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On Embeddings of Countable Generalized Soluble Groups into Two-Generated Groups

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TO PROFESSOR H. S. MIKAELIAN, MY FATHER

Strengthening a theorem of L. G. Kovács and B. H. Neumann on embeddings of countable SN^*- and SI^*-groups into two-generated SN^*- and SI^*-groups, we establish embeddability of fully ordered countable SN-, SN^*-, SI-, and SI^*- groups into appropriate fully ordered two-generated groups of the same type. Moreover, for an arbitrary non-trivial word set $V \subseteq F_\infty$ the mentioned two-generated group can be chosen such that the verbal V-subgroup of the latter contains the order isomorphic copy of an initial countable group. These embeddings are subnormal but not, in general, normal.

Key Words: embeddings of groups; ordered groups; generalized soluble groups; normal and subnormal systems; ascending and descending series; two-generated groups.

1. INTRODUCTION

The starting point of this paper is the theorem of L. G. Kovács and B. H. Neumann on generalized soluble groups establishing the embeddability of an arbitrary countable SI^*-group (SN^*-group) into a two-generated SI^*-group (SN^*-group) [KN_{65}] (see Section 2, Notations and Conventions).

The importance of embeddings of countable soluble, generalized soluble groups into two-generated soluble, generalized soluble groups is explained not only by the attractivity of the theorem of G. Higman, B. H. Neumann, and Hanna Neumann [HNN_{47}] on the embeddability of an arbitrary countable group into an appropriate two-generated group ("Probably most

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famous of all embedding theorems" [R_{TG}]), but also by the fact that abelian and nilpotent groups are not, in general, embeddable into two-generated abelian and, respectively, nilpotent groups [NN_{59}, N_{68}].

In 1959 Hanna Neumann and B. H. Neumann showed that every countable soluble group G is embeddable into a soluble two-generated group H [NN_{59}]. In 1960 B. H. Neumann, solving a problem he posed himself, showed that if the (soluble) group G is fully ordered, the two-generated (soluble) group H can be chosen fully ordered such that G is order isomorphic to its image in H [N_{60}]. (And, moreover, the image of G lies in the second derived subgroup of H.)

In 1964 L. G. Kovács and B. H. Neumann extended the result of 1959 for the case of generalized soluble groups: each countable SI^*-group (SN^*-group) G is embeddable into a two-generated SI^*-group (SN^*-group) H.

It is very natural to ask whether one can "add full order" to this embedding as well, that is, *whether the given countable fully ordered SI^*-group (or SN^*-group) G is embeddable into a two-generated fully ordered SI^*-group (SN^*-group) H such that the initial group G is order isomorphic to its image in H*.

Our first aim here is to give a positive answer to this question: every countable fully ordered SI^*-group (or SN^*-group) G possesses a subnormal embedding of the type mentioned. Moreover, the analog of this holds for the case where the soluble normal (or subnormal) series of G is not ascending but descending; then the corresponding two-generated group H can be chosen to have a descending normal (subnormal) series. In fact we will prove something more general: each countable fully ordered group G with a soluble normal (subnormal) system (that is a SI- or SN-group) is subnormally embeddable into a two-generated fully ordered SI-group (or, respectively, into a two-generated fully ordered SN-group) H, such that G is order isomorphic to its image in H. These embeddings are presented by statements (i), (ii), and (iii) of Theorem 1 (Section 3).

Thorem 2 (Section 3) adds *"verbality"* to the embedding obtained. The embeddings of [NN_{59}, N_{60}] embed the given countable group not only into a two-generated group, but also into the second derived subgroup of the latter. Taking into account the intensive development of the theory of varieties of groups after the 1960s, it is a question of independent interest whether for an arbitrary non-trivial word set $V \subseteq F_\infty$ the given countable group is embeddable into a two-generated group H, such that the image $\widetilde{G}$ of G under this embedding lies in $V(H)$ (that is, whether the image of G can occupy an "arbitrarily little part of H"). These questions are solved positively in [M_{00}], and in the current investigation we prove that the requirement $\widetilde{G} \subseteq V(H)$ can be combined with requirements of statements (i), (ii), and (iii) of Theorem 1. In fact the method of our embedding was originally developed for purposes of verbal subnormal and normal embeddings [M_{00}, M_{02}, HM_{00}], and the current paper is an adaptation of our construction to the elegant method of L. G. Kovács and B. H. Neumann.

Let us note that subnormal embeddability of countable groups into two-generated groups (without requirements of full order, solubility, or verbality) was first established by R. Dark in [D_{68}] (see also [H_{74}]).

Since all embeddings we construct here are subnormal, we consider the question whether the embeddings we discuss can be normal or not. This question in general has a negative answer (Theorem 3, Examples 1, 2, and 3 in Section 5).

Examples 4 and 5 (Section 6) closing the paper are of a different nature. They show that analogs of results obtained are not, in general, true for some other classes of generalized soluble and generalized nilpotent groups (N-groups, ZA-groups).

We would like to announce here our recent result concerning non-locally soluble SI^*-groups [M_{sub2}]. An example of a group of such type is constructed by P. Hall [H_{61}] and, independently, by L. G. Kovács and B. H. Neumann [KN_{65}]. And this seems to be the only example of such a group in the literature. The verbal embedding construction of the current paper together with ideas from [O_{70}, H_{61}, KN_{65}] enables us to constuct countably and, even, continuously infinite sets of SI^*-groups which are not locally soluble. Further applications of the method provide us with continuously infinite sets of, say, SN^*-groups which are not SI^*-groups, etc. (see [M_{sub2}] for details).

2. NOTATIONS AND CONVENTIONS

Since the terms we use have no standard notation in the literature (fully ordered groups are sometimes called O-groups; SN-, SI-groups are called RN-, RI-groups in some texts[1]; normal series are sometimes called invariant series (reserving the word "normal" for what we mean by "subnormal"), etc.), let us begin with a list of main notations and definitions, referring the reader to the basic literature for details.

The set $\{G_\delta; \delta \in \Delta\}$ of subgroups of G is a *soluble subnormal system* if (1) it contains $\{1\}$ and G; (2) it is linearly ordered by inclusion; (3) it is *closed* (that is, contains unions and intersections of its elements); (4) it satisfies the condition $G_\delta \lhd G_\delta^\wedge$, where $G_\delta^\wedge$ is the intersection of all elements

[1]In particular in those authored by A. G. Kuroš and S. N. Černikov (see, for instance, in [K_{TG}]). The role of these authors is in this area so important that the classes of generalized soluble groups are sometimes called Kuroš–Černikov classes.

of $\{G_\delta; \delta \in \Delta\}$ greater than G_δ; and (5) every factor $G_\delta^\wedge/G_\delta$ is abelian. If, moreover, all of the subgroups G_δ are normal in G, $\{G_\delta; \delta \in \Delta\}$ is said to be a *soluble normal system* of G. The group G is an *SI*-group (*SN*-group) if it possesses a soluble normal (subnormal) system. The condition $G_\delta \neq G_\delta^\wedge$ is equivalent to the fact that the elements of $\{G_\delta; \delta \in \Delta\}$ can be well ordered; that is, G is an SI^*-group (or a SN^*-group) and possesses a soluble normal (or subnormal) *ascending series* $\{G_\delta; \delta \in \Delta\}$ of some ordinal length α. If, to the contrary, $G_\delta \neq G_\delta^\vee$ holds, where $G_\delta^\vee$ is the union of all elements of $\{G_\delta; \delta \in \Delta\}$ less than G_δ, then G possesses a soluble normal (or subnormal) *descending series* $\{G_\delta; \delta \in \Delta\}$ of some length β, where β is some inverse ordinal.[2] All systems and series constructed in this paper are *soluble*; so this fact is always assumed, even if it is not specially mentioned. Further information on generalized soluble groups can be found in [K_{TG}, R_{FC}, $\mathrm{KM}_{\mathrm{TG}}$].

We reserve the German letters $\mathfrak{A}$ and $\mathfrak{N}_c$ for varieties of all abelian groups and of all nilpotent groups of class at most c, respectively. For each non-trivial word set V there exist nilpotent groups N which are not contained in the variety $\mathfrak{V}$ corresponding to the word set V. Let us denote by $\mu(V)$ the least possible class of nilpotency of such nilpotent groups N. For information on varieties of groups we refer to the book of Hanna Neumann [N_{VG}].

Since we use wreath products repeatedly, let us reserve Greek lowercase letters for elements of the base group, and Roman lowercase letters for elements of the operating ("active") group. Information on the wreath product can be found in [N_{64}, $\mathrm{KM}_{\mathrm{TG}}$].

The group G is *fully ordered* if a transitive binary relation $<$ on G is defined such that for each $a, b \in G$ one and only one of the alternatives $a < b$, $a = b$, and $b < a$ holds, and if $a < b$ then $ac < bc$ and $ca < cb$ hold for arbitrary $c \in G$. The groups A and B are *order isomorphic* if there exists an order-preserving isomorphism $f\colon A \to B$. For an ordered group G we denote by G^+ and G^- the sets of "positive" and "negative" elements: an element x is positive if $1 < x$, and x is negative if $x < 1$. For information on ordered groups we refer to the papers of B. H. Neumann [N_{49}] and Levi [L_{42}, L_{43}] or to the book of Fuchs [F_{OS}].

If we have an isomorphic embedding $\beta\colon G \to H$, we avoid details immaterial to our purposes and use in proofs the same notation for the group G and its image in H. And if in the same situation the group G is ordered and if we have built an order relation on H such that its reduction on $\beta(G)$

[2]Some authors use "symmetric" notations: SN' for SN^*, and $SN^{\grave{}}$ for the class of *SN*-groups with well-ordered soluble descending subnormal series (in analogy, SI' and $SI^{\grave{}}$ for the cases of ascending and descending normal series; see for, example, [S_{GA}]). Adoption of such a notation would shorten our theorems slightly. We, nevertheless, use here the most common notation and terms.

makes the latter order isomorphic to G, we will use the same sign $<$ for the orders on G and on H.

3. THE MAIN RESULTS

The analog of the following theorem is also true for general groups (without requirements of full order or of solubility or generalized solubility on G or on H) and is presented in [M_{00}, M_{02}]. To avoid repetitions, we formulate our main result for the case of generalized soluble groups.

THEOREM 1. (i) *Every countable SI-group* (*SN-group*) *G is subnormally embeddable into a two-generated SI-group* (*SN-group*) *H.*

(ii) *If, moreover, G possesses a soluble ascending normal* (*subnormal*) *series of length α, that is, if G is an SI^*-group* (*SN^*-group*), *then H can be chosen to have an ascending normal* (*subnormal*) *series of length at most $\lambda_1 = \alpha + 2$.*
And if G possesses a soluble descending normal (*subnormal*) *series of length β* (*where β is an inverse ordinal number*), *then H can be chosen to have a descending normal* (*subnormal*) *series of length at most $\lambda_2 = \beta + 2$.*
In general the values of λ_1 and λ_2 are the best possible.

(iii) *If, moreover, G is a fully ordered group, the group H of statements* (i) *and* (ii) *can be chosen fully ordered in such a way that G is order isomorphic to its image $\widetilde{G}$ in H.*

The embeddings mentioned in statements (i), (ii), and (iii) of Theorem 1 are in fact not only into H but also into the second derived subgroup of the latter. Moreover:

THEOREM 2. *Let G be as in Theorem* 1. *Then for an arbitrary non-trivial word set $V \subseteq F_\infty$ the group H of statements* (i) *and* (iii) *of Theorem* 1 *can be constructed in such a way that the isomorphic copy $\widetilde{G}$ of G lies in the verbal subgroup $V(H)$.*
This is, in general, not true for statement (ii) *of Theorem* 1. *Nevertheless the condition $\widetilde{G} \subseteq V(H)$ can be satisfied by an embedding in the sense of* (ii) *if λ_1 and λ_2 are replaced by $\lambda_1 = \alpha + 2 + \mu(V)$ and $\lambda_2 = \beta + 2 + \mu(V)$.*

Remark 1. It follows from (ii) that every soluble countable (fully ordered) group of length, say, l is subnormally embeddable into a soluble two-generated (fully ordered) group of length at most $l + 2$ [NN_{59}, N_{60}]; and, moreover, for an arbitrary but fixed non-trivial word set V the corresponding two-generated group H (of length at most $l + 2 + \mu(V)$) can be chosen such that the image $\widetilde{G}$ of G lies in $V(H)$ and is of order isomorphic to G [M_{02}].

Remark 2. Since $\beta + j = \beta$ for an arbitrary infinite inverse ordinal number β and for an arbitrary integer j, we obtain that the two-generated group H of the statements (ii) and (iii) of Theorem 1 has a descending series of the same length β as that of G, if G possesses a descending series of infinite inverse ordinal type β.

Remark 3. Note that the value of $\mu(V)$ does not depend on the initial group G. Once found for the given word set V, it can be used in constructions for all countable (fully ordered, generalized soluble) groups G.

4. THE MAIN EMBEDDING CONSTRUCTION

To avoid consideration of many situations similar to each other, it is more convenient to build the main subnormal verbal embedding construction described in Theorem 2 for a given non-trivial word set V and for a fully ordered countable group G possessing a soluble system. Statements (ii) and (iii) of Theorem 1 will be obtained then as simplifications of the main construction. This will bring our proof closer to that of [M$_{02}$] and will enable us to refer to that paper and to avoid superfluous repetitions of details of proof. Nevertheless, the proof here will be detailed enough, and we will be able to see the group H, the structure of its full order, as well as the normal or subnormal system of H in explicit form.

First, for the non-trivial word set V, we have to find a fully ordered torsion free nilpotent group S with a non-trivial positive element $a \in V(S)$ (see Lemma 2 in [M$_{02}$] for details). We take $S = F_k(\mathfrak{N}_c)$ to be a free nilpotent group of some class $c = \mu(V)$ and some rank k such that $S \notin \mathfrak{V}$, where $\mathfrak{V} = \mathrm{var}(F_\infty / V(F_\infty))$ is the variety corresponding to V. First we order the factors S_j / S_{j+1} of the lower central series

$$S = S_1 \geq S_2 \geq \cdots \geq S_{c+1} = \{1\}$$

of S lexicographically as direct powers of finitely many copies of infinite cycles. Next we continue this full order on the whole group S by defining the sets S^+ and S^- of positive and negative elements of S.

We use a useful criterion of F. W. Levi [L$_{42}$] for fully ordered groups. The group A is fully ordered if and only if it can be presented as a union

$$A = A^- \cup \{1\} \cup A^+$$

such that A^- and A^+ are semigroups, and for arbitrary $a \in A$

$$a^{-1} \cdot A^+ \cdot a \subseteq A^+$$

holds. If the given group A is presented in the form mentioned one can set for $a, b \in A$

$$a < b \quad \text{if and only if} \quad a^{-1} b \in A^+.$$

In our situation if

$$S_{c-i}^{+} = \{x \in S_{c-i} \mid 1 < x(\mathrm{mod} S_{c-i+1})\}$$

and

$$S_{c-i}^{-} = \{x \in S_{c-i} \mid x < 1(\mathrm{mod} S_{c-i+1})\},$$

we set

$$S^{+} = \bigcup_{i=0}^{c-1} S_{c-i}^{+} \quad \text{and} \quad S^{-} = \bigcup_{i=0}^{c-1} S_{c-i}^{-},$$

and define for $x, y \in S$

$$x < y \quad \text{if and only if} \quad x^{-1}y \in S^{+}.$$

Finally we take an arbitrary non-trivial element $a \in V(S) \neq \{1\}$. In any case we can assume a to be positive, for we are always in position to replace our order relation $<$ by the inverse relation $<^{-1}$.

As an element of $V(S)$ our element a has the presentation

$$a = (v_1(a_{11}, \dots, a_{1t_1}))^{\varepsilon_1} \cdots (v_d(a_{d1}, \dots, a_{dt_d}))^{\varepsilon_d},$$

where $\varepsilon_i = \pm 1, v_i \in V, a_{ij} \in S$ $(i = 1, \dots, d; j = 1, \dots, t_i)$. Now let us take a (not necessarily countable, ordered, or generalized soluble group) G and consider the complete wreath product

$$G \operatorname{Wr} S$$

with the base group G^S. The latter contains elements χ_g defined as

$$\chi_g(s) = \begin{cases} g, & \text{if } s = a^i, i = 0, 1, 2, \dots, \\ 1, & \text{if } s \in S \backslash \{a^i \mid i = 0, 1, 2, \dots\}. \end{cases}$$

(Here a is that found above.) Denote by $T = T(G, V)$ the following subgroup of $G \operatorname{Wr} S$:

$$T = \langle \chi_g, a_{ij} \mid g \in G; i = 1, \dots, d; j = 1, \dots, t_i \rangle. \tag{1}$$

LEMMA 1. *Let V be an arbitrary non-trivial word set*:

1. *If G is a fully ordered SI-group (SN-group) and $T = T(G, V)$ is that constructed above, then T is an SI-group (SN-group) of the same cardinality as G, the group G can be subnormally embedded in T such that its image lies in $V(T)$, and T can be fully ordered in such a way that G is order isomorphic to its image in T.*

2. *If, moreover, G possesses an ascending normal (subnormal) series of length α, the group T possesses an ascending normal (subnormal) series of length at most $\alpha + \mu(V)$.*

And if G possesses a descending normal (subnormal) series of length β, the group T possesses a descending normal (subnormal) series of length at most $\beta + \mu(V)$.

Proof. Clearly $V(T)$ is non-trivial, for it contains the element a (and all of its powers). First let us show that $V(T)$ contains the first copy of G in $G \operatorname{Wr} S$. We have

$$a^{\chi_g} = \chi_g^{-1} a \chi_g = a\left(\chi_g^{-1}\right)^a \chi_g \in V(T).$$

But as computations show,

$$\left(\left(\chi_g^{-1}\right)^a \chi_g\right)(s) = \begin{cases} g, & \text{if } s = 1 = a^0, \\ 1, & \text{if } s = a, a^2, a^3, \ldots, \\ 1, & \text{if } s \in S \backslash \{a^i \mid i = 0, 1, 2, \ldots\}. \end{cases}$$

Thus $(\chi_g^{-1})^a \chi_g = \varphi_g \in G_0$, where φ_g is the element of the first copy of G in G^S corresponding to $g \in G$. Thus $\varphi_g = a^{-1} a^{\chi_g} \in V(T)$.

Clearly $\operatorname{card}(T) = \operatorname{card}(G)$.

Now let us take an appropriate soluble normal or subnormal system $\{G_\delta; \delta \in \Delta\}$ of G and construct a system of the same type for T.

Since cartesian powers $\{\prod_{s\in S} G_\delta; \delta \in \Delta\}$ do not in general form a normal or subnormal system in the cartesian power $\prod_{s \in S} G = G^S$ (the condition $G^S = \bigcup_{s\in S}(\prod_{s\in S} G_\delta)$ may fail), we have to chose a "small part" of G^S. Denote

$$L = T \cap G^S.$$

Each element $\chi_g \in G^S$ has the value g for some $s = a^0, a^1, a^2, \ldots$ and has the value 1 for all other elements s of S, in particular, for all s such that $s < 1$. Since conjugations of χ_g by elements from S simply "shift" the coordinates of χ_g, and since multiplication of elements of the base group is "coordinate by coordinate," we can first assert that for arbitrary $\tau \in L$ there exists an s' such that for all $s < s'$, $\tau(s) = 1$ holds.

The situation is not so simple for the "right part" of τ. We cannot hope that all values $\tau(s)$ are trivial (or even equal to each other) for all sufficiently large[3] elements s.

Let us consider an element $\tau \in L$. According to (1) there exist elements

$$\chi_{g_1}, \ldots, \chi_{g_p}$$

[3]As in the proof in [KN_{65}].

of G^S and elements

$$a_{i_1 j_1}, \dots, a_{i_q j_q}$$

of S such that

$$\tau \in \langle \chi_{g_1}, \dots, \chi_{g_p}; a_{i_1 j_1}, \dots, a_{i_q j_q} \rangle. \tag{2}$$

Denote by $\delta(\tau)$ an ordinal such that the subgroup $G_{\delta(\tau)}$ contains all the finitely many elements $g_1, ..., g_p$. In spite of the fact that the coordinates of τ can take unpredictably different values, nevertheless, they all belong to $G_{\delta(\tau)}$. Therefore τ belongs to $M_{\delta(\tau)}$, where the latter is the intersection of L and Cartesian power $\prod_{s \in S} G_{\delta(\tau)}$ of copies of $G_{\delta(\tau)}$.

It is easy to verify that, if subgroups G_δ form a soluble subnormal system in G, then subgroups

$$M_\delta = L \cap \prod_{s \in S} G_\delta$$

form a soluble subnormal system in L. And if the elements G_δ form a soluble normal system in G, the elements M_δ form a soluble normal system in L (in this case elements M_δ are normal even in T).

The factor group T/L is of nilpotency class at most $\mu(V)$; so to continue the normal (subnormal) system $\{M_\delta; \delta \in \Delta\}$ to a system for the whole group T we simply add to $\{M_\delta; \delta \in \Delta\}$ the pre-images of any central series of T/L under the natural endomorphism $T \to T/L$. Let us denote the system obtained by $\{M_\delta; \delta \in \Delta'\}$.

It is clear that if the soluble system $\{G_\delta; \delta \in \Delta\}$ is a soluble ascending normal (subnormal) series (of length α) or a soluble descending normal (subnormal) series (of length β, β is an inverse ordinal), then $\{M_\delta; \delta \in \Delta'\}$ will form an ascending or, respectively, descending series of T of the same type and of length $\alpha + \mu(V)$ or, respectively, $\beta + \mu(V)$.

Finally we continue the full order of G to a full order of T as follows. If $s_1\chi_1, s_2\chi_2 \in T$ (here $s_1, s_2 \in \langle a_{ij} \mid i = 1, \dots, d; j = 1, \dots, t_i \rangle$) and $s_1\chi_1 \neq s_2\chi_2$ then

$$s_1\chi_1 < s_2\chi_2$$

if and only if $s_1 < s_2$ (according to full order defined on S) or if $s_1 = s_2$ and $\chi_1(s_0) < \chi_2(s_0)$ (according to full order of G), where s_0 is the least element of S for which $\chi_1(s_0) = \chi_2(s_0)$. This relation is a full order on T, coinciding on the first copy of G in the base group with initial order of G (see details in Lemma 2 in [M_{02}]). Lemma 1 is proved. ∎

To continue our construction we need a subnormal embedding of T into the derived subgroup of some group D of the same cardinality as T, such that

a. the full order of T can be continued on D;

b. if T is an SI-group (SN-group), D can be chosen to be an SI-group (SN-group);

c. if T has a soluble ascending normal (subnormal) series of length $\alpha + \mu(V)$, then D has a soluble ascending normal (subnormal) series of length at most $\alpha + \mu(V) + 1$;

d. and if T has a soluble descending normal (subnormal) series of length $\beta + \mu(V)$, then D has a soluble descending normal (subnormal) series of length at most $\beta + \mu(V) + 1$.

Let us note that if we simply apply Lemma 1 to T for the case where V consists of a commutator word $w = [x_1, x_2]$, we get an embedding with properties described above with one possible exception. We will get slightly greater values for lengths of ascending and descending series of the group constructed,

$$\alpha + \mu(V) + \mu(w) = \alpha + \mu(V) + 2$$

and

$$\beta + \mu(V) + \mu(w) = \beta + \mu(V) + 2,$$

respectively (clearly $\mu(w) = 2$). To obtain values $\alpha + \mu(V) + 1$ and $\beta + \mu(V) + 1$ for later purposes we have to use another construction frequently used in the literature.

LEMMA 2. *Every fully ordered group T can be subnormally embedded into a fully ordered group D of the same cardinality as T, which belongs to variety* $\operatorname{var}(T) \cdot \mathfrak{A}$ *and which satisfies conditions* a, b, c, *and* d *listed previously.*

Proof. Consider the complete wreath product $T \operatorname{Wr} C$, where $C = \langle c \rangle$ is an infinite cycle and denote by ψ_g the element of the first copy of T in base group $(T)^C$ corresponding to g. In addition, define

$$\pi_g(c^i) = \begin{cases} g, & \text{if } i \geq 0, \\ 1, & \text{if } i < 0. \end{cases}$$

Then $[\pi_{g^{-1}}, c] = \psi_g$, and the first copy of T lies in the derived subgroup of the group

$$D = \langle \pi_g, c \mid g \in T \rangle.$$

Embed T into D by the rule $g \mapsto \psi_g$, for all $g \in G$. Let us construct the normal (subnormal) system of D. Elements ζ of intersection

$$Y = D \cap (T)^C$$

clearly have the following properties:

1. there exists an integer i_1 such that for arbitrary $i < i_1$, $\zeta(c^i) = 1$ holds;
2. there exists an integer i_2 such that for arbitrary $i', i'' > i_2$, $\zeta(c') = \zeta(c'')$ holds.

Thus every element ζ has only finitely many different coordinates from T. Therefore intersections K_δ of cartesian powers $\prod_{c^i \in C} M_\delta$ (of members M_δ of a normal (subnormal) system of T) with Y form a normal (subnormal) system $\{K_\delta; \delta \in \Delta'\}$ of Y. (If M_δ is normal in T, then K_δ is normal even in D.) To get the corresponding normal (subnormal) system $\{K_\delta; \delta \in \Delta''\}$ of D it remains to add one more member, namely D, to the system built for Y.

Clearly if $\{M_\delta; \delta \in \Delta'\}$ is an ascending or descending series for T, then the system $\{K_\delta; \delta \in \Delta''\}$ also is an ascending or descending series for D.

To continue the full order of T on D we set for two non-equal elements of D

$$c^i \pi_i < c^j \pi_j$$

(here i and j are integers, π_i and π_j are arbitrary elements of Y) if and only if $i < j$ or if $i = j$ and $\pi_i(c^k) < \pi_j(c^k)$, where c^k is the least power of c for which $\pi_i(c^k)$ and $\pi_j(c^k)$ are different. (See details in the proof of Lemma 4 in [M_{02}].) ∎

Next we take another infinite cycle $Z = \langle z \rangle$ and embed D into the complete wreath product $D \operatorname{Wr} Z$, onto the first copy of D in the base group D^Z. Denote by W the direct power $\overline{\prod}_{z \in Z} D$ of copies of D. The direct powers

$$W_\delta = \overline{\prod_{z \in Z}} K_\delta, \qquad \delta \in \Delta''$$

of elements K_δ do form a normal (subnormal) system for W (each element of W has only finitely many non-trivial coordinates). This system $\{W_\delta; \delta \in \Delta''\}$ is an ascending or descending series, provided that $\{K_\delta; \delta \in \Delta''\}$ is an ascending or, respectively, descending series.

Now assume the initial group G and, therefore, the group D to be countable:

$$D = \{d_0, d_1, \ldots, d_n, \ldots;\ n \in \mathbb{N}\}.$$

Define an element ω in D^Z:

$$\omega(z^i) = \begin{cases} d_k, & \text{if } i = 2^k,\ k = 0, 1, 2, \ldots, \\ 1, & \text{if } i \in \mathbb{Z} \backslash \{2^k \mid k = 0, 1, 2, \ldots\}. \end{cases}$$

For arbitrary d_n (that is, for every n) $\omega^{(z^{-2^n})}(1) = d_n$ holds. So for each pair d_n and d_m we have

$$[\omega^{(z^{-2^n})}, \omega^{(z^{-2^m})}](1) = [d_n, d_m]. \tag{3}$$

Furthermore, for arbitrary $j \neq 0$,

$$[\omega^{(z^{-2^n})}, \omega^{(z^{-2^m})}](z^j) = 1 \tag{4}$$

(for details see [M_{02}]). Thus every element of the derived subgroup D' belongs to the derived subgroup of a two-generated group,

$$H = \langle \omega, z \rangle.$$

Thus H contains as subnormal subgroups isomorphic copies of T and, therefore, of G (and the latter lies in $V(H)$).

Consider the intersection $B = H \cap W$. Intersections

$$B_\delta = W_\delta \cap H, \qquad \delta \in \Delta'',$$

form a soluble normal (subnormal) system for B. This system $\{B_\delta; \delta \in \Delta''\}$ can be continued to a normal (subnormal) system $\{B_\delta; \delta \in \Delta'''\}$ for the whole group H. We simply add one more member: the group H itself. For, first, it is clear that W_δ are normal in B, and then they are normal in H, too; and, second, the derived subgroup H' is contained in W' according to (3) and (4).

It remains to continue the full order of D on the group H. Assume $z^i\sigma$ to be any element of H (i is an integer and σ belongs to $H \cap D^Z$). It is clear that for each such σ there is an integer i_3 such that for all $i < i_3$ $\sigma(i) = 1$ holds. Thus for arbitrary non-equal elements $z^i\sigma_i$ and $z^j\sigma_j$ we can set

$$z^i\sigma_i < z^j\sigma_j$$

if and only if $i < j$ or if $i = j$ and $\sigma_i(z^k) < \sigma_j(z^k)$, where z^k is the least power of z such that $\sigma_i(z^k) \neq \sigma_j(z^k)$ (see [M_{02}] for verification).

The embedding promised in Theorem 2 is built. Moreover, this construction provides suitable embedding for statements (i) and (iii) of Theorem 1. In fact, to build an embedding for statements (i) and (iii) (without verbality), one could shorten the proof. Nevertheless we will not describe that version, to avoid unnecessary repetition.

To complete the proof of statement (ii) of Theorem 1 it only remains to confirm the values $\alpha + 2$ and $\beta + 2$ promised in (ii). Since for this case we are free from the requirement $\widetilde{G} \subseteq V(H)$, we embed the group G directly into the group D of Lemma 2 in the same way as the group T is embedded in D. The rest of the proof remains unchanged.

We conclude the proof with an illustration of the assertion of Theorem 2 regarding the impossibility of values $\lambda_1 = \alpha + 2$ and $\lambda_2 = \beta + 2$ for the general case of verbal embeddings.

If V contains the commutator word $\delta_t(x_1, \ldots, x_{2^t})$,

$$\delta_0 = x, \qquad \delta_t(x_1, \ldots, x_{2^t}) = [\delta_{t-1}(x_1, \ldots, x_{2^{t-1}}), \delta_{t-1}(x_{2^{t-1}+1}, \ldots, x_{2^t})],$$

$t = 0, 1, 2, \ldots$, then for $t > 2$ the two-generated group H (constructed for this V and for a countable soluble group G of length l) cannot be of length $l+2$ because, if it is the case, the verbal subgroup $V(H)$ of H is of length at most $l+2-t$. But such a group cannot contain a subgroup of length l.

Theorems 1 and 2 are proved. ∎

Remark 4. It is interesting to note that if the initial group G is torsion free, the group H constructed is also torsion free. Of course, we cannot prove the analog of this for the case of embeddings of periodical groups into periodical groups, for the latter cannot be fully ordered.

5. NORMAL EMBEDDINGS, CONNECTIONS WITH A PROBLEM OF H. HEINEKEN, EXAMPLES

All embeddings we construct here are subnormal. It is a question of independent interest whether these embeddings can be normal or not.

An uncomplicated example can show that there are countable (even finite) groups which cannot be normally embedded into a two-generated group.

EXAMPLE 1. Assume that the countable group G is the normal subgroup of the two-generated group $H = \langle x, y \rangle$. Then each element $h \in H$ operates by conjugation on G as an automorphism f_h of the latter. This defines an isomorphism of $H/C_H(G)$ onto some subgroup H^* of $\mathrm{Aut}\,(G)$. The isomorphism maps $GC_H(G)/C_H(G)$ onto $\mathrm{Inn}\,(G)$. Thus $\mathrm{Inn}\,(G)$ is contained in a two-generated subgroup $H^* = \langle f_x, f_y \rangle$ of $\mathrm{Aut}\,(G)$.

Now let us take G to be an arbitrary complete group which is countable (or finite) but which cannot be generated by two elements. Since in this case

$$G = \mathrm{Inn}\,(G) = \mathrm{Aut}\,(G),$$

we get a contradiction: G cannot be contained in a two-generated subgroup of itself.

As concrete examples of G one can take one of the following: $G = \mathrm{Aut}\,(F)$, where F is a free countable group of rank greater than 1 [DF_{75}, R_{89}]; one of the complete Mathieu groups [$\mathrm{BH}_{\mathrm{EG}}$]; and $G = \mathrm{Aut}\,(L)$, where L is an arbitrary non-abelian finite simple group, such that $\mathrm{Aut}\,(L)$ cannot be generated by two elements [R_{TG}, K_{TG}, R_{89}], etc.

The question of whether the embedding of Theorem 2 can be normal leads us to another consideration. We advisedly avoided using the fact that G is a countable group in the first part of the proof of Theorem 1. The thing is that Lemma 1 for arbitrary non-trivial V establishes a subnormal embedding (with mentioned properties) of defect 2 of an infinite (fully ordered, generalized soluble) group G into an appropriate group T of the same cardinality as G, such that the image of G lies in $V(T)$. It is natural to ask whether this defect can be reduced to 1 or not, that is, whether this embedding can be normal or not. So this time we stress the *verbality* of embedding, not the fact that the embedding is *into a two-generated group H*.

The problem of normal verbal embeddings has been formulated (and solved for finite p-groups) by H. Heineken in 1992 (see [H_{92}]). B. Eick has extended the construction of H. Heineken for the case of all finite groups [E_{97}]. The answer for the general case is given by our Main Theorem in [HM_{00}].

The following result, which is based on Lemma 1 of the current paper and on the Main Theorem mentioned, answers the question set above and strengthens Theorem 3 in [M_{02}] (see also [M_{98}]).

THEOREM 3. (i) *For an arbitrary fully ordered SI-group* (*SN-group*) G *there exists a fully ordered SI-group* (*SN-group*) F *of the same cardinality as* G *with a subnormal subgroup* $\widetilde{G}$ *of defect* 2, *which is order-isomorphic to* G *and lies in* $V(F)$.

(ii) *If* G *possesses a soluble ascending or descending normal* (*subnormal*) *series of length* α, F *can be chosen to have a series of the same type of length at most* $\alpha + \mu(V)$.

(iii) *The defect of embeddings mentioned in* (i) *and* (ii) *cannot in general be made smaller*: *the embedding mentioned cannot in general be normal.*

Proof. The embeddings of (i) and (ii) are constructed in Lemma 1. The statement (iii) can be illustrated by Examples 2 and 3. ∎

Remark 5. Note that statement (iii) of Theorem 3 is enough to assert that the subnormal embedding of Theorem 2 cannot in general be normal.

EXAMPLE 2. Set $G = F_n$ to be an arbitrary free group of rank $n > 1$, and set V to be an identity satisfied in the group

$$\mathrm{Aut}\,(F_n(\mathfrak{W})),$$

where $\mathfrak{W}$ is an arbitrary locally finite variety. Then G can be fully ordered and is an SN-group (and an SI-group), but it does not have a normal embedding of the mentioned type for the word set V (see [M_{02}]).

EXAMPLE 3. We can build even an example of a nilpotent group G which does not possess a normal embedding of the mentioned type. Set $G = F_n(\mathfrak{N}_c)$ to be the nilpotent free group of rank $n > 1$ and of class c; then the group $\mathrm{Aut}\,(F_n(\mathfrak{N}_c))$ has a non-trivial identity $w \equiv 1$. Thus it suffices to take $V = \{w\}$ (see [M_{02}] for details).

Concerning the problem of normal verbal embeddings, we restrict ourselves by Theorem 3 and these examples. We would like to announce here our recent paper [$M_{\mathrm{sub}1}$] which gives a criterion under which there exist normal embeddings of the mentioned type for soluble and generalized soluble groups (even without the requirement of verbality of embedding).

6. EXAMPLES OF GROUPS WHICH ARE NOT EMBEDDABLE INTO TWO-GENERATED GROUPS

The analog of Theorem 1 does not exist for some other classes of generalized soluble and generalized nilpotent groups. Let us consider two examples of it.

EXAMPLE 4. Consider N-groups (each subgroup of which can be included in an ascending subnormal series). N-groups are locally nilpotent [P_{51}]. Thus a two-generated N-group has to be nilpotent and, therefore, cannot contain an infinitely generated countable nilpotent (or even abelian) subgroup G.

EXAMPLE 5. For the very same reason the analog of Theorem 1 does not hold for ZA-groups (for groups with a central ascending series). These groups are also locally nilpotent; let us transform the proof of this statement (due to A. I. Mal'cev [M_{49}]) for our case of two-generated ZA-groups.

Let us take an arbitrary infinitely generated ZA-group G and assume that G is embedded into a two-generated ZA-group H. The latter possesses an ascending central series

$$\{1\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\gamma = H = \langle x, y \rangle. \tag{5}$$

That H is nilpotent is clear for $\gamma = 1$. Assume, furthermore, that we have proved the nilpotency of H for all $\nu < \gamma$. Certainly γ is not a limit ordinal, for there is a κ such that $x, y \in H_\kappa$. Thus there exist a limit ordinal θ and an integer i such that $\gamma = \theta + i$. Consider all possible commutator words of type

$$[\dots[[h_1, h_2], h_3]\dots, h_{i+1}], \tag{6}$$

where each of the elements $h_1, \dots, h_{i+1}$ takes one of the values x or y. All possible values of words (6) belong to H_θ and their number is finite. Thus there exists a $H_{\theta'}$ properly contained in H_θ and containing all values of (6). It is possible to build an ascending central series of length $\theta' + i < \theta + i$ of the group H in the following way. We take all members of central series (5) from $\{1\}$ to $H_{\theta'}$ and define new members

$$H'_{\theta'+1}, H'_{\theta'+2}, \dots,$$

such that $H'_{\theta'+j+1}/H'_{\theta'+j}$ is the center of $H/H'_{\theta'+j}$, $j = 0, 1, 2, \dots$ ($H'_{\theta'} = H_{\theta'}$). The process will terminate in, at most, i steps.

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