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Riemann surfaces with real forms which have maximal cyclic symmetry

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Abstract

We determine all the Klein surfaces which have a cyclic automorphism group of the maximum possible order, and find their topological types. We also compute their full automorphism groups and show that, except for a finite number of exceptions, they coincide with the full automorphism groups of their Riemann double covers. Explicit algebraic equations of the surfaces and the formulae of their real forms and automorphisms are also given.

Keywords: Riemann surface; Klein surface; Real form; Automorphism group

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Introduction

The study of conformal automorphism groups of Riemann surfaces started at the end of the nineteenth century with the works of Schwarz, Hurwitz, Harnack, Klein, Riemann and others. Wiman [21] showed that the maximum order of a cyclic group acting on a Riemann surface of genus $g \geqslant 2$ is $4g+2$. Using Fuchsian groups, this was also proved by Harvey [14] later on. A surface for which this bound is attained, which exists for each g, is unique within its genus, and admits no more automorphisms.

It is natural to ask analogous questions concerning cyclic groups of automorphisms of Klein surfaces. A *Klein surface* is the quotient space of a Riemann surface S under the action of an anticonformal involution $\tau : S \to S$. The Riemann surface S is called the *Riemann double cover* of the Klein surface, and the isomorphism class of the Klein surface is a *real form* of S. The maximum order of a cyclic group acting on a Klein surface of algebraic genus $g \geqslant 2$ is $2g+2$ if g is even and $2g$ if g is odd, see [5,17]. The corresponding bound is attained in every genus, although not in every topological type. In this paper we determine all Klein surfaces having a maximal cyclic automorphism group. The topological types of such surfaces are also given. We show that, unlike the case of Riemann surfaces, a Klein surface attaining this bound is not unique within its genus and, in addition, it *always* admits more automorphisms. In fact, we show that the moduli space of Riemann surfaces having a real form with maximal cyclic symmetry depends on a real parameter. This parametrization is explicitly described. We also prove that, except for a finite number of exceptions, the full automorphism group of the real form coincides with that of its Riemann double cover, both being dihedral.

As with Riemann surfaces, Klein surfaces have an interesting algebraic counterpart. Alling and Greenleaf showed in [2] that the algebraic objects naturally associated to Klein surfaces are algebraic function fields in one variable over $\mathbb{R}$, that is, *real algebraic curves.* The explicit computation of the algebraic curve associated to the surface is a subject of increasing research. In this paper we exhibit algebraic equations of the surfaces we deal with and also the formulae of their real forms and automorphisms.

1. Preliminaries

A compact Klein surface is the quotient space of a compact Riemann surface S under the action of an anticonformal involution $\tau : S \to S$. The *algebraic genus* of the Klein surface $S/\langle\tau\rangle$ is defined to be the genus of its Riemann double cover. In the same way as Riemann surfaces of genus $g \geqslant 2$ are uniformized by Fuchsian groups, Klein surfaces of algebraic genus $g \geqslant 2$ are uniformized by *non-euclidean crystallographic groups*, NEC groups for short. These are the discrete subgroups Λ of the group $\mathrm{PGL}(2,\mathbb{R})$ of orientation preserving or reversing isometries of the hyperbolic plane U such that the quotient space U/Λ is compact. The first presentations for NEC groups appeared in [20] and their structure was clarified by the introduction of signatures in [15].

The *signature* of an NEC group Γ is a collection of non-negative integers and symbols, represented by $\sigma(\Gamma)$, which collects algebraic and topological features of Γ. In this paper we deal mainly with NEC groups whose signature is of the form

$$\bigl(0; +; [m_1, \dots, m_r]; \bigl\{(n_1, \dots, n_s)\bigr\}\bigr).$$

A presentation for an NEC group having this signature is the following: it has *generators* $x_1, \dots, x_r$ (elliptic isometries) and $c_0, \dots, c_s$ (reflections), and *defining relations* $x_i^{m_i} = 1$ for $i = 1, \dots, r$, $c_i^2 = (c_{j-1}c_j)^{n_j} = 1$ for $i = 0, \dots, s$ and $j = 1, \dots, s$, and $x_1 x_2 \cdots x_r c_0 (x_1 \cdots x_r)^{-1} = c_s$. A set of generators like this will be called a *set of canonical generators.*

An NEC group Λ is a *surface group* if its non-identity orientation preserving elements act fixed point free. Its name comes from the fact that each compact Klein surface X of algebraic genus $g \geqslant 2$ can be written as the quotient U/Λ for some surface group Λ. The topological features of X are reflected in the signature of Λ in the following way: if $X = U/\Lambda$ has topological genus γ and k boundary components, then

$$\sigma(\Lambda) = \bigl(\gamma; \pm; [-]; \bigl\{(-), \overset{k}{\dots}, (-)\bigr\}\bigr),$$

where a "+" appears if X is orientable and a "−" if it is non-orientable. Recall that the algebraic genus g of X equals $\alpha\gamma + k - 1$ with $\alpha = 2$ if X is orientable and $\alpha = 1$ otherwise. For a fixed g, the *topological type* or the *species* $\mathrm{spc}(X)$ of X is defined to be

$$\mathrm{spc}(X) = \begin{cases} 0, & \text{if } X \text{ has empty boundary (and is non-orientable);} \\ k, & \text{if } X \text{ has } k > 0 \text{ boundary components and is orientable;} \\ -k, & \text{if } X \text{ has } k > 0 \text{ boundary components and is non-orientable.} \end{cases}$$

The number of boundary components satisfies $k \leqslant g + 1$ with $k \equiv g + 1 \bmod 2$ if X is orientable, and $k \leqslant g$ otherwise. The surfaces we deal with in this paper turn out to have an "extremal" number of boundary components, as we shall see.

If a compact Klein surface X is written as U/Λ, then a finite group G is a group of automorphisms of X if and only if there exists an NEC group Γ containing Λ as a normal subgroup such that $G \cong \Gamma/\Lambda$. Conversely, if there exists an epimorphism $\theta : \Gamma \to G$ from an NEC group Γ onto a finite group G such that its kernel is a surface NEC group, then the quotient $X = U/\ker\theta$ has the structure of a compact Klein surface, and G acts as a group of automorphisms on it. Such an epimorphism is called *smooth.* In this situation, G is the full automorphism group $\mathrm{Aut}\, X$ of X if and only if Γ is the normalizer in $\mathrm{PGL}(2, \mathbb{R})$ of the surface group Λ, and hence G fails to be the full automorphism group of X if and only if Γ is properly contained with finite index in another NEC group Γ' which also normalises Λ.

A monomorphism $r : \Gamma \to \mathrm{PGL}(2, \mathbb{R})$ is said to be *type-preserving* if it maps reflections and glide reflections to reflections and glide reflections respectively, and if it maps elliptic and hyperbolic elements to elliptic and hyperbolic elements, respectively. Given an NEC group Γ, we let $R(\Gamma)$ denote the set of type-preserving monomorphisms $r : \Gamma \to \mathrm{PGL}(2, \mathbb{R})$ such that $r(\Gamma)$ is also an NEC group.

A *normal pair* is a pair (σ, σ') of NEC signatures such that for each NEC group Γ with signature σ there exists an NEC group Γ' with signature σ' which contains Γ as a proper normal subgroup, and such that each monomorphism $r \in R(\Gamma)$ is the restriction of a monomorphism $r' \in R(\Gamma')$. Such a signature σ is said to be *non-maximal*. The complete list of normal pairs of NEC signatures is given in [4], and it relies upon the list of normal pairs of Fuchsian signatures given by Singerman in [19]. Normal pairs play a key role in the problem of deciding whether a given group of automorphisms of a Klein surface is its full automorphism group.

Instead of viewing a Klein surface as the quotient space $S/\langle\tau\rangle$ of a Riemann surface S under an anticonformal involution τ, it is often more convenient to see it as a pair (S, τ). This way, the automorphisms of the Klein surface are identified with the conformal automorphisms of S which are compatible with τ, see [2]:

$$\mathrm{Aut}(S, \tau) = \{f \in \mathrm{Aut}(S)\colon\ f\tau = \tau f\}.$$

We shall call this group, with an abuse of language, the *centralizer* of τ in $\mathrm{Aut}(S)$. A real form of a given Riemann surface S may be viewed as the conjugacy class of an anticonformal involution $\tau : S \to S$ within the full group $\mathrm{Aut}^{\pm}(S)$ of conformal and anticonformal automorphisms of S. The *species* of a real form is the species of any of its representatives τ, which, in turn is defined to be the species of the Klein surface (S, τ). The real forms correspond to the non-isomorphic Klein surfaces whose double cover is the given Riemann surface S. In algebraic terms, they correspond to non-birationally equivalent real algebraic curves with isomorphic complexification.

Throughout this paper C_n, D_m, A_n, and S_n will denote the cyclic group of order n, the dihedral group of order $2m$, and the alternating and symmetric groups on n letters respectively. All surfaces to be considered here are compact with (algebraic) genus $g \geqslant 2$.

We begin by studying the Klein surfaces with an automorphism of the maximum possible order, which is $2g + 2$ or $2g$ depending on the parity of g. We show that they always admit more automorphisms, computing explicitly their full automorphism groups. We also determine their Riemann double covers. It turns out that all such Klein surfaces are different real forms of the same family of Riemann surfaces, namely, the family of Riemann surfaces with maximal dihedral symmetry.

2. Case g even

The maximal order of a cyclic group of automorphisms that a Klein surface of even algebraic genus g may admit is $2g + 2$, see [5,17]. In addition, $C_{2g+2} = \Gamma/\Lambda$ where Γ has either signature $\sigma_1 = (0; +; [2, g + 1]; \{(-)\})$ or $\sigma_2 = (0; +; [g + 1]; \{(2, 2)\})$. Writing $C_{2g+2} = \langle A, B \mid A^{g+1} = B^2 = [A, B] = 1\rangle$, we see that the unique smooth epimorphisms $\theta_i : \Gamma \to C_{2g+2}$ are the following: if Γ has signature σ_1, then either $\theta_1(x_1) = B$, $\theta_1(x_2) = A$ and $\theta_1(c) = B$, or $\theta_2(x_1) = B$, $\theta_2(x_2) = A$, and $\theta_2(c) = 1$. If Γ has signature σ_2, then $\theta_3(x_1) = A$, $\theta_3(c_0) = B$, $\theta_3(c_1) = 1$, and $\theta_3(c_2) = B$. It is easy to see that $\ker\theta_1$ has signature $(g + 1; -; [-]; \{-\})$, $\ker\theta_2$ has signature $(g/2; +; [-]; \{(-)\})$, and $\ker\theta_3$ has signature $(0; +; [-]; \{(-), \overset{g+1}{\ldots}, (-)\})$ (see [9,10]). Hence, the Klein surfaces we are dealing with have topological types 0, 1, or $g+1$. Observe that these are extremal values of the number of boundary components.

We claim that in all cases θ_i *always extends to a smooth epimorphism* $\theta' : \Gamma' \to G'$ *where* Γ' *has signature* $\sigma' = (0; +; [-]; \{(2, 2, 2, g+1)\})$ *and* $G' \cong D_{2g+2}$. In such a case, observe that $\ker\theta_i = \ker\theta'$ since $|\Gamma' : \Gamma| = |D_{2g+2} : C_{2g+2}|$, and so $D_{2g+2} = \Gamma'/\ker\theta_i$ is a group of automorphisms of the Klein surface $X = U/\ker\theta_i$ larger than $C_{2g+2} = \Gamma/\ker\theta_i$.

To show the claim, we consider first the case of θ_1. Observe that the pair (σ_1, σ') is a *normal pair*, see [4], so we may choose a group Γ' and a set of generators $\{c_0', c_1', c_2', c_3', c_4'\}$ for it such that the expression of the generators x_1, x_2 and c of Γ in terms of the c_i' is the following: $x_1 = c_2'c_3'$, $x_2 = c_3'c_4'$, $c = c_1'$. Let us write

$$G' := \left\langle A, B, C \;\middle|\; A^{g+1} = B^2 = C^2 = [A, B] = [B, C] = (CA)^2 = 1 \right\rangle.$$

Clearly,

$$G' = \langle A, C\rangle \times \langle B\rangle = D_{g+1} \times C_2 = D_{2g+2}.$$

It is then easy to prove that the assignment $c_0' \mapsto CA$, $c_1' \mapsto B$, $c_2' \mapsto BC$, $c_3' \mapsto C$, and $c_4' \mapsto CA$ is a well-defined smooth epimorphism $\theta' : \Gamma' \to D_{2g+2}$ which extends θ_1. This proves our claim for the epimorphism θ_1. A similar proof also works for θ_2 and θ_3.

We have shown that if X admits a cyclic group of automorphisms of order $2g+2$, then it has species 0, 1, or $g+1$ and actually admits D_{2g+2} as an automorphism group. In the next theorem we prove that D_{2g+2} is the full automorphism group $\operatorname{Aut} X$ of X, and also that, except for the Accola–Maclachlan curve (whose full automorphism group is well-known to have order $8g+8$, see [1,16]) $\operatorname{Aut} X$ coincides with the full group $\operatorname{Aut} S$ of conformal automorphisms of the Riemann double cover S of X. We also give algebraic equations for the surfaces and their automorphisms, and determine which real form realizes each species. Observe that, in algebraic terms, the equality $\operatorname{Aut} X = \operatorname{Aut} S$ means that the maximal cyclic symmetry of the real algebraic curve X implies that its complexification S admits no more automorphisms than the real ones.

The surfaces we are dealing with in this paper turn out to be hyperelliptic. For notational convenience we denote by $\rho : (z, w) \mapsto (z, -w)$ the hyperelliptic involution.

Theorem 2.1. *Let* (S, τ) *be a Klein surface of even algebraic genus* g *which admits a cyclic automorphism group* C_{2g+2} *of the maximum possible order. Then* $\operatorname{spc}(S, \tau) = 0, 1,$ *or* $g+1$. *Assume that* S *is not the Accola–Maclachlan curve. Then*

$$\operatorname{Aut}(S, \tau) = \operatorname{Aut} S = D_{2g+2}.$$

The surface S *is given by* $w^2 = (z^{g+1} - \lambda^{g+1})(z^{g+1} - 1/\lambda^{g+1})$ *where* $0 < \lambda < 1$ *or* $\lambda = e^{i\alpha}$ *with* $0 < \alpha < \pi/(2g+2)$. *Let* $\tau_1(z, w) = (1/\bar{z}, \overline{w}/\bar{z}^{g+1})$.

(i) *If* $0 < \lambda < 1$, *then either* $\tau = \tau_1$, *which has species* 0, *or* $\tau = \tau_1\rho$, *which has species* 1.

(ii) *If* $\lambda = e^{i\alpha}$ *with* $0 < \alpha < \pi/(2g+2)$, *then either* $\tau = \tau_1$ *or* $\tau = \tau_1\rho$; *both real forms have species* $g+1$.

In either case, $\mathrm{Aut}(S,\tau)$ *is generated by* $u : (z,w) \mapsto (z \cdot e^{2\pi i/(g+1)}, -w)$ *and* $v : (z,w) \mapsto (1/z, w/z^{g+1})$.

Proof. For the first equalities, since $D_{2g+2} \subset \mathrm{Aut}(S,\tau) \subset \mathrm{Aut}\, S$, it suffices to show that $\mathrm{Aut}\, S = D_{2g+2}$. The maximum order of a dihedral group of automorphisms of a Riemann surface of even genus g is $4g+4$, and so the Riemann surface S attains this bound. Such surfaces have been studied in [8, Section 4], and there it is proven that for all but a finite number of exceptions, D_{2g+2} is, indeed, the full automorphism group of the Riemann surface. One of the exceptions is the Accola–Maclachlan curve, which is excluded in the statement of the theorem. The other exception occurs in genus 2, namely it is the Riemann surface given by the equation $w^2 = z(z^4-1)$; however, this surface has no real form with D_6 as its automorphism group, see [11, Proposition 7.3].

We now deal with equations. A Riemann surface S_λ of even genus g which admits a dihedral group of automorphisms of the maximum order $4g+4$ is of the form described in the theorem for some

$$\lambda \in \left\{|\lambda| < 1\colon\ 0 < \arg(\lambda) < \pi/(g+1)\right\} \cup \{\mathrm{Im}\,\lambda = 0\colon\ 0 < \mathrm{Re}\,\lambda < 1\} \\ \cup \left\{e^{i\theta}\colon\ 0 < \theta \leqslant \pi/(2g+2)\right\},$$

and such a dihedral group is generated by the automorphisms u and v described also in the theorem, see [8, Section 4]. Hence the surface S has the above form and $\mathrm{Aut}\, S$ the above generators (if S is different from the mentioned exceptions). We now have to find out whether it admits a real form with species 0, 1, or $g+1$ whose centralizer in $\mathrm{Aut}\, S$ is $\langle u, v\rangle$.

It turns out that if S_λ admits a real form, then either $\mathrm{Im}(\lambda) = 0$ or $|\lambda| = 1$ or $\arg(\lambda) = \pi/(2g+2)$, see [8, Theorem 4.1]. If $\mathrm{Im}(\lambda) = 0$, then S_λ has four real forms, but only τ_1 and $\tau_1\rho$ have non-negative species; indeed, $\mathrm{spc}(\tau_1) = 0$ and $\mathrm{spc}(\tau_1\rho) = 1$ [7, Theorem 3.4.7(a)]. In addition, both real forms commute with u and v and so $\mathrm{Aut}(S,\tau_1) = \mathrm{Aut}(S,\tau_1\rho) = D_{2g+2}$. If $|\lambda| = 1$, then S_λ has also four real forms, and now all of them have non-negative species [7, Theorem 3.4.7(a)]; however, only τ_1 and $\tau_1\rho$, both with species $g+1$, commute with u and v. Finally, if $\arg(\lambda) = \pi/(2g+2)$, then S_λ admits a unique real form but it has negative species, see [7, Theorem 3.3.1].

Remarks 2.2.

(1) The Accola–Maclachlan curve $w^2 = z^{2g+2} + 1$ corresponds to $\lambda = e^{\pi i/(2g+2)}$ and it also admits τ_1 as a real form (see [7, Theorem 3.4.7(b)]). Moreover, amongst its four real forms, τ_1 is the unique one whose centralizer in $\mathrm{Aut}\, S$ is dihedral of order $4g+4$ and, in fact, it is generated by the automorphisms u and v of the theorem (see, e.g., [8] for an explicit description of the automorphisms of the Accola–Maclachlan curve; its real forms have also been described in [3]). That is, $\mathrm{Aut}(S,\tau_1) = D_{2g+2}$. However, the equality $\mathrm{Aut}\, S = D_{2g+2}$ is no longer true since $|\mathrm{Aut}\, S| = 8g+8$.

(2) In [8] it is shown that different values of λ give rise to non-isomorphic Riemann surfaces S_λ. Hence, the assignment $\lambda \mapsto S_\lambda$ gives a parametrization between $\{0 < \lambda < 1\} \cup \{e^{i\alpha}\colon\ 0 < \alpha < \pi/(2g+2)\} \cup \{e^{\pi i/(2g+2)}\}$ and the moduli space of Riemann surfaces of even genus which have a real form with maximal cyclic symmetry. Observe that this space has two connected components.

3. Case *g* odd

The maximal order of a cyclic group of automorphisms that a Klein surface of odd algebraic genus may admit is $2g$, see [5,17]. In this case $C_{2g} = \Gamma/\Lambda$ where Γ has signature either $\sigma_1 = (1; -; [2, 2g]; \{-\})$ or $\sigma_2 = (0; +; [2, 2g]; \{(-)\})$ or $\sigma_3 = (0; +; [2g]; \{(2, 2)\})$. (A presentation of an NEC group with signature σ_1 is $\langle d, x_1, x_2 \mid x_1x_2d^2 = x_1^2 = x_2^{2g} = 1\rangle$.) By analyzing the smooth epimorphisms $\theta_i : \Gamma \to C_{2g}$ which an NEC group Γ may admit, it is easy to see that their kernels have signatures $(g+1; -; [-]; \{-\})$, $((g-1)/2; +; [-]; \{(-), (-)\})$, or $(1; -; [-]; \{(-), \overset{g}{\ldots}, (-)\})$. Hence, the species of the Klein surfaces we are dealing with in this case are 0, 2, or $-g$. Observe that 0 and g are extremal values of the number of boundary components within the non-orientable surfaces.

It can be shown that in all cases θ_i always extends to a smooth epimorphism $\theta' : \Gamma' \to D_{2g}$ where $|\Gamma' : \Gamma| = |D_{2g} : C_{2g}| = 2$. Therefore, D_{2g} is a group of automorphisms of the Klein surface $U/\ker\theta_i$ larger than C_{2g}. The proof of this fact is similar to that in the case of even genus, and it relies upon the existence, for each $i = 1, 2, 3$ of a normal pair (σ_i, σ'). Indeed, for $i = 1$ take $\sigma' = (0; +; [2]; \{(2, 2g)\})$, while for $i = 2, 3$ take $\sigma' = (0; +; [-]; \{(2, 2, 2, 2g)\})$ [4].

Remark 3.1. The above shows, together with the results in the even genus, that the maximum order of a dihedral group acting on a compact Klein surface of algebraic genus $g \geqslant 2$ is $2g+2$ if g is even and $2g$ if g is odd. Both bounds are attained, by Klein surfaces with species 0, 1, or $g+1$ in the case of even genus, and by Klein surfaces with species 0, 2, or $-g$ in the case of odd genus. These bounds coincide with those obtained in [13] for the case of Klein surfaces with species 0.

Equations for the surfaces of odd genus which attain these bounds appear in the next theorem, where we show that, except for a finite number of exceptions, D_{2g} is also the full automorphism group of both the Klein surface and its Riemann double cover. Recall that $\rho : (z, w) \mapsto (z, -w)$ denotes the hyperelliptic involution.

Theorem 3.2. *Let (S, τ) be a Klein surface of odd algebraic genus g which admits a cyclic automorphism group C_{2g} of the maximum possible order. Then* $\mathrm{spc}(S, \tau) = 0, 2,$ *or* $-g$. *Assume that S is not any of the following surfaces*: $w^2 = z(z^{2g} + 1)$, $w^2 = z(z^{10} + 11z^5 - 1)$, $w^2 = z^8 + 14z^4 + 1$. *Then*

$$\mathrm{Aut}(S, \tau) = \mathrm{Aut}\, S = D_{2g}.$$

The surface S is given by $w^2 = z(z^g - \lambda^g)(z^g - 1/\lambda^g)$ where $0 < \lambda < 1$, or $\lambda = e^{i\alpha}$ with $0 < \alpha < \pi/(2g)$, or $\lambda = re^{\pi i/2g}$ with $0 < r < 1$. Let $\tau_1(z, w) = (1/\bar{z}, \overline{w}/\bar{z}^{g+1})$ and $\tau_2(z, w) = (-1/\bar{z}, i\overline{w}/\bar{z}^{g+1})$.

(i) *If $0 < \lambda < 1$, then either $\tau = \tau_1$, which has species 2, or $\tau = \tau_1\rho$, which has species 0.*
(ii) *If $\lambda = e^{i\alpha}$ with $0 < \alpha < \pi/(2g)$, then either $\tau = \tau_1$ or $\tau = \tau_1\rho$; both real forms have species $-g$.*
(iii) *If $\lambda = re^{\pi i/2g}$ with $0 < r < 1$, then either $\tau = \tau_2$ or $\tau = \tau_2\rho$; both real forms have species 0.*

In any case, $\mathrm{Aut}(S, \tau)$ is generated by $u : (z, w) \mapsto (z \cdot e^{2\pi i/g}, w \cdot e^{\pi i/g})$ and $v : (z, w) \mapsto (1/z, w/z^{g+1})$.

Proof. The proof is similar to that of Theorem 2.1. The Riemann surface S attains the bound of the maximum order of a dihedral group acting on odd genus g, namely $4g$. Excluding the three exceptions mentioned in the theorem, all such surfaces have D_{2g} as their full automorphism group. In addition, they are of the form S_λ described in the theorem and $\mathrm{Aut}\, S_\lambda$ is generated by u and v, see [8].

If S_λ admits a real form, then either $\mathrm{Im}(\lambda) = 0$ or $|\lambda| = 1$ or $\arg(\lambda) = \pi/(2g)$. If $\mathrm{Im}(\lambda) = 0$, then S_λ has four real forms, but only τ_1, which has species 2, and $\tau_1\rho$, which has species 0, commute with u and v [7, Theorem 3.4.7(d)]. If $|\lambda| = 1$, then S_λ also has four real forms, and again τ_1 and $\tau_1\rho$, both with species $-g$, are the unique ones which commute with u and v [7, Theorem 3.4.7(d)]. Finally, if $\arg(\lambda) = \pi/(2g)$, then S_λ admits four real forms, but only τ_2 and $\tau_2\rho$ have allowable species, see [7, Theorem 3.3.2], and commute with u and v. □

Remarks 3.3.

(1) The curve $S : w^2 = z(z^{2g} + 1)$, which corresponds to $\lambda = e^{\pi i/2g}$, has four real forms, but only τ_1, which has species $-g$, and τ_2, which has species 0, have allowable species [7, Theorem 3.4.7(e)]. In addition, their centralizer in $\mathrm{Aut}\, S$ is precisely the group generated by u and v. Hence $\mathrm{Aut}(S, \tau_i) = D_{2g}$ for $i = 1, 2$; however, the equality $\mathrm{Aut}\, S = D_{2g}$ is no longer true since this curve satisfies $|\mathrm{Aut}\, S| = 8g$.
(2) The curve $S : w^2 = z(z^{10} + 11z^5 - 1)$ (of genus 5) have four real forms, but only $\tau : (z, w) \mapsto (-1/\bar{z}, \overline{w}/\bar{z}^6)$ and $\tau\rho$ have allowable species; indeed, both have species 0 [7, Theorem 3.8.9(b)]. It turns out that both real forms commute with the full automorphism group of the curve, which is $A_5 \times C_2$ (see [12] for an explicit description of the automorphisms of this curve). That is, $\mathrm{Aut}(S, \tau) = \mathrm{Aut}(S, \tau\rho) = \mathrm{Aut}\, S = A_5 \times C_2$.
(3) The curve $S : w^2 = z^8 + 14z^4 + 1$ (of genus 3) has six real forms, four of them with allowable species [7, Theorem 3.7.9(b)]. The centralizer of two of these real forms is $D_4 \times C_2$, so the corresponding Klein surfaces do not have a cyclic automorphism group of the maximum order. The other two are $\tau : (z, w) \mapsto (-1/\bar{z}, \overline{w}/\bar{z}^4)$ and $\tau\rho$, both with species 0; it turns out that both commute with the full automorphism group of the curve, which is $S_4 \times C_2$ (see [12] for an explicit description of the automorphisms of this curve). Therefore, $\mathrm{Aut}(S, \tau) = \mathrm{Aut}(S, \tau\rho) = \mathrm{Aut}\, S = S_4 \times C_2$.

(4) In [8] it is shown that different values of λ give rise to non-isomorphic Riemann surfaces S_λ. Hence, the assignment $\lambda \mapsto S_\lambda$ gives a parametrization between $\{0 < \lambda < 1\} \cup \{e^{i\alpha}\colon\ 0 < \alpha \leqslant \pi/(2g)\} \cup \{re^{\pi i/2g}\colon\ 0 < r < 1\}$ and the moduli space of Riemann surfaces of odd genus > 5 which have a real form with maximal cyclic symmetry. Observe that this space has two connected components.

Remark 3.4. For each $g \geqslant 2$ let $\nu(g)$ denote the order of the largest group of automorphisms of a Riemann surface of genus g. The existence of the Accola–Maclachlan curve shows that $\nu(g) \geqslant 8g + 8$ for all g. In fact, this is the best bound which holds for all g. The computation of $\nu(g)$ for the case of Klein surfaces of algebraic genus g has also been studied. For each $g \geqslant 2$ May in [18] gave examples of surfaces with species $g + 1$ and $-g$ with $4g + 4$ and $4g$ automorphisms respectively, see also [6] (compare this with (ii) in Theorems 2.1 and 3.2). Hence, in the case of bordered Klein surfaces, we have $\nu(g) \geqslant 4g + 4$ if the surfaces are orientable and $\nu(g) \geqslant 4g$ if they are non-orientable. He also showed that these are the best bounds which hold for all g. The surfaces $(S_\lambda, \tau_1\rho)$ occurring in (i) in Theorem 2.1 are examples of surfaces with species 1 which also guarantee the bound $\nu(g) \geqslant 4g+4$ in the case of even genus, while the surfaces (S_λ, τ_1) occurring in (i) in Theorem 3.2 are examples of surfaces with species 2 which also guarantee the bound $\nu(g) \geqslant 4g$ in the case of odd genus.

Some results concerning $\nu(g)$ in the case of Klein surfaces with species 0 have been obtained by Conder, Maclachlan, Todorovic Vasiljevic, and Wilson in [13]. They show that for each g odd there exists such a Klein surface with a dihedral group D_{2g} of automorphisms, and so $\nu(g) \geqslant 4g$ for g odd. These surfaces are therefore those $(S_\lambda, \tau_1\rho)$ appearing in (i) in Theorem 3.2 or those (S_λ, τ_2) or $(S_\lambda, \tau_2\rho)$ appearing in (ii) in the same theorem.

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