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An extension of $U(\mathfrak{gl}_n)$ related to the alternating group and Galois orders

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ABSTRACT

In 2010, V. Futorny and S. Ovsienko gave a realization of $U(\mathfrak{gl}_n)$ as a subalgebra of the ring of invariants of a certain noncommutative ring with respect to the action of $S_1 \times S_2 \times \cdots \times S_n$, where S_j is the symmetric group on j variables. An interesting question is what a similar algebra would be in the invariant ring with respect to a product of alternating groups. In this paper we define such an algebra, denoted $\mathscr{A}(\mathfrak{gl}_n)$, and show that it is a Galois ring. For $n = 2$, we show that it is a generalized Weyl algebra, and for $n = 3$ provide generators and a list of verified relations. We also discuss some techniques to construct Galois orders from Galois rings. Additionally, we study categories of finite-dimensional modules and generic Gelfand-Tsetlin modules over $\mathscr{A}(\mathfrak{gl}_n)$. Finally, we discuss connections between the Gelfand-Kirillov Conjecture, $\mathscr{A}(\mathfrak{gl}_n)$, and the positive solution to Noether's problem for the alternating group.

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1. Introduction

The study of algebra-subalgebra pairs is an important technique used in the representation theory of Lie algebras [16], [2]. In 2010, Futorny and Ovsienko focused on so called semicommutative pairs $\Gamma \subset \mathscr{U}$, where $\mathscr{U}$ is an associative (noncommutative) $\mathbb{C}$-algebra and Γ is an integral domain [7]. This situation generalizes the pair $(\Gamma, U(\mathfrak{gl}_n))$ where Γ is the *Gelfand-Tsetlin* subalgebra $\Gamma = \mathbb{C}\langle \cup_{k=1}^n Z(U(\mathfrak{gl}_k))\rangle$ [12], [2]. *Galois rings* and *Galois orders* were originally defined and studied by Futorny and Ovsienko in [7] and [8]. They form a collection of algebras that contains many important examples including: *generalized Weyl algebras* defined by independently by Bavula [1] and Rosenberg [19] in the early nineties, the universal enveloping algebra of $\mathfrak{gl}_n$, shifted Yangians and finite W-algebras [6], Coulomb branches [21], and $U_q(\mathfrak{gl}_n)$ [5]. Their structures and representations have been studied in [4], [9], [14], and [18].

In [7], Futorny and Ovsienko described $U(\mathfrak{gl}_n)$ as the subalgebra of the ring of invariants of a certain noncommutative ring with respect to the action of $S_1 \times S_2 \times \cdots \times S_n$, where S_j is the symmetric group on j variables such that $U(\mathfrak{gl}_n)$ was a Galois order with respect to its Gelfand-Tsetlin subalgebra Γ.

We recall in Galois theory, given a Galois extension L/K with $\mathrm{Gal}(L/K) = G$ the subgroups $\widetilde{G}$ of G correspond to intermediate fields $\widetilde{K}$ with $\mathrm{Gal}(L/\widetilde{K}) = \widetilde{G}$ with normal subgroups of particular interest. Since S_n has only one normal subgroup for $n \geq 5$, one might wonder what the object similar to $U(\mathfrak{gl}_n)$ would be if we considered the invariants with respect to the normal subgroup $A_1 \times A_2 \times \cdots \times A_n$, where A_j is the alternating group on j variables. This paper describes such an algebra, denoted by $\mathscr{A}(\mathfrak{gl}_n)$ (see Definition 2.1). This provides the first natural example of a Galois ring whose ring Γ is not a semi-Laurent polynomial ring, that is, a tensor product of polynomial rings and Laurent polynomial rings. Additionally, our symmetry group $A_1 \times A_2 \times \cdots \times A_n$ is not a complex reflection group. Our algebra $\mathscr{A}(\mathfrak{gl}_n)$ is an extension of $U(\mathfrak{gl}_n)$ by $n-1$ elements $\mathcal{V}_2, \ldots, \mathcal{V}_n$. In Proposition 2.2, we prove some properties of $\mathscr{A}(\mathfrak{gl}_n)$ that are quite similar to $U(\mathfrak{gl}_n)$. For example, it is shown that the "Weyl Group" of $\mathscr{A}(\mathfrak{gl}_n)$ is the alternating group A_n, in the sense that there is a natural extension $\widetilde{\varphi}_{\mathrm{HC}}$ of the Harish-Chandra homomorphism $\varphi_{\mathrm{HC}}\colon Z(U(\mathfrak{gl}_n)) \to S(\mathfrak{h}) \cong \mathbb{C}[x_1, \ldots, x_n]$, such that

$$\widetilde{\varphi}_{\mathrm{HC}}\colon Z(\mathscr{A}(\mathfrak{gl}_n)) \xrightarrow{\cong} \mathbb{C}[x_1, \ldots, x_n]^{A_n}.$$

Moreover, there is a chain of subalgebras $\mathscr{A}(\mathfrak{gl}_1) \subset \mathscr{A}(\mathfrak{gl}_2) \subset \cdots \subset \mathscr{A}(\mathfrak{gl}_n)$. In Section 3, we give multiple descriptions of $\mathscr{A}(\mathfrak{gl}_2)$ and prove it is realizable as a Galois order. Example 4.2 shows that $\mathscr{A}(\mathfrak{gl}_n)$ is not a Galois order for $n \geq 3$. The rest of Section 4 lists a set of generators and some verified relations for $\mathscr{A}(\mathfrak{gl}_3)$, but this list may be incomplete. In Section 5, we show that the category of finite-dimensional modules in not semi-simple, that is we show the existence of non-simple irreducible module, and classify simple finite-dimensional weight modules. In Section 6, we provide a technique to turn a general Galois ring into a Galois order that is related to localization (see

Theorem 6.2). We use this to prove that a family of simple examples are Galois orders (see Corollary 6.8) and that a localization of $\mathscr{A}(\mathfrak{gl}_n)$ is a (co-)principal Galois order over the localized $\widetilde{\Gamma}$ (see Definition 1.13 and Corollary 6.11). We use this localization to construct canonical Gelfand-Tsetlin modules over $\mathscr{A}(\mathfrak{gl}_n)$ in Section 7. Finally, in Section 8, we compute the division ring of fractions and prove, that for $n \leq 5$, $\mathscr{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov conjecture (see [13]). For the latter, we use Maeda's positive solution to Noether's problem for the alternating group A_5 [17], and Futorny-Schwarz's Theorem 1.1 in [10].

1.1. Galois orders

Galois orders were introduced in [7]. We will be following the set up from [14]. Let Λ be an integrally closed domain, G a finite subgroup of $\mathrm{Aut}(\Lambda)$, and $\mathscr{M}$ a submonoid of $\mathrm{Aut}(\Lambda)$. We will adhere to the following assumptions for the entire paper:

$$\begin{aligned} &\text{(A1)} \quad (\mathscr{M}\mathscr{M}^{-1}) \cap G = 1_{\mathrm{Aut}\,\Lambda} && (\textit{separation}) \\ &\text{(A2)} \quad \forall g \in G, \forall \mu \in \mathscr{M} : {}^{g}\mu = g \circ \mu \circ g^{-1} \in \mathscr{M} && (\textit{invariance}) \\ &\text{(A3)} \quad \Lambda \text{ is Noetherian as a module over } \Lambda^G && (\textit{finiteness}) \end{aligned}$$

Let $L = \mathrm{Frac}(\Lambda)$ and $\mathscr{L} = L\#\mathscr{M}$, the skew monoid ring, which is defined as the free left L-module on $\mathscr{M}$ with multiplication given by $a_1\mu_1 \cdot a_2\mu_2 = (a_1\mu_1(a_2))(\mu_1\mu_2)$ for $a_i \in L$ and $\mu_i \in \mathscr{M}$. As G acts on Λ by automorphisms, we can easily extend this action to L, and by (A2), G acts on $\mathscr{L}$. So we consider the following G-invariant subrings $\Gamma = \Lambda^G$, $K = L^G$, and $\mathscr{K} = \mathscr{L}^G$.

A benefit of these assumptions is the following lemma.

Lemma 1.1 *([14], Lemma 2.1 (ii), (iv) & (v)).*

(i) $K = \mathrm{Frac}(\Gamma)$.
(ii) Λ *is the integral closure of* Γ *in* L.
(iii) Λ *is a finitely generated* Γ*-module and a Noetherian ring.*

What follows are some definitions and propositions from [7].

Definition 1.2 *([7]).* A finitely generated Γ-subring $\mathscr{U} \subseteq \mathscr{K}$ is called a *Galois* Γ*-ring* (or *Galois ring with respect to* Γ) if $K\mathscr{U} = \mathscr{U}K = \mathscr{K}$.

Definition 1.3. Let $u \in \mathscr{L}$ such that $u = \sum_{\mu\in\mathscr{M}} a_\mu \mu$. The *support of* u *over* $\mathscr{M}$ is the following:

$$\operatorname{supp} u = \left\{ \mu \in \mathscr{M} \;\middle|\; a_\mu \neq 0 \text{ for } u = \sum_{\mu\in\mathscr{M}} a_\mu \mu \right\}$$

Proposition 1.4 *([7], Proposition 4.1). Assume a* Γ*-ring* $\mathscr{U} \subseteq \mathscr{K}$ *is generated by* $u_1, \ldots, u_k \in \mathscr{U}$*.*

(1) *If* $\bigcup_{i=1}^{k} \operatorname{supp} u_i$ *generate* $\mathscr{M}$ *as a monoid, then* $\mathscr{U}$ *is a Galois ring.*
(2) *If* $L\mathscr{U} = L\#\mathscr{M}$*, then* $\mathscr{U}$ *is a Galois ring.*

Theorem 1.5 *([7], Theorem 4.1 (4)). Let* $\mathscr{U}$ *be a Galois* Γ*-ring. Then the center* $Z(\mathscr{U})$ *of the algebra* $\mathscr{U}$ *equals* $\mathscr{U} \cap K^{\mathscr{M}}$*, where* $K^{\mathscr{M}} = \{k \in K \mid \mu(k) = k \ \forall \mu \in \mathscr{M}\}$

Definition 1.6 *([7]).* A Galois Γ-ring $\mathscr{U}$ in $\mathscr{K}$ is a *left* (respectively *right*) *Galois* Γ*-order in* $\mathscr{K}$ if for any finite-dimensional left (respectively right) K-subspace $W \subseteq \mathscr{K}$, $W \cap \mathscr{U}$ is a finitely generated left (respectively right) Γ-module. A Galois Γ-ring $\mathscr{U}$ in $\mathscr{K}$ is a *Galois* Γ*-order in* $\mathscr{K}$ if $\mathscr{U}$ is a left and right Galois Γ-order in $\mathscr{K}$.

Definition 1.7 *([2]).* Let $\Gamma \subset \mathscr{U}$ be a commutative subalgebra. Γ is called a *Harish-Chandra subalgebra* in $\mathscr{U}$ if for any $u \in \mathscr{U}$, $\Gamma u \Gamma$ is finitely generated as both a left and as a right Γ-module.

Let $\mathscr{U}$ be a Galois ring and $e \in \mathscr{M}$ the unit element. We denote $\mathscr{U}_e = \mathscr{U} \cap Le$.

Theorem 1.8 *([7], Theorem 5.2). Assume that* $\mathscr{U}$ *is a Galois ring,* Γ *is finitely generated and* $\mathscr{M}$ *is a group.*

(1) *Let* $m \in \mathscr{M}$*. Assume* $m^{-1}(\Gamma) \subseteq \Lambda$ *(respectively* $m(\Gamma) \subseteq \Lambda$*). Then* $\mathscr{U}$ *is right (respectively left) Galois order if and only if* $\mathscr{U}_e$ *is an integral extension of* Γ*.*
(2) *Assume that* Γ *is a Harish-Chandra subalgebra in* $\mathscr{U}$*. Then* $\mathscr{U}$ *is a Galois order if and only if* $\mathscr{U}_e$ *is an integral extension of* Γ*.*

The following are some useful results from [14].

Proposition 1.9 *([14], Proposition 2.14).* Γ *is maximal commutative in any left or right Galois* Γ*-order* $\mathscr{U}$ *in* $\mathscr{K}$*.*

Lemma 1.10 *([14], Lemma 2.16). Let* $\mathscr{U}_1$ *and* $\mathscr{U}_2$ *be two Galois* Γ*-rings in* $\mathscr{K}$ *such that* $\mathscr{U}_1 \subseteq \mathscr{U}_2$*. If* $\mathscr{U}_2$ *is a Galois* Γ*-order, then so too is* $\mathscr{U}_1$*.*

It is common to write elements of L on the right side of elements of $\mathscr{M}$.

Definition 1.11. For $X = \sum_{\mu \in \mathscr{M}} \mu \alpha_\mu \in \mathscr{L}$ and $a \in L$ defines the *evaluation of* X *at* a to be

$$X(a) = \sum_{\mu \in \mathscr{M}} \mu(\alpha_\mu \cdot a) \in L.$$

Similarly defined is *co-evaluation* by

$$X^{\dagger}(a) = \sum_{\mu \in \mathscr{M}} \alpha_{\mu} \cdot (\mu^{-1}(a)) \in L$$

The following was independently defined by [20] called the *universal ring*.

Definition 1.12. The *standard Galois* Γ*-order* is as follows:

$$\mathscr{K}_{\Gamma} := \{X \in \mathscr{K} \mid X(\gamma) \in \Gamma \ \forall \gamma \in \Gamma\}.$$

Similarly we define the *co-standard Galois* Γ*-order* by

$$_{\Gamma}\mathscr{K} := \{X \in \mathscr{K} \mid X^{\dagger}(\gamma) \in \Gamma \ \forall \gamma \in \Gamma\}.$$

Definition 1.13. Let $\mathscr{U}$ be a Galois Γ-ring in $\mathscr{K}$. If $\mathscr{U} \subseteq \mathscr{K}_{\Gamma}$ (resp. $\mathscr{U} \subseteq {}_{\Gamma}\mathscr{K}$), then $\mathscr{U}$ is called a *principal* (resp. *co-principal*) *Galois* Γ*-order*.

In [14] it was shown that any (co-)principal Galois Γ-order is a Galois order in the sense of Definition 1.6.

2. Defining the extension

2.1. Galois order realization of $U(\mathfrak{gl}_n)$

We first recall the realization of $U(\mathfrak{gl}_n)$ as a Galois Γ-order from [7]. Let $\Lambda = \mathbb{C}[x_{ki} \mid 1 \leq i \leq k \leq n]$ the polynomial algebra in indeterminates x_{ki}, $\mathbb{S}_n = S_1 \times S_2 \times \cdots \times S_n$, and $\Gamma = \Lambda^{\mathbb{S}_n} = \mathbb{C}[e_{ki} \mid 1 \leq i \leq k \leq n]$. Here

$$e_{ki} = e_{ki}(x_{k1}, \ldots, x_{kk}) = \sum_{1 \leq j_1 < \cdots < j_i \leq k} x_{kj_1} \cdots x_{kj_i} \tag{1}$$

are the elementary symmetric polynomials. Also, let $L = \mathrm{Frac}(\Lambda)$ and $K = \mathrm{Frac}(\Gamma)$. Now, we construct a skew monoid ring. Let $\mathscr{M}$ be the subgroup of $\mathrm{Aut}(\Lambda)$ generated by $\{\delta^{ki}\}_{1 \leq i \leq k \leq n-1}$, where δ^{ki} is defined by

$$\delta^{ki}(x_{\ell j}) = x_{\ell j} - \delta_{\ell k}\delta_{ij}. \tag{2}$$

We observe that $\mathscr{M} \cong \mathbb{Z}^{n(n-1)/2}$. Let $\mathscr{L} = L\#\mathscr{M}$ and $\mathscr{K} = (L\#\mathscr{M})^{\mathbb{S}_n}$. In [7], the authors describe an embedding $\varphi\colon U(\mathfrak{gl}_n) \to \mathscr{K}$ defined by

$$\varphi(E_k^{\pm}) = \sum_{i=1}^{k} (\delta^{ki})^{\pm 1} a_{ki}^{\pm}, \qquad \varphi(E_{kk}) = \sum_{j=1}^{k} (x_{kj} + j - 1) - \sum_{i=1}^{k-1} (x_{k-1,i} + i - 1), \tag{3}$$

where

$$a_{ki}^{\pm} = \mp \frac{\prod_{j=1}^{k\pm 1}(x_{k\pm 1,j} - x_{ki})}{\prod_{j\neq i}(x_{kj} - x_{ki})}, \tag{4}$$

and $E_k^+ = E_{k,k+1}$, $E_k^- = E_{k+1,k}$ where E_{ij} denotes the matrix units. Let $U_n = \varphi(U(\mathfrak{gl}_n))$. The algebra U_n is a Galois Γ-order.

2.2. Defining $\mathscr{A}(\mathfrak{gl}_n)$

Let $\mathbb{A}_n = A_1 \times A_2 \times \cdots \times A_n$ and

$$\widetilde{\Gamma} = \Lambda^{\mathbb{A}_n} = \mathbb{C}[e_{ki}, \mathcal{V}_\ell \mid 1 \leq i \leq k \leq n, 2 \leq \ell \leq n]. \tag{5}$$

Here

$$\mathcal{V}_\ell = \mathcal{V}_\ell(x_{\ell 1}, \ldots, x_{\ell\ell}) = \prod_{i<j}(x_{\ell i} - x_{\ell j}) \tag{6}$$

denotes the Vandermonde polynomial in the ℓ variables $x_{\ell 1}, \ldots, x_{\ell\ell}$. Abstractly, $\widetilde{\Gamma}$ is isomorphic to the following quotient

$$\mathbb{C}[T_{ki}, V_\ell \mid 1 \leq i \leq k \leq n, 2 \leq \ell \leq n]/(V_\ell^2 - D_\ell(T_{\ell 1}, \ldots, T_{\ell\ell}) \mid 2 \leq \ell \leq n),$$

where T_{ki}, V_ℓ are indeterminates and $D_\ell(T_{\ell 1}, \ldots, T_{\ell\ell})$ is the Vandermonde discriminant. Also, let $\widetilde{K} = \mathrm{Frac}(\widetilde{\Gamma})$ and $\widetilde{\mathscr{K}} = (L\#\mathscr{M})^{\mathbb{A}_n}$.

Definition 2.1. Consider the following extension of $U(\mathfrak{gl}_n)$, denoted $\mathscr{A}(\mathfrak{gl}_n)$, defined as the subalgebra of $\widetilde{\mathscr{K}}$ generated by $U_n \cup \{\mathcal{V}_2, \mathcal{V}_3, \cdots, \mathcal{V}_n\}$. Explicitly, $\mathscr{A}(\mathfrak{gl}_n)$ is the subalgebra of $\mathscr{L}$ generated by

$$\begin{array}{ll} X_k^{\pm} = \sum_{i=1}^{k}(\delta^{ki})^{\pm 1} a_{ki}^{\pm} & \text{for } k = 1, \ldots, n-1, \\ X_{kk} = \sum_{j=1}^{k}(x_{kj} + j - 1) - \sum_{i=1}^{k-1}(x_{k-1,i} + i - 1) & \text{for } k = 1, \ldots, n, \\ \mathcal{V}_k = \mathcal{V}_k(x_{k1}, \ldots, x_{kk}) = \prod_{i<j}(x_{ki} - x_{kj}) & \text{for } k = 1, \ldots, n-1, \end{array}$$

where $a_{ki}^{\pm}$ are defined in (4).

The following proposition lists some basic properties of $\mathscr{A}(\mathfrak{gl}_n)$.

Proposition 2.2.

(i) *$U(\mathfrak{gl}_n) \cong U_n \subset \mathscr{A}(\mathfrak{gl}_n)$.*
(ii) *$\mathscr{A}(\mathfrak{gl}_n)$ is a Galois $\widetilde{\Gamma}$-ring.*

(iii) $\mathcal{V}_n$ *is central in* $\mathscr{A}(\mathfrak{gl}_n)$.
(iv) $Z(\mathscr{A}(\mathfrak{gl}_n)) \cong \mathbb{C}[x_1, \dots, x_n]^{A_n}$.
(v) *There is a chain of subalgebras* $\mathscr{A}(\mathfrak{gl}_1) \subset \mathscr{A}(\mathfrak{gl}_2) \subset \cdots \subset \mathscr{A}(\mathfrak{gl}_n)$.
(vi) $\mathscr{A}(\mathfrak{gl}_n)$ *is the minimal extension of* $U(\mathfrak{gl}_n)$ *with properties (iv) and (v).*

Proof. (i) Clear because φ is injective and $\mathscr{A}(\mathfrak{gl}_n)$ contains $\varphi(E_k^{\pm}), \varphi(E_k k)$.
(ii) Define $\mathscr{X}$ as follows:

$$\mathscr{X} = \{X_i^{\pm}, X_{ii}, \mathcal{V}_j \mid 1 \le i \le n, 2 \le j \le n\}.$$

Since $X_i^{\pm} \in \mathscr{X}$, it is clear that $\bigcup_{u \in \mathscr{X}} \operatorname{supp} u$ generates $\mathscr{M}$. Thus, $\mathscr{A}(\mathfrak{gl}_n)$ is a Galois $\widetilde{\Gamma}$-ring for every $n \ge 1$ by Proposition 1.4.
(iii) As δ^{ki} fixes $x_{\ell j}$ iff $\ell \ne k$ and $k \ne n$, it follows that $\mathcal{V}_n$ is central in $\mathscr{A}(\mathfrak{gl}_n)$.
(iv) We first show that $Z(\mathscr{A}(\mathfrak{gl}_n) = \mathbb{C}\langle Z(U_n), \mathcal{V}_n\rangle$. $\mathbb{C}\langle Z(U_n), \mathcal{V}_n)\rangle \subseteq Z(\mathscr{A}(\mathfrak{gl}_n))$ is clear. Next, we observe that $\mathscr{A}(\mathfrak{gl}_n) \subset (L'\#\mathscr{M})^{\mathbb{A}_n}$, where $L' = \mathbb{C}(x_{ki} \mid 1 \le i \le k \le n-1)[x_{n1}, \dots, x_{nn}]$. By Theorem 1.5, we have

$$Z(\mathscr{A}(\mathfrak{gl}_n)) = \mathscr{A}(\mathfrak{gl}_n) \cap \widetilde{K}^{\mathscr{M}} \subseteq (L'\#\mathscr{M})^{\mathbb{A}_n} \cap \widetilde{K}^{\mathscr{M}} \subseteq \mathbb{C}\langle Z(U_n), \mathcal{V}_n\rangle.$$

Consider the Harish-Chandra homomorphism $\varphi_{\mathrm{HC}}\colon Z(U(\mathfrak{gl}_n)) \to \mathbb{C}[x_1, \dots, x_n]^{S_n}$. We can define an extension of this map $\widetilde{\varphi}_{\mathrm{HC}}\colon Z(\mathscr{A}(\mathfrak{gl}_n)) \to \mathbb{C}[x_1, \dots, x_n]$ as follows:

$$\widetilde{\varphi}_{\mathrm{HC}}(X) = \begin{cases} \varphi_{\mathrm{HC}}(\varphi^{-1}(X)), & X \in Z(U_n), \\ \displaystyle\prod_{1=i<j=n} (x_i - x_j), & X = \mathcal{V}_n. \end{cases} \tag{7}$$

In conjunction with Chevalley's Theorem (see [15]), φ_{HC} provides an isomorphism with $\mathbb{C}[x_1, \dots, x_n]^{S_n}$. The claim follows by recalling that $\mathbb{C}[x_1, \dots, x_n]^{A_n}$ is generated by the symmetric polynomials and the Vandermonde polynomial.
(v) Follows from the definition.
(vi) We prove this result by induction on n. Since $\mathscr{A}(\mathfrak{gl}_1) = U(\mathfrak{gl}_1)$, the base step is clear. Assuming the claim holds for $\mathscr{A}(\mathfrak{gl}_{n-1})$, now consider an extension $\mathscr{A}$ of $U(\mathfrak{gl}_n)$ satisfying (iv) and (v). By (v), $\mathscr{A}$ contains $\mathcal{V}_\ell$ for $\ell = 1, \dots, n-1$, and it contains $U(\mathfrak{gl}_n)$ by definition. From (iv) we get an element $\mathcal{V}$ that is central in $\mathscr{A}$ that maps to $\prod_{i<j}(x_i - x_j) \in \mathbb{C}[x_1, \dots, x_n]^{A_n}$. This allows us to define an isomorphism $\tau\colon \mathscr{A} \to \mathscr{A}(\mathfrak{gl}_n)$ by sending $\{U(\mathfrak{gl}_n), \mathcal{V}_\ell \mid \ell = 1, \dots, n-1\}$ to themselves and $\mathcal{V} \mapsto \mathcal{V}_n$. □

Remark 1. In [11] another Galois algebra is described in the invariants of a Weyl algebra with respect to a single alternating group in Corollary 24 in [11].

3. The structure of $\mathscr{A}(\mathfrak{gl}_2)$

In this section, we find a presentation for $\mathscr{A}(\mathfrak{gl}_2)$ as an extension of $U(\mathfrak{gl}_2)$ and as a generalized Weyl algebra as well as prove that it is a Galois $\widetilde{\Gamma}$-order.

Lemma 3.1.

(i) *$\mathcal{V}_2$ commutes with every element of U_2.*
(ii) *$\mathscr{A}(\mathfrak{gl}_2) = U_2 \oplus (U_2 \cdot \mathcal{V}_2)$*

Proof. (i) Follows by Proposition 2.2 (iii).
(ii) Since $\mathcal{V}_2$ commutes with everything in U_2,

$$\mathscr{A}(\mathfrak{gl}_2) = \left\{ \sum_{j=0}^{\infty} u_j \mathcal{V}_2^j \mid u_j \in U_2, \text{ at most finitely many } u_j \neq 0 \right\}.$$

Since $\mathcal{V}_2^2 \in U_2$, $\mathscr{A}(\mathfrak{gl}_2) = U_2 + U_2 \cdot \mathcal{V}_2$. Now consider $(12)_2 := ((1),(12)) \in \mathbb{S}_2$ acting on $\mathscr{L}$ by automorphisms. We have,

$$(12)_2|_{U_2} = \mathrm{Id}\,|_{U_2}, \quad (12)_2|_{U_2 \cdot \mathcal{V}_2} = (-1) \cdot \mathrm{Id}\,|_{U_2 \cdot \mathcal{V}_2}.$$

This implies that $\mathscr{A}(\mathfrak{gl}_2) = U_2 \oplus (U_2 \cdot \mathcal{V}_2)$. $\square$

Definition 3.2. The *k-th Gelfand invariant for $\mathfrak{gl}_n$* is defined as follows

$$c_{nk} = \sum_{(i_1, i_2, \ldots, i_d) \in [n]^d} E_{i_1,i_2} E_{i_2,i_3} \cdots E_{i_{d-1},i_d} E_{i_d,i_1}.$$

There are n such Gelfand invariants for $\mathfrak{gl}_n$, and they generate the center of $U(\mathfrak{gl}_n)$.

We now give a presentation for $\mathscr{A}(\mathfrak{gl}_2)$ in terms of $U(\mathfrak{gl}_2)$.

Proposition 3.3. *There is an isomorphism*

$$\widetilde{\varphi}\colon \frac{U(\mathfrak{gl}_2)[T_2]}{(T_2^2 - (-c_{21}^2 + 2c_{22} + 1))} \to \mathscr{A}(\mathfrak{gl}_2), \tag{8}$$

where T_2 is an indeterminate and c_{2i} are the Gelfand invariants for $\mathfrak{gl}_2$. Explicitly

$$\widetilde{\varphi}|_{U(\mathfrak{gl}_2)} = \varphi, \quad \widetilde{\varphi}(T_2) = \mathcal{V}_2, \tag{9}$$

where φ is the embedding from (3).

Proof. Let $p(T_2) = T_2^2 - (-c_{21}^2 + 2c_{22} - 1) \in U(\mathfrak{gl}_2)[T_2]$. Since $p(T_2)$ is degree two, $U(\mathfrak{gl}_2)[T_2]/(p(T_2))$ is free of rank 2 as a left $U(\mathfrak{gl}_2)$-module with basis $\{1, \overline{T_2}\}$ where $\overline{T_2} = T_2 + (p(T_2))$. It follows Lemma 3.1 (ii) that $\mathscr{A}(gl_2)$ is also free of rank 2 with basis $\{1, \mathcal{V}_2\}$ via the isomorphism φ in (3). Therefore there is an isomorphism $\widetilde{\varphi}\colon U(\mathfrak{gl}_2)[T_2]/(p(T_2)) \to \mathscr{A}(\mathfrak{gl}_2)$ as $U(\mathfrak{gl}_2)$-modules sending 1 to 1 and $\overline{T_2}$ to $\mathcal{V}_2$. Thus, it suffices to show that $\widetilde{\varphi}(\overline{T_2}^2) = \mathcal{V}_2^2$.

To show this, we calculate the images of c_{2i} under φ:

$$\begin{aligned}
\varphi(c_{21}) &= \varphi\big(E_{11} + E_{22}\big) = (x_{11}) + (x_{21} + x_{22} - x_{11} + 1) = x_{21} + x_{22} + 1,\\
\varphi(c_{22}) &= \varphi\big(E_{11}^2 + E_1^+ E_1^- + E_1^- E_1^+ + E_{22}^2\big) = x_{21}^2 + x_{22}^2 + x_{21} + x_{22}.
\end{aligned}$$

As such,

$$\begin{aligned}
\widetilde{\varphi}(\overline{T_2}^2) = \widetilde{\varphi}(-c_{21}^2 + 2c_{22} + 1) &= -\varphi(c_{21})^2 + 2\varphi(c_{22}) + 1\\
&= -(x_{21} + x_{22} + 1)^2 + 2(x_{21}^2 + x_{22}^2 + x_{21} + x_{22}) + 1\\
&= (x_{21} - x_{22})^2 = \mathcal{V}_2^2.
\end{aligned}$$

Therefore, $\widetilde{\varphi}$ is an algebra isomorphism. □

Theorem 3.4. *$\mathscr{A}(\mathfrak{gl}_2)$ is a Galois $\widetilde{\Gamma}$-order.*

Proof. We first observe that $\mathscr{A}(\mathfrak{gl}_2)$ is a Galois $\widetilde{\Gamma}$-ring by Proposition 2.2 (ii). To prove that $\mathscr{A}(\mathfrak{gl}_2)$ is a Galois $\widetilde{\Gamma}$-order, we will use Theorem 1.8. Since Γ is a Harish-Chandra subalgebra of $U(\mathfrak{gl}_2)$, $\widetilde{\Gamma}$ is a Harish-Chandra subalgebra of $\mathscr{A}(\mathfrak{gl}_2)$. Since $\mathbb{A}_2$ is a group, all we need to show is that $\widetilde{\Gamma}$ is maximal commutative in $\mathscr{A}(\mathfrak{gl}_2)$. This is clear because Γ is maximal commutative in U_2, and $\widetilde{\Gamma}$ is just an extension by a central element by Proposition 3.3. $\widetilde{\Gamma}$ is maximal commutative in $\mathscr{A}(\mathfrak{gl}_2)$; therefore, $\mathscr{A}(\mathfrak{gl}_2)$ is a Galois $\widetilde{\Gamma}$-order. □

The following shows that $\mathscr{A}(\mathfrak{gl}_2)$ is a generalized Weyl algebra [1], which gives another way to show it is a Galois order [7]. First we recall the definition of a generalized Weyl algebra.

Definition 3.5 *([1]).* Let D be a ring, σ a ring automorphism of D, and t a central element of D. The *generalized Weyl algebra* of rank 1, $D(\sigma, t)$ is a ring generated by the ring D and two elements x and y subject to the following relations:

$$xd = \sigma(d)x \text{ and } yd = \sigma^{-1}(d)y \text{ for all } d \in D; \tag{10}$$

$$yx = t \text{ and } xy = \sigma(t). \tag{11}$$

Proposition 3.6. *$\mathscr{A}(\mathfrak{gl}_2)$ is isomorphic to the generalized Weyl algebra $\widetilde{\Gamma}(\sigma, t)$, where $\sigma = \delta^{11}$, $t = -e_{22} + e_{11}e_{21} - e_{11}^2$, and $\widetilde{\Gamma}$ is defined in (5).*

Proof. Recall that $\mathscr{A}(\mathfrak{gl}_2)$ is the subalgebra generated by $\widetilde{\Gamma}, X_1^{\pm}$ (see Definition 2.1). We define $\psi\colon \widetilde{\Gamma}(\sigma,t) \to \mathscr{A}(\mathfrak{gl}_2)$ by

$$x \mapsto X_1^+,\ y \mapsto X_1^-, \text{ and } \gamma \mapsto \gamma \text{ for all } \gamma \in \widetilde{\Gamma}.$$

One can verify that the defining relations (10) and (11) are preserved by ψ, making it well-defined. Clearly ψ is surjective. For injectivity, we note ψ is graded, when $\widetilde{\Gamma}(\sigma,t)$ and $\mathscr{A}(\mathfrak{gl}_2)$ are equipped with the $\mathbb{Z}$-gradations determined by

$$\deg X_1^{\pm} = \pm 1,\ \ \deg \gamma = 0\ \forall \gamma \in \widetilde{\Gamma},\ \ \deg x = 1,\ \ \deg y = -1. \tag{12}$$

As such, $\ker \psi$ is a graded ideal. But $\widetilde{\Gamma} \cap (\ker \psi) = 0$. Since the only graded ideal I in a generalized Weyl algebra $D(\sigma,t)$, where D is a domain, with $D \cap I = 0$ is $I = 0$, we get $\ker \psi = 0$. $\square$

We observe the following interesting property of $\mathscr{A}(\mathfrak{gl}_2)$ that we prove does not hold for general n (see Proposition 4.3).

Proposition 3.7. *$\mathscr{A}(\mathfrak{gl}_2)$ has the property that $(\mathscr{A}(\mathfrak{gl}_2))^{\mathbb{S}_2} = U_2$.*

Proof. This becomes clear when we consider the direct sum decomposition shown in Lemma 3.1 (ii). Consider $a + b\mathcal{V}_2 \in \mathscr{A}(\mathfrak{gl}_2)$:

$$\begin{aligned} a + b\mathcal{V}_2 \in (\mathscr{A}(\mathfrak{gl}_2))^{\mathbb{S}_2} &\iff (12)_2(a + b\mathcal{V}_2) = a + b\mathcal{V}_2 \\ &\iff a - b\mathcal{V}_2 = a + b\mathcal{V}_2 \\ &\iff b = 0 \\ &\iff a + b\mathcal{V}_2 = a \in U_2. \end{aligned}$$

Therefore, $(\mathscr{A}(\mathfrak{gl}_2))^{\mathbb{S}_2} = U_2$. $\square$

4. The structure of $\mathscr{A}(\mathfrak{gl}_3)$

Based on the result of the previous section, the next logical step is to see if similar results hold for $\mathfrak{gl}_n$ with $n \geq 3$. We will continue using the notation of the images of the generators of the $U(\mathfrak{gl}_n)$ as before. As such:

$$X_i^{\pm} := \varphi(E_i^{\pm}) \quad \text{and} \quad X_{ii} := \varphi(E_{ii}).$$

4.1. Non-polynomial rational functions in $\mathscr{A}(\mathfrak{gl}_3)$

Unlike in $U(\mathfrak{gl}_3)$ and $\mathscr{A}(\mathfrak{gl}_2)$, we can construct non-polynomial rational functions in $\mathscr{A}(\mathfrak{gl}_3)$. It follows that for $n \geq 3$, $\mathscr{A}(\mathfrak{gl}_n)$ is not a Galois $\widetilde{\Gamma}$-order, and the invariant property of $\mathscr{A}(\mathfrak{gl}_2)$ does not hold.

Lemma 4.1. *The following identity holds in* $\mathscr{A}(\mathfrak{gl}_3)$*:*

$$\pm[X_2^{\pm}, \mathcal{V}_2] = (\delta^{21})^{\pm 1} a_{21}^{\pm} - (\delta^{22})^{\pm 1} a_{22}^{\pm}. \tag{13}$$

Proof. To show this, consider $\mathcal{V}_2 X_2^{\pm}$:

$$\begin{aligned}\mathcal{V}_2 X_2^{\pm} &= (x_{21} - x_{22})((\delta^{21})^{\pm 1} a_{21}^{\pm} + (\delta^{22})^{\pm 1} a_{22}^{\pm}) \\ &= (\delta^{21})^{\pm 1} a_{21}^{\pm}(x_{21} \pm 1 - x_{22}) + (\delta^{22})^{\pm 1} a_{22}^{\pm}(x_{21} - x_{22} \mp 1) \\ &= X_2^{\pm}\mathcal{V}_2 \pm ((\delta^{21})^{\pm 1} a_{21}^{\pm} - (\delta^{22})^{\pm 1} a_{22}^{\pm}).\end{aligned}$$

Therefore, $\pm[X_2^{\pm}, \mathcal{V}_2] = (\delta^{21})^{\pm 1} a_{21}^{\pm} - (\delta^{22})^{\pm 1} a_{22}^{\pm}$. $\square$

Let us denote the element described in (13) by $\widetilde{X}_2^{\pm}$. We define the following:

$$\begin{array}{ll} A_{21}^{+} := \frac{1}{2}(X_2^{+} + \widetilde{X}_2^{+}) = \delta^{21} a_{21}^{+} & A_{21}^{-} := \frac{1}{2}(X_2^{-} + \widetilde{X}_2^{-}) = (\delta^{21})^{-1} a_{21}^{-} \\ A_{22}^{+} := \frac{1}{2}(X_2^{+} - \widetilde{X}_2^{+}) = \delta^{22} a_{22}^{+} & A_{22}^{-} := \frac{1}{2}(X_2^{-} - \widetilde{X}_2^{-}) = (\delta^{22})^{-1} a_{22}^{-} \end{array}$$

By their definition, it is clear that they are in $\mathscr{A}(\mathfrak{gl}_3)$.

The following example shows that if $n \geq 3$, then $\widetilde{\Gamma}$ is not maximal commutative; hence, $\mathscr{A}(\mathfrak{gl}_n)$ is not a Galois $\widetilde{\Gamma}$-order by Proposition 1.9.

Example 4.2. The following element belongs to $\mathscr{A}(\mathfrak{gl}_n)$ for $n \geq 3$:

$$A_{21}^{+} A_{21}^{-} = -\frac{\prod_{i=1}^{3}(x_{3i} - x_{21} + 1)}{(x_{22} - x_{21} + 1)} \cdot \frac{x_{11} - x_{21}}{x_{22} - x_{21}}.$$

This is a rational function; hence, it lies in $\mathrm{Cent}_{\mathscr{A}(\mathfrak{gl}_3)}(\widetilde{\Gamma})$.

The following rather surprising fact shows that the property in Proposition 3.7 does not hold for larger n.

Proposition 4.3. *For* $n \geq 3$, $\mathscr{A}(\mathfrak{gl}_n)^{\mathbb{S}_n} \supsetneq U_n$.

Proof. The fact that $U_n \subset \mathscr{A}(\mathfrak{gl}_n)^{\mathbb{S}_n}$ is obvious by definition. To show the containment is strict, we recall that because U_n is a Galois Γ-order, it is known that $U_n \cap K = \Gamma$. Therefore, we consider $\mathscr{A}(\mathfrak{gl}_n)^{\mathbb{S}_n} \cap K$. Since $U_3 \subseteq U_n$ for every $n \geq 3$, it suffices to show that $\mathscr{A}(\mathfrak{gl}_3)^{\mathbb{S}_3} \cap K \supsetneq \Gamma$.

The object to prove this claim is constructed in the same way as in Example 4.2. It is quickly observed that

$$A_{21}^{+} A_{21}^{-} A_{22}^{+} A_{22}^{-} = \frac{\prod_{i=1}^{3}(x_{3i} - x_{21} + 1)}{(x_{22} - x_{21} + 1)} \cdot \frac{x_{11} - x_{21}}{x_{22} - x_{21}} \cdot \frac{\prod_{i=1}^{3}(x_{3i} - x_{22} + 1)}{(x_{21} - x_{22} + 1)} \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22}}$$

is invariant under the action of $\mathbb{S}_3$. This element is clearly not in Γ, so this element is in $\mathscr{A}(\mathfrak{gl}_3)^{\mathbb{S}_3} \cap K \setminus \Gamma$, thereby proving the claim. $\square$

4.2. Generators and relations for $n = 3$

Based on the previous subsection, we determine a set of generators and some verified relations for $\mathscr{A}(\mathfrak{gl}_3)$. However, we do not know if this constitutes a presentation, that is, this may be an incomplete list.

Proposition 4.4. *The algebra $\mathscr{A}(\mathfrak{gl}_3)$ is generated by $\{X_{11}, X_{22}, X_{33}, A_{11}^{\pm}, A_{21}^{\pm}, A_{22}^{\pm}, \mathcal{V}_2, \mathcal{V}_3\}$, where $A_{ij}^{\pm} := (\delta^{ij})^{\pm 1} a_{ij}^{\pm}$, $\mathcal{V}_2 = x_{21} - x_{22}$, and $\mathcal{V}_3 = \prod_{i<j}(x_{3i} - x_{3j})$. What follows is a list of known relations:*

- (i) $[\mathcal{V}_3, X] = 0$ *for all* $X \in \mathscr{A}(\mathfrak{gl}_3)$ *(i.e. $\mathcal{V}_3$ is central in $\mathscr{A}(\mathfrak{gl}_3)$),*
- (ii) $[X, Y] = 0$ *for all* $X, Y \in \mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{X_{11}, X_{22}, X_{33}, \mathcal{V}_2, \mathcal{V}_3\}$,
- (iii) $[h, A_{ij}^{\pm}] = \pm\alpha_{ij}(h) A_{ij}^{\pm}$ *for all* $h \in \mathfrak{h}$ *and* $1 \leq j \leq i \leq 2$, *where* $\alpha_{ij}(h)$ *are given by the following matrix:*

$$
\begin{array}{c c}
 & \begin{array}{ccccc} X_{11} & X_{22} & X_{33} & \mathcal{V}_2 & \mathcal{V}_3 \end{array} \\
\begin{array}{c} \alpha_{11} \\ \alpha_{21} \\ \alpha_{22} \end{array} & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix}
\end{array},
$$

- (iv) $[A_{21}^{\pm}, A_{22}^{\mp}] = 0$,
- (v) $[A_{11}^{\pm}, A_{2i}^{\mp}] = 0$ *for* $i = 1, 2$,
- (vi) $[A_{11}^{+}, A_{11}^{-}] = X_{11} - X_{22}$,
- (vii) $[A_{21}^{+}, A_{21}^{-}] + [A_{22}^{+}, A_{22}^{-}] = X_{22} - X_{33}$,
- (viii) $[A_{11}^{\pm}, [A_{11}^{\pm}, A_{2i}^{\pm}]] = 0$ *for* $i = 1, 2$,
- (ix) $A_{22}^{\pm} \mathcal{V}_2 A_{21}^{\pm} = A_{21}^{\pm} \mathcal{V}_2 A_{22}^{\pm}$.

Proof. Any of the relations involving only elements from $U(\mathfrak{gl}_3)$ (such as (vi)) follow from $U(\mathfrak{gl}_3)$ relations by recalling that $\{X_{11}, X_{22}, X_{33}, A_{11}^{+}, A_{11}^{-}\} \in \mathscr{A}(\mathfrak{gl}_3)$ correspond to $\{E_{11}, E_{22}, E_{33}, E_{12}, E_{21}\} \in U(\mathfrak{gl}_3)$. All that remains is to prove the relations involving new elements.

(i) This follows from Proposition 2.2 (iii).

(ii) This follows by observing that each is an element of $\widetilde{\Gamma}$ which is a commutative ring.

(iii) By the statement at the beginning of this proof and (i), we only need to check the second two rows and the second to last column. Each is proved in an identical manner, we provide one below:

$$
\begin{aligned}
\mathcal{V}_2 \cdot A_{21}^{+} &= (x_{21} - x_{22}) \cdot -\delta^{21} \frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21}} \\
&= -\delta^{21} \frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21}} \cdot (x_{21} - x_{22} + 1)
\end{aligned}
$$

$$= A_{21}^{+}\mathcal{V}_2 + A_{21}^{+}.$$

Thus, $[\mathcal{V}_2, A_{21}^{+}] = A_{21}^{+} = \alpha_{21}(\mathcal{V}_2)A_{21}^{+}$.
(iv) Consider the following calculation:

$$\begin{aligned}
A_{21}^{+}A_{22}^{-} &= -\delta^{21}\frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21}} \cdot (\delta^{22})^{-1}\frac{x_{11} - x_{22}}{x_{21} - x_{22}} \\
&= -\delta^{21}(\delta^{22})^{-1}\frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21} - 1} \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22}} \\
&= -\delta^{21}(\delta^{22})^{-1}\frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21}} \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22} + 1} \\
&= (\delta^{22})^{-1}\frac{x_{11} - x_{22}}{x_{21} - x_{22}} \cdot -\delta^{21}\frac{\prod_{i=1}^{3} x_{3i} - x_{21}}{x_{22} - x_{21}} \\
&= A_{22}^{-}A_{21}^{+}.
\end{aligned}$$

The other relation is proved similarly.
(v) Consider the following calculation:

$$\begin{aligned}
A_{11}^{+}A_{22}^{-} &= -\delta^{11}(x_{21} - x_{11})(x_{22} - x_{11}) \cdot (\delta^{22})^{-1}\frac{x_{11} - x_{22}}{x_{21} - x_{22}} \\
&= -\delta^{11}(\delta^{22})^{-1}(x_{21} - x_{11})(x_{22} - x_{11} - 1) \cdot \frac{x_{11} - x_{22}}{x_{21} - x_{22}} \\
&= -\delta^{11}(\delta^{22})^{-1}(x_{21} - x_{11})(x_{22} - x_{11}) \cdot \frac{x_{11} - x_{22} + 1}{x_{21} - x_{22}} \\
&= (\delta^{22})^{-1}\frac{x_{11} - x_{22}}{x_{21} - x_{22}} \cdot -\delta^{11}(x_{21} - x_{11})(x_{22} - x_{11}) \\
&= A_{22}^{-}A_{11}^{+}.
\end{aligned}$$

The other relations are proved similarly.
(vii) We consider the relation $[E_{23}, E_{32}] = E_{22} - E_{33}$ mapped under φ from (3):

$$\begin{aligned}
X_{22} - X_{33} &= [X_2^{+}, X_2^{-}] \\
&= [A_{21}^{+} + A_{22}^{+}, A_{21}^{-} + A_{22}^{-}] \\
&= [A_{21}^{+}, A_{21}^{-}] + [A_{21}^{+}, A_{22}^{-}] + [A_{22}^{+}, A_{21}^{-}] + [A_{22}^{+}, A_{22}^{-}] \\
&= [A_{21}^{+}, A_{21}^{-}] + [A_{22}^{+}, A_{22}^{-}] \quad \text{by (iv)}.
\end{aligned}$$

This demonstrates that (vii) holds.
(viii) We observe that

$$\begin{aligned} A^-_{11}A^-_{22} &= (\delta^{11})^{-1} \cdot (\delta^{22})^{-1} \frac{x_{11}-x_{22}}{x_{21}-x_{22}} \\ &= (\delta^{11}\delta^{22})^{-1} \frac{x_{11}-x_{22}}{x_{21}-x_{22}} \\ &= (\delta^{22})^{-1} \frac{x_{11}-x_{22}+1}{x_{21}-x_{22}} \cdot (\delta^{11})^{-1} \\ &= A^-_{22}A^-_{11} - (\delta^{11}\delta^{22})^{-1} \frac{1}{x_{21}-x_{22}} \\ [A^-_{11}, A^-_{22}] &= -(\delta^{11}\delta^{22})^{-1} \frac{1}{x_{21}-x_{22}}, \end{aligned}$$

which has no x_{11}'s and as such commutes with A^-_{11}. Thus, $[A^-_{11}, [A^-_{11}, A^-_{22}]] = 0$. The others are proved identically.

(ix) We prove this by direct computation as follows:

$$\begin{aligned} A^\pm_{22}\mathcal{V}_2A^\pm_{21} &= (\delta^{21}\delta^{22})^{\pm 1} \frac{\prod_{i=1}^{2\pm 1}(x_{2\pm 1,i}-x_{21})(x_{2\pm 1,i}-x_{22})}{x_{21}-x_{22}} \\ &= -(\delta^{21})^{\pm 1} \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{21} \cdot (\delta^{22})^{\pm 1} \frac{\prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{22}}{x_{21}-x_{22}} \\ &= (\delta^{21})^{\pm 1} \prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{21} \cdot \frac{\mathcal{V}_2}{-\mathcal{V}_2} (\delta^{22})^{\pm 1} \frac{\prod_{i=1}^{2\pm 1} x_{2\pm 1,i} - x_{22}}{x_{21}-x_{22}} \\ &= A^\pm_{21}\mathcal{V}_2A^\pm_{22}. \end{aligned}$$

This verifies that relation (ix) holds. □

Open Problem 1. Determine whether the relations in Proposition 4.4 constitute a presentation for the algebra $\mathscr{A}(\mathfrak{gl}_3)$.

5. Finite-dimensional modules over $\mathscr{A}(\mathfrak{gl}_n)$

Since, as was shown in Section 4, $\mathscr{A}(\mathfrak{gl}_n)$ is not a Galois $\widetilde{\Gamma}$-order, techniques different from [8] are required to study representations of $\mathscr{A}(\mathfrak{gl}_n)$.

If we consider the case of $n = 2$, we recall that $\mathscr{A}(\mathfrak{gl}_2) \cong U(\mathfrak{gl}_2)[T_2]/(T_2^2 - (-c_{21}^2 + 2c_{22} + 1))$. As such, it makes sense to consider the induction and restriction functors between the categories of $\mathscr{A}(\mathfrak{gl}_2)$-modules and $U(\mathfrak{gl}_2)$-modules.

By applying the restriction functor to a given finite-dimensional simple module, we see that it decomposes to a direct sum of finite-dimensional simple $U(\mathfrak{gl}_2)$-modules, so the induction functor should help us to construct all of the possible finite-dimensional simple $\mathscr{A}(\mathfrak{gl}_2)$-modules.

Proposition 5.1. *The finite-dimensional simple $\mathscr{A}(\mathfrak{gl}_2)$-modules are characterized by ordered pairs $(\lambda_2, \varepsilon_2)$, where $\lambda_2 := (\lambda_{21}, \lambda_{22}) \in \mathbb{C}^2$ is a dominant integral weight for $U(\mathfrak{gl}_2)$ (i.e. $\lambda_{21} - \lambda_{22} \in \mathbb{Z}_{\geq 0}$) and $\varepsilon_2 \in \{1, -1\}$.*

Proof. Recall that every finite-dimensional simple $U(\mathfrak{gl}_2)$-module is characterized by a weight denoted by a pair of complex numbers $\lambda_2 = (\lambda_{21}, \lambda_{22})$ with $\lambda_{21} - \lambda_{22} \in \mathbb{Z}_{\geq 0}$; we will denote this module by $V(\lambda_2)$. We can induce such a module $V(\lambda_2)$ to a $\mathscr{A}(\mathfrak{gl}_2)$-module as follows,

$$\mathscr{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2).$$

So, it is important to describe $\mathscr{A}(\mathfrak{gl}_2)$ as a right $U(\mathfrak{gl}_2)$-module. By Proposition 3.3:

$$\mathscr{A}(\mathfrak{gl}_2) \cong \frac{U(\mathfrak{gl}_2)[T_2]}{(T_2^2 - (-c_{21}^2 + 2c_{22} + 1))} \cong U(\mathfrak{gl}_2) \oplus T_2 U(\mathfrak{gl}_2)$$

as right $U(\mathfrak{gl}_2)$-modules. Thus:

$$\begin{aligned}\mathscr{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2) &\cong \big(U(\mathfrak{gl}_2) \oplus T_2 U(\mathfrak{gl}_2)\big) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2) \\ &\cong \big(U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)\big) \oplus \big(T_2 U(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)\big) \\ &\cong \big(1 \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)\big) \oplus \big(T_2 \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)\big).\end{aligned}$$

As such, we can determine the action of T_2 on this modules now. For $v \in V(\lambda_2)$, we have that $T_2.(1 \otimes v) = T_2 \otimes v$, and $T_2.(T_2 \otimes v) = T_2^2 \otimes v = 1 \otimes T_2^2.v = (\lambda_{21} - \lambda_{22})^2(1 \otimes v)$. Thus, T_2 can be characterized by the following matrix:

$$\begin{bmatrix} 0 & (\lambda_{21} - \lambda_{22})^2 I \\ I & 0 \end{bmatrix} \cong \begin{bmatrix} (\lambda_{21} - \lambda_{22}) I & 0 \\ 0 & -(\lambda_{21} - \lambda_{22}) I \end{bmatrix},$$

so we can see that $\mathscr{A}(\mathfrak{gl}_2) \otimes_{U(\mathfrak{gl}_2)} V(\lambda_2)$ decomposes into the two eigenspaces of the action of T_2: $V(\lambda_2, +1) := \langle (\lambda_{21} - \lambda_{22})(1 \otimes v) + T_2 \otimes v \mid v \in V(\lambda_2) \rangle$ and $V(\lambda_2, -1) := \langle -(\lambda_{21} - \lambda_{22})(1 \otimes v) + T_2 \otimes v \mid v \in V(\lambda_2) \rangle$ both of which are clearly simple. It is also clear that as vector spaces $V(\lambda_2, \pm 1) \cong V(\lambda_2)$.

Conversely, if we have a finite-dimensional simple $\mathscr{A}(\mathfrak{gl}_2)$-module V restricted to a $U(\mathfrak{gl}_2)$-module, it must remain simple, as T_2 is a central element. As such, $V \cong V(\lambda_2)$ for some weight λ_2. Thus, $V \cong V(\lambda_2, \varepsilon_2)$ for some $\varepsilon_2 \in \{\pm 1\}$. □

Next, we classify a collection of finite-dimensional simple weight modules over $\mathscr{A}(\mathfrak{gl}_n)$.

Definition 5.2. Let $V(\lambda_n)$ be a weight module of $U(\mathfrak{gl}_n)$, we extend it to a module for $\mathscr{A}(\mathfrak{gl}_n)$, denoted $V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2)$, by describing the actions of each $\mathcal{V}_k$ for $k = 2, 3, \ldots, n$ as follows:

$$\mathcal{V}_n.v = \varepsilon_n \prod_{i\le j}(\lambda_{ni} - \lambda_{nj} + j - i)v,$$

with $\varepsilon_n = \pm 1$. Recall that when we restrict $V(\lambda_n)$ to a $U(\mathfrak{gl}_k)$ module, the number of simple $U(\mathfrak{gl}_k)$ modules it decomposes into is the same as the number of ways to fill in the k-th row of a Gelfand-Tsetlin pattern with top row λ_n. Denote this number by $r_{\lambda_n,k}$. Then let $\mathcal{V}_k$ act diagonallizably on a $v = (v_1, \dots, v_{r_{\lambda_n,k}}) \in V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_2)$ by the following $r_{\lambda_n,k} \times r_{\lambda_n,k}$ matrix,

$$\begin{pmatrix} \varepsilon_{k,1}\prod_{i\le j}(\lambda^1_{ki} - \lambda^1_{kj} + j - i) & 0 & \cdots & 0 \\ 0 & \varepsilon_{k,2}\prod_{i\le j}(\lambda^2_{ki} - \lambda^2_{kj} + j - i) & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & & \varepsilon_{k,r_{\lambda_n,k}}\prod_{i\le j}(\lambda^{r_{\lambda_n,k}}_{ki} - \lambda^{r_{\lambda_n,k}}_{kj} + j - i) \end{pmatrix},$$

where λ^ℓ_{ki} denotes the ki entry from the ℓ-th pattern in the decomposition of v as a $U(\mathfrak{gl}_k)$-module, and $\varepsilon_k = (\varepsilon_{k,1}, \varepsilon_{k,2}, \dots, \varepsilon_{k,r_{\lambda_n,k}}) \in \{\pm 1\}^{r_{\lambda_n,k}}$.

Theorem 5.3. *Every finite-dimensional simple module over $\mathscr{A}(\mathfrak{gl}_n)$, on which $\mathcal{V}_2, \dots, \mathcal{V}_{n-1}$ act diagonallizably, is of the form $V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_2)$ (see Definition 5.2), where $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn})$ is a dominant integral weight of $U(\mathfrak{gl}_n)$, $\varepsilon_j \in \{\pm 1\}^{r_{\lambda_n,j}}$, with $r_{\lambda_n,j}$ denoting the number of ways to fill the j-th row of Gelfand-Tsetlin pattern with fixed top row λ_n, and $j = 2, 3, \dots, n$.*

Proof. We prove this by induction on n. For the base case, $n = 3$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{A}(\mathfrak{gl}_3)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & \mathscr{A}(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \\ \downarrow & & \downarrow \\ U(\mathfrak{gl}_3)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} & \longrightarrow & U(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \end{array},$$

where each arrow is the restriction functor. If we consider a simple $V \in \mathscr{A}(\mathfrak{gl}_3)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}$ and its image in the bottom right corner, we see that $V \cong \bigoplus_{\lambda_3}\bigoplus_{\lambda_2} V(\lambda_2)_{\lambda_3} \in U(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}$, where λ_3 and λ_2 are weights for $U(\mathfrak{gl}_3)$ and $U(\mathfrak{gl}_2)$, respectively, by the semi-simplicity of $U(\mathfrak{gl}_3)$ and $U(\mathfrak{gl}_2)$. Moreover, $V(\lambda_2)_{\lambda_3}$'s are the components of the restriction of $V(\lambda_3)$ to $U(\mathfrak{gl}_2)$. We know that $\mathcal{V}_2$ must have a diagonal action by assumption. As such, we have $V \cong \bigoplus_{\lambda_3}\bigoplus_{\lambda_2} V(\lambda_2, \varepsilon_2)_{\lambda_3}$ in the upper right corner by Proposition 5.1, where $\varepsilon_2 = \varepsilon_2(\lambda_2)$ depends λ_2. This is because otherwise the dimensions of the λ_2 weight spaces would not match. Since $\mathcal{V}_2$ acts diagonally, $\mathcal{V}_3$ is central, and the diagram commutes, it follows that $V \cong V(\lambda_3, \varepsilon_3, \varepsilon_2) \in \mathscr{A}(\mathfrak{gl}_3)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}$, where ε_3 is determined as in Proposition 5.1, and $\varepsilon_2 = \{\varepsilon_2(\lambda_2)\}_{\lambda_2}$ is indexed by the number $r_{\lambda_3,2}$.

To finish the induction we look at a similar diagram:

$$\begin{tikzcd}
\mathscr{A}(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \arrow[r] \arrow[d] & \mathscr{A}(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{A}(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \arrow[d] \\
U(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \arrow[r] & U(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}} \arrow[r] & \cdots \arrow[r] & U(\mathfrak{gl}_2)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}
\end{tikzcd}$$

Following the image of a simple $V \in \mathscr{A}(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}$ and using identical arguments, we observe that:

$$V \cong \bigoplus_{\lambda_n} \bigoplus_{\lambda_{n-1}} V(\lambda_{n-1})_{\lambda_n} \in U(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}.$$

By the induction hypothesis,

$$V \cong \bigoplus_{\lambda_n} \bigoplus_{\lambda_{n-1}} V(\lambda_{n-1}, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2)_{\lambda_n} \in \mathscr{A}(\mathfrak{gl}_{n-1})\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}.$$

Finally by $\mathcal{V}_n$ central, $\mathcal{V}_j$ acting diagonally for $j = 2, \ldots, n-1$, and the diagram commuting, it follows that $V \cong V(\lambda_n, \varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_2)$. $\square$

The following example demonstrates that $\mathscr{A}(\mathfrak{gl}_n)\text{-}\underline{\mathrm{Mod}}^{\mathrm{f.d.}}$ is not semi-simple for every $n \geq 2$.

Example 5.4. We recall that $\mathcal{V}_2^2$ must act diagonally on any $\mathscr{A}(\mathfrak{gl}_2)$-module V because $\mathrm{Res}_{U(\mathfrak{gl}_2)}^{\mathscr{A}(\mathfrak{gl}_2)} V$ can be viewed as a direct sum of irreducible $U(\mathfrak{gl}_2)$-modules and $\mathcal{V}_2^2$ is a quadratic polynomial of Gelfand invariants in $U(\mathfrak{gl}_2)$. Let $V = V(0) \oplus V(0)$, where $U(\mathfrak{gl}_2)$ acts trivially. This means that $\mathcal{V}_2^2$ must act as Id_V. We define the following action of $\mathcal{V}_2$

$$\mathcal{V}_2.\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

with $0 \neq \alpha \in \mathbb{C}$. It is clear then that $\mathcal{V}_2^2$ acts as the identity on V, but the subrepresentation $W = \{(v_1, 0) \mid v_1 \in V(0)\}$ is not a direct summand of V as a $\mathscr{A}(\mathfrak{gl}_2)$-module.

6. A technique for creating Galois orders from Galois rings via localization

In this section, we describe a technique that allows us to turn a Galois ring into a Galois order involving localization. We use this technique on a toy example and a localized version of $\mathscr{A}(\mathfrak{gl}_n)$ denoted $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ (see Definition 6.10).

6.1. The general result

We recall that Proposition 1.9 states that Γ is maximal commutative in a Galois Γ-order. We observe that for a general Galois Γ-ring $\mathscr{U}$, while Γ might not be maximal commutative, its centralizer $C_{\mathscr{U}}(\Gamma)$ in $\mathscr{U}$ will be [7]. This can be seen from the following remark:

Remark 2. For Galois Γ-ring $\mathscr{U}$, the centralizer of Γ in $\mathscr{U}$, denoted $C_{\mathscr{U}}(\Gamma)$, is equal to $\mathscr{U} \cap K$.

First we define a subring of L that is needed in our result.

Definition 6.1. Let $\mathscr{U}$ be a subalgebra of $\mathscr{L}$. We define the *ring of coefficients* of $\mathscr{U}$:

$$D_{\mathscr{U}} := \langle \alpha \in L \mid \exists X \in \mathscr{U} \text{ such that } \alpha \text{ is a left coefficient of some } \mu \in \operatorname{supp}_{\mathscr{M}} X \rangle_{\text{ring}}.$$

Similarly, we define the *opposite ring of coefficients* of $\mathscr{U}$, denoted $D_{\mathscr{U}}^{\text{op}}$, using right coefficients.

Now for the result.

Theorem 6.2. *Let G be arbitrary and $\mathscr{U}$ be a Galois Γ-ring in $(L\#\mathscr{M})^G$. If $C = C_{\mathscr{U}}(\Gamma)$ is the G invariants of the localization of Λ with respect to a set that is $\mathscr{M}$-invariant, that is $C = (S^{-1}\Lambda)^G$, where S is $\mathscr{M}$-invariant, and $D_{\mathscr{U}}$ is a finitely generated module over C, then $\mathscr{U}$ is a Galois C-order in $(L\#\mathscr{M})^G$. Moreover, if $D_{\mathscr{U}} \subseteq S^{-1}\Lambda$ (resp. $D_{\mathscr{U}}^{\text{op}} \subset S^{-1}\Lambda$), then $\mathscr{U}$ is a (co-)principal Galois C-order.*

Proof. First, we find a Λ' such that $(\Lambda', G, \mathscr{M})$ satisfies the assumptions in Section 1.1. We define $\Lambda' = \overline{C}$, the integral closure of C in L. We observe that $C = (S^G)^{-1}\Gamma$. As such, C is a localization, and it follows that:

$$\overline{C} = (S^G)^{-1}\overline{\Gamma} = S^{-1}\Lambda. \tag{14}$$

Since S is $\mathscr{M}$-invariant and $\overline{C}$ is integral over C, it follows that $\mathscr{M}$ and G are subgroups of $\operatorname{Aut}(\Lambda')$. The first two assumptions clearly hold, and the third follows by $\Lambda' = S^{-1}\Lambda$.

We have that $\mathscr{U}$ is a Galois C-ring since it is a Galois Γ-ring and $\operatorname{Frac}(C) = \operatorname{Frac}(\Gamma) = K$. All that remains is to show that $\mathscr{U}$ is a Galois C-order. We consider $W \subset \mathscr{L}$ a finite-dimensional left L-subspace and aim to show that $W \cap \mathscr{U}$ is finitely generated as a left C-module. W has a finite basis $w_1, \ldots, w_n$ such that:

$$W = \{\sum \alpha_i w_i \mid \alpha_i \in L\}.$$

Note that for each i, $w_i = \sum_{\mu \in \mathscr{M}} \beta_{i,\mu}\mu$; as such, since C is a localization of a Noetherian ring and therefore Noetherian, WLOG we can assume $w_i = \mu_i$ for some $\mu_i \in \mathscr{M}$. Hence:

$$W = \sum_i L\mu_i.$$

So, $W \cap \mathscr{U} \subset \sum_i D_{\mathscr{U}} \mu_i$, and is therefore finitely generated. A similar argument justifies the claim if W is instead a right L-module. Therefore, U is a Galois C-order.

If additionally we assume $D_{\mathscr{U}} \subset S^{-1}\Lambda$, we need to show that $X(c) \in C$ for all $X \in \mathscr{U}$ and $c \in C$. So, we consider an arbitrary $c \in C$ and $X \in \mathscr{U}$. By Lemma 2.19 in [14], it follows that $X(c) \in K$. Since $C = (S^G)^{-1}\Gamma$, it follows that $X(c) \in S^{-1}\Lambda$. As such:

$$X(c) \in S^{-1}\Lambda \cap K = (S^{-1}\Lambda)^G = C. \tag{15}$$

Thus $X(c) \in C$. If instead $D_{\mathscr{U}}^{\mathrm{op}} \subset S^{-1}\Lambda$, a similar argument shows that $X^\dagger(c) \in C$, thereby proving the claim. □

The above theorem also gives an alternate proof to one direction of Corollary 2.15 in [14].

6.2. A toy example

In this subsection, we provide a family of simple examples of Galois rings to which Theorem 6.2 can be applied.

Let $\Lambda = \mathbb{C}[x]$ the polynomial algebra in one indeterminate x, $\delta \in \operatorname{Aut}\Lambda$ such that $\delta(x) = x - 1$, $\mathscr{M} = \langle\delta\rangle_{\mathrm{grp}}$, and G the trivial group. Then, let $\mathscr{L} = L\#\mathscr{M}$ be the skew-monoid ring and $f(x) \in \mathbb{C}[x]$ such that $f(0) \neq 0$. We define $X, Y \in \mathscr{L}$ such that:

$$X := \delta\frac{f(x)}{x} \qquad \text{and} \qquad Y := \delta^{-1}. \tag{16}$$

Let $U_f = \mathbb{C}\langle\Lambda, X, Y\rangle_{\mathrm{alg}}$ and $C_{U_f}(\Lambda)(= C_{U_f})$ the centralizer of Λ in U_f. We note, as G is trivial, that $\Lambda = \Gamma$. First, we will show that U_f is Galois Γ-ring.

Proposition 6.3. *The algebra U_f is a Galois Γ-ring in $L\#\mathscr{M}$.*

Proof. This immediately follows from Proposition 1.4 letting $\mathscr{X} = \{X, Y\}$. □

In order to apply Theorem 6.2, we need to describe C_{U_f}. The next three lemmas are used to do just that.

Lemma 6.4. *For any $f(x)$ such that $f(0) \neq 0$, we have $\frac{1}{x}, \frac{1}{x-1} \in C_{U_f}$.*

Proof. First, we show that $\frac{1}{x} \in C_{U_f}$. Now, $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_0 \neq 0$ by assumption. As such:

$$\begin{aligned} \frac{f(x)}{x} &= a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots a_1 + \frac{a_0}{x} \\ &\Rightarrow \frac{1}{x} = a_0^{-1}\left(\frac{f(x)}{x} - (a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots a_1)\right). \end{aligned}$$

This shows that $\frac{1}{x} \in C_{U_f}$. To see that $\frac{1}{x-1} \in C_{U_f}$, we follow a similar division algorithm argument with $\frac{f(x-1)}{x-1}$. □

Lemma 6.5. *For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 1$, we have* $\frac{1}{x+k} \in C_{U_f}$.

Proof. Let m be the order of $(x+k)$ in $\prod_{j=0}^{k-1} f(x+j)$. Then consider the following:

$$
\begin{aligned}
Y^{k+1}(XY)^m X^{k+1} &= \delta^{-k-1}\left(\frac{f(x-1)}{x-1}\right)^m \delta^{k+1} \prod_{j=0}^{k} \frac{f(x+j)}{x+j} \\
&= \left(\frac{f(x+k)}{x+k}\right)^m \prod_{j=0}^{k} \frac{f(x+j)}{x+j} \\
&= \left(\frac{f(x+j)}{x+j}\right)^{m+1} \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j}
\end{aligned}
$$

As such, there are m factors of $(x+k)$ in the numerator and $m+1$ factors in the denominator. Thus, multiplying by $\prod_{j=0}^{k-1}(x+j)$ and using a division algorithm argument, it follows that $\frac{1}{x+k} \in C_{U_f}$. □

Lemma 6.6. *For any $f(x)$ such that $f(0) \neq 0$ and $k \geq 2$, we have* $\frac{1}{x-k} \in C_{U_f}$.

Proof. Let m be the order of $(x-k)$ in $\prod_{j=1}^{k-1} f(x-j)$. Then consider the following:

$$
\begin{aligned}
X^k(YX)^m Y^k &= \delta^k \prod_{j=0}^{k-1} \frac{f(x+j)}{x+j} \left(\frac{f(x)}{x}\right)^m \delta^{-k} \\
&= \prod_{j=0}^{k-1} \frac{f(x+j-k)}{x+j-k} \left(\frac{f(x-k)}{x-k}\right)^m \\
&= \left(\frac{f(x-k)}{x-k}\right)^{m+1} \prod_{\ell=1}^{k-1} \frac{f(x-\ell)}{x-\ell}.
\end{aligned}
$$

As such, there are m factors of $(x-k)$ in the numerator and $m+1$ factors in the denominator. Thus, multiplying by $\prod_{j=1}^{k-1}(x-j)$ and using a division algorithm argument, it follows that $\dfrac{1}{x-k} \in C_{U_f}$. $\square$

Proposition 6.7. *If $f(x)$ is a polynomial such that $f(0) \neq 0$, then $C_{U_f} = \mathbb{C}[x]\left[\frac{1}{x+k} \;\middle|\; k \in \mathbb{Z}\right]$.*

Proof. $C_{U_f} \supseteq \mathbb{C}[x]\left[\frac{1}{x+k} \;\middle|\; k \in \mathbb{Z}\right]$ by Lemmas 6.4, 6.5, and 6.6. To show the reverse inclusion, we observe that for $Z \in C_{U_f}$, Z must be of "degree 0" with regards to δ that is:

$$
\begin{aligned}
Z &= \sum_{k=1}^{m} g_k(x) \prod_{n=0}^{\infty} (X^n Y^n)^{k_{-n}} (Y^n X^n)^{k_n} \\
&= \sum_{k=1}^{m} g_k(x) \prod_{\ell=-\infty}^{\infty} \left(\frac{f(x+\ell)}{(x+\ell)}\right)^{k_\ell} \\
&= \sum_{k=1}^{m} G_k(x) \prod_{\ell=-\infty}^{\infty} \frac{1}{(x+\ell)^{k_\ell}} \in \mathbb{C}[x]\left[\frac{1}{x+k} \;\middle|\; k \in \mathbb{Z}\right],
\end{aligned}
$$

where $k_\ell \neq 0$ for at most finitely many terms. Thus $C_{U_f} \subseteq \mathbb{C}[x]\left[\frac{1}{x+k} \;\middle|\; k \in \mathbb{Z}\right]$. $\square$

We can now prove that U_f is a Galois C_{U_f}-order using Theorem 6.2.

Corollary 6.8. *The algebra U_f is a principal and co-principal Galois C_{U_f}-order in $L\#\mathscr{M}$.*

Proof. Proposition 6.7 gives us that the main supposition of Theorem 6.2. All that remains to show is $D_{U_f}, D_{U_f}^{\mathrm{op}} \subset S^{-1}\Lambda = C_{U_f}$ in this case. However, this is clear since U_f is generated by X, Y, and Λ. $\square$

6.3. Localizing $\mathscr{A}(\mathfrak{gl}_n)$

In this subsection, we construct a localization of $\mathscr{A}(\mathfrak{gl}_n)$ denoted $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$, to which Theorem 6.2 can be applied.

In order to construct this localization, we describe shifted Vandermonde polynomials using the following notation:

Notation. Let $\mathcal{V}_k$ be the Vandermonde in the x_{ki} variables. Let $l := (l_1, l_2, \ldots, l_{k-1}) \in \mathbb{Z}^{k-1}$. We denote the ($l$-)shifted $\mathcal{V}_k$ as follows:

$$\mathcal{V}_{k,l} := \prod_{i<j}(x_{ki} - x_{kj} + \sum_{n=i}^{j-1} l_n).$$

This notation makes sense because for $i < j$:

$$x_{ki} - x_{kj} = (x_{ki} - x_{k,i+1}) + (x_{k,i+1} - x_{k,i+2}) + \cdots + (x_{k,j-1} - x_{kj}).$$

Therefore, any shift of $\mathcal{V}_k$ is uniquely determined by the shifts of $x_{ki} - x_{k,i+1}$ for $i = 1, 2, \ldots, k-1$.

Now to construct our localization.

Definition 6.9. Let $S := \langle \mathcal{V}_{k,l} \mid l \in \mathbb{Z}^{k-1};\ k = 2, \ldots, n-1\rangle_{\text{monoid}}$. We observe that S is a multiplicatively closed set in Λ, and $\mathscr{A}(\mathfrak{gl}_n) \subset (S^{-1}\Lambda \# \mathscr{M})^{\mathbb{A}_n}$. We also note that S is the smallest $\mathscr{M}$-invariant multiplicatively closed set that contains $\mathcal{V}_2, \ldots, \mathcal{V}_{n-1}$.

As Example 4.2 demonstrates, $C_{\mathscr{A}(\mathfrak{gl}_n)}(\widetilde{\Gamma}) \subset (S^{-1}\Lambda)^{\mathbb{A}_n}$. It is not known if this containment is strict, so this motivates the construction of the following localization of $\mathscr{A}(\mathfrak{gl}_n)$.

Definition 6.10. Our new algebra of interest in $\widetilde{\mathscr{K}}$ is $\widetilde{\mathscr{A}}(\mathfrak{gl}_n) := \mathbb{C}\langle U_n, (S^{-1}\Lambda)^{\mathbb{A}_n}\rangle_{\text{alg}}$. Notice this coincides with the definitions of $\mathscr{A}(\mathfrak{gl}_2)$ for $n = 2$.

Remark 3. It follows from Lemma 2.10 in [14] that $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ is a Galois $\widetilde{\Gamma}$-ring since it contains $\mathscr{A}(\mathfrak{gl}_n)$. Moreover, $C_{\widetilde{\mathscr{A}}(\mathfrak{gl}_n)}(\widetilde{\Gamma}) = (S^{-1}\Lambda)^{\mathbb{A}_n}$ as well.

Remark 4. In $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$, relation (ix) from Section 4.2 can be rewritten either as

(ix)′ $[A_{21}^{\pm}, A_{22}^{\pm}] = \dfrac{\pm 2}{\mathcal{V}_2 \pm 1} A_{21}^{\pm} A_{22}^{\pm}$, or

(ix)″ $A_{22}^{\pm} A_{21}^{\pm} = \dfrac{\mathcal{V}_2 \mp 1}{\mathcal{V}_2 \pm 1} A_{21}^{\pm} A_{22}^{\pm}$.

Corollary 6.11. *The subalgebra $\widetilde{\mathscr{A}}(\mathfrak{gl}_n) \subset \widetilde{K}$ is both a principal and co-principal Galois $(S^{-1}\Lambda)^{\mathbb{A}_n}$-order.*

Proof. It is clear by construction that $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ satisfies the main supposition of Theorem 6.2. Also, it follows from the definition of the $a_{ki}^{\pm}$'s in (4) that $D_{\widetilde{\mathscr{A}}(\mathfrak{gl}_n)}, D^{\text{op}}_{\widetilde{\mathscr{A}}(\mathfrak{gl}_n)} \subseteq S^{-1}\Lambda$. We can therefore apply Theorem 6.2. $\square$

In [21], it was shown that every (co-)principal Galois order has a corresponding (co-)principal flag order. This leads us to the following:

Open Problem 2. What is the corresponding (co-)principal flag order of $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$?

7. (Generic) Gelfand-Tsetlin modules over $\mathscr{A}(\mathfrak{gl}_n)$

7.1. Some general results

Following the techniques in [3] and [14], we construct canonical simple Gelfand-Tsetlin modules over $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$. We need the following additional assumptions for these next two results:

(A4) Λ is finitely generated over an algebraically closed field $\Bbbk$ of characteristic 0,
(A5) G and M act by $\Bbbk$-algebra homomorphisms on Λ.

Let $\hat{\Gamma}$ be the set of all Γ-characters (i.e., $\Bbbk$-algebra homomorphisms $\xi\colon \Gamma \to \Bbbk$).

Definition 7.1. Let $\mathscr{U}$ be a Galois Γ-ring in $\mathscr{K}$. A left $\mathscr{U}$-modules V is said to be a *Gelfand-Tsetlin module (with respect to* Γ*)* if Γ acts locally finitely on V. Equivalently:

$$V = \bigoplus_{\xi \in \hat{\Gamma}} V_\xi, \qquad V_\xi = \{v \in V \mid (\ker \xi)^N v = 0, N \gg 0\}.$$

Similarly, one can define a right Gelfand-Tsetlin modules.

The details for the following lemma can be found in [2].

Lemma 7.2. *Let $\mathscr{U}$ be a Galois Γ-ring in $\mathscr{K}$.*

(i) *Any submodule and any quotient of a Gelfand-Tsetlin module is a Gelfand-Tsetlin module.*
(ii) *Any $\mathscr{U}$-module generated by generalized weight vectors is a Gelfand-Tsetlin module.*

Theorem 7.3 *([14], Theorem 3.3 (ii)). Let $\xi \in \hat{\Gamma}$ be any character. If $\mathscr{U}$ is a co-principal Galois Γ-order in $\mathscr{K}$, then the left cyclic $\mathscr{U}$-module $\mathscr{U}\xi$ has a unique simple quotient $V'(\xi)$. Moreover, $V'(\xi)$ is a Gelfand-Tsetlin over $\mathscr{U}$ with $V'(\xi)_\xi \neq 0$ and is called the canonical simple left Gelfand-Tsetlin $\mathscr{U}$-module associated to ξ.*

7.2. The case of $\mathscr{A}(\mathfrak{gl}_n)$

We note that for $n \geq 3$ that $\widetilde{\Lambda}$ is not finitely generated as a $\mathbb{C}$-algebra. This prevents us from using all of the results as is, but all is not lost. The main arguments of Theorem 7.3 rest on:

$$\mathrm{Hom}_\Gamma(\Gamma/\mathfrak{m}, \Gamma^*) \cong \mathrm{Hom}_\Bbbk(\Gamma/\mathfrak{m} \otimes_\Gamma \Gamma, \Bbbk) \cong \Bbbk.$$

If we want a similar result for $S^{-1}\widetilde{\Gamma}$ we need to recall that every maximal ideal $\mathfrak{m}$ of $S^{-1}\widetilde{\Gamma}$ is of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime (not necessarily maximal) ideal of $\widetilde{\Gamma} \setminus S$. Therefore we have the following result.

Theorem 7.4. *Let ξ be a character of $S^{-1}\widetilde{\Gamma}$ such that $\ker \xi = S^{-1}\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of $\widetilde{\Gamma}$. Then the left cyclic module $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)\xi$ has a unique simple quotient $V'(\xi)$ which is a Gelfand-Tsetlin module over $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ with $V'(\xi)_\xi \neq 0$.*

Proof. The key difference in this proof compared to Theorem 7.3 is observing that

$$S^{-1}\widetilde{\Gamma}/S^{-1}\mathfrak{m} \cong S^{-1}(\widetilde{\Gamma}/\mathfrak{m}) \cong \Bbbk.$$

Otherwise, the proof follows the same structure. □

Since $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ is created by localizing $\widetilde{\Gamma}$ and Λ, we can view any $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$-module V as a $\mathscr{A}(\mathfrak{gl}_n)$-module by precomposing with the embedding $\iota\colon \mathscr{A}(\mathfrak{gl}_n) \hookrightarrow \widetilde{\mathscr{A}}(\mathfrak{gl}_n)$.

8. Gelfand-Kirillov conjecture for $\mathscr{A}(\mathfrak{gl}_n)$

In this section we will discuss for which n's the algebras $\mathscr{A}(\mathfrak{gl}_n)$ and $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ satisfy the Gelfand-Kirillov Conjecture. This is related to the Noncommutative Noether Problem for the alternating group A_n, as discussed in [10].

An algebra A is said to satisfy Gelfand-Kirillov Conjecture if it is birationally equivalent to a Weyl algebra. That is its skew-field of fractions is isomorphic to a skew Weyl field.

Lemma 8.1. $\mathrm{Frac}(\widetilde{\mathscr{A}}(\mathfrak{gl}_n)) = \mathrm{Frac}(\mathscr{A}(\mathfrak{gl}_n))$.

Proof. This follows because $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ is created by localizing $\widetilde{\Gamma}$ and Λ. □

Hence, $\widetilde{\mathscr{A}}(\mathfrak{gl}_n)$ and $\mathscr{A}(\mathfrak{gl}_n)$ either both will or will not satisfy the Gelfand-Kirillov Conjecture for each n.

Proposition 8.2. *For every n,*

$$\mathrm{Frac}(\mathscr{A}(\mathfrak{gl}_n)) \cong \mathrm{Frac}\left(\mathbb{C}(x_1,\ldots,x_n)^{A_n} \otimes \bigotimes_{k=1}^{n-1} \left(\mathrm{Frac}(\mathscr{W}_k(\mathbb{C}))\right)^{A_k}\right),$$

where $\mathscr{W}_k(\mathbb{C})$ is the k-dimensional Weyl algebra over $\mathbb{C}$.

Proof. It is clear by construction that:

$$\mathrm{Frac}(\mathscr{A}(\mathfrak{gl}_n)) = \mathrm{Frac}(\mathscr{L}^{\mathbb{A}_n}) = \mathrm{Frac}((L\#\mathscr{M})^{\mathbb{A}_n}). \tag{17}$$

Since $L = \mathrm{Frac}(\Lambda)$:

$$\mathrm{Frac}((L\#\mathscr{M})^{\mathbb{A}_n}) \cong \mathrm{Frac}((\Lambda\#\mathscr{M})^{\mathbb{A}_n}). \tag{18}$$

We now recall that $\mathscr{M}$ is generated by δ^{ki}'s and δ^{ki} fixes $x_{\ell j}$ if $\ell \neq k$. As such, we have:

$$\mathrm{Frac}((\Lambda\#\mathscr{M})^{\mathbb{A}_n}) \cong \mathrm{Frac}((\Lambda_n \otimes \bigotimes_{k=1}^{n-1} \Lambda_k\#\mathscr{M}_k)^{\mathbb{A}_n}), \tag{19}$$

where $\Lambda_k = \mathbb{C}[x_{k1}, \ldots, x_{kk}] \subset \Lambda$ and $\mathscr{M}_k = \langle \delta^{ki} \mid 1 \leq i \leq k\rangle_{\mathrm{grp}} \leq \mathscr{M}$. Now, the k-th component of $\mathbb{A}_n$ acts only on the k-th component of the tensor product. Therefore:

$$\mathrm{Frac}((\Lambda_n \otimes \bigotimes_{k=1}^{n-1} \Lambda_k\#\mathscr{M}_k)^{\mathbb{A}_n}) \cong \mathrm{Frac}(\Lambda_n^{A_n} \otimes \bigotimes_{k=1}^{n-1} (\Lambda_k\#\mathscr{M}_k)^{A_k}). \tag{20}$$

Finally, since A_k is finite for each k we have:

$$\mathrm{Frac}(\Lambda_n^{A_n} \otimes \bigotimes_{k=1}^{n-1} (\Lambda_k\#\mathscr{M}_k)^{A_k}) \cong \mathrm{Frac}\left((\mathrm{Frac}(\Lambda_n))^{A_n} \otimes \bigotimes_{k=1}^{n-1} (\mathrm{Frac}(\Lambda_k\#\mathscr{M}_k))^{A_k}\right). \tag{21}$$

Combining the equations (17)-(21), we have:

$$\mathrm{Frac}(\mathscr{A}(\mathfrak{gl}_n)) \cong \mathrm{Frac}\left((\mathrm{Frac}(\Lambda_n))^{A_n} \otimes \bigotimes_{k=1}^{n-1} (\mathrm{Frac}(\Lambda_k\#\mathscr{M}_k))^{A_k}\right). \tag{22}$$

We finish the proof by observing that $\mathrm{Frac}(\Lambda_n) \cong \mathbb{C}(x_1, \ldots, x_n)$ and $\Lambda_k\#\mathscr{M}_k \cong \mathscr{W}_k(\mathbb{C})$ by sending $\delta^{ki}x_{ki} \mapsto X_i$ and $(\delta^{ki})^{-1} \mapsto Y_i$. $\square$

We recall for readers both the classical Noether's problem and the noncommutative Noether's problem as stated in [10]. The classical problem asks, given a finite group G and a rational function field $\Bbbk(x_1, \ldots, x_n)$ over a field $\Bbbk$ such that G acts linearly on $\Bbbk(x_1, \ldots, x_n)$, is $\Bbbk(x_1, \ldots, x_n)^G$ a purely transcendental extension of $\Bbbk$. The noncommutative problem exchanges the rational function field with the skew field of fractions of a Weyl algebra and asks if the G invariants are the skew field of some purely transcendental extension of $\Bbbk$.

Theorem 8.3 *(Theorem 1.1 in [10]). If G satisfies the Commutative Noether's problem, then G satisfies the Noncommutative Noether's Problem.*

Noether's problem for A_n is still open for $n \geq 5$. However, we obtain the following result:

Theorem 8.4. *If the alternating groups $A_1, A_2, \ldots, A_n$ provide a positive solution to Noether's problem, then $\mathscr{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov conjecture.*

Proof. If A_k satisfies Noether's problem, then $\mathrm{Frac}(\mathscr{W}_k(\mathbb{C}))^{A_k} \cong \mathrm{Frac}(\mathscr{W}_k(\mathbb{C}))$. The rest follows from Proposition 8.2. $\square$

Hence, as a corollary to Theorem 8.4 and Maeda's results in [17], we have:

Corollary 8.5. *For $n \leq 5$, $\mathscr{A}(\mathfrak{gl}_n)$ satisfies the Gelfand-Kirillov Conjecture.*

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