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# Convergence of Rees valuations of sequences of one-fibered domains



Matthew Toeniskoetter

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## ABSTRACT

We show that for an infinite sequence  $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$  of local normalized quadratic transforms of analytically unramified one-fibered Noetherian local domains, the corresponding sequence of Rees valuations converges. This extends a result of [4], where the authors show that for an infinite sequence of local quadratic transforms of regular local rings, the order valuations converge.

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## 1. Introduction

The purpose of this paper is to extend known results about valuations associated to infinite sequences of blow-ups to a more general setting. For a sequence of local quadratic transforms of 2-dimensional regular local rings, or more generally a sequence of local normalized quadratic transforms of 2-dimensional Noetherian local domains, the union itself is the associated valuation ring.

**Proposition 1.1.** [1, p. 337] *Let  $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$  be a sequence of local normalized quadratic transforms of 2-dimensional analytically unramified Noetherian local domains. Then  $\bigcup_{n \geq 0} R_n$  is a valuation domain.*

E-mail address: [mtoeniskoetter@fau.edu](mailto:mtoeniskoetter@fau.edu).

Abhyankar proves this result for sequences of 2-dimensional regular local rings, and Lipman notes in [7, Theorem 2.1] that Abhyankar’s proof applies also to sequences of 2-dimensional normal Noetherian local domains.

In dimension  $\geq 3$ , the union of local normalized quadratic transforms often fails to be a valuation domain, even for regular local rings. For instance, take the example of Shannon in [13]. If  $R$  is a 3-dimensional regular local ring with  $\mathfrak{m} = (x, y, z)R$ , and  $\nu$  is a valuation such that  $\nu(x) = 1$ ,  $\nu(y) = \alpha$ , and  $\nu(z) > 1 + \alpha$ , where  $\alpha > 0$  is irrational, then the union of the sequence of local quadratic transforms of  $R$  along  $\nu$  is a 2-dimensional integrally closed non-Noetherian domain that is not a valuation domain.

If  $R$  is a regular local ring, it has an associated order valuation  $\text{ord}_R$ , where  $\text{ord}_R(x) = \sup\{n \geq 0 \mid x \in \mathfrak{m}^n\}$  for  $x \in R$ . Associated to a sequence  $\{R_n\}_{n \geq 0}$  of local quadratic transforms of regular local ring, there is a corresponding sequence  $\{\text{ord}_{R_n}\}_{n \geq 0}$  of order valuations. The union  $\bigcup_{n \geq 0} R_n$  often fails to be a valuation ring, but in [4], the authors show that the sequence of order valuations converge in the following sense:

**Definition 1.2.** Let  $\{\nu_n\}_{n \geq 0}$  be a sequence of valuations on a field  $K$ . The sequence *converges* (to a valuation  $\nu$ ) if for every nonzero  $f \in K$ , the sign of  $\nu_n(f)$  is eventually constant (and equals the sign of  $\nu(f)$ ).

Notice that if a sequence  $\{\nu_n\}_{n \geq 0}$  of valuations converges, then the corresponding valuation ring can be written as the directed union of intersections of valuation rings,

$$V = \bigcup_{m \geq 0} \bigcap_{n \geq m} V_n.$$

The set of valuations on a field  $K$  containing a domain  $R$  of  $K$  is naturally endowed with the Zariski topology and the associated patch topology, also called the constructible topology. Definition 1.2 of convergence is equivalent to the topological definition of convergence in the patch topology. The patch topology is compact and Hausdorff; there exists at least one limit point of any infinite set of valuations, and if a sequence converges, it converges to a unique limit point.

We recall the main results of [4] in the following theorem, and we reference [5] for a thorough investigation of the structure of the rings and valuations contained therein:

**Theorem 1.3.** Let  $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$  be an infinite sequence of local quadratic transforms of regular local rings and let  $(S, \mathfrak{m}_S)$  denote the directed union. Then:

1. There exists a minimal proper Noetherian overring of  $S$ , say  $T$ , where  $T$  is a localization of  $R_n$  for  $n \gg 0$ . We call  $T$  the Noetherian hull of the sequence.
2. The sequence of order valuations  $\{\text{ord}_{R_n}\}_{n \geq 0}$  converges to a valuation  $\nu$  with valuation ring  $V$ . We call  $\nu$  the boundary valuation of the sequence.
3.  $S = T \cap V$ .

In this paper, we extend the result on the convergence of order valuations to sequences of Noetherian local domains that are not necessarily regular. Our proof uses normality in essential ways, and local quadratic transforms of normal Noetherian domains are not necessarily integrally closed. As in the 2-dimensional case Proposition 1.1, we consider a sequence of local normalized quadratic transforms. In general, the integral closure of a Noetherian domain is not necessarily module-finite and potentially not even Noetherian. To ensure good behavior under integral closure, we assume in addition that  $R_0$  is analytically unramified, which is equivalent to the condition that for any essentially finitely generated birational extension  $S$  of  $R_0$ , the integral closure of  $S$  is a finite  $S$ -module. The condition that a normal Noetherian local domain  $R_0$  be analytically unramified is a mild one, satisfied by rings that arise naturally in geometric contexts.

We also impose the condition that  $\mathfrak{m}_n$  is *one-fibered* for  $n \gg 0$ , which means that  $\mathfrak{m}_n$  has precisely one Rees valuation, say  $\text{Rees } \mathfrak{m}_n = \{\nu_n\}$ .

With these additional assumptions, we obtain the main result of this paper:

**Theorem 1.4.** *Let  $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$  be an infinite sequence of normalized local quadratic transforms of analytically unramified Noetherian local domains whose maximal ideals  $\mathfrak{m}_n$  are one-fibered and let  $\nu_n$  denote the unique Rees valuation of  $\mathfrak{m}_n$ . Then the sequence of Rees valuations  $\{\nu_n\}_{n \geq 0}$  converges.*

Every regular local ring is analytically unramified, every local quadratic transform of a regular local ring is itself regular and hence normal, and every regular local ring  $(R, \mathfrak{m})$  is one-fibered with  $\text{Rees } \mathfrak{m} = \{\text{ord}_R\}$ . Thus this result is a direct generalization of the result for sequences of local quadratic transforms of regular local rings.

## 2. Preliminaries

Throughout this section, let  $(R, \mathfrak{m})$  be a Noetherian local domain with quotient field  $K$  except where noted.

The ring  $R$  is *analytically unramified* if its completion  $\hat{R}$  is reduced and *analytically irreducible* if  $\hat{R}$  is a domain. Rees characterizes analytically unramified Noetherian local domains in [11]:  $R$  is analytically unramified if and only if for every finitely generated birational extension  $S$  of  $R$ , the integral closure  $\overline{S}$  of  $S$  is a finite  $S$ -module.

A *local quadratic transform* of  $R$  is a local ring  $(S, \mathfrak{m}_S)$  birationally dominating  $R$  on the blow-up  $\text{Proj } R[\mathfrak{m}t]$  of  $\mathfrak{m}$ . Algebraically, the ring  $S$  is of the form  $R[\frac{\mathfrak{m}}{x}]_{\mathfrak{n}}$  for some  $x \in \mathfrak{m}$  and some maximal ideal  $\mathfrak{n}$  in  $R[\frac{\mathfrak{m}}{x}]$  containing  $\mathfrak{m}$ , where  $\mathfrak{m}_S = \mathfrak{n}S$ .

Analogously, by a *local normalized quadratic transform*, we mean a local ring  $(T, \mathfrak{m}_T)$  birationally dominating  $R$  on the normalized blow-up  $\text{Proj } \overline{R[\mathfrak{m}t]}$  of  $\mathfrak{m}$ . A local normalized quadratic transform  $T$  is a localization at a maximal ideal of the integral closure  $\overline{S}$  of a local quadratic transform  $S$ .

A local quadratic transform  $(S, \mathfrak{m}_S)$  of a Noetherian local domain is a Noetherian local domain, and if  $R$  is regular or analytically unramified, then  $S$  is regular or analytically

unramified, respectively. If  $R$  is analytically unramified, then a local normalized quadratic transform  $T$  is a localization of a finite  $S$ -module, so it is also an analytically unramified Noetherian local domain.

For an ideal  $I$  of  $R$ , the *Rees valuations* of  $I$  are the unique minimal finite set of rank 1 discrete valuations on  $K$  that determine the integral closures of the powers of  $I$ . The Rees valuations of  $I$  correspond to the minimal primes of  $I$  in the normalized blow-up  $\text{Proj } \overline{R[It]}$ .

Every proper nonzero ideal has at least one Rees valuation. An ideal  $I$  of  $R$  is called *one-fibered* if it has precisely one Rees valuation.

Sally shows in [12] that if  $R$  is analytically unramified and admits a one-fibered  $\mathfrak{m}$ -primary ideal, then  $R$  is analytically irreducible, i.e.  $\hat{R}$  is a domain. See [15] and [14] for more on Rees valuations and one-fibered ideals.

If a Noetherian local ring  $R$  is analytically irreducible, then it is formally equidimensional. Thus  $R$  is universally catenary [8, Theorem 31.6].

For an ideal  $I$  in a (not necessarily local) ring  $R$ , denote by  $e_R(I)$  the multiplicity of the ideal  $I$  over the ring  $R$ , namely

$$e_R(I) = \lim_{n \rightarrow \infty} \frac{\lambda_R(R/I^n)}{d!},$$

where  $\lambda_R(-)$  denotes length as an  $R$ -module and  $d = \dim R$ . For the local ring  $(R, \mathfrak{m})$ , denote  $e(R) = e_R(\mathfrak{m})$ .

We recall necessary formulas for multiplicity.

**Proposition 2.1.** [15, Theorem 11.2.7] *Let  $S$  be a module-finite birational extension of  $R$  and let  $\mathfrak{q} \subset R$  be an  $\mathfrak{m}$ -primary ideal. Then*

$$e_R(\mathfrak{q}) = \sum_{\substack{\mathfrak{n} \in \text{Max } S \\ \dim S_{\mathfrak{n}} = \dim R}} e_{S_{\mathfrak{n}}}(\mathfrak{q}S_{\mathfrak{n}})[S/\mathfrak{n} : R/\mathfrak{m}].$$

Under the notation of the previous proposition, let  $\mathfrak{n}$  denote a maximal ideal of  $S$  with  $\dim S_{\mathfrak{n}} = \dim R$ . Since  $\mathfrak{m}S_{\mathfrak{n}} \subset \mathfrak{n}S_{\mathfrak{n}}$ , it follows that  $e_{S_{\mathfrak{n}}}(\mathfrak{m}S_{\mathfrak{n}}) \geq e(S_{\mathfrak{n}})$ . Since  $e(R) \geq e_{S_{\mathfrak{n}}}(\mathfrak{m}S_{\mathfrak{n}})$  by the proposition, it follows that  $e(R) \geq e(S_{\mathfrak{n}})$ . We record this statement in a corollary.

**Corollary 2.2.** *If  $S$  is a localization at a maximal ideal of a module-finite birational extension of  $R$  with  $\dim S = \dim R$ , then  $e(S) \leq e(R)$ .*

The multiplicity of the Noetherian local domain is a coarse measure to how far it deviates from being regular; if the ring  $R$  is unmixed, then  $R$  is regular if and only if  $e(R) = 1$  [9, Theorem 40.6]. One may attempt to improve the multiplicity of singularities by blowing them up. Though multiplicity need not strictly decrease in general even after an arbitrary number of blow-ups, Bennett [2, Theorem (0)] and Hironaka [6, Theorem I]

showed that for permissible blow-ups, multiplicity cannot increase. We state the special case of this theorem that we'll apply in the proof of the main theorem:

**Proposition 2.3.** *Let  $(R', \mathfrak{m}')$  be a local quadratic transform of  $(R, \mathfrak{m})$  with  $\dim R' = \dim R$ . Then  $e(R) \geq e(R')$ .*

Let  $\mathfrak{p}$  be a height 1 prime ideal of  $R$ , let  $A = R[\frac{\mathfrak{m}}{x}]$  be an affine chart of the blow-up  $\text{Proj } R[\mathfrak{m}t]$ , and consider a height 1 prime ideal  $\mathfrak{q}$  of  $A$  lying over a height 1 prime ideal  $\mathfrak{p}$  of  $R$ . Notice that  $x \notin \mathfrak{p}$ , since  $xA = \mathfrak{m}A$  lies over  $\mathfrak{m}$ . Since  $A$  is birational over  $R$  and  $R_{\mathfrak{p}}$  is a DVR, it follows that  $A_{\mathfrak{q}} = R_{\mathfrak{p}}$ , so by permutability of localization and residue class formation,  $A/\mathfrak{q}$  is birational over  $R/\mathfrak{p}$ . Let  $\mathfrak{m} = (x, y_1, \dots, y_r)$  and let  $R[T_1, \dots, T_r] \twoheadrightarrow A$  be a presentation, where  $T_i$  maps to  $\frac{y_i}{x}$ . Then consider the composition with the quotient map  $A \twoheadrightarrow A/\mathfrak{q}$ , call this composition  $\phi$ . Certainly  $\mathfrak{p}R[T_1, \dots, T_r] \subset \ker \phi$ , so  $\phi$  induces a map  $(R/\mathfrak{p})[T_1, \dots, T_r] \twoheadrightarrow A/\mathfrak{q}$ . Moreover, the image of  $T_i$  in the domain  $A/\mathfrak{q}$  satisfies the relation  $\phi(T_i)\overline{x} = \overline{y_i}$ . Since  $\overline{x} \neq 0$ , it follows that  $\phi(T_i) = \frac{\overline{y_i}}{\overline{x}}$  in the quotient field of  $A/\mathfrak{q}$ , which equals the quotient field of  $R/\mathfrak{p}$ . We conclude that  $\phi$  is a presentation of the affine chart of the blow-up of the maximal ideal  $\overline{\mathfrak{m}}$  of  $R/\mathfrak{p}$ , and in particular,  $A/\mathfrak{q}$  is an affine chart of the blow-up of  $\overline{\mathfrak{m}}$  of  $R/\mathfrak{p}$ .

Let  $(R', \mathfrak{m}')$  be a local quadratic transform of  $R$  with  $\dim R = \dim R'$  and let  $\mathfrak{p}'$  be a height 1 prime ideal of  $R'$  that lies over a height 1 prime  $\mathfrak{p}$  of  $R$ . By the preceding paragraph, it follows that  $R'/\mathfrak{p}'$  is a local quadratic transform of  $R/\mathfrak{p}$  (cf. [3, Corollary 7.15]). Assume in addition that  $R$  is analytically unramified and universally catenary (for instance, if  $\mathfrak{m}$  is one-fibered) and let  $(R_1, \mathfrak{m}_1)$  be a local normalized quadratic transform of  $R$  such that  $R_1$  dominates  $R'$ . Let  $\mathfrak{p}_1$  be the unique height 1 prime ideal of  $R_1$  lying over  $\mathfrak{p}$ . Since  $R_1$  is a localization of a finite  $R'$ -module, it follows that  $R_1/\mathfrak{p}_1$  is a localization of a finite  $(R'/\mathfrak{p}')$ -module. Since  $R_1$  to  $R'$  is a birational extension and  $(R_1)_{\mathfrak{p}_1}$  is a DVR, it follows that  $(R_1)_{\mathfrak{p}_1} = R'_{\mathfrak{p}'}$ , so once again, by permutability of localization and residue class formation, the extension  $R_1/\mathfrak{p}_1$  to  $R'/\mathfrak{p}'$  is birational. The dimension formula for universally catenary Noetherian domains implies that  $\dim R' = \dim R$ , and since  $R_1$  and  $R'$  are universally catenary,  $\dim R'/\mathfrak{p}' = \dim R_1/\mathfrak{p}_1 = \dim R - 1$ . Then Corollary 2.2 implies that  $e(R_1/\mathfrak{p}_1) \leq e(R/\mathfrak{p})$ . We record this as a lemma to our main theorem.

**Lemma 2.4.** *Let  $R$  be a universally catenary analytically unramified Noetherian local domain. Let  $(R_1, \mathfrak{m}_1)$  be a normalized local quadratic transform of  $(R, \mathfrak{m})$  with  $\dim R_1 = \dim R$  and let  $\mathfrak{p}_1$  be a height 1 prime of  $R_1$  lying over a height 1 prime  $\mathfrak{p}$  of  $R$ . Then*

$$e(R_1/\mathfrak{p}_1) \leq e(R/\mathfrak{p}).$$

For an  $\mathfrak{m}$ -primary ideal  $\mathfrak{q}$ , Rees showed that the degree function of  $\mathfrak{q}$  is completely determined by the Rees valuations of  $\mathfrak{q}$ . We recall the definition of the degree function of  $\mathfrak{q}$  and state Rees's result.

**Definition 2.5.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring and let  $\mathfrak{q} \subset R$  be an  $\mathfrak{m}$ -primary ideal. Then the degree function  $d_{\mathfrak{q}}(-) : R \setminus \{0\} \rightarrow \mathbb{Z}$  of  $\mathfrak{q}$  is defined by

$$d_{\mathfrak{q}}(x) = e_{R/xR}((\mathfrak{q} + xR)/xR).$$

**Theorem 2.6** ([10, Theorem 2.3]). Let  $(R, \mathfrak{m})$  be a Noetherian local domain, let  $\mathfrak{q} \subset R$  be an  $\mathfrak{m}$ -primary ideal, and let  $\text{Rees } \mathfrak{q} = \{\nu_1, \dots, \nu_r\}$ . Then there are positive integers  $d_1, \dots, d_r$  such that

$$d_{\mathfrak{q}}(x) = \sum_{i=1}^r d_i \nu_i(x).$$

If in addition  $R$  is normal, then by the associative formula for multiplicity, the divisor of  $xR$  determines the degree function  $d_{\mathfrak{q}}(x)$ :

**Proposition 2.7** ([10, p. 5]). Let  $(R, \mathfrak{m})$  be an integrally closed local domain, let  $\mathfrak{q} \subset R$  be an  $\mathfrak{m}$ -primary ideal, let  $x \in R$  be nonzero, and let

$$xR = \mathfrak{p}_1^{(d_1)} \cdots \mathfrak{p}_r^{(d_r)}$$

be a primary decomposition of  $xR$ . Then

$$d_{\mathfrak{q}}(x) = \sum_{i=1}^r d_i d_{\mathfrak{q}}(\mathfrak{p}_i),$$

where

$$d_{\mathfrak{q}}(\mathfrak{p}_i) = e_{R/\mathfrak{p}_i}((\mathfrak{q} + \mathfrak{p}_i)/\mathfrak{p}_i).$$

### 3. Main result

We fix the following notation:

**Setting 3.1.** Let  $(R_0, \mathfrak{m}_0)$  be an analytically unramified normal Noetherian local domain, and for all  $n \geq 0$ , let  $(R_{n+1}, \mathfrak{m}_{n+1})$  be a local normalized quadratic transform of  $(R_n, \mathfrak{m}_n)$ . By replacing  $R_0$  with  $R_n$  for some sufficiently large  $n$ , assume that  $d := \dim R_0 = \dim R_n$  for all  $n \geq 0$ .

**Theorem 3.2.** Assume the notation of Setting 3.1. If  $\mathfrak{m}_n$  is one-fibered for  $n \gg 0$ , then the corresponding sequence of Rees valuations converges.

**Proof.** Replace  $R_0$  with  $R_n$  for a sufficiently large  $n$  to assume that  $\mathfrak{m}_n$  is one-fibered for all  $n \geq 0$ . Since  $\mathfrak{m}_n$  is one-fibered, it follows that  $R_n$  is analytically irreducible and

hence universally catenary. Let  $\nu_n$  denote the unique Rees valuation of  $\mathfrak{m}_n$  and denote by  $d_n(-)$  the degree function of  $\mathfrak{m}_n$  on  $R_n$ . By Theorem 2.6, there is an integer constant  $c_n > 0$  such that  $d_n(x) = c_n \nu_n(x)$  for all  $x \in R_n$ .

Let  $f, g \in R_0$ . We aim to show that the sign of  $\nu_n(f) - \nu_n(g)$  is constant for  $n \gg 0$ .

Write primary decompositions for  $fR_n$  and  $gR_n$ , say

$$fR_n = \bigcap_{i=1}^r \mathfrak{p}_i^{(\alpha_i)}, \quad gR_n = \bigcap_{i=1}^s \mathfrak{q}_i^{(\beta_i)},$$

where the  $\mathfrak{p}_i$  and  $\mathfrak{q}_j$  are height 1 primes of  $R_n$ . Re-arrange the  $\mathfrak{p}_i$  and  $\mathfrak{q}_j$  such that  $\mathfrak{p}_i = \mathfrak{q}_i$  for  $i \leq t$  and  $\mathfrak{p}_i \neq \mathfrak{q}_j$  for  $i, j > t$ .

Then, as in Proposition 2.7,

$$c_n \nu_n(f) = d_n(f) = \sum_{i=1}^r \alpha_i d_n(\mathfrak{p}_i), \quad c_n \nu_n(g) = d_n(g) = \sum_{i=1}^s \beta_i d_n(\mathfrak{q}_i).$$

Then there is a “common part” of  $f, g$ , namely

$$D_n = \sum_{1 \leq i \leq t} \min\{\alpha_i, \beta_i\} d_n(\mathfrak{p}_i).$$

Denote  $a_n$  and  $b_n$  to be the reduced sums,

$$\begin{aligned} a_n &= c_n \nu_n(f) - D_n = \sum_{i=1}^t (\alpha_i - \min\{\alpha_i, \beta_i\}) d_n(\mathfrak{p}_i) + \sum_{i=t+1}^r \alpha_i d_n(\mathfrak{p}_i), \\ b_n &= c_n \nu_n(g) - D_n = \sum_{i=1}^t (\beta_i - \min\{\alpha_i, \beta_i\}) d_n(\mathfrak{p}_i) + \sum_{i=t+1}^s \beta_i d_n(\mathfrak{q}_i). \end{aligned}$$

Suppose without loss of generality that  $\nu_n(f) \leq \nu_n(g)$ , or equivalently, that  $a_n \leq b_n$ . We claim that  $a_{n+1} \leq a_n$ .

To see this claim, consider the primary decompositions of  $fR_{n+1}$  and  $gR_{n+1}$ . Denote by  $\mathfrak{d}$  the center of the Rees valuation  $\nu_n$  on  $R_{n+1}$ . Denote  $\mathfrak{p}'_i$  ( $\mathfrak{q}'_i$ ) to be the unique height 1 prime of  $R_{n+1}$  lying over  $\mathfrak{p}_i$  ( $\mathfrak{q}_i$ ) if it exists and the ring  $R_{n+1}$  itself otherwise. Then

$$fR_{n+1} = \mathfrak{d}^{\nu_n(f)} \cap \bigcap_{i=1}^r (\mathfrak{p}'_i)^{(\alpha_i)}, \quad gR_{n+1} = \mathfrak{d}^{\nu_n(g)} \cap \bigcap_{i=1}^s (\mathfrak{q}'_i)^{(\beta_i)}.$$

And again,

$$c_{n+1} \nu_{n+1}(f) = \nu_n(f) d_{n+1}(\mathfrak{d}) + \sum_{i=1}^r \alpha_i d_{n+1}(\mathfrak{p}'_i),$$

$$c_{n+1}\nu_{n+1}(g) = \nu_n(g)d_{n+1}(\mathfrak{d}) + \sum_{i=1}^s \beta_i d_{n+1}(\mathfrak{q}'_i),$$

where  $d_{n+1}(R_{n+1}) = 0$ .

Lemma 2.4 implies that  $d_{n+1}(\mathfrak{p}'_i) \leq d_n(\mathfrak{p}_i)$  and  $d_{n+1}(\mathfrak{q}'_i) \leq d_n(\mathfrak{q}_i)$  for all  $i$ . By assumption,  $\nu_n(f) \leq \nu_n(g)$ , so  $\nu_n(f) - \min\{\nu_n(f), \nu_n(g)\} = 0$ , so the term  $(\nu_n(f) - \min\{\nu_n(f), \nu_n(g)\})d_{n+1}(\mathfrak{d})$  vanishes in the expression for  $a_{n+1}$ . Thus

$$\begin{aligned} a_{n+1} &= \sum_{i=1}^t (\alpha_i - \min\{\alpha_i, \beta_i\})d_{n+1}(\mathfrak{p}'_i) + \sum_{i=t+1}^r \alpha_i d_{n+1}(\mathfrak{p}'_i) \\ &\leq \sum_{i=1}^t (\alpha_i - \min\{\alpha_i, \beta_i\})d_n(\mathfrak{p}_i) + \sum_{i=t+1}^r \alpha_i d_n(\mathfrak{p}_i) \\ &= a_n \end{aligned}$$

Thus we have shown that  $\{\min\{a_n, b_n\}\}_{n \geq 0}$  is a decreasing sequence of nonnegative integers, so it stabilizes to some nonnegative integer value for  $n \gg 0$ , say for  $n \geq N$ .

Suppose without loss of generality that  $a_N \leq b_N$ . Then  $a_n = a_N$  and  $a_n \leq b_n$  for all  $n \geq N$ . If  $a_m = b_m$  for any  $m \geq N$ , then  $b_n = b_m$  for all  $n \geq m$ , so  $a_n = b_n$  for all  $n \gg 0$ . In any case, the sign of  $a_n - b_n$  is eventually constant, hence the sign of  $v_n(f) - v_n(g)$  is eventually constant.  $\square$

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