(\frac{1}{t}\right)$, $t \in \mathbb{C}$, which will be used in the proof of Theorem 16 above.

Denote by $L := f^+\left(\mathcal{O}_{\mathbb{C}}\left(\dfrac{1}{t}\right)\right) = \mathcal{O}_V\left(\dfrac{1}{f}\right)$ the $\mathcal{D}_V$-module generated by its G'-invariant homogeneous sections $e_{-m} := f^{-m}$ (where m is a non negative integer: $m \in \mathbb{Z}_{\geq 0}$) satisfying the following equations obtained from (36), (37): $(0 \leq k \leq d)$

$$f e_{-m} = e_{-m+1}, \tag{53}$$

$$\Delta e_{-m} = -cm(-m+\lambda_1)\cdots(-m+\lambda_{d-1})\, e_{-m-1}, \ \text{ with } \lambda_{m+1} \geq \lambda_m, \tag{54}$$

$$f_k \Delta_k e_{-m} = -c_k m(-m+\lambda_1)\cdots(-m+\lambda_{d-k-1})\, e_{-m}. \tag{55}$$

In particular, we note that

$$\Delta e_{-m} = 0 \quad \text{for } m = \lambda_0 = 0, \ \ m = \lambda_j, 1 \leqslant j \leqslant d-1, \lambda_j \in \mathbb{Z}_{\geq 0} \text{ or } \lambda_j \in \frac{1}{2}\mathbb{Z}_{\geq 0}. \tag{56}$$

Let $L_m \subset L$ be submodules generated by e_{-m} $(m = 0, 1, \cdots, d)$ in $\mathcal{O}_V\left(\dfrac{1}{f}\right)$. One has the chain

$$L_0 := \mathcal{O}_V \subset L_1 := \mathcal{D}_V f^{-1} \subset \cdots \subset L_d := \mathcal{D}_V f^{-d}. \tag{57}$$

We have the following lemma:

Lemma 19. *Assume $\lambda_m \leqslant \lambda_{m+1}$, $0 \leqslant m \leqslant d-1$, the submodule L_{λ_m} generated by $e_{-\lambda_m}$ in $\mathcal{O}\left(\dfrac{1}{f}\right)$ does not contain the sections $e_{-\lambda_k}$ for $k > m$; in particular*

$$L_{\lambda_{m+1}} \neq L_{\lambda_m}.$$

Proof. Let $s \in L_m$ be an homogeneous G'-invariant section in L_m:

$$s = Pe_{-m} \quad \text{with} \quad P \in \mathcal{D}_V. \tag{58}$$

The action of the group G' on the section s is defined as follows: for any $g \in G'$

$$gs = gP \cdot ge_{-m}. \tag{59}$$

Since the sections s and e_{-m} are G'-invariant (i.e. $gs = s$ and $ge_{-m} = e_{-m}$) the previous equality becomes

$$s = gP \cdot e_{-m}. \tag{60}$$

Using (58) and (60) we get

$$gP \cdot e_{-m} = Pe_{-m} \Longleftrightarrow gP = P. \tag{61}$$

This means that P is a G'-invariant differential operator ($P \in \overline{\mathcal{A}}$). Then the section s can be written as

$$s = Pe_{-m} \quad \text{with} \quad P \in \bar{\mathcal{A}} \tag{62}$$

an homogeneous G'-invariant differential operator in $\bar{\mathcal{A}}$.

If we denote by $\deg s$, $\deg P$ the homogeneity degrees of s and P respectively, that is $\theta s = (\deg s)\, s$, $[\theta, P] = (\deg P)\, P$, then if $\deg s < -m$, $\deg P < 0$, we have

$$P = Q\Delta \tag{63}$$

(with Q a differential operator such that $\deg Q = \deg P + 1$) and

$$s = Q\Delta e_{-m}. \tag{64}$$

Since $\Delta e_{-\lambda_m} = 0$ for $0 \leqslant m \leqslant d-1$ (see (56)), we obtain for $s \in L_{\lambda_m}$

$$s = Pe_{-\lambda_m} = Q\Delta e_{-\lambda_m} = 0. \tag{65}$$

In particular, the submodule L_{λ_m} does not contain $e_{-\lambda_{(m+1)}}$ and $L_{\lambda_{m+1}} \neq L_{\lambda_m}$ for $0 \leqslant m \leqslant d-1$. $\square$

Now, we are interested in the following successive quotient modules L_m/L_{m-1}. Note that the submodule L_m is generated by the invariant homogeneous sections $e_0, e_{-1}, \cdots, e_{-m}$. Denote by $\widetilde{e}_{-m} := e_{-m} \bmod L_{m-1}$ the class of e_{-m} modulo L_{m-1}. We can see that the quotient module L_m/L_{m-1} is generated by one only element $\widetilde{e}_{-m}$ homogeneous of degree $-dm$ satisfying the following relations:

$$L_m/L_{m-1} := \begin{cases} \text{one generator} \quad \widetilde{e}_{-m} := e_{-m} \quad \mod \ L_{m-1} \\ \theta\widetilde{e}_{-m} = -dm\widetilde{e}_{-m} \\ \Delta\widetilde{e}_{-m} = 0 \end{cases}$$

We have the following result:

Proposition 20. *Assume $\lambda_{m-1} \leqslant \lambda_m$, the successive quotients $L_{\lambda_m}/L_{\lambda_{m-1}}$ are simple holonomic $\mathcal{D}_V$-modules of multiplicity 1 supported by $\overline{V}_{d-m}$ respectively for $0 \leqslant m \leqslant d$.*

Proof. We will work with an algebraic point of view. Actually, since the concerning $\mathcal{D}$-modules are regular holonomic it becomes the same to consider the algebraic case or the analytic one. Let us consider the affine variety V with the stratification $(V_r)_{0\leqslant r\leqslant d}$. We want to show what follows:

Let j be the inclusion in V of the open set $\Omega = V\backslash\overline{V}_{d-1}$ where $\overline{V}_{d-1}$ is the hypersurface defined by $f = 0$. Let us give a filtration on the direct image $j_*\mathcal{O}_\Omega$ by the sub-$\mathcal{D}$-modules L_{λ_m} generated by the $e_{-\lambda_m} = f^{-\lambda_m}$, $0 \leqslant m \leqslant d$. Actually the global sections of the direct image $j_*\mathcal{O}$ are the quotients of polynomials by any power of f. So it is a $\mathcal{D}_V$-module generated by $f^{-\lambda_d}$. Then we know that $j_*\mathcal{O}_\Omega = L_{\lambda_d}$, and we would like the successive quotients $L_{\lambda_m}/L_{\lambda_{m-1}}$ to be simple modules, and that $L_{\lambda_m}/L_{\lambda_{m-1}}$ (where $L_{\lambda_{-1}=0}$) to be with support on $\overline{V}_{d-m}$, and moreover that on V_{d-m} the quotients $L_{\lambda_m}/L_{\lambda_{m-1}}$ correspond to the trivial $\mathcal{D}$-module $\mathcal{O}$ by the Kashiwara equivalence (see [18]).

We proceed by induction on d, the result being clear for $d = 0, 1$. Let us prove the case d, assuming the induction hypothesis.

In the neighborhood of a point of V_r, the space V (with its stratification) is homeomorphic to the space $V_r \times [V$ for the value $d - r$ of $d]$. Actually, it is locally isomorphic (i.e. an étale morphism). The induction hypothesis thus ensures that everything goes as we want out of V_0. It remains to see that

1) $L_{\lambda_r}/L_{\lambda_{r-1}}$ does not contain any sections with support in $\{0\}$, nor quotient submodules with support in $\{0\}$, for $r < d$.
2) $L_{\lambda_d}/L_{\lambda_{d-1}}$ is the Dirac module in $\{0\}$.

The group G acts linearly on V. In the language of $\mathcal{D}$-modules, this is an horizontal action that is there is an isomorphism of $\mathcal{D}$-modules on $V \times G$ (see Definition 14). Denote by Q the largest quotient (sub-object) of $L_{\lambda_r}/L_{\lambda_{r-1}}$ supported by $\{0\}$. On Q this gives the structure

$$Q \sim \delta_{\{0\}} \otimes W \tag{66}$$

where $\delta_{\{0\}}$ is the Dirac module in $\{0\}$ equipped with an obvious action and W is a vector space with a trivial action. An homotopy argument asserts that the $\mathcal{D}$-module whose sections are with schematic support $\{0\}$ corresponds by Riemann–Hilbert to the sheaf with support $\{0\}$ on which G acts trivially. So, if Q_0 is the subspace of sections with schematic support in $\{0\}$, then we obtain that for a section $q \in Q_0$,

$$q \prod dx_{ij} \tag{67}$$

is invariant under the action of G. Now, because G is a reductive group and since we are in the algebraic case, the section $q \in Q_0$ lifts into a section of the module L_{λ_r} with the same variance. It thus appears that the section $q \in Q_0$ lifts into $f(x)^{-\lambda_d}$. Since $f(x)^{-\lambda_d}$ is not in L_{λ_d} (see Lemma 19) then 1) follows and 2) also: the section q cannot be the image of $f(x)^{-\lambda_d}$, thus Q_0 is of dimension one. $\square$

Let us consider the quotient modules of $\mathcal{O}_V(\frac{1}{f})$ by the $L_{\lambda_{m-1}}$ for $m = 1, \cdots, d$:

$$Q_m := \mathcal{O}_V\left(\frac{1}{f}\right)/L_{\lambda_{m-1}} = \mathcal{O}_V\left(\frac{1}{f}\right)/\mathcal{D}_V f^{-\lambda_{(m-1)}}. \tag{68}$$

Then each $\mathcal{D}_V$-modules Q_m is also generated by its invariant homogeneous sections, and we can see that: for $\lambda_{m-1} \leqslant \lambda_m$ $(1 \leqslant m \leqslant d)$ there exists a Jordan–Hölder chain:

$$Q_m := \mathcal{O}_V\left(\frac{1}{f}\right)/L_{\lambda_{m-1}},\ L_{\lambda_{m-1}/L_{\lambda_{m-2}}},\ L_{\lambda_{m-2}/L_{\lambda_{m-3}}},\ \cdots,\ L_{\lambda_2}/L_{\lambda_1},\ L_{\lambda_1}/L_{\lambda_0},\quad L_{\lambda_0} \tag{69}$$

(with $L_{\lambda_0} = L_0 = 0$) supported respectively by

$$\overline{V}_{d-m},\quad \overline{V}_{d-(m-1)},\quad \overline{V}_{d-(m-2)},\quad \cdots,\quad \overline{V}_{d-2},\quad \overline{V}_{d-1},\quad \overline{V}_d = V$$

$(1 \leqslant m \leqslant d-1)$.

So we have the following lemma:

Lemma 21. *The modules* $Q_m := \mathcal{O}\left(\frac{1}{f}\right)/L_{\lambda_{m-1}}$ *are supported by* $\overline{V}_{d-m}$ $(1 \leqslant m \leqslant d-1)$.

We have the following proposition:

Proposition 22. *Any section* $s \in \Gamma\left(V\backslash\overline{V}_{d-m-1}, Q_m\right)$ *of the* $\mathcal{D}_V$*-module* Q_m *in the complement of* $\overline{V}_{d-m-1}$ *extends to* V $(m = 1, \cdots, d-1)$.

Proof. We should note that $\overline{V}_{d-m}$ is smooth out of $\overline{V}_{d-m-1}$, and is a normal variety along $\overline{V}_{d-m-1}$ for $m = 1, \cdots, d-1$ (see [5]). Recall that $\lambda_{m-1} \leqslant \lambda_m$. Consider the chain

$$\begin{aligned} Q_m := \mathcal{O}_V\left(\frac{1}{f}\right)/L_{\lambda_{m-1}},\ \ L_{\lambda_d}/L_{\lambda_{m-1}},\ \ L_{\lambda_{d-1}}/L_{\lambda_{m-1}},\ \cdots,\\ L_{\lambda_{m-2}}/L_{\lambda_{m-1}},\ \ L_{\lambda_{m-1}}/L_{\lambda_{m-1}} = 0. \end{aligned} \tag{70}$$

Put $R_k := L_{\lambda_k}/L_{\lambda_{m-1}}$ for $k = m-1, \cdots, d$. The $\mathcal{D}_V$-module L_{λ_m} (resp. Q_m) is the union of modules L_{λ_k}, $0 \leqslant k \leqslant m$ (resp. R_k, $m-1 \leqslant k \leqslant d$) such that the associated graded modules $\mathrm{gr}\,(Q_m)$ is the sum of quotient modules

$$\mathrm{gr}\,(Q_m) = \bigoplus_{k=m-1}^{d} R_k/R_{k-1} \simeq \bigoplus_{k=m-1}^{d} L_{\lambda_k}/L_{\lambda_{k-1}}. \tag{71}$$

Recall that we have denoted $\widetilde{e}_{-\lambda_k} := e_{-\lambda_k} \bmod L_{\lambda_{k-1}}$ $(1 \leq k \leq m)$, then

$$\mathrm{gr}\,(Q_m) = \bigoplus_{k=m-1}^{d} \mathcal{O}_{T^*_{V_{d-m}}V}\widetilde{e}_{-\lambda_k}. \tag{72}$$

In this case the property of extension here is true for functions because $\overline{V}_{d-m}$ is normal along $\overline{V}_{d-m-1}$. $\square$

We are now ready to prove the Theorem 16 saying that the $\mathcal{D}_V$-modules studied here are generated by G'-invariant sections, except for few special cases. In the proof we are using a descending induction on orbits; but some few "smaller" orbits in V are not as nice as the open orbits because their stabilizers in G' are disconnected.

5.3. *Proof of Theorem 16*

Recall that the irreducible multiplicity free representation (G, V) has a Zariski open dense orbit Ω, and a relative invariant f (i.e., there exists a character $\chi \in \mathcal{X}(G)$ such that $g \cdot f = \chi(g)f$ for $g \in G$) which is a G'-invariant homogeneous polynomial of degree d such that $\mathbb{C}[V]^{G'} \simeq \mathbb{C}[f]$. In this case, we know from V.G. Kac [17] that G has finitely many orbits, namely $d+1$ orbits. We denote by $\overline{V_k}$ the closure of the G-orbits V_k for $0 \leqslant k \leqslant d$ with $V_0 = \{0\}$. Let us consider again f as the mapping $f : V \longrightarrow \mathbb{C}$, $x \mapsto f(x)$, and $\overline{V}_{d-1}$ the hypersurface defined by $f = 0$, then we have $\Omega := V \backslash \overline{V}_{d-1}$ the complement in V of $\overline{V}_{d-1}$.

Let $\mathcal{M}$ be a holomorphic regular holonomic $\mathcal{D}_V$-module in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. One sets

$$\mathcal{M}^{G'} := \mathcal{D}_V\{m_1, \cdots, m_p \in \Gamma(V, \mathcal{M})^{G'} \text{ such that } \dim_{\mathbb{C}}\mathbb{C}[\theta]m_j < \infty \text{ for } 1 \leqslant j \leqslant p\}$$

the submodule of $\mathcal{M}$ generated, over $\mathcal{D}_V$, by finitely many homogeneous global sections, which are invariant under the action of G'. We will see successively that the quotient module $\mathcal{M}/\mathcal{M}^{G'}$ is supported by $\overline{V}_{d-m}$ $(0 \leqslant m \leqslant d-1)$, and the monodromy is trivial since the orbits $\overline{V}_{d-m}\backslash\overline{V}_{d-m-1}$ are simply-connected (except for special cases described in Remark 1).

First, we claim that on the open dense orbit Ω, we have the equality $\mathcal{M} = \mathcal{M}^{G'}$.

Indeed, let $j : \Omega \longrightarrow V$ be the open embedding. The restriction $\mathcal{M}_\Omega := j^+(\mathcal{M})$ is a G'-equivariant $\mathcal{D}_\Omega$-module. Notice that, if we denote again by f the mapping $f : V \longrightarrow \mathbb{A}^1$, this identifies Ω/G with $\mathbb{G}_m = \mathbb{A}^1\backslash\{0\}$. The generic stabilizers H in G' of points in Ω are connected (see Appendix C, Remark), so the G'-equivariant $\mathcal{D}_\Omega$-module $\mathcal{M}_\Omega$ is the pullback by f of a $\mathcal{D}_{\Omega/G}$-module $\mathcal{N}$ on Ω/G:

$$\mathcal{M}_\Omega = f^+(\mathcal{N}) \quad \text{with} \quad \mathcal{N} \text{ a } \mathcal{D}_{\Omega/G}\text{-module.} \tag{73}$$

Thus on Ω, the G'-invariant sections of $\mathcal{M}_\Omega$, i.e., $\Gamma(\Omega, \mathcal{M}_\Omega)^{G'}$ (which are exactly the inverse images by f of $\Gamma(\mathbb{G}_m, \mathcal{N})$ the sections on $\mathbb{G}_m$ of $\mathcal{N}$) generate $\Gamma(\Omega, \mathcal{M}_\Omega)$ as a $\Gamma(\Omega, \mathcal{O}_\Omega)$-module:

$$\Gamma(\Omega, \mathcal{M}_\Omega)^{G'} = f^{-1}(\Gamma(\mathbb{G}_m, \mathcal{N})), \tag{74}$$

and

$$\Gamma(\Omega, \mathcal{M}_\Omega) = \Gamma(\Omega, \mathcal{O}_\Omega)\left\{\Gamma(\Omega, \mathcal{M}_\Omega)^{G'}\right\} = \Gamma(\Omega, \mathcal{O}_\Omega)\left\{f^{-1}(\Gamma(\mathbb{G}_m, \mathcal{N}))\right\}. \tag{75}$$

Now, for every section $m \in \Gamma(\Omega, \mathcal{M}_\Omega)$, one can find a sufficiently large integer $N \gg 0$ such that the section obtained by multiplication by f^N, that is,

$$mf^N \in \Gamma(\Omega, \mathcal{M}_\Omega) \tag{76}$$

extends to a global section of $\mathcal{M}$ (see Lemma 17), i.e., the section mf^N lifts to a global section

$$\widetilde{mf^N} \in \Gamma(V, \mathcal{M}). \tag{77}$$

If m is a G'-invariant section on Ω ($m \in \Gamma(\Omega, \mathcal{M}_\Omega)^{G'}$), so is mf^N, i.e.,

$$mf^N \in \Gamma(\Omega, \mathcal{M}_\Omega)^{G'}. \tag{78}$$

Then, according to the Lemma 18, we can choose this lifting section $\widetilde{mf^N}$ to be G'-invariant:

$$\widetilde{mf^N} \in \Gamma(V, \mathcal{M})^{G'}. \tag{79}$$

Thus, by (75) (and since the mapping f is invertible on Ω), the image of $\Gamma(V, \mathcal{M})^{G'}$ in $\Gamma(\Omega, \mathcal{M}_\Omega)^{G'}$ generates $\Gamma(\Omega, \mathcal{M}_\Omega)^{G'}$ as a $\Gamma(\Omega, \mathcal{O}_\Omega)$-module.

Since Ω is an affine variety, we see that the restriction of $\mathcal{M}^{G'}$ to Ω equals $\mathcal{M}_\Omega$:

$$j^+\left(\mathcal{M}^{G'}\right) = \mathcal{M}_\Omega. \tag{80}$$

Hence on Ω, the quotient module $\mathcal{M}/\mathcal{M}^{G'}$ is zero, namely

$$\mathcal{M}/\mathcal{M}^{G'} = 0 \quad \text{on} \quad \Omega, \tag{81}$$

and its support lies in the hypersurface $\overline{V}_{d-1}$:

$$\text{supp}\left(\mathcal{M}/\mathcal{M}^{G'}\right) \subset \overline{V}_{d-1}. \tag{82}$$

Now, since we already know that $\mathcal{M}$ is a G'-equivariant $\mathcal{D}_V$-module (see Remark 15), then $\mathcal{M}^{G'}$ is also G'-equivariant, hence such is the quotient module $\mathcal{M}/\mathcal{M}^{G'}$. Moreover, since $\overline{V}_{d-1}$ has a finite number of G'-orbits, which, except for few special cases (see Remark 1, (5)), are simply-connected (see J. Haris [12, Theorem 1, p. 85] for case (7), J. Igusa [16] for case (6), J. Milnor [30] for cases (4) and (8), Nang [32, Prop. 3, p. 194] for case (5), Nang [33, Prop. 1, p. 117] for case (3), [34–36]), we will obtain that $\mathcal{M}/\mathcal{M}^{G'}$ is supported by the closure of the G'-orbits, i.e.,

$$\text{Supp}\left(\mathcal{M}/\mathcal{M}^{G'}\right) \subset \overline{V}_k \quad \text{for } 0 \le k \le d-2. \tag{83}$$

Indeed, since $\mathcal{M}$ is supported by the hypersurface $\overline{V}_{d-1}$, then $\mathcal{M}$ is locally isomorphic out of $\overline{V}_{d-2}$ to a direct sum of copies of the Dirac module on $\overline{V}_{d-1}$ that is

$$Q_1 = \mathcal{O}_V(1/f)/\mathcal{O}_V. \tag{84}$$

As the monodromy here is trivial (because the orbits are simply-connected) then $\mathcal{M}$ is isomorphic, globally outside $\overline{V}_{d-2}$ to a direct sum of a finite number of copies of Q_1, i.e.

$$\mathcal{M}_{|_{V\setminus\overline{V}_{d-2}}} \simeq \bigoplus_{i=1}^{N} Q_{1|_{V\setminus\overline{V}_{d-2}}}. \tag{85}$$

One knows that Q_1 is naturally generated by its G'-invariant global sections. The relation (85) implies that $\mathcal{M}$ is generated out of $\overline{V}_{d-2}$ by invariant sections $s_1, \cdots, s_p$ in $\Gamma\left(V\setminus\overline{V}_{d-2},\, Q_1\right)^{G'}$, i.e.

$$\mathcal{M}_{|_{V\setminus\overline{V}_{d-2}}} = \mathcal{D}_V < s_1, \cdots, s_p > \qquad \text{with } s_j \in \Gamma\left(V\setminus\overline{V}_{d-2},\, Q_1\right)^{G'}. \tag{86}$$

The Proposition 22 says that the sections $(s_j)_{1\leqslant j\leqslant p}$ extend to global sections $(\sigma_j)_{1\leqslant j\leqslant p} \in \Gamma\left(V,\, Q_1\right)^{G'}$:

$$\mathcal{M}_{|_{V\setminus\overline{V}_{d-2}}} = \mathcal{D}_V < \sigma_1, \cdots, \sigma_p >_{|_{V\setminus\overline{V}_{d-2}}} \quad \text{with } \sigma_j \in \Gamma\left(V,\ Q_1\right)^{G'}. \tag{87}$$

Now taking the restriction on $V\setminus\overline{V}_{d-2}$ of the quotient of $\mathcal{M}$ by $\mathcal{M}^{G'}$ gives

$$\left(\mathcal{M}/\mathcal{M}^{G'}\right)_{|_{V\setminus\overline{V}_{d-2}}} \simeq \mathcal{M}_{|_{V\setminus\overline{V}_{d-2}}}/\mathcal{M}^{G'}_{|_{V\setminus\overline{V}_{d-2}}} = 0 \quad \text{that is} \tag{88}$$

$$\operatorname{Supp}\left(\mathcal{M}/\mathcal{M}^{G'}\right) \subset \overline{V}_{d-2}. \tag{89}$$

In the same way by recurrence on m, if $\mathcal{M}$ is with support on $\overline{V}_{d-m}$, $(0 \leqslant m \leqslant d-1)$ then there is a morphism $\mathcal{M} \longrightarrow Q_m^N$ which is an isomorphism out of $\overline{V}_{d-m-1}$, such that $\mathcal{M}/\mathcal{M}^{G'}$ is with support on $\overline{V}_{d-m-1}$ because the submodules of Q_m are also generated by their invariant homogeneous sections.

Finally, if $\mathcal{M}$ is supported by V_0 (the Dirac module with support at the origin) then the result is obvious.

6. Equivalence of categories

In this section, we establish the main result of this paper: Theorem 25.

Recall that $\overline{\mathcal{A}} = \mathbb{C}\langle f, \Delta, \theta, \Omega_1, \cdots, \Omega_{r-1}\rangle$ is the algebra of G'-invariant differential operators. Since the Euler vector field θ belongs to $\overline{\mathcal{A}}$, we can decompose the algebra $\overline{\mathcal{A}}$ under the adjoint action of θ:

$$\overline{\mathcal{A}} = \bigoplus_{k\in\mathbb{N}} \overline{\mathcal{A}}[k], \quad \overline{\mathcal{A}}[k] = \{P \in \overline{\mathcal{A}} : \ [\theta, P] = kP\} \tag{90}$$

and we can check that

$$\forall\, k, l \in \mathbb{N}, \quad \overline{\mathcal{A}}[k] \cdot \overline{\mathcal{A}}[l] \subset \overline{\mathcal{A}}[k+l], \tag{91}$$

so $\overline{\mathcal{A}}$ is a graded algebra.

Recall also that $\mathcal{J} \subset \overline{\mathcal{A}}$ is the annihilator of $\mathbb{C}[f]$. We have denoted $\overline{\mathcal{J}}$ the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ defined by the relations (33), (34), (35), (36) of Proposition 11:

$$\begin{aligned}
[\theta, f] &= df, \\
[\theta, \Delta] &= -d, \Delta, \\
f\Delta &= c\frac{\theta}{d}(\frac{\theta}{d} + \lambda_1)\cdots(\frac{\theta}{d} + \lambda_{d-1}), \\
\Delta f &= c(\frac{\theta}{d} + 1)(\frac{\theta}{d} + \lambda_1 + 1)\cdots(\frac{\theta}{d} + \lambda_{d-1} + 1).
\end{aligned}$$

We put $\mathcal{A}$ the quotient of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$: $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ (see Corollary 12).

Now, since $\overline{\mathcal{J}}$ is an ideal of $\overline{\mathcal{A}}$ it decomposes also under the adjoint action of θ:

$$\overline{\mathcal{J}} = \bigoplus_{k\in\mathbb{N}} \overline{\mathcal{J}}\left[k\right], \quad \overline{\mathcal{J}}\left[k\right] = \overline{\mathcal{J}} \cap \overline{\mathcal{A}}\left[k\right]. \tag{92}$$

Note that $\overline{\mathcal{J}}$ is an homogeneous ideal of the graded algebra $\overline{\mathcal{A}}$, thus the quotient algebra $\mathcal{A} = \overline{\mathcal{A}}/\overline{\mathcal{J}}$ is naturally graded by

$$\mathcal{A}\left[k\right] := \left(\overline{\mathcal{A}}/\overline{\mathcal{J}}\right)\left[k\right] = \overline{\mathcal{A}}\left[k\right]/\overline{\mathcal{J}}\left[k\right]. \tag{93}$$

As in the introduction, we denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category whose objects are finitely generated left $\mathcal{A}$-modules T such that for each $s \in T$, the $\mathbb{C}$-vector space spanned by the set $\{\theta^n s \;/\; n \geq 1\}$ is finite dimensional. Equivalently the category consisting of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] u < \infty$ for any u in T. In other words, T is a direct sum of finite dimensional $\mathbb{C}$-vector spaces:

$$T = \bigoplus_{\alpha\in\mathbb{C}} T_\alpha, \quad T_\alpha := \bigcup_{p\in\mathbb{N}} \ker\left(\theta-\alpha\right)^p \;\; (\text{with } \dim_{\mathbb{C}} T_\alpha < \infty) \tag{94}$$

equipped with the endomorphisms f, θ, Δ of degree d, 0, $-d$, respectively and satisfying the relations (33), (34), (35), (36) of Proposition 11 with $(\theta - \alpha)$ being a nilpotent operator on each T_α.

Recall that $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ stands for the category consisting of holomorphic regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in Λ the union of conormal bundles to the orbits for the action of G on the complex vector space V.

Let $\mathcal{M}$ be an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$, denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma(V, \mathcal{M})$ consisting of G'-invariant homogeneous global sections u in $\mathcal{M}$ such that $\dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] u < \infty$:

$$\Psi(\mathcal{M}) := \left\{ u \in \Gamma(V, \mathcal{M})^{G'},\; \dim_{\mathbb{C}} \mathbb{C}\left[\theta\right] u < \infty \right\}. \tag{95}$$

We are going to show that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ except in the special cases.

Let $(\sigma_{1,} \cdots, \sigma_p) \in \Gamma(V, \mathcal{M})^{G'}$ be a finite family of homogeneous invariant global sections generating the $\mathcal{D}_V$-module $\Psi(\mathcal{M})$ (see Theorem 16):

$$\Psi(\mathcal{M}) := \mathcal{D}_V \left\langle \sigma_{1,} \cdots, \sigma_p \right\rangle. \tag{96}$$

We are going to see that the family $(\sigma_{1,} \cdots, \sigma_p)$ generates also $\Psi(\mathcal{M})$ as an $\mathcal{A}$-module: indeed, an invariant section $\sigma \in \Psi(\mathcal{M})$ can be written as

$$\sigma = \sum_{j=1}^{p} q_j(X, D)\, \sigma_j \quad \text{where} \quad q_j \in \mathcal{D}_V. \tag{97}$$

Let G_c be the compact maximal subgroup of G' and denote by $\widetilde{q_j} := \int_{G_c} g \cdot q_j dg$ the average of q_j over G_c. Then, the average $\widetilde{q_j}$ belongs to the algebra $\overline{\mathcal{A}}$ (i.e., $\widetilde{q_j} \in \overline{\mathcal{A}}$). Now, denote by f_j the class of $\widetilde{q_j}$ modulo $\overline{\mathcal{J}}$:

$$f_j \ := \ \widetilde{q_j} \mod \overline{\mathcal{J}} \quad \text{that is} \quad f_j \in \mathcal{A}. \tag{98}$$

Therefore, we also have

$$\sigma = \sum_{j=1}^{p} \widetilde{q_j}\sigma_j = \sum_{j=1}^{p} f_j\sigma_j \quad \text{with } f_j \in \mathcal{A}. \tag{99}$$

This last means that

$$\Psi(\mathcal{M}) := \mathcal{A}\langle \sigma_1, \cdots, \sigma_p \rangle, \tag{100}$$

and $\Psi(\mathcal{M})$ is an $\mathcal{A}$-module. Moreover, according to Theorem 7 ii), we have

$$\Psi(\mathcal{M}) = \bigoplus_{\alpha \in \mathbb{C}} \Psi(\mathcal{M})_\alpha \tag{101}$$

where

$$\Psi(\mathcal{M})_\alpha := [\Psi(\mathcal{M})] \bigcap \left[\bigcup_{p \in \mathbb{N}} \ker(\theta - \alpha)^p \right] \quad (\text{with } \dim_{\mathbb{C}} \Psi(\mathcal{M})_\alpha < \infty) \tag{102}$$

is the finite dimensional $\mathbb{C}$-vector space of homogeneous global sections of degree $\alpha \in \mathbb{C}$ in $\Psi(\mathcal{M})$. Finally, we can check that

$$\mathcal{A}[k]\,\Psi(\mathcal{M})_\alpha \subset \Psi(\mathcal{M})_{\alpha+k} \qquad \text{for all } k \in \mathbb{N},\ \alpha \in \mathbb{C}. \tag{103}$$

So, $\Psi(\mathcal{M})$ is a graded $\mathcal{A}$-module of finite type for the Euler vector field θ thanks to (100)–(103). This means that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$.

Conversely, let T be an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, one associates to it the $\mathcal{D}_V$-module

$$\Phi(T) := \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \tag{104}$$

where $\mathcal{M}_0 := \mathcal{D}_V/\overline{\mathcal{J}}$. Then $\Phi(T)$ is an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$.

Thus, we have defined two functors

$$\Psi : \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}), \tag{105}$$

$$\Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V). \tag{106}$$

We need the two following lemmas:

Lemma 23. *The canonical morphism*

$$T \longrightarrow \Psi(\Phi(T)),\ t \longmapsto 1 \otimes t \tag{107}$$

is an isomorphism, and defines an isomorphism of functors

$$\mathrm{Id}_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})} \longrightarrow \Psi \circ \Phi, \tag{108}$$

except in special cases

Proof. Let (G, V) be a Capelli type representation different from the special cases. We have set $\mathcal{M}_0 := \mathcal{D}_V/\overline{\mathcal{J}}$. Denote by ε (the class of $1_{\mathcal{D}}$ modulo $\overline{\mathcal{J}}$) the canonical generator of $\mathcal{M}_0$. Recall that G_c is the compact maximal subgroup of G'. Let $h \in \mathcal{D}_V$, denote by $\widetilde{h} \in \overline{\mathcal{A}}$ its average on G_c and by φ the class of $\widetilde{h}$ modulo $\overline{\mathcal{J}}$, that is, $\varphi \in \mathcal{A}$.

Since ε is G'-invariant, we get $\widetilde{h\varepsilon} = \widetilde{h}\varepsilon = \varepsilon\varphi$. Moreover, we have $\widetilde{h}\varphi = 0$ if and only if $\widetilde{h} \in \overline{\mathcal{J}}$, in other words $\varphi = 0$. Therefore, the average operator (over G_c)

$$\mathcal{D}_V \longrightarrow \overline{\mathcal{A}}, \quad h \longmapsto \widetilde{h}$$

induces a surjective morphism of $\mathcal{A}$-modules $v : \mathcal{M}_0 \longrightarrow \mathcal{A}$. More generally, for any $\mathcal{A}$-module T in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v \otimes 1_T$ is surjective

$$v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A} \bigotimes_{\mathcal{A}} T = T \tag{109}$$

which is the left inverse of the morphism

$$u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T,\ t \longmapsto \varepsilon \otimes t, \tag{110}$$

that is, $(v \otimes 1_T) \circ (\varepsilon \otimes 1_T) = v(\varepsilon) = 1_T$. This means that the morphism u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T = \Phi(T)$, that is, $\Psi(\Phi(T))$: indeed if $\sigma = \sum_{i=1}^{p} h_i \otimes t_i$ is an invariant section in $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, we may replace each h_i by its average $\widetilde{h}_i \in \mathcal{A}$, then we get

$$\sigma = \sum_{i=1}^{p} \widetilde{h}_i \otimes t_i = \varepsilon \otimes \sum_{i=1}^{p} \widetilde{h}_i t_i \ \in\ \varepsilon \otimes T, \tag{111}$$

that is, $\sum_{i=1}^{p} \widetilde{h}_i t_i \in T$. Therefore, the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines an isomorphism of functors. $\square$

Next, we note the following:

Lemma 24. *The canonical morphism*

$$w : \Phi\left(\Psi\left(\mathcal{M}\right)\right) \longrightarrow \mathcal{M} \tag{112}$$

is an isomorphism and defines an isomorphism of functors

$$\Phi \circ \Psi \longrightarrow \mathrm{Id}_{\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)}, \tag{113}$$

except in special cases.

Proof. As in the Theorem 16, except in the special cases, the $\mathcal{D}_V$-module $\mathcal{M}$ is generated by a finite family of invariant sections $(\sigma_i)_{i=1,\cdots,p} \in \Psi\left(\mathcal{M}\right)$ so that the morphism w is surjective. Now, consider $\mathcal{Q}$ the kernel of the morphism $w : \Phi\left(\Psi\left(\mathcal{M}\right)\right) \longrightarrow \mathcal{M}$. It is also generated over $\mathcal{D}_V$ by its invariant sections , that is, by $\Psi\left(\mathcal{Q}\right)$. Then we get

$$\Psi\left(\mathcal{Q}\right) \subset \Psi\left[\Phi\left(\Psi\left(\mathcal{M}\right)\right)\right] = \Psi\left(\mathcal{M}\right) \tag{114}$$

where we used $\Psi \circ \Phi = Id_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})}$ (see the preceding Lemma 23). Since the morphism $\Psi\left(\mathcal{M}\right) \longrightarrow \mathcal{M}$ is injective $(\Psi\left(\mathcal{M}\right) \subset \Gamma\left(V,\ \mathcal{M}\right))$, we obtain $\Psi\left(\mathcal{Q}\right) = 0$. Therefore $\mathcal{Q} = 0$ (because $\Psi\left(\mathcal{Q}\right)$ generates $\mathcal{Q}$). $\square$

This section ends by Theorem 25 established by means of the preceding lemmas.

Theorem 25. *Let (G, V) be a representation of Capelli type with a one-dimensional quotient, except in special cases. Then the functors Φ and Ψ induce equivalence of categories*

$$\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V) \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}). \tag{115}$$

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Appendix A. Representations of Capelli type with one-dimensional quotient

	$(G,\ V)$	$\deg f$	$b(s)$
(1)	$(SO(n)\times\mathbb{C}^*,\ \mathbb{C}^n)$	2	$(s+1)(s+\frac{n}{2})$
(2)	$(GL(n),\ S^2\mathbb{C}^n)$	n	$\prod_{i=1}^n(s+\frac{i+1}{2})$
(3)	$(GL(n),\ \Lambda^2\mathbb{C}^n),\ n$ even	$\frac{n}{2}$	$\prod_{i=1}^n(s+2i-1)$
(4)	$(GL(n)\times SL(n),\ M_n(\mathbb{C}))$	n	$\prod_{i=1}^n(s+i)$
(5)	$(Sp(n)\times GL(2),\ (\mathbb{C}^{2n})^2)$	2	$(s+1)(s+2n)$
(6)	$(SO(7)\times\mathbb{C}^*,\ spin=\mathbb{C}^8)$	2	$(s+1)(s+4)$
(7)	$(G_2\times\mathbb{C}^*,\ \mathbb{C}^7)$	2	$(s+1)(s+\frac{7}{2})$
(8)	$(GL(4)\times Sp(2), M_4(\mathbb{C}))$	4	$(s+1)(s+2)(s+3)(s+4)$

Appendix B. Generic isotropy subgroups G_{X_0} for representations of Capelli type

	$(G,\ V)$	G_{X_0} := isotropy subgroup at generic point $X_0\in V\backslash f^{-1}(0)$
(1)	$(SO(n)\times\mathbb{C}^*,\ \mathbb{C}^n)$	$SO(1)\times SO(n-1)$
(2)	$(GL(n),\ S^2\mathbb{C}^n)$	$O(n)$
(3)	$(GL(n),\ \Lambda^2\mathbb{C}^n),\ n$ even	$Sp(\frac{n}{2})$
(4)	$(GL(n)\times SL(n),\ M_n(\mathbb{C}))$	$Sp(1)\times Sp(n-1)$
(5)	$(Sp(n)\times GL(2),\ (\mathbb{C}^{2n})^2)$	$SL(n)$
(6)	$(SO(7)\times\mathbb{C}^*,\ spin=\mathbb{C}^8)$	$SO(1)\times SO(6)$
(7)	$(G_2\times\mathbb{C}^*,\ \mathbb{C}^7)$	
(8)	$(GL(4)\times Sp(2), M_4(\mathbb{C}))$	

(see A. Sasada [42, (1), (2), (3), (13), (15), pp. 79–83] or Sato–Kimura [44, (1), (2), (3), (13), (15), pp. 144–145]).

Appendix C. Generic isotropy subgroups H for derived subgroups G' of the group G

	$(G',\ V)$	$H =$ isotropy subgroup at a generic point $X_0 \in V \backslash f^{-1}(0)$
(1)	$(SO(n),\ \mathbb{C}^n)$	$SO(1) \times SO(n-1)$
(2)	$(SL(n),\ S^2\mathbb{C}^n)$	$SO(n)$
(3)	$(SL(n),\ \Lambda^2\mathbb{C}^n),\ n$ even	$Sp(\frac{n}{2})$
(4)	$(SL(n) \times SL(n),\ M_n(\mathbb{C}))$	$Sp(1) \times Sp(n-1)$
(5)	$(Sp(n) \times SL(2),\ (\mathbb{C}^{2n})^2)$	$SL(n)$
(6)	$(SO(7),\ spin = \mathbb{C}^8)$	$SO(1) \times SO(6)$
(7)	$(G_2,\ \mathbb{C}^7)$	
(8)	$(SL(4) \times Sp(2), M_4(\mathbb{C}))$	

Remark. The generic isotropy[3] subgroups H of (G', V) are connected.

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