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Equivariant localization of $\bar{D}$-modules on the flag variety of the symplectic group of degree 4 ☆

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Abstract

We will show on the flag variety of the symplectic group of degree 4 over a field of positive characteristic $p \geqslant 5$ that the direct image under the Frobenius morphism of any invertible sheaf defined by a p-regular weight is tilting. In particular, the derived localization theorem holds on the flag variety for the modules of finite type over the endomorphism ring of the direct image under the Frobenius morphism of the structure sheaf, which is locally a central reduction of the ring of crystalline differential operators.

Keywords: D-modules; Algebraic groups; Representation theory

0. Introduction

After the spectacular success of Bezrukavnikov, Mirkovic and Rumynin [9] in establishing the derived localization theorem for the sheaf $\mathcal{D}$ of algebras of crystalline differential operators on the flag variety in positive characteristic, we started in [16] an investigation into such for a central reduction $\bar{\mathcal{D}}$ of $\mathcal{D}$ to find it to hold on the projective spaces and on the flag variety of SL_3. The present work tests and verifies the derived localization theorem for $\bar{\mathcal{D}}$ on the flag variety of Sp_4.

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For a reductive algebraic group G in positive characteristic p the representation theory of G_1T, G_1 the Frobenius kernel of G and T a maximal torus of G, carries much information on the representation theory of G itself and is often more accessible. In turn, a G_1T-module is the same as a T-equivariant $\mathrm{Dist}(G_1)$-module with $\mathrm{Dist}(G_1)$ the algebra of distributions on G_1, which is a central reduction of the universal enveloping algebra $\mathbf{U}$ of the Lie algebra of G. On the flag variety $\mathcal{B} = G/B$ of G, B a Borel subgroup of G, the ideal of the Frobenius center of $\mathbf{U}$ to obtain $\mathrm{Dist}(G_1)$ also centrally reduces $\mathcal{D}$ to yield $\bar{\mathcal{D}} = \mathcal{M}od_{\mathcal{B}^{(1)}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$ the sheaf of endomorphism algebras of the structure sheaf of $\mathcal{B}$ over the structure sheaf of the Frobenius twist $\mathcal{B}^{(1)}$ of $\mathcal{B}$. The derived localization theorem for $\bar{\mathcal{D}}$ follows from Beilinson's lemma [5,6] if the dual $(F_*\mathcal{O}_\mathcal{B})^\vee$ of the direct image of the structure sheaf of $\mathcal{B}$ under the absolute Frobenius endomorphism F on $\mathcal{B}$ is tilting. In this paper we will describe for the invertible sheaf $\mathcal{L}(\mu)$ on $\mathcal{B}$ associated to any p-regular weight μ of the Borel subgroup B how the direct image $F_*\mathcal{L}(\mu)$ of $\mathcal{L}(\mu)$ under the Frobenius endomorphism decomposes on the flag variety of $G = \mathrm{SL}_2, \mathrm{SL}_3, \mathrm{Sp}_4$, and show that such $F_*\mathcal{L}(\mu)$ is tilting iff $p \geqslant h$ the Coxeter number of G, to obtain the localization theorem for $\bar{\mathcal{D}}(\mu) = \mathcal{M}od_{\mathcal{B}^{(1)}}(\mathcal{L}(\mu), \mathcal{L}(\mu))$; $(F_*\mathcal{L}(\mu))^\vee$ is isomorphic to $F_*\mathcal{L}(2(p-1)\rho - \mu)$, ρ a half sum of the positive roots of G. The decomposition of $F_*\mathcal{L}(\mu)$ is obtained by the determination of the structure on the associated Humphreys–Verma G_1B-module of highest weight μ, building on [4].

Throughout the paper $\Bbbk$ will denote an algebraically closed field of positive characteristic p. The sheaf of crystalline differential operators on a smooth variety X over $\Bbbk$ coincides with Berthelot's sheaf $\mathcal{D}^{(0)}_{X/\Bbbk}$ of the $\Bbbk$-algebras of arithmetic differential operators of level 0 on X over $\Bbbk$. Let $\mathcal{D}^{(m)}_{X/\Bbbk}$ be the sheaf of $\Bbbk$-algebras of arithmetic differential operators on X over $\Bbbk$ of level m [7]. We let F_X be the absolute Frobenius endomorphism of X and $F^{m+1}_{X/\Bbbk} : X \to X^{(m+1)}$ be the $(m+1)$st Frobenius morphism relative to $\Bbbk$. Then

$$\bar{\mathcal{D}}^{(m)}_{X/\Bbbk} = \mathcal{M}od_{\mathcal{O}_{X^{(m+1)}}}(\mathcal{O}_X, \mathcal{O}_X)$$

is a central reduction of $\mathcal{D}^{(m)}_{X/\Bbbk}$. One has

$$\varinjlim_{m\to\infty} \mathcal{D}^{(m)}_{X/\Bbbk} \simeq \varinjlim_{m\to\infty} \bar{\mathcal{D}}^{(m)}_{X/\Bbbk} \simeq \mathcal{D}iff_{X/\Bbbk}$$

the sheaf of $\Bbbk$-algebras of classical differential operators on X over $\Bbbk$, so we will denote $\mathcal{D}iff_{X/\Bbbk}$ by $\mathcal{D}^{(\infty)}_{X/\Bbbk}$ or $\bar{\mathcal{D}}^{(\infty)}_{X/\Bbbk}$. For a $\Bbbk$-linear space V we let V^* denote the $\Bbbk$-dual of V while on a variety X if $\mathcal{V}$ is an $\mathcal{O}_X$-module, $\mathcal{V}^\vee$ will denote the $\mathcal{O}_X$-dual of $\mathcal{V}$. Unless otherwise specified, $\otimes$ will denote the tensor product over $\Bbbk$. We let $\otimes_X$ denote the tensor product over the structure sheaf of X. For short we will write "iff" to mean "if and only if."

In application to the representation theory it will be desirable to have the localization theorem for $\bar{\mathcal{D}}^{(0)}_{\mathcal{B}/\Bbbk}$ to be T-equivariant. To this end, we will first write down some details on the formalities of equivariant $\mathcal{D}^{(m)}$- and $\bar{\mathcal{D}}^{(m)}$-modules.

1. Arithmetic enveloping algebras

1.1. Let C be a commutative ring and G an affine group scheme over C. Put $A = C[G]$, $\mathfrak{m}_G = \ker(\varepsilon_G) \lhd A$ with ε_G the counit of G, $\mathrm{Dist}_n(G) = \{\mu \in \mathbf{Mod}_C(A, C) \mid \mu(\mathfrak{m}_G^{n+1}) = 0\}$,

$n \in \mathbb{N}$, $I_A = \ker(\mathrm{diag}^\sharp = \mathrm{mult}\colon A \otimes_C A \to A)$, and $P_G^n = P_{G/C}^n = (A \otimes_C A)/I_A^{n+1}$, $\mathit{Diff}_G^n = \mathbf{Mod}_A(P_G^n, A)$. Let $\mathrm{Dist}(G) = \bigcup_{n\in\mathbb{N}} \mathrm{Dist}_n(G)$ be the algebra of distributions over C and $\mathit{Diff}_G = \mathit{Diff}_{G/C} = \bigcup_{n\in\mathbb{N}} \mathit{Diff}_G^n$ the C-algebra of differential operators of A over C.

Let $\eta \in \mathbf{Sch}_C(G \times_C G, G \times_C G)^\times$ be an automorphism of C-schemes on $G \times_C G$ such that $(x, y) \mapsto (x, yx)$. If $\Delta_G\colon A \to A \otimes_C A$ is the comultiplication on A, the comorphism $\eta^\sharp$ of η may be described by

$$\eta^\sharp\colon a \otimes b \mapsto a \sum_i b_i' \otimes b_i \quad \text{with } \Delta_G(b) = \sum_i b_i \otimes b_i'. \tag{1}$$

It induces a commutative diagram of A-linear bijections [23, 4.4.2]

$$\begin{tikzcd} A \otimes_C A & A \otimes_C A \arrow[l, "\eta^\sharp"', "\sim"] \\ A \otimes_C \mathfrak{m}_G \arrow[u, hook] & I_A, \arrow[l, dashed, "\sim"] \arrow[u, hook] \end{tikzcd} \tag{2}$$

and hence also an A-linear bijection for each $n \in \mathbb{N}$

$$\eta_n\colon P_A^n \to A \otimes_C \left(A/\mathfrak{m}_G^{n+1}\right). \tag{3}$$

In turn, dualizing η_n yields

Proposition. *Assume G is infinitesimally flat over C* [19, I.7.9]. *$\forall n \in \mathbb{N}$, η_n induces an A-linear bijection*

$$A \otimes_C \mathrm{Dist}_n(G) \to \mathit{Diff}_G^n \quad \text{via } a \otimes \mu \mapsto a(\mu \otimes_C A) \circ \Delta_G.$$

1.2. Let G be a linear algebraic group over $\Bbbk$. For a smooth variety X over $\Bbbk$ we observed in [16] that each $\mathcal{D}_{X/\Bbbk}^{(m)}$, $m \in \mathbb{N}$, admits a presentation

$$\begin{aligned} \mathcal{D}_{X/\Bbbk}^{(m)} \simeq \mathrm{T}_\Bbbk\big(\mathcal{D}\mathit{iff}_{X/\Bbbk}^{2p^m-1}\big)\big/\big(&\lambda - \lambda 1_{\mathcal{O}_X}, \delta \otimes \delta' - \delta' \otimes \delta - [\delta, \delta'], \delta \otimes \delta'' - \delta\delta'' \mid \\ &\lambda \in \Bbbk,\ \delta'' \in \mathcal{D}\mathit{iff}_{X/\Bbbk}^{p^m-1};\ \delta, \delta' \in \mathcal{D}\mathit{iff}_{X/\Bbbk}^{p^m}\big), \end{aligned}$$

after which we introduce the mth arithmetic enveloping algebra of G to be

$$\begin{aligned} \mathbf{U}^{(m)} = \mathrm{T}_\Bbbk\big(\mathrm{Dist}_{2p^m-1}(G)\big)\big/\big(&\lambda - \lambda\varepsilon_G, \mu \otimes \mu' - \mu' \otimes \mu - [\mu, \mu'], \mu'' \otimes \mu - \mu''\mu \\ &\forall\lambda \in \Bbbk,\ \forall\mu, \mu' \in \mathrm{Dist}_{p^m}(G),\ \forall\mu'' \in \mathrm{Dist}_{p^m-1}(G)\big). \end{aligned}$$

As G is always infinitesimally flat over $\Bbbk$, the presentation of $D_G^{(m)} = \Gamma(G, \mathcal{D}_{G/\Bbbk}^{(m)})$ yields a commutative diagram of $\Bbbk[G]$-modules

$$\begin{tikzcd}
\Bbbk[G] \otimes_\Bbbk \mathrm{Dist}(G) \arrow[r, "\sim"] & \mathit{Diff}_G \\
\Bbbk[G] \otimes_\Bbbk \mathbf{U}^{(m)} \arrow[u] \arrow[r, two heads] & D_G^{(m)} \arrow[u]
\end{tikzcd}$$

with $\varinjlim_m \mathbf{U}^{(m)} \simeq \mathrm{Dist}(G)$ and the bottom horizontal map surjective.

1.3. Assume now that G is a Chevalley $\mathbb{Z}$-group scheme. Thus $\mathbb{Z}[G]$ is free over $\mathbb{Z}$ [19, II.1.1], and G is infinitesimally flat over $\mathbb{Z}$ [19, II.1.12.1]. In particular [19, I.7.4], for each commutative ring C

$$\mathrm{Dist}(G) \otimes_\mathbb{Z} C \simeq \mathrm{Dist}(G_C) \quad \text{with } G_C = G \otimes_\mathbb{Z} C.$$

In the notation of [19, II.1.12] with a Chevalley system $(X_{\pm\alpha}, H_i \mid \alpha \in R^+,\ i \in [1, \mathrm{rk}\, G])$ one has from [19, I.7.8 and 7.4.2] that

$$\mathrm{Dist}_n(G) \text{ is free over } \mathbb{Z} \text{ of basis } \frac{e^k}{k!}\binom{h}{r}\frac{f^\ell}{\ell!}, \quad |k| + |r| + |\ell| \leqslant n, \tag{1}$$

where

$$\frac{e^k}{k!} = \prod_{\alpha \in R^+} \frac{X_\alpha^{k_\alpha}}{k_\alpha!}, \qquad \binom{h}{r} = \prod_{i=1}^{\mathrm{rk}\, G} \binom{H_i}{r_i}, \qquad \frac{f^\ell}{\ell!} = \prod_{\alpha \in R^+} \frac{X_{-\alpha}^{\ell_\alpha}}{\ell_\alpha!}, \qquad |k| = \sum_{\alpha \in R^+} k_\alpha,$$
$$|r| = \sum_{i=1}^{\mathrm{rk}\, G} r_i, \qquad |\ell| = \sum_{\alpha \in R^+} \ell_\alpha.$$

Let $\mathfrak{g}_{\mathbb{Z}_{(p)}} = \mathrm{Lie}(G) \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$, $\mathbf{U}(\mathfrak{g}_{\mathbb{Z}_{(p)}})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, and $\mathbf{U}_{\mathbb{Z}_{(p)}}^{(m)}$ the mth arithmetic enveloping algebra of $G_{\mathbb{Z}_{(p)}}$, $m \in \mathbb{N}$.

Lemma. *There are natural imbeddings of $\mathbb{Z}_{(p)}$-algebras*

$$\mathbf{U}(\mathfrak{g}_{\mathbb{Z}_{(p)}}) \simeq \mathbf{U}_{\mathbb{Z}_{(p)}}^{(0)} \leqslant \mathbf{U}_{\mathbb{Z}_{(p)}}^{(m)} \leqslant \mathrm{Dist}(G_{\mathbb{Z}_{(p)}}).$$

Proof. Let us first check the isomorphism $\mathbf{U}(\mathfrak{g}_{\mathbb{Z}_{(p)}}) \simeq \mathbf{U}_{\mathbb{Z}_{(p)}}^{(0)}$. As

$$\mathrm{Dist}_1(G_{\mathbb{Z}_{(p)}}) = \mathbb{Z}_{(p)}\varepsilon_{G_{\mathbb{Z}_{(p)}}} \oplus \mathfrak{g}_{\mathbb{Z}_{(p)}},$$

there is a homomorphism of $\mathbb{Z}_{(p)}$-algebras $\phi : \mathbf{U}(\mathfrak{g}_{\mathbb{Z}_{(p)}}) \to \mathbf{U}_{\mathbb{Z}_{(p)}}^{(0)}$ such that $x \mapsto x$, $\forall x \in \mathfrak{g}_{\mathbb{Z}_{(p)}}$. There is also a homomorphism of $\mathbb{Z}_{(p)}$-algebras $\psi : \mathbf{U}_{\mathbb{Z}_{(p)}}^{(0)} \to \mathbf{U}(\mathfrak{g}_{\mathbb{Z}_{(p)}})$ such that $\gamma\varepsilon_{G_{\mathbb{Z}_{(p)}}} \mapsto \gamma$, $\forall \gamma \in \mathbb{Z}_{(p)}$, $x \mapsto x$, $\forall x \in \mathfrak{g}_{\mathbb{Z}_{(p)}}$, which is an inverse of ϕ.

To see that the natural $\mathbb{Z}_{(p)}$-homomorphisms $\mathbf{U}^{(0)}_{\mathbb{Z}_{(p)}} \to \mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}} \to \mathrm{Dist}(G_{\mathbb{Z}_{(p)}})$ are both injective, we have only to verify the injectivity of the latter. By (1)

$$\mathrm{Dist}_{2p^m-1}(G_{\mathbb{Z}_{(p)}}) = \coprod_{|k|+|r|+|\ell| \leqslant 2p^m-1} \mathbb{Z}_{(p)} \frac{e^k}{k!}\binom{h}{r}\frac{f^\ell}{\ell!}.$$

If $n \in]p^m, 2p^m-1]$, write $n = p^m + n'$. Then $\forall \alpha \in R$, $\forall i \in [1, \mathrm{rk}\, G]$,

$$\frac{X_\alpha^{p^m}}{p^m!}\frac{X_\alpha^{n'}}{n'!} = \binom{n}{p^m}\frac{X_\alpha^n}{n!} \quad \text{and} \quad \binom{H_i}{p^m}\binom{H_i}{n'} \in \binom{n}{p^m}\binom{H_i}{n} + \sum_{r<n}\mathbb{Z}\binom{H_i}{r}$$

with

$$\binom{n}{p^m} \in (\mathbb{Z}_{(p)})^\times \quad \text{as} \binom{n}{p^m} \equiv 1 \mod p.$$

For the latter see [18, 26.1]. It follows from the commutator relations in [18, §26] that $\mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}}$ is a $\mathbb{Z}_{(p)}$-span of elements of the form

$$\left\{\prod_{\alpha \in R^+} \frac{X_\alpha^{k_\alpha^0}}{k_\alpha^0!}\left(\frac{X_\alpha^{p^m}}{p^m!}\right)^{k_\alpha^1}\right\}\left\{\prod_{i=1}^{\mathrm{rk}\, G}\binom{H_i}{r_i^0}\binom{H_i}{p^m}^{r_i^1}\right\}\left\{\prod_{\alpha \in R^+}\frac{X_{-\alpha}^{\ell_\alpha^0}}{\ell_\alpha^0!}\left(\frac{X_{-\alpha}^{p^m}}{p^m!}\right)^{\ell_\alpha^1}\right\},$$

where $k_\alpha^0, r_i^0, \ell_\alpha^0 \in [0, p^m-1]$; $k_\alpha^1, r_i^1, \ell_\alpha^1 \in \mathbb{N}$. As those elements remain linearly independent over $\mathbb{Z}_{(p)}$ in $\mathrm{Dist}(G_{\mathbb{Z}_{(p)}}) \simeq \mathbf{U}_{\mathbb{Z}_{(p)}}$ the Kostant $\mathbb{Z}_{(p)}$-form of $\mathbf{U}$ over $\mathbb{Z}_{(p)}$, the natural map $\mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}} \to \mathrm{Dist}(G_{\mathbb{Z}_{(p)}})$ is injective. □

1.4. Keep the assumptions of 1.3. Berthelot's ring of arithmetic differential operators can be defined over $\mathbb{Z}_{(p)}$ when the variety is defined over $\mathbb{Z}_{(p)}$, and admits a presentation of the same type. Thus let $D^{(m)}_{G_{\mathbb{Z}_{(p)}}} = \Gamma(X, \mathcal{D}^{(m)}_{G_{\mathbb{Z}_{(p)}}})$. As $\mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}]$ is free over $\mathbb{Z}_{(p)}$, one obtains from 1.1 a commutative diagram of $\mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}]$-linear maps

$$\begin{tikzcd}
\mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}] \otimes_{\mathbb{Z}_{(p)}} \mathrm{Dist}(G_{\mathbb{Z}_{(p)}}) \arrow[r, "\sim"] & \mathit{Diff}_{G_{\mathbb{Z}_{(p)}}} \\
\mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}] \otimes_{\mathbb{Z}_{(p)}} \mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}} \arrow[r] \arrow[u, hook] & D^{(m)}_{G_{\mathbb{Z}_{(p)}}}. \arrow[u]
\end{tikzcd}$$

It follows that the bottom horizontal map is injective. It is also surjective by the presentation of $D^{(m)}_{G_{\mathbb{Z}_{(p)}}}$ and by 1.1. We have thus obtained

Proposition. *$\forall m \in \mathbb{N}$, there is a commutative diagram of natural $\mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}]$-linear maps*

$$\begin{tikzcd} \mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}] \otimes_{\mathbb{Z}_{(p)}} \mathrm{Dist}(G_{\mathbb{Z}_{(p)}}) \arrow[r, "\sim"] & \mathit{Diff}_{G_{\mathbb{Z}_{(p)}}} \\ \mathbb{Z}_{(p)}[G_{\mathbb{Z}_{(p)}}] \otimes_{\mathbb{Z}_{(p)}} \mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}} \arrow[u, hook] \arrow[r, "\sim"'] & D^{(m)}_{G_{\mathbb{Z}_{(p)}}} \arrow[u, hook] \end{tikzcd}$$

such that

$$\varinjlim_m \mathbf{U}^{(m)}_{\mathbb{Z}_{(p)}} \simeq \mathrm{Dist}(G_{\mathbb{Z}_{(p)}}).$$

1.5. Let G be a reductive algebraic group over $\Bbbk$. As G carries a structure split over $\mathbb{Z}$, we obtain

Corollary. *$\forall m \in \mathbb{N}$, there is a commutative diagram of $\Bbbk[G]$-modules*

$$\begin{tikzcd} \Bbbk[G] \otimes_{\Bbbk} \mathrm{Dist}(G) \arrow[r, "\sim"] & \mathit{Diff}_G \\ \Bbbk[G] \otimes_{\Bbbk} \mathbf{U}^{(m)} \arrow[u] \arrow[r, "\sim"'] & D^{(m)}_G \arrow[u] \end{tikzcd}$$

with

$$\varinjlim_m \mathbf{U}^{(m)} \simeq \mathrm{Dist}(G).$$

2. Equivariant $\mathcal{D}$-modules

Let G be a linear algebraic group over $\Bbbk$, X a smooth variety over $\Bbbk$ with G-action $\mathfrak{a}: G \times_{\Bbbk} X \to X$, and $\mathcal{D}_X \in \{\mathcal{D}^{(m)}_{X/\Bbbk}, \bar{\mathcal{D}}^{(m)}_{X/\Bbbk} \mid m \in [0, \infty]\}$. Put $D_X = \Gamma(X, \mathcal{D}_X)$. We will suppress $\Bbbk$ in $\times_{\Bbbk}$ and $\otimes_{\Bbbk}$. For a morphism $f: X \to Y$ of smooth varieties we will write $\mathcal{D}_{f\to}$ for the inverse image $f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ under f of $\mathcal{D}_Y$ and put $D_{f\to} = \Gamma(X, \mathcal{D}_{f\to})$. The sheaf $\mathcal{D}_{f\to}$ is equipped with a structure of $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule [8, 2.1.1], [20, 2.4], so that $D_{f\to}$ forms a (D_X, D_Y)-bimodule.

2.1. We begin with a few preliminary lemmas; most of them are straightforward and we will omit the proofs. If Y is another smooth variety over $\Bbbk$, one has an isomorphism of sheaves of $\Bbbk$-algebras

$$\mathcal{D}_{X\times Y} \to \mathcal{D}_X \boxtimes \mathcal{D}_Y \quad \text{via } 1 \mapsto 1 \otimes 1, \tag{1}$$

taking the global sections of which yields an isomorphism of $\Bbbk$-algebras

$$D_{X\times Y} \simeq D_X \otimes D_Y.$$

2.2. Let $\eta \in \mathbf{Sch}_{\Bbbk}(X, Y)^{\times}$ be an invertible morphism of $\Bbbk$-schemes from X to Y. A systematic use of $D_{\eta\to}$, suggested by Tanisaki, is a key throughout this section.

Lemma.

(i) *There is a natural isomorphism of right D_Y-modules $D_{\eta\to} \simeq D_Y$.*
(ii) *Let Z be another smooth $\Bbbk$-variety. $\forall \xi \in \mathbf{Sch}_{\Bbbk}(Y, Z)$, there is an isomorphism of (D_X, D_Z)-bimodules*

$$D_{\xi\circ\eta\to} \simeq D_{\eta\to} \otimes_{D_Y} D_{\xi\to}.$$

(iii) *There is an isomorphism of functors from the category $\mathrm{qc}(\mathcal{D}_Y)$ of quasi-coherent $\mathcal{D}_Y$-modules to the category $D_X\mathbf{Mod}$ of left D_X-modules*

$$\Gamma(X, ?) \circ \eta^* \simeq (D_{\eta\to} \otimes_{D_Y} ?) \circ \Gamma(Y, ?) : \mathrm{qc}(\mathcal{D}_Y) \to D_X\mathbf{Mod}.$$

2.3. Let Y' be another smooth $\Bbbk$-variety and let $\mathfrak{p}_2 : Y \times X \to X$, $\mathfrak{p}_{23} : Y' \times Y \times X \to Y \times X$ be the projections.

Lemma. *Let $\eta_1 \in \mathbf{Sch}_{\Bbbk}(Y \times X, Y \times X)^{\times}$ and $\eta_2 \in \mathbf{Sch}_{\Bbbk}(Y' \times Y \times X, Y' \times Y \times X)^{\times}$.*

(i) *There are an isomorphism of right D_X-modules*

$$D_{\mathfrak{p}_2\circ\eta_1\to} \simeq \Gamma(Y, \mathcal{O}_Y) \otimes D_X \simeq D_{\mathfrak{p}_2\to}$$

and an isomorphism of $(D_{Y\times X}, D_X)$-bimodules

$$D_{\mathfrak{p}_2\circ\eta_1\to} \simeq D_{\eta_1\to} \otimes_{D_{Y\times X}} D_{\mathfrak{p}_2\to}.$$

(ii) *If Z is another smooth $\Bbbk$-variety, $\forall \xi_1 \in \mathbf{Sch}_{\Bbbk}(X, Z)$ and $\forall \xi_2 \in \mathbf{Sch}_{\Bbbk}(Y \times X, Z)$, there are an isomorphism of (D_X, D_Z)-bimodules*

$$D_{\xi_1\circ\mathfrak{p}_2\to} \simeq D_{\mathfrak{p}_2\to} \otimes_{D_X} D_{\xi_1\to}$$

and an isomorphism of $(D_{Y'\times Y\times X}, D_Z)$-bimodules

$$D_{\mathfrak{p}_{23}\circ\eta_2\to} \otimes_{D_{Y\times X}} D_{\xi_2\to} \simeq D_{\xi_2\circ\mathfrak{p}_{23}\circ\eta_2\to}.$$

(iii) *There are isomorphisms of functors $\mathrm{qc}(\mathcal{D}_X) \to D_{Y\times X}\mathbf{Mod}$*

$$\Gamma(Y \times X, ?) \circ \mathfrak{p}_2^* \simeq (D_{\mathfrak{p}_2\to} \otimes_{D_X} ?) \circ \Gamma(X, ?),$$
$$\Gamma(Y \times X, ?) \circ (\mathfrak{p}_2 \circ \eta_1)^* \simeq (D_{\mathfrak{p}_2\circ\eta_1\to} \otimes_{D_X} ?) \circ \Gamma(X, ?),$$

and of functors $\mathrm{qc}(\mathcal{D}_{Y\times X}) \to D_{Y'\times Y\times X}\mathbf{Mod}$

$$\Gamma(Y' \times Y \times X, ?) \circ \mathfrak{p}_{23}^* \simeq (D_{\mathfrak{p}_{23}\to} \otimes_{D_{Y\times X}} ?) \circ \Gamma(Y \times X, ?),$$
$$\Gamma(Y' \times Y \times X, ?) \circ (\mathfrak{p}_{23} \circ \eta_2)^* \simeq (D_{\eta_2\circ\mathfrak{p}_{23}\to} \otimes_{D_{Y\times X}} ?) \circ \Gamma(Y \times X, ?).$$

2.4.

Lemma. *Let $\eta \in \mathbf{Sch}_{\Bbbk}(X, X)^{\times}$. There are an isomorphism of left D_X-modules*

$$D_{\eta\to} \simeq D_X,$$

and an isomorphism of $(\mathcal{D}_X, D_X)$-bimodules

$$\mathcal{D}_{\eta\to} \simeq \mathcal{D}_X \otimes_{D_X} D_{\eta\to}.$$

Proof. $\forall \delta \in \mathcal{D}_X$, define $\delta^{\eta} \in \mathcal{D}_X$ to be

$$\delta^{\eta} = \begin{cases} (\eta^{\sharp})^{-1} \circ \delta \circ \eta^{\sharp} & \text{if } \mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}, \\ (\eta^{\sharp})^{-1} \circ \delta \circ (\eta \times \eta)^{\sharp} & \text{if } \mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}, \end{cases} \tag{1}$$

where $(\eta \times \eta)^{\sharp} : \mathcal{P}_{X/\Bbbk,(m)} \to \mathcal{P}_{X/\Bbbk,(m)}$ is a natural morphism induced by η [7, 2.1.1, Remark 2.1.3, 2.1.4.3]. Then $\forall \delta' \in \mathcal{D}_X$,

$$(\delta\delta')^{\eta} = \delta^{\eta}\delta'^{\eta}. \tag{2}$$

In the case of $\mathcal{D}^{(m)}_{X/\Bbbk}$ one may write $\delta^{\eta} = (\eta^{\sharp})^{-1} \circ \delta \circ \eta^{\sharp}$ on $\mathcal{D}iff^{2p^m-1}_{X/\Bbbk}$, which can be extended to the whole of $\mathcal{D}^{(m)}_{X/\Bbbk}$ using the presentation [16]. But both sides are computed inside $\mathcal{D}iff_{X/\Bbbk}$, and hence agree by (2).

Identifying $D_{\eta\to}$ with D_X as right D_X-modules, the structure of (D_X, D_X)-bimodule on $D_{\eta\to}$ reads as

$$\delta_1 \cdot \delta \cdot \delta_2 = \delta_1^{\eta}\delta\delta_2. \tag{3}$$

In particular, there is an isomorphism of left D_X-modules

$$D_{\eta\to} \to D_X \quad \text{via } \delta \mapsto \delta^{\eta^{-1}}, \tag{4}$$

and hence the first assertion. Sheafifying (3) yields the second. □

2.5.

Lemma. *Assume Y is affine and let $\mathfrak{p}_2 : Y \times X \to X$ be the projection. $\forall \eta \in \mathbf{Sch}_{\Bbbk}(Y \times X, Y \times X)^{\times}$, there is a functorial isomorphism*

$$(\mathfrak{p}_2 \circ \eta)^* \circ (\mathcal{D}_X \otimes_{D_X} ?) \simeq (\mathcal{D}_{Y\times X} \otimes_{D_{Y\times X}} ?) \circ (D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} ?) : D_X\mathbf{Mod} \to \mathrm{qc}(\mathcal{D}_{Y\times X}).$$

2.6. Back now to the G-variety X let

$$\begin{tikzcd} G \times X \arrow[drr, "\mathfrak{p}"'] & & G \times G \times X \arrow[ll, "\mathfrak{p}_{23}"'] \arrow[rr, "\mathfrak{p}_{13}"] \arrow[d, "\mathfrak{p}_3"'] & & G \times X \arrow[dll, "\mathfrak{p}"] \\ & & X & & \end{tikzcd}$$

be the projections, and let $\hat{\mathfrak{a}} \in \mathbf{Sch}_{\Bbbk}(G \times X, G \times X)^{\times}$ such that $(g, x) \mapsto (g, gx)$. By the preceding we have the following isomorphisms:

$$D_{\mathfrak{a}\to} \simeq \Bbbk[G] \otimes D_X \simeq D_{\mathfrak{p}\to} \quad \text{as right } D_X\text{-modules}, \tag{1}$$

$$D_{\hat{\mathfrak{a}}\to} \simeq D_{G\times X} \quad \text{as right } D_{G\times X}\text{-modules}, \tag{2}$$

$$D_{\mathfrak{a}\to} \simeq D_{\hat{\mathfrak{a}}\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} \quad \text{as } (D_{G\times X}, D_X)\text{-bimodules}, \tag{3}$$

$$D_{\mathrm{mult}\times X\to} \simeq D_{G\times\mathfrak{a}\to} \simeq D_{\mathfrak{p}_{23}\to} \simeq D_{\mathfrak{p}_{13}\to} \simeq \Bbbk[G] \otimes D_{G\times X} \quad \text{as right } D_{G\times X}\text{-modules}, \tag{4}$$

$$\begin{aligned} D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to} &\simeq D_{\mathfrak{a}\circ(\mathrm{mult}\times X)\to} \simeq D_{G\times\mathfrak{a}\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to}, \\ D_{G\times\mathfrak{a}\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} &\simeq D_{\mathfrak{p}\circ(G\times\mathfrak{a})\to} \simeq D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to}, \\ D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} &\simeq D_{\mathfrak{p}\circ(\mathrm{mult}\times X)\to} \simeq D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to}, \end{aligned} \tag{5}$$

all as $(D_{G\times G\times X}, D_X)$-bimodules,

$$\begin{aligned} &\Gamma(G \times X, ?) \circ \hat{\mathfrak{a}}^* \simeq (D_{\hat{\mathfrak{a}}\to} \otimes_{D_{G\times X}} ?) \circ \Gamma(G \times X, ?) \\ &\quad \text{as functors } \mathrm{qc}(\mathcal{D}_{G\times X}) \to D_{G\times X}\mathbf{Mod}, \end{aligned} \tag{6}$$

$$\begin{aligned} &\Gamma(G \times X, ?) \circ \mathfrak{p}^* \simeq D_{\mathfrak{p}\to} \otimes_{D_X} \Gamma(X, ?), \quad \Gamma(G \times X, ?) \circ \mathfrak{a}^* \simeq D_{\mathfrak{a}\to} \otimes_{D_X} \Gamma(X, ?) \\ &\quad \text{both as functors } \mathrm{qc}(\mathcal{D}_X) \to D_{G\times X}\mathbf{Mod}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\Gamma(G \times G \times X, ?) \circ \mathfrak{p}_{23}^* \simeq (D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} ?) \circ \Gamma(G \times X, ?), \\ &\Gamma(G \times G \times X, ?) \circ (\mathrm{mult} \times X)^* \simeq (D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} ?) \circ \Gamma(G \times X, ?), \\ &\Gamma(G \times G \times X, ?) \circ (G \times \mathfrak{a})^* \simeq (D_{G\times\mathfrak{a}\to} \otimes_{D_{G\times X}} ?) \circ \Gamma(G \times X, ?) \end{aligned} \tag{8}$$

all as functors $\mathrm{qc}(\mathcal{D}_{G\times X}) \to D_{G\times G\times X}\mathbf{Mod}$,

$$D_{\hat{\mathfrak{a}}\to} \simeq D_{G\times X} \quad \text{as left } D_{G\times X}\text{-modules}, \tag{9}$$

$$\mathcal{D}_{\hat{\mathfrak{a}}\to} \simeq \mathcal{D}_{G\times X} \otimes_{D_{G\times X}} D_{\hat{\mathfrak{a}}\to} \quad \text{as } (\mathcal{D}_{G\times X}, D_{G\times X})\text{-bimodules}, \tag{10}$$

$$\begin{aligned} \mathfrak{p}^*(\mathcal{D}_X \otimes_{D_X} ?) &\simeq (\mathcal{D}_{G\times X} \otimes_{D_{G\times X}} ?) \circ (D_{\mathfrak{p}\to} \otimes_{D_X} ?), \\ \mathfrak{a}^*(\mathcal{D}_X \otimes_{D_X} ?) &\simeq (\mathcal{D}_{G\times X} \otimes_{D_{G\times X}} ?) \circ (D_{\mathfrak{a}\to} \otimes_{D_X} ?) \end{aligned} \tag{11}$$

both as functors $D_X\mathbf{Mod} \to \mathrm{qc}(\mathcal{D}_{G\times X})$, and

$$\begin{aligned}
\mathfrak{p}_{23}^* \circ (\mathcal{D}_{G\times X} \otimes_{D_{G\times X}} ?) &\simeq (\mathcal{D}_{G\times G\times X} \otimes_{D_{G\times G\times X}} ?) \circ (D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} ?),\\
(\mathrm{mult}\times X)^* \circ (\mathcal{D}_{G\times X} \otimes_{D_{G\times X}} ?) &\simeq (\mathcal{D}_{G\times G\times X} \otimes_{D_{G\times G\times X}} ?) \circ (D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} ?),\\
(G\times \mathfrak{a})^* \circ (\mathcal{D}_{G\times X} \otimes_{D_{G\times X}} ?) &\simeq (\mathcal{D}_{G\times G\times X} \otimes_{D_{G\times G\times X}} ?) \circ (D_{G\times \mathfrak{a}\to} \otimes_{D_{G\times X}} ?) \qquad (12)
\end{aligned}$$

all as functors $D_{G\times X}\mathbf{Mod} \to \mathrm{qc}(\mathcal{D}_{G\times G\times X})$.

2.7.

Definition. We say $\mathcal{M} \in \mathrm{qc}(\mathcal{D}_X)$ is G-equivariant iff $\mathcal{M}$ is equipped with a $\mathcal{D}_{G\times X}$-linear isomorphism $\Delta_{\mathcal{M}} : \mathfrak{a}^*\mathcal{M} \to \mathfrak{p}^*\mathcal{M}$ verifying the cocycle condition

(c)
$$\begin{tikzcd}
(\mathrm{mult}\times X)^*\mathfrak{a}^*\mathcal{M} \arrow[r, "(\mathrm{mult}\times X)^*\Delta_{\mathcal{M}}"] \arrow[d, "\sim"'] & (\mathrm{mult}\times X)^*\mathfrak{p}^*\mathcal{M} \arrow[ddddd, "\sim"] \\
\{\mathfrak{a}\circ(\mathrm{mult}\times X)\}^*\mathcal{M} \arrow[d, equal] & \\
\{\mathfrak{a}\circ(G\times\mathfrak{a})\}^*\mathcal{M} \arrow[d, "\sim"'] & \\
(G\times\mathfrak{a})^*\mathfrak{a}^*\mathcal{M} \arrow[d, "(G\times\mathfrak{a})^*\Delta_{\mathcal{M}}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
(G\times\mathfrak{a})^*\mathfrak{p}^*\mathcal{M} \arrow[d, "\sim"'] & \\
\big(\mathfrak{p}\circ(G\times\mathfrak{a})\big)^*\mathcal{M} \arrow[d, equal] & \{\mathfrak{p}\circ(\mathrm{mult}\times X)\}^*\mathcal{M} \arrow[d, equal] \\
(\mathfrak{a}\circ\mathfrak{p}_{23})^*\mathcal{M} \arrow[d, "\sim"'] & (\mathfrak{p}\circ\mathfrak{p}_{23})^*\mathcal{M} \\
\mathfrak{p}_{23}^*\mathfrak{a}^*\mathcal{M} \arrow[r, "\mathfrak{p}_{23}^*\Delta_{\mathcal{M}}"'] & \mathfrak{p}_{23}^*\mathfrak{p}^*\mathcal{M}. \arrow[u, "\sim"']
\end{tikzcd}$$

We will denote the category of G-equivariant quasi-coherent $\mathcal{D}_X$-modules by $\mathrm{qc}_G(X)$.

$\forall\phi \in \mathbf{Sch}_{\Bbbk}(X,Y)$, there is an isomorphism of $\mathcal{D}_X$-modules $\phi^*\mathcal{O}_Y = \mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{O}_Y \simeq \mathcal{O}_X$ via $f\otimes 1 \mapsto f$, and hence $\mathcal{O}_X$ is naturally equipped with a structure of G-equivariant $\mathcal{D}_X$-module. For each $\mu \in \mathrm{Dist}_{2p^m-1}(G)$ let $\delta_\mu = (\mu\otimes\Bbbk[G])\circ\Delta_G$ be the image of μ in D_G under 1.2, and let $\delta_{\mu,X}$ be the image of μ in $\mathit{Diff}_{X/\Bbbk}^{2p^m-1} \leqslant D_X$ induced by the structure of G-equivariant $\mathcal{O}_X$-module on $\mathcal{O}_X$, see 2.9 (4). Then we have in $D_{\mathfrak{a}\to} \simeq \Bbbk[G]\otimes D_X$

$$\delta_\mu \cdot (1\otimes 1) = 1\otimes\delta_{\mu,X}. \qquad (1)$$

2.8. We say $\mathcal{M} \in \mathrm{qc}(\mathcal{D}_X)$ is quasi-G-equivariant iff $\mathcal{M}$ is equipped with an $\mathcal{O}_G \boxtimes \mathcal{D}_X$-linear isomorphism $\mathfrak{a}^*\mathcal{M} \to \mathfrak{p}^*\mathcal{M}$ verifying the cocycle condition (c). E.g., $\mathcal{D}_X$ itself is naturally quasi-G-equivariant, but that structure is not G-equivariant in general; define an $\mathcal{O}_{G\times X}$-linear morphism $\theta : \mathfrak{a}^*\mathcal{D}_X \to \mathfrak{p}^*\mathcal{D}_X$ via

$$\mathcal{O}_{G\times X} \otimes_{\mathfrak{a}^{-1}\mathcal{O}_X} \mathfrak{a}^{-1}\mathcal{D}_X \ni f \otimes \delta \mapsto f(1 \boxtimes \delta)^{\hat{\mathfrak{a}}^{-1}} \cdot (1 \boxtimes 1) \in \mathcal{O}_G \boxtimes \mathcal{D}_X = \mathcal{D}_{\mathfrak{p}\to}, \tag{1}$$

regarding $1 \boxtimes \delta$ as an element of $\mathcal{D}_G \boxtimes \mathcal{D}_X \simeq \mathcal{D}_{G\times X}$. For each $g \in G(\Bbbk)$ let $i_g \in \mathbf{Sch}(X, G \times X)$ be the closed immersion $x \mapsto (g, x)$, and let $\imath_g : \mathfrak{p}^*\mathcal{D}_X \to i_{g*}i_g^*\mathfrak{p}^*\mathcal{D}_X \simeq i_{g*}\mathcal{D}_X$ be the adjunction. Then at each point $(g, x) \in (G \times X)(\Bbbk)$ $\forall f \in \mathcal{O}_{G\times X,(g,x)}$ and $\forall \delta \in \mathcal{D}_{X,gx}$,

$$\imath_g \circ \theta(f \otimes \delta) = i_g^\sharp(f)\mathrm{Ad}_X(g)\delta, \tag{2}$$

where $\mathrm{Ad}_X(g)\delta = \delta^{\mathfrak{a}(g^{-1},?)}$ with $\mathfrak{a}(g^{-1}, ?) \in \mathbf{Sch}_\Bbbk(X, X)^\times$ such that $x \mapsto \mathfrak{a}(g^{-1}, x) = g^{-1}x$; for $g_1, g_2 \in G$ one has $\mathrm{Ad}_X(g_1g_2)\delta = \mathrm{Ad}_X(g_2)\mathrm{Ad}_X(g_1)\delta$. Likewise

$$\imath_g\big((1 \otimes \delta)^{\hat{\mathfrak{a}}} \cdot (1 \otimes 1)\big) = \mathrm{Ad}_X\big(g^{-1}\big)\delta. \tag{3}$$

Using these identities, one checks that θ equips $\mathcal{D}_X$ with a structure of quasi-G-equivariant $\mathcal{D}_X$-module.

2.9. Let $\mathcal{M} \in \mathrm{qc}(\mathcal{D}_X)$. Put

$$\mathcal{D}_X(\mathcal{M}) = \begin{cases} \mathcal{D}^{(m)}_{X/\Bbbk}(\mathcal{M}) = \varinjlim_n \mathcal{M}od_X(\mathcal{P}^n_{X/\Bbbk,(m)} \otimes_X \mathcal{M}, \mathcal{M}) & \text{if } \mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}, \\ \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}(\mathcal{M}) = \mathcal{M}od_{X^{(m+1)}}(\mathcal{M}, \mathcal{M}) & \text{if } \mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}. \end{cases}$$

In the case $\mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}$ let $(\varepsilon_n)_n$ be the associated stratification on $\mathcal{M}$, and $\theta_n \in \mathbf{Mod}_X(\mathcal{M}, \mathcal{M} \otimes_X \mathcal{P}^n_{X/\Bbbk,(m)})$ corresponding to ε_n under the Frobenius reciprocity. By [7, 2.3.2] one has a commutative diagram of $\mathcal{O}_X$-rings

$$\begin{tikzcd} \delta_n \arrow[rr, mapsto] & & (\mathcal{M} \otimes_X \delta_n) \circ \varepsilon_n \arrow[dd, mapsto] \\ \mathcal{D}^{(m)}_{X/\Bbbk} \arrow[r] \arrow[dr] & \mathcal{D}^{(m)}_{X/\Bbbk}(\mathcal{M}) \arrow[d] & \\ & \mathcal{M}od_\Bbbk(\mathcal{M}, \mathcal{M}) & (\mathcal{M} \otimes_X \delta_n) \circ \theta_n \end{tikzcd} \tag{1}$$

$\forall \delta_n \in \mathcal{D}^{(m)}_{X/\Bbbk,n} = (\mathcal{P}^n_{X/\Bbbk,(m)})^\vee$. Let $\mathcal{D}iff^n_{X/\Bbbk}(\mathcal{M}) = \mathcal{M}od_X(\mathcal{P}^n_{X/\Bbbk,(\infty)} \otimes_X \mathcal{M}, \mathcal{M})$, $n \in \mathbb{N}$, be the sheaf of differential operators of $\mathcal{M}$ of order $\leqslant n$. As $\mathcal{P}^{2p^m-1}_{X/\Bbbk,(m)} \simeq \mathcal{P}^{2p^m-1}_{X/\Bbbk,(\infty)}$, one has under (1)

$$\begin{tikzcd} \mathcal{D}^{(m)}_{X/\Bbbk} \arrow[r] & \mathcal{D}^{(m)}_{X/\Bbbk}(\mathcal{M}) \\ \mathcal{D}iff^{2p^m-1}_{X/\Bbbk} \arrow[u, hook] \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{D}iff^{2p^m-1}_{X/\Bbbk}(\mathcal{M}) \arrow[u, hook] \\ \delta \arrow[r, mapsto] & \delta_{\mathcal{M}}. \end{tikzcd}$$

Thus, if we put

$$\hat{\mathcal{D}}^{(m)}_{X/\Bbbk}(\mathcal{M}) = \mathrm{T}_{\Bbbk}\big(\mathcal{D}\mathit{iff}^{\,2p^m-1}_{X/\Bbbk}(\mathcal{M})\big)/\big(\lambda - \lambda\,\mathrm{id}_{\mathcal{M}}, \delta\otimes\delta' - \delta'\otimes\delta - [\delta,\delta'], \delta''\otimes\delta - \delta''\delta \mid \lambda\in\Bbbk,\ \delta,\delta'\in\mathcal{D}\mathit{iff}^{\,p^m}_{X/\Bbbk}(\mathcal{M}),\ \delta''\in\mathcal{D}\mathit{iff}^{\,p^m-1}_{X/\Bbbk}(\mathcal{M})\big),$$

using the presentation of $\mathcal{D}_X$ yields a commutative diagram

$$\begin{tikzcd} \mathcal{D}^{(m)}_{X/\Bbbk} \arrow[r, dashed] \arrow[dr] & \hat{\mathcal{D}}^{(m)}_{X/\Bbbk}(\mathcal{M}) \arrow[d] \\ & \mathcal{D}^{(m)}_{X/\Bbbk}(\mathcal{M}). \end{tikzcd}$$

Assume now that $\mathcal{M}$ is G-equivariant with the structure morphism $\Delta_{\mathcal{M}}$. If $D_X(\mathcal{M}) = \Gamma(X, \mathcal{D}_X(\mathcal{M}))$, taking the global sections of the morphism of $\mathcal{O}_X$-rings $\mathcal{D}_X \to \mathcal{D}_X(\mathcal{M})$ yields a homomorphism of $\Bbbk$-algebras

$$\begin{tikzcd} D_X \arrow[r] & D_X(\mathcal{M}) \\ \mathit{Diff}^{\,2p^m-1}_{X/\Bbbk} \arrow[u, hook] \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & \mathit{Diff}^{\,2p^m-1}_{X/\Bbbk}(\mathcal{M}). \arrow[u, hook] \end{tikzcd} \tag{2}$$

On the other hand, $\Delta_{\mathcal{M}}$ defines as in [12, II.4.5, 4.6] a commutative diagram of $\Bbbk$-algebras, except for the ones involving the 3rd row which are only $\Bbbk$-linear:

$$\begin{tikzcd} \mathrm{Dist}(D) \arrow[r] & \mathit{Diff}_{X/\Bbbk}(\mathcal{M}) \\ \mathrm{Dist}(G_{m+1})^{\mathrm{op}} \arrow[u, hook] \arrow[r] & \bar{D}^{(m)}_{X/\Bbbk}(\mathcal{M}) \arrow[u, hook] \\ \mathrm{Dist}_{2p^m-1}(G) \arrow[u, hook] \arrow[r] \arrow[d, hook] & \mathit{Diff}^{\,2p^m-1}_{X/\Bbbk}(\mathcal{M}) \arrow[u, hook] \arrow[d, hook] \\ (\mathbf{U}^{(m)})^{\mathrm{op}} \arrow[r] \arrow[uu, bend left, two heads] & D^{(m)}_{X/\Bbbk}(\mathcal{M}), \end{tikzcd} \tag{3}$$

where the second (respectively the fourth) row is relevant according to the case $\mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}$ (respectively $\mathcal{D}^{(m)}_{X/\Bbbk}$). In particular, one has a commutative diagram

$$\begin{tikzcd}
\mathrm{Dist}(G_{m+1})^{\mathrm{op}} \arrow[r] & \bar{D}^{(m)}_{X/\Bbbk} \\
\mathrm{Dist}_{2p^m-1}(G) \arrow[u, hook] \arrow[r] \arrow[d, hook] & \mathit{Diff}^{2p^m-1}_{X/\Bbbk} \arrow[u, hook] \arrow[d, hook] \\
(\mathbf{U}^{(m)})^{\mathrm{op}} \arrow[r] & D^{(m)}_{X/\Bbbk}.
\end{tikzcd} \tag{4}$$

Put $M = \Gamma(X, \mathcal{M})$. Taking the global sections of $\Delta_{\mathcal{M}}$ equips M with a structure of G^{op}-module, i.e., of left $\Bbbk[G]$-comodule Δ_M, via

$$\begin{tikzcd}
\Gamma(G \times X, \mathfrak{a}^*\mathcal{M}) \arrow[rr, "\Gamma(G\times X, \Delta_{\mathcal{M}})"] \arrow[d, dash, "\sim"'] & & \Gamma(G \times X, \mathfrak{p}^*\mathcal{M}) \arrow[d, dash, "\sim"] \\
D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[d, dash, "\sim"'] & \circlearrowleft & D_{\mathfrak{p}\to} \otimes_{D_X} M \arrow[d, dash, "\sim"] \\
\Bbbk[G] \otimes M & & \Bbbk[G] \otimes M. \\
M \arrow[u, hook] \arrow[urr, dashed, "\Delta_M"'] & &
\end{tikzcd}$$

Proposition. *The structure of G^{op}-module on M is such that*

(i) *The actions of* $\mathrm{Dist}(G_{m+1})^{\mathrm{op}}$ *if* $\mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}$ *(respectively* $(\mathbf{U}^{(m)})^{\mathrm{op}}$ *if* $\mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}$*), on M restricting* (3) *and* (4) *to* $\mathrm{Dist}_{2p^m-1}(G)$ *coincide*

$$\begin{tikzcd}
\mathrm{Dist}_{2p^m-1}(G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & D_X(\mathcal{M}). \\
D_X \arrow[ur] &
\end{tikzcd}$$

(ii) *If we let G^{op} act on D_X via* Ad_X *from* 2.8*, then the action of D_X on M is G-linear*:

$$g(\delta v) = \big(\mathrm{Ad}_X(g)\delta\big)(gv) \quad \forall g \in G,\ \delta \in D_X,\ v \in M.$$

2.10.

Definition. A G-equivariant D_X-module, or a (G, D_X)-module for short, is a D_X-module M equipped with a structure of G^{op}-module such that:

(EQ1) the actions of $\mathrm{Dist}_{2p^m-1}(G)$ on M induced by the structure of G^{op}-module and by the $\Bbbk$-algebra homomorphism $\mathrm{Dist}(G_{m+1})^{\mathrm{op}} \to D_X$ if $\mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}$ (respectively $(\mathbf{U}^{(m)})^{\mathrm{op}} \to D_X$ if $\mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}$) coincide: the commutative diagram of 2.9 (i) holds;
(EQ2) the action of D_X is G-linear in the sense of 2.9 (ii).

We will denote the category of (G, D_X)-modules by $(G, D_X)\mathbf{Mod}$.

2.11. A converse of 2.9 holds. For that, however, we first need a lemma.

Lemma. *Let $M \in (G, D_X)\mathbf{Mod}$ with $\Delta_M : M \to \Bbbk[G] \otimes M$ the left comodule map making M into a G^{op}-module. Define a $\Bbbk[G]$-linear map $\hat{\Delta}_M$ by commutative diagram*

$$\begin{tikzcd}
\Bbbk[G] \otimes M \arrow[r, "\sim"] \arrow[d, "\Bbbk[G]\bar{\otimes}\Delta_M"'] & \big(\Bbbk[G] \otimes D_X\big) \otimes_{D_X} M \arrow[r, "\sim"] & D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[d, dashed, "\hat{\Delta}_M"] \\
\Bbbk[G] \otimes M \arrow[r, "\sim"] & \big(\Bbbk[G] \otimes D_X\big) \otimes_{D_X} M \arrow[r, "\sim"] & D_{\mathfrak{p}\to} \otimes_{D_X} M.
\end{tikzcd}$$

Then $\hat{\Delta}_M$ is a $D_{G\times X}$-linear bijection verifying the cocycle condition:

$$\text{(C)}\quad \begin{tikzcd}
D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[rr, "D_{\mathrm{mult}\times X\to}\otimes_{D_{G\times X}}\hat{\Delta}_M"] \arrow[d, "\sim"'] & & D_{\mathrm{mult}\times X\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} \otimes_{D_X} M \arrow[dddd, "\sim"] \\
D_{\mathfrak{a}\circ(\mathrm{mult}\times X)\to} \otimes_{D_X} M \arrow[d, "\sim"'] & & \\
D_{G\times\mathfrak{a}\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[d, "D_{G\times\mathfrak{a}\to}\otimes_{D_{G\times X}}\hat{\Delta}_M"'] \arrow[r, phantom, "\circlearrowleft"] & \, & \\
D_{G\times\mathfrak{a}\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} \otimes_{D_X} M \arrow[d, "\sim"'] & & \\
D_{\mathfrak{a}\circ\mathfrak{p}_{23}\to} \otimes_{D_X} M \arrow[d, "\sim"'] & & D_{\mathfrak{p}\circ(\mathrm{mult}\times X)\to} \otimes_{D_X} M \\
D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[rr, "D_{\mathfrak{p}_{23}\to}\otimes_{D_{G\times X}}\hat{\Delta}_M"'] & & D_{\mathfrak{p}_{23}\to} \otimes_{D_{G\times X}} D_{\mathfrak{p}\to} \otimes_{D_X} M. \arrow[u, "\sim"']
\end{tikzcd}$$

Proof. The bijectivity of $\hat{\Delta}_M$ follows from two identities

$$\begin{aligned}&\big(\Bbbk[G] \bar{\otimes} \sigma_G \otimes M\big) \circ \big(\Bbbk[G] \otimes \Delta_M\big) \circ \big(\Bbbk[G] \bar{\otimes} \Delta_M\big)\\&\quad = \mathrm{id}_{\Bbbk[G]\otimes M} = \big(\Bbbk[G] \bar{\otimes} \Delta_M\big) \circ \big(\Bbbk[G] \bar{\otimes} \sigma_G \otimes M\big) \circ \big(\Bbbk[G] \otimes \Delta_M\big),\end{aligned}$$

where σ_G is the antipode on $\Bbbk[G]$.

We check next the $D_{G\times X} \simeq D_G \otimes D_X$-linearity of $\hat{\Delta}_M$. By 1.2 the $\Bbbk[G]$-algebra D_G is generated by $\mathrm{Dist}_{2p^m-1}(G)$ through

$$\mathrm{Dist}_{2p^m-1}(G) \to D_G \quad \text{via } \mu \mapsto \big(\mu \otimes \Bbbk[G]\big) \circ \Delta_G,$$

where Δ_G is the comultiplication on $\Bbbk[G]$. By definition $\hat{\Delta}_M$ is already $\Bbbk[G]$-linear; note that the structure of $\Bbbk[G]$-module on both $D_{\mathfrak{a}\to} \otimes_{D_X} M$ and $D_{\mathfrak{p}\to} \otimes_{D_X} M$ is given by the multiplication to the left. Denoting the image of $\mu \in \mathrm{Dist}_{2p^m-1}(G)$ in D_G by δ_μ, one has

$$\begin{aligned}\hat{\Delta}_M\big(\delta_\mu \cdot (1 \otimes v)\big) &= \hat{\Delta}_M(1 \otimes \delta_{\mu,X} v) \quad \text{by 2.7 (1)}\\ &= \hat{\Delta}_M(1 \otimes \mu v) \quad \text{by (EQ1).}\end{aligned}$$

If $\Delta_M(v) = \sum_i f_i \otimes v_i$, $f_i \in \Bbbk[G]$, $v_i \in M$, on the other hand,

$$\delta_\mu \hat{\Delta}_M(1 \otimes v) = \delta_\mu \Delta_M(v) = \sum_i \delta_\mu(f_i) \otimes v_i.$$

$\forall g \in G(\Bbbk)$, one has

$$(g \otimes M)\Delta_M(\mu v) = g(\mu v) = g(\mu \otimes M)\Delta_M(v) = g \sum_i \mu(f_i) v_i = \sum_i \mu(f_i) g v_i$$

while

$$\begin{aligned}&(g \otimes M) \sum_i \delta_\mu(f_i) \otimes v_i\\ &\quad = \sum_i g\big(\delta_\mu(f_i)\big) v_i = \sum_i g\big(\big(\mu \otimes \Bbbk[G]\big)\Delta_G(f_i)\big) v_i\\ &\quad = \sum_i (\mu \otimes g)\Delta_G(f_i) v_i = (\mu \otimes g \otimes M) \sum_i \Delta_G(f_i) \otimes v_i\\ &\quad = (\mu \otimes g \otimes M) \sum_i f_i \otimes \Delta_M(v_i) \quad \text{as } (\Delta_G \otimes M) \circ \Delta_M = \big(\Bbbk[G] \otimes \Delta_M\big) \circ \Delta_M\\ &\quad = \sum_i \mu(f_i) g v_i,\end{aligned}$$

and hence $\hat{\Delta}_M(1 \otimes \mu v) = \Delta_M(\mu v) = \delta_\mu \hat{\Delta}_M(1 \otimes v)$, as desired.

We examine next the D_X-linearity of $\hat{\Delta}_M$. Let $\delta \in D_X$ and write $\delta \cdot (1 \otimes 1) = \sum_j a_j \otimes \delta_j$ in $\Bbbk[G] \otimes D_X \simeq D_{\mathfrak{a}\to}$. Then in $\Bbbk[G] \otimes M \simeq D_{\mathfrak{p}\to} \otimes_{D_X} M$

$$\hat{\Delta}_M\big(\delta \cdot (1 \otimes v)\big) = \hat{\Delta}_M\left(\sum_j a_j \otimes \delta_j v\right) = \sum_j a_j \Delta_M(\delta_j v)$$

while

$$\begin{aligned}
\delta \hat{\Delta}_M(1 \otimes v) &= \delta \sum_i f_i \otimes v_i = \sum_i \delta f_i \cdot (1 \otimes v_i) \\
&= \sum_i f_i \delta \cdot (1 \otimes v_i) \quad \text{as } \Bbbk[G] \text{ and } D_X \text{ commute in } D_{G \times X} \simeq D_G \otimes D_X \\
&= \sum_i f_i \otimes \delta v_i.
\end{aligned}$$

Then $\forall g \in G(\Bbbk)$,

$$\begin{aligned}
(g \otimes M)\hat{\Delta}_M\big(\delta \cdot (1 \otimes v)\big) &= \sum_j a_j(g) g(\delta_j v) = g\left(\sum_j a_j(g)\delta_j v\right) \\
&= g\big(\mathrm{Ad}_X\big(g^{-1}\big)\delta v\big) \quad \text{by 2.8 (3)}
\end{aligned}$$

while

$$\begin{aligned}
(g \otimes M)\delta \hat{\Delta}_M(1 \otimes v) &= \sum_i f_i(g)\delta v_i = \delta \sum_i f_i(g)\delta v_i = \delta g v \\
&= g\big(\mathrm{Ad}_X\big(g^{-1}\big)\delta\big)v \quad \text{by (EQ2)},
\end{aligned}$$

and hence $\hat{\Delta}_M(\delta \cdot (1 \otimes v)) = \delta \hat{\Delta}_M(1 \otimes v)$, as desired. We have now verified that $\hat{\Delta}_M$ is $D_{G \times X}$-linear.

Finally, to check the cocyclicity, if we write $\Delta_M(v_i) = \sum_j f_{ij} \otimes m_{ij}$,

$$\sum_i \Delta_G(f_i) \otimes v_i = (\Delta_G \otimes M) \circ \Delta_M(v) = \big(\Bbbk[G] \otimes \Delta_M\big) \circ \Delta_M(v) = \sum_{i,j} f_i \otimes f_{ij} \otimes v_{ij},$$

and hence the diagram (C) commutes, as desired. □

2.12. We have thus obtained

Proposition. *$\forall M \in D_X$**Mod**, a $\Bbbk$-linear map $\Delta_M : M \to \Bbbk[G] \otimes M$ makes M into a (G, D_X)-module iff $\hat{\Delta}_M$ is a $D_{G \times X}$-linear bijection verifying the cocycle condition.*

2.13.

Corollary. *The functor $\mathcal{D}_X \otimes_{D_X} ? : D_X$**Mod** $\to \mathrm{qc}(\mathcal{D}_X)$ restricts to a functor (G, D_X)**Mod** $\to \mathrm{qc}_G(\mathcal{D}_X)$.*

2.14. Finally, let us formulate a derived version of 2.9 and 2.13. Recall first from [11, VI.2.1] that any $\mathcal{M} \in \mathrm{qc}(\mathcal{D}_X)$ admits a mono $\mathcal{M} \rightarrowtail \mathcal{I}$ in $\mathrm{qc}(\mathcal{D}_X)$ with $\mathcal{I}$ injective in $\mathcal{D}_X\mathbf{Mod}$. In particular, any injective in $\mathrm{qc}(\mathcal{D}_X)$ remains injective in $\mathcal{D}_X\mathbf{Mod}$. As $\mathrm{qc}(\mathcal{D}_X)$ is thick in $\mathcal{D}_X\mathbf{Mod}$ by [13, 2.2.2], one obtains

$$\mathrm{D}^+\big(\mathrm{qc}(\mathcal{D}_X)\big) \simeq \mathrm{D}^+_{\mathrm{qc}}(\mathcal{D}_X). \tag{1}$$

Also,

$$\mathrm{hd}\big(\mathrm{qc}(\mathcal{D}_X)\big) \leqslant \begin{cases} 2\dim X & \text{if } \mathcal{D}_X = \bar{\mathcal{D}}^{(m)}_{X/\Bbbk}, \\ 3\dim X & \text{if } \mathcal{D}_X = \mathcal{D}^{(m)}_{X/\Bbbk}. \end{cases} \tag{2}$$

For the projection $\mathfrak{p}_2 : Y \times X \to X$ with any Y the functor $\mathfrak{p}_2^* : \mathrm{qc}(\mathcal{D}_X) \to \mathrm{qc}(\mathcal{D}_{Y\times X})$ is exact, so therefore is $(\mathfrak{p} \circ \eta)^* \simeq \eta^* \circ \mathfrak{p}^*$ for each $\eta \in \mathbf{Sch}_{\Bbbk}(Y \times X, Y \times X)^\times$. As the cohomological dimension of $\Gamma(Y \times X, ?)$ is $\leqslant \dim(Y \times X)$ by the Grothendieck vanishing, the functor

$$\mathbb{R}\Gamma(Y \times X, ?) \circ (\mathfrak{p}_2 \circ \eta)^* : \mathrm{D}^b\big(\mathrm{qc}(\mathcal{D}_X)\big) \to \mathrm{D}^b(D_{Y\times X})$$

makes sense [21, Ex. I.23].

Lemma. *Assume Y is affine. $\forall \eta \in \mathbf{Sch}_{\Bbbk}(Y \times X, Y \times X)^\times$, there is a functorial isomorphism*

$$\mathbb{R}\Gamma(Y \times X, ?) \circ (\mathfrak{p}_2 \circ \eta)^* \simeq (D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} ?) \circ \mathbb{R}\Gamma(X, ?) : \mathrm{D}^b\big(\mathrm{qc}(\mathcal{D}_X)\big) \to \mathrm{D}^b(D_{Y\times X}).$$

Proof. If $\mathcal{I}$ is an injective $\mathcal{D}_X$-module, $\forall i \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{H}^i\big(Y \times X, (\mathfrak{p}_2 \circ \eta)^*\mathcal{I}\big) \simeq \mathrm{H}^i\big(Y \times X, \eta^*\mathfrak{p}_2^*\mathcal{I}\big) \simeq \mathrm{H}^i\big(Y \times X, \big(\eta^{-1}\big)_*\mathfrak{p}_2^*\mathcal{I}\big) \\
&\quad \simeq \mathrm{H}^i\big(Y \times X, \mathfrak{p}_2^*\mathcal{I}\big) \quad \text{by the degeneracy of the Leray spectral sequence as } \big(\eta^{-1}\big)_* \text{ is exact} \\
&\quad \simeq \mathrm{H}^i(Y \times X, \mathcal{O}_Y \boxtimes \mathcal{I}) \\
&\quad \simeq \Bbbk[Y] \otimes \mathrm{H}^i(X, \mathcal{I}) \quad \text{by the Künneth formula as } Y \text{ is affine} \\
&\quad = 0 \quad \text{if } i \geqslant 1 \text{ as } \mathcal{I} \text{ is flasque [21, II.2.4.6].}
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{R}\Gamma(Y \times X, ?) \circ (\mathfrak{p}_2 \circ \eta)^* \simeq \mathbb{R}\big\{\Gamma(Y \times X, ?) \circ (\mathfrak{p}_2 \circ \eta)^*\big\} \\
&\quad \simeq \mathbb{R}\big\{(D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} ?) \circ \Gamma(X, ?)\big\} \quad \text{by 2.3 (iii)} \\
&\quad \simeq (D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} ?) \circ \mathbb{R}\Gamma(X, ?) \quad \text{as } D_{\mathfrak{p}_2\circ\eta\to} \simeq \Bbbk[Y] \otimes D_X \text{ by 2.3 (i) is flat over } D_X. \qquad \square
\end{aligned}$$

2.15.

Remark. (i) As $D_{\mathfrak{U}}$ is Noetherian for each affine open $\mathfrak{U}$ of X by [7, 2.2.5], $\mathcal{D}_X$ is a sheaf of coherent rings [7, 3.1]. Then by [17, I.4.4.4] and a result of Kleiman [15, Ex. III.6.8]

$\mathrm{D}^b(\mathrm{coh}(\mathcal{D}_X)) \simeq \mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ the full subcategory of the bounded derived category $\mathrm{D}^b(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules consisting of the complexes whose cohomologies are all coherent. One can thus define

$$\begin{tikzcd}
\mathrm{D}^b\big(\mathrm{coh}(\mathcal{D}_X)\big) \arrow[rr, dashed] \arrow[d, hook] & & \mathrm{D}^b(D_X). \\
\mathrm{D}^b(\mathcal{D}_X) \arrow[urr, "\mathbb{R}\Gamma(X,?)"'] \arrow[ur, phantom, "\circlearrowleft"] & &
\end{tikzcd}$$

(ii) In case X is projective, as $\bar{D}^{(m)}_{X/\Bbbk}$ is finite-dimensional, one has

$$\begin{tikzcd}
\mathrm{D}^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}^{(m)}_{X/\Bbbk}\big)\big) \arrow[r, "\mathbb{R}\Gamma(X,?)"] \arrow[dr, dashed] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & \mathrm{D}^b_{\mathrm{fin}}\big(\bar{D}^{(m)}_{X/\Bbbk}\big) \arrow[d, dash, "\sim"] \\
& \mathrm{D}^b\big(\bar{D}^{(m)}_{X/\Bbbk}\mathbf{mod}\big),
\end{tikzcd}$$

where $\mathrm{D}^b_{\mathrm{fin}}(\bar{D}^{(m)}_{X/\Bbbk})$ is the full subcategory of $\mathrm{D}^b(\bar{D}^{(m)}_{X/\Bbbk})$ consisting of the complexes whose cohomologies are all of finite type over $\bar{D}^{(m)}_{X/\Bbbk}$.

(iii) Assume $X = \mathcal{B} = G/B$ a flag variety with G a semisimple group and B a Borel subgroup of G, and that $\mathcal{D}_{\mathcal{B}} = \mathcal{D}^{(0)}_{\mathcal{B}/\Bbbk}$. According to [9, 3.1.8], for $p > h$ the Coxeter number of G,

$$\mathbb{R}\Gamma(\mathcal{B}, \mathcal{M}^\bullet) \in \mathrm{D}^b(D_{\mathcal{B}}\mathbf{mod}) \quad \forall \mathcal{M}^\bullet \in \mathrm{D}^b\big(\mathrm{coh}(\mathcal{D}_{\mathcal{B}})\big).$$

2.16.

Definition. For $* \in \{-, b\}$ we say $\mathcal{M} \in \mathrm{D}^*(\mathrm{qc}(\mathcal{D}_X))$ is G-equivariant iff there is $\Delta_{\mathcal{M}} \in \mathrm{D}^*(\mathrm{qc}(\mathcal{D}_{G\times X}))(\mathfrak{a}^*\mathcal{M}, \mathfrak{p}^*\mathcal{M})^\times$ verifying the cocycle condition in $\mathrm{D}^*(\mathrm{qc}(\mathcal{D}_{G\times G\times X}))$. If $\mathcal{V} \in \mathrm{D}^*(\mathrm{qc}(\mathcal{D}_X))$ is another G-equivariant object, we say $\phi \in \mathrm{D}^*(\mathrm{qc}(\mathcal{D}_X))(\mathcal{M}, \mathcal{V})$ is G-equivariant iff in $\mathrm{D}^*(\mathrm{qc}(\mathcal{D}_{G\times X}))$

$$\begin{tikzcd}
\mathfrak{a}^*\mathcal{M} \arrow[r, "\mathfrak{a}^*\phi"] \arrow[d, "\Delta_{\mathcal{M}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{a}^*\mathcal{V} \arrow[d, "\Delta_{\mathcal{V}}"] \\
\mathfrak{p}^*\mathcal{M} \arrow[r, "\mathfrak{p}^*\phi"'] & \mathfrak{a}^*\mathcal{V}.
\end{tikzcd}$$

We will denote the category of G-equivariant objects and morphisms of $\mathrm{D}^*(\mathrm{qc}(\mathcal{D}_X))$ by $\mathrm{D}^*_G(\mathrm{qc}(\mathcal{D}_X))$.

We say $M \in \mathrm{D}^*(D_X)$ is G-equivariant iff M is equipped with $\hat{\Delta}_M \in \mathrm{D}^*(D_{G\times X})(D_{\mathfrak{a}\to} \otimes_{D_X} M, D_{\mathfrak{p}\to} \otimes_{D_X} M)^\times$ verifying the cocycle condition in $\mathrm{D}^*(D_{G\times G\times X})$. If $V \in \mathrm{D}^*(D_X)$ is also G-equivariant, we say $f \in \mathrm{D}^*(D_X)(M, V)$ is G-equivariant iff in $\mathrm{D}^*(D_{G\times X})$

$$\begin{tikzcd} D_{\mathfrak{a}\to} \otimes_{D_X} M \arrow[r, "D_{\mathfrak{a}\to}\otimes_{D_X} f"] \arrow[d, "\hat{\Delta}_M"'] \arrow[dr, phantom, "\circlearrowright"] & D_{\mathfrak{a}\to} \otimes_{D_X} V \arrow[d, "\hat{\Delta}_V"] \\ D_{\mathfrak{p}\to} \otimes_{D_X} M \arrow[r, "D_{\mathfrak{p}\to}\otimes_{D_X} f"'] & D_{\mathfrak{p}\to} \otimes_{D_X} V. \end{tikzcd}$$

We will denote the category of G-equivariant objects and morphisms of $\mathrm{D}^*(D_X)$ by $\mathrm{D}^*_G(D_X)$.

2.17.

Proposition. *The functor* $\mathbb{R}\Gamma(X, ?) : \mathrm{D}^b(\mathrm{qc}(\mathcal{D}_X)) \to \mathrm{D}^b(D_X)$ *restricts to a functor*

$$\mathrm{D}^b_G\big(\mathrm{qc}(\mathcal{D}_X)\big) \to \mathrm{D}^b_G(D_X).$$

Proof. By 2.14

$$\hat{\Delta}_{\mathbb{R}\Gamma(X,\mathcal{M})} := \mathbb{R}(G \times X, \Delta_{\mathcal{M}}) : D_{\mathfrak{a}\to} \otimes_{D_X} \mathbb{R}\Gamma(X, \mathcal{M}) \to D_{\mathfrak{p}\to} \otimes_{D_X} \mathbb{R}\Gamma(X, \mathcal{M})$$

defines a structure morphism on $\mathbb{R}\Gamma(X, \mathcal{M})$. □

2.18.

Remark. (i) If X is projective, one obtains likewise

$$\mathbb{R}\Gamma(X, ?) : \mathrm{D}^b_G\big(\mathrm{coh}\big(\bar{\mathcal{D}}^{(m)}_{X/\Bbbk}\big)\big) \to \mathrm{D}^b_G\big(\bar{D}^{(m)}_{X/\Bbbk}\mathbf{mod}\big).$$

(ii) In case $X = \mathcal{B}$, the main theorem of [9] implies for $p > h$ that

$$\mathbb{R}\Gamma(X, ?) : \mathrm{D}^b_G\big(\mathrm{coh}\big(\mathcal{D}^{(0)}_{\mathcal{B}/\Bbbk}\big)\big) \to \mathrm{D}^b_G\big(D^{(0)}_{\mathcal{B}/\Bbbk}\mathbf{mod}\big)$$

is a triangulated equivalence.

2.19.

Lemma. *Assume Y is affine. For each $\eta \in \mathbf{Sch}_\Bbbk(Y \times X, Y \times X)^\times$ there is a functorial isomorphism*

$$(\mathfrak{p}_2 \circ \eta)^* \circ \big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} ?\big) \simeq \big(\mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} ?\big) \circ (D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} ?) : \mathrm{D}^-(D_X) \to \mathrm{D}^-\big(\mathrm{qc}(\mathcal{D}_{Y\times X})\big).$$

Proof. Recall from 2.3 (i) that $D_{\mathfrak{p}_2\circ\eta\to} \simeq \Bbbk[Y] \otimes D_X$ is flat over D_X.

If $P \overset{\mathbf{qis}}{\to} \Bbbk[Y]$ (respectively $Q \overset{\mathbf{qis}}{\to} M$) is a flat resolution of $\Bbbk[Y]$ (respectively M) in $D_Y\mathbf{Mod}$ (respectively $D_X\mathbf{Mod}$), then

$$\begin{aligned}
&\mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} (D_{\mathfrak{p}_2\to} \otimes_{D_X} M) \simeq (\mathcal{D}_Y \boxtimes \mathcal{D}_X) \otimes^{\mathbb{L}}_{D_Y \otimes D_X} \big(\Bbbk[Y] \otimes M\big)\\
&\quad\simeq (\mathcal{D}_Y \boxtimes \mathcal{D}_X) \otimes^{\mathbb{L}}_{D_Y \otimes D_X} (P \otimes Q)\\
&\quad\simeq (\mathcal{D}_Y \otimes_{D_Y} P) \boxtimes (\mathcal{D}_X \otimes_{D_X} Q)\\
&\quad\simeq \big((\mathcal{O}_Y \otimes_{\Bbbk[Y]} D_Y) \otimes_{D_Y} P\big) \boxtimes (\mathcal{D}_X \otimes_{D_X} Q) \quad \text{as } Y \text{ is affine}\\
&\quad\simeq (\mathcal{O}_Y \otimes_{\Bbbk[Y]} P) \boxtimes (\mathcal{D}_X \otimes_{D_X} Q)\\
&\quad\simeq \mathcal{O}_Y \boxtimes (\mathcal{D}_X \otimes_{D_X} Q) \quad \text{as } D_Y \text{ is locally free, hence flat over } \Bbbk[Y]\\
&\quad\simeq \mathfrak{p}^*(\mathcal{D}_X \otimes_{D_X} Q) \simeq \mathfrak{p}^*\big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M\big).
\end{aligned}$$

Thus

$$\big(\mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} ?\big) \circ (D_{\mathfrak{p}_2\to} \otimes_{D_X} ?) \simeq \mathfrak{p}_2^* \circ \big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} ?\big). \tag{1}$$

Then

$$\begin{aligned}
&\mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} (D_{\mathfrak{p}_2\circ\eta\to} \otimes_{D_X} M) \simeq \mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} (D_{\eta\to} \otimes_{D_{Y\times X}} D_{\mathfrak{p}_2\to} \otimes_{D_X} M) \quad \text{by 2.3 (i)}\\
&\quad\simeq (\mathcal{D}_{Y\times X} \otimes_{D_{Y\times X}} D_{\eta\to}) \otimes^{\mathbb{L}}_{D_{Y\times X}} (D_{\mathfrak{p}_2\to} \otimes_{D_X} M)\\
&\qquad \text{as } D_{\eta\to} \text{ is flat as left } D_{Y\times X}\text{-module by 2.4}\\
&\quad\simeq \mathcal{D}_{\eta\to} \otimes^{\mathbb{L}}_{D_{Y\times X}} (D_{\mathfrak{p}_2\to} \otimes_{D_X} M) \quad \text{by 2.4 again}\\
&\quad\simeq (\eta^*\mathcal{D}_{Y\times X}) \otimes^{\mathbb{L}}_{D_{Y\times X}} D_{\mathfrak{p}_2\to} \otimes_{D_X} M \simeq \eta^*\big(\mathcal{D}_{Y\times X} \otimes^{\mathbb{L}}_{D_{Y\times X}} D_{\mathfrak{p}_2\to} \otimes_{D_X} M\big)\\
&\quad\simeq \eta^*\mathfrak{p}_2^*\big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M\big) \quad \text{by (1)}\\
&\quad\simeq (\mathfrak{p}_2 \circ \eta)^*\big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M\big),
\end{aligned}$$

hence the assertion. □

2.20.

Proposition. *The functor* $\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} ?: \mathrm{D}^-(D_X) \to \mathrm{D}^-(\mathrm{qc}(\mathcal{D}_X))$ *induces a functor* $\mathrm{D}^-_G(D_X) \to \mathrm{D}^-_G(\mathrm{qc}(\mathcal{D}_X))$.

Proof. If $\hat{\Delta}_M \in \mathrm{D}^-(D_{G\times X})(D_{\mathfrak{a}\to} \otimes_{D_X} M, D_{\mathfrak{p}\to} \otimes_{D_X} M)$ is the structure morphism of $M \in \mathrm{D}^-_G(D_X)$, $\mathcal{D}_{G\times X} \otimes^{\mathbb{L}}_{D_{G\times X}} \hat{\Delta}_M$ equips $\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M$ with a structure of $\mathrm{D}^-_G(\mathrm{qc}(\mathcal{D}_X))$ by 2.18:

$$\begin{tikzcd}
\mathfrak{a}^*\big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M\big) \arrow[r, dashed] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{p}^*\big(\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} M\big) \arrow[d, "\sim"] \\
\mathcal{D}_{G\times X} \otimes^{\mathbb{L}}_{D_{G\times X}} (D_{\mathfrak{a}\to} \otimes_{D_X} M) \arrow[r, "\mathcal{D}_{G\times X} \otimes^{\mathbb{L}}_{D_{G\times X}} \hat{\Delta}_M"'] & \mathcal{D}_{G\times X} \otimes^{\mathbb{L}}_{D_{G\times X}} (D_{\mathfrak{p}\to} \otimes_{D_X} M),
\end{tikzcd}$$

verifying the cocycle condition by 2.6 (12). □

2.21.

Remark. If $\operatorname{gl\,dim} D_X < \infty$, $\mathcal{D}_X \otimes^{\mathbb{L}}_{D_X} ?$ restricts to

$$\mathrm{D}^b_G(D_X) \to \mathrm{D}^b_G\big(\mathrm{qc}(\mathcal{D}_X)\big).$$

If, moreover, D_X is coherent, it further restricts to

$$\mathrm{D}^b_G(D_X\mathbf{mod}) \to \mathrm{D}^b_G\big(\mathrm{coh}(\mathcal{D}_X)\big).$$

To see that the structure morphism $\mathcal{D}_{G\times X} \otimes^{\mathbb{L}}_{D_{G\times X}} \hat{\Delta}_M$ stays in $\mathrm{D}^b(\mathrm{qc}(\mathcal{D}_{G\times X}))$, as $D_{\mathfrak{p}\to} \simeq \Bbbk[G] \boxtimes D_X$ as left $D_G \boxtimes D_X$-module, it is enough to have $\mathrm{pd}_{D_G}(\Bbbk[G]) < \infty$; $\Bbbk[G]$ is projective over $\bar{D}^{(m)}_{G/\Bbbk}$ by [14, 1.8.1] while $\operatorname{gl\,dim} D^{(m)}_{G/\Bbbk} = 2\dim G$ by [7, Remark, p. 221].

2.22.

Corollary. *When there holds a derived localization theorem, if D_X is coherent, the localization induces G-equivariant equivalences*

$$\mathrm{D}^b_G(D_X\mathbf{mod}) \underset{\mathbb{R}\Gamma(X,?)}{\overset{\mathcal{D}_X\otimes^{\mathbb{L}}_{D_X}?}{\rightleftarrows}} \mathrm{D}^b_G\big(\mathrm{coh}(\mathcal{D}_X)\big)$$

quasi-inverse to each other.

Proof. When the derived localization theorem holds, one will have

$$\mathrm{lgl\ dim}(D_X) = \mathrm{hd}(D_X) \leqslant 2\dim X$$

by [21, Ex. I.17] and [15, Ex. III.6.4, 6.5]. $\square$

3. Decomposition of the direct images of invertible sheaves under the Frobenius morphism

3.1. Let G be a simply connected simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, B a Borel subgroup of G, T a maximal torus of B, $\Lambda = \mathbf{Grp}_{\Bbbk}(B, \mathrm{GL}_1)$ the weight group of B, R the root system of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, R^s the set of simple roots of R^+, Λ^+ the set of dominant weights of Λ, and $\Lambda_1 = \{\lambda \in \Lambda^+ \mid \langle\lambda, \alpha^\vee\rangle < p\ \forall\alpha \in R^s\}$. Let $\mathcal{B} = G/B$ and $F = F_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ be the absolute Frobenius endomorphism on $\mathcal{B}$. If $F_{\mathcal{B}/\Bbbk} : \mathcal{B} \to \mathcal{B}^{(1)}$ is the Frobenius morphism on $\mathcal{B}$ relative to $\Bbbk$, G_1 the Frobenius kernel of G, and if $q : G/B \to G/G_1B$ is the natural morphism, one has a commutative diagram of schemes

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F_{\mathcal{B}}"] \arrow[d, "q"'] \arrow[dr, "F_{\mathcal{B}/\Bbbk}"] & \mathcal{B} \arrow[r, "\text{structure}"] & \mathrm{Spec}(\Bbbk) \\
G/G_1B \arrow[r, "\sim"'] & \mathcal{B}^{(1)} \arrow[u, "\phi"] \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{Spec}(\Bbbk), \arrow[u, "F_{\Bbbk}"']
\end{tikzcd}$$

where ϕ is an isomorphism of schemes. Put for simplicity $\bar{\mathcal{B}} = G/G_1B$. If M_1 (respectively M_2) is a G-(respectively B-)module,

$$\phi_*\mathcal{L}_{\bar{\mathcal{B}}}\bigl(M_1 \otimes_{\Bbbk} M_2^{[1]}\bigr) \simeq M_1 \otimes_{\Bbbk} \mathcal{L}_{\mathcal{B}}(M_2). \tag{1}$$

Let $\hat{\nabla} = \mathrm{ind}_B^{G_1B}$ be the Humphreys–Verma induction functor from the category of B-modules to the category of G_1B-modules. $\forall \mu \in \Lambda$, one has an isomorphism of $\mathcal{O}_{\mathcal{B}}$-modules

$$F_*\mathcal{L}(\mu) \simeq \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\bigl(\hat{\nabla}(\mu)\bigr).$$

We will denote the G_1T-socle series on $\hat{\nabla}(\mu)$ by $0 = \mathrm{soc}^0(\hat{\nabla}(\mu)) < \mathrm{soc}^1(\hat{\nabla}(\mu)) < \mathrm{soc}^2(\hat{\nabla}(\mu)) < \cdots$, and put $\mathrm{soc}_i(\hat{\nabla}(\mu)) = \mathrm{soc}^i(\hat{\nabla}(\mu))/\mathrm{soc}^{i-1}(\hat{\nabla}(\mu))$. Then each socle layer $\mathrm{soc}_i(\hat{\nabla}(\mu))$ is equipped with a structure of G_1B-module to admit a decomposition

$$\mathrm{soc}_i\bigl(\hat{\nabla}(\mu)\bigr) \simeq \coprod_{\lambda\in\Lambda_1} L(\lambda) \otimes_{\Bbbk} G_1\mathbf{Mod}\bigl(L(\lambda), \mathrm{soc}_i\bigl(\hat{\nabla}(\mu)\bigr)\bigr)$$

as G_1B-modules. Put $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu)) = L(\lambda) \otimes_{\Bbbk} G_1\mathbf{Mod}(L(\lambda), \mathrm{soc}_i(\hat{\nabla}(\mu)))$ and $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1 = G_1\mathbf{Mod}(L(\lambda), \mathrm{soc}_i(\hat{\nabla}(\mu)))^{[-1]}$. Recall that we say $\mu \in \Lambda$ is p-regular iff $p \nmid \langle \mu + \rho, \alpha^\vee \rangle$ $\forall \alpha \in R$. We will show in this section

Proposition. *Let $G \in \{\mathrm{SL}_2(\Bbbk), \mathrm{SL}_3(\Bbbk), \mathrm{Sp}_4(\Bbbk)\}$ and assume $p \geqslant h$ the Coxeter number of G. Then for any p-regular $\mu \in \Lambda$ one has a decomposition*

$$F_*\mathcal{L}_{\mathcal{B}}(\mu) \simeq \coprod_{i\in\mathbb{N}} \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\bigl(\mathrm{soc}_i\bigl(\hat{\nabla}(\mu)\bigr)\bigr) \simeq \coprod_i \coprod_{\lambda\in\Lambda_1} \mathcal{L}_{\mathcal{B}}\bigl(\mathrm{soc}_{i,\lambda}\bigl(\hat{\nabla}(\mu)\bigr)^1\bigr) \otimes_{\Bbbk} L(\lambda)$$

with $\mathbf{Mod}_{\mathcal{B}}(\mathcal{L}_{\mathcal{B}}(\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1), \mathcal{L}_{\mathcal{B}}(\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1)) \simeq \Bbbk$ for each nonzero $\mathcal{L}_{\mathcal{B}}(\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1)$. In particular, all nonzero $\mathcal{L}_{\mathcal{B}}(\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1)$ are indecomposable, and by Krull–Schmidt–Azumaya the decomposition of $F_*\mathcal{L}_{\mathcal{B}}(\mu)$ into indecomposables is unique up to isomorphism.

3.2. Let us first make some general preparations. In this subsection G may be any reductive group over $\Bbbk$. In what follows we will often suppress the subscripts from $\mathcal{L}$. Recall first from [19, II.4.5 and 5.4] the vanishing theorems of Kempf and Andersen:

$$\mathrm{H}^i\bigl(\mathcal{B}, \mathcal{L}(\nu)\bigr) = 0 \quad \forall \nu \in \Lambda^+ \text{ and } i \geqslant 1 \tag{1}$$

while

$$\mathrm{H}^\bullet\bigl(\mathcal{B}, \mathcal{L}(\nu)\bigr) = 0 \quad \text{if } \nu \in \bigl(-\rho + \Lambda^+\bigr) \setminus \Lambda^+, \tag{2}$$

and from [19, II.5.5] the Borel–Weil–Bott theorem for small dominant weights:

$$\mathrm{H}^i\bigl(\mathcal{B}, \mathcal{L}(w \cdot \nu)\bigr) \simeq \delta_{i\ell(w)}\mathrm{H}^0\bigl(\mathcal{B}, \mathcal{L}(\nu)\bigr) \quad \forall \nu \in -\rho + \Lambda^+ \text{ with } \bigl\langle \nu + \rho, \alpha^\vee \bigr\rangle \leqslant p \ \forall \alpha \in R^+. \tag{3}$$

To rearrange the structure of $F_*\mathcal{L}(\mu)$, the following simple observation is useful. Let $\mathcal{M}$ be an $\mathcal{O}_{\mathcal{B}}$-module with filtration $\mathcal{M} > \mathcal{M}^1 > \mathcal{M}^2$. If $\mathcal{M}^1/\mathcal{M}^2$ is a direct summand of $\mathcal{M}^1$ and also of $\mathcal{M}/\mathcal{M}^2$, then $\mathcal{M}^1/\mathcal{M}^2$ is a direct summand of $\mathcal{M}$. Precisely, given a diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\hat{\pi}"] & \mathcal{M}/\mathcal{M}^2 \arrow[d, shift left, "\pi'"] \\ \mathcal{M}^1 \arrow[u, hook, "j"] \arrow[r, shift left, "\pi"] & \mathcal{M}^1/\mathcal{M}^2 \arrow[l, shift left, "i"] \arrow[u, hook, shift left, "\bar{j}"] \end{tikzcd}$$

such that $\hat{\pi} \circ j = \bar{j} \circ \pi$ and $\pi \circ i = \mathrm{id}_{\mathcal{M}^1/\mathcal{M}^2} = \pi' \circ \bar{j}$, one has

$$\pi' \circ \hat{\pi} \circ j \circ i = \pi' \circ \bar{j} \circ \pi \circ i = \mathrm{id}_{\mathcal{M}^1/\mathcal{M}^2}. \tag{4}$$

To determine the structure of B-module on each $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1$, recall from [19, II.5.20] that $\forall \lambda, \mu \in \Lambda$,

$$\mathrm{Ext}^1_B(\lambda, \mu) \simeq \begin{cases} \Bbbk & \text{if } \lambda - \mu = p^r\alpha \ \exists r \in \mathbb{N}, \alpha \in R^s, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Let α be a simple root, P_α the minimal standard parabolic subgroup of G associated with α, and ω the corresponding fundamental weight.

Lemma. [19, II.5.2] *Assume $p \geqslant 3$.*

(i) $\mathrm{H}^0(P_\alpha/B, \mathcal{L}(\omega))$ *is (up to isomorphism) a unique B-extension of $\omega - \alpha$ by ω.*
(ii) $\mathrm{H}^0(P_\alpha/B, \mathcal{L}(2\omega))$ *is a unique B-extension of $2\omega - 2\alpha$ by $\omega \otimes_\Bbbk \mathrm{H}^0(P_\alpha/B, \mathcal{L}(\omega))$, and a unique B-extension of $(\omega - \alpha) \otimes_\Bbbk \mathrm{H}^0(P_\alpha/B, \mathcal{L}(\omega))$ by 2ω.*

Proof. The unicity follows from (5). □

3.3. We next make some reductions. If $\lambda \in \Lambda$ and $M \in B\mathbf{Mod}$, one has an isomorphism of G_1B-modules

$$\big(L(\lambda) \otimes M^{[1]}\big)^* \simeq L(\lambda)^* \otimes \big(M^{[1]}\big)^* \simeq L(\lambda)^* \otimes \big(M^*\big)^{[1]}.$$

Also $M^{[1]}$ and hence M is determined uniquely by $L(\lambda) \otimes M^{[1]}$ as

$$M^{[1]} \simeq G_1\mathbf{Mod}\big(L(\lambda), L(\lambda) \otimes M^{[1]}\big).$$

Now let $\mu \in \Lambda$ be p-regular and assume that the decomposition 3.1 holds:

$$\mathcal{L}_{\tilde{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big) \simeq \coprod_i \mathcal{L}_{\tilde{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\big).$$

If rad_i denotes the ith radical layer of the radical series,

$$\begin{aligned}
\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho-\mu\big)\big) &\simeq \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)^*\big) \quad \text{by [19, II.9.2]}\\
&\simeq \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big)^\vee \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\big)^\vee \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)^*\big)\\
&\simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{rad}_{i-1}\big(\hat{\nabla}(\mu)^*\big)\big) \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{rad}_{i-1}\big(\hat{\nabla}\big(2(p-1)\rho-\mu\big)\big)\big)\\
&\simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_{\dim\mathcal{B}-i+2}\big(\hat{\nabla}\big(2(p-1)\rho-\mu\big)\big)\big) \quad \text{by [3, 5.4, 5.6].} \qquad (1)
\end{aligned}$$

Also, by the unicity there is an isomorphism of B-modules

$$\mathrm{soc}_{\dim\mathcal{B}-i+2,-w_0\lambda}\big(\hat{\nabla}\big(2(p-1)\rho-\mu\big)\big)^1 \simeq \big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big)^*. \qquad (2)$$

For each $\nu \in \Lambda$ the functor $? \otimes_{\Bbbk} p\nu : G_1T\mathbf{Mod} \to G_1T\mathbf{Mod}$ is an equivalence with quasi-inverse $? \otimes_{\Bbbk} p(-\nu)$, preserving $G_1B\mathbf{Mod}$ under restriction. Thus for each $\lambda \in \Lambda_1$ and for each $i \in \mathbb{N}$ there is an isomorphism of G_1B-modules

$$\mathrm{soc}_i\big(\hat{\nabla}(\mu+p\nu)\big) \simeq \mathrm{soc}_i\big(\hat{\nabla}(\mu)\otimes p\nu\big) \simeq \mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\otimes p\nu$$

and an isomorphism of B-modules

$$G_1\mathbf{Mod}\big(L(\lambda), \mathrm{soc}_i\big(\hat{\nabla}(\mu+p\nu)\big)\big) \simeq G_1\mathbf{Mod}\big(L(\lambda), \mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\big)\otimes p\nu,$$

and hence an isomorphism of B-modules

$$\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu+p\nu)\big)^1 \simeq \mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1 \otimes \nu. \qquad (3)$$

It will then follow from the decomposition for $\mathcal{L}_{\bar{\mathcal{B}}}(\mu)$ that

$$\begin{aligned}
\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu+p\nu)\big) &\simeq \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big)\otimes_{\bar{\mathcal{B}}}\mathcal{L}_{\bar{\mathcal{B}}}(p\nu) \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\big)\otimes_{\bar{\mathcal{B}}}\mathcal{L}_{\bar{\mathcal{B}}}(p\nu)\\
&\simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\otimes p\nu\big) \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu+p\nu)\big)\big). \qquad (4)
\end{aligned}$$

Assume now that μ and ν belong to the same alcove. Then the translation functor $\mathrm{T}_\mu^\nu : G_1T\mathbf{Mod} \to G_1T\mathbf{Mod}$ is an exact equivalence with quasi-inverse T_ν^μ, preserving $G_1B\mathbf{Mod}$ under restriction [19, II.9.19]. Let $\lambda \in \Lambda_1$ with $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu)) \neq 0$. Fix a weight $\eta \in \Lambda$ of $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1$; in particular, $\lambda + p\eta \in W_p \cdot \mu$. If $\eta' \in \Lambda$ is another weight of $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(\mu))^1$, then

$$\eta' \in \eta + \mathbb{Z}R. \qquad (5)$$

If $\lambda' \in \Lambda_1$ with $\lambda' + p\eta \in W_p \cdot \nu$, i.e., $\mathrm{T}_\mu^\nu(L(\lambda)\otimes p\eta) \simeq L(\lambda')\otimes p\eta$, one has an isomorphism of G_1B-modules

$$\mathrm{T}_\mu^\nu\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)\big) \simeq \mathrm{soc}_{i,\lambda'}\big(\hat{\nabla}(\nu)\big) \simeq L(\lambda')\otimes\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big)^{[1]}. \qquad (6)$$

3.4. Let us first consider the case $\mu = 2(p-1)\rho$ lying in the top alcove of the bottom dominant box $\Pi^+ = \{\xi \in \lambda \mid \langle \xi + \rho, \alpha^\vee \rangle \in]0, p[\ \forall \alpha \in R^s\}$. We will show that for each $i, j \in \mathbb{N}$ with $i > j$ and for each $\lambda, \nu \in \Lambda_1$

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}\big(2(p-1)\rho\big)\big)^1\big), \mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{j,\nu}\big(\hat{\nabla}\big(2(p-1)\rho\big)\big)^1\big)\big) = 0. \tag{1}$$

Then the decomposition 3.1 will follow from 3.2 (4). In turn, by 3.3 (4) and (6) the decomposition will hold for any $\mu + p\xi$, $\xi \in \Lambda$, with μ contained in the top alcove of Π^+, and hence by 3.3 (1) and (2) also for any $\mu + p\xi$, $\xi \in \Lambda$, with μ in the bottom dominant alcove.

At this moment Proposition 3.1 for $G = \mathrm{SL}_2(\mathbb{k})$ or $\mathrm{SL}_3(\mathbb{k})$ follows from [16]; the assertion (1) has been proved there. We will thus assume $G = \mathrm{Sp}_4(\mathbb{k})$ throughout the rest of this section with the simple roots α_1 and α_2, α_1 short. Let ω_1 and ω_2 be the corresponding fundamental weights. Then the socle layers of $\hat{\nabla}(2(p-1)\rho)$ are given as follows by [3], [4, 3.1], where we abbreviate $\mathrm{soc}_i(\hat{\nabla}(2(p-1)\rho))$ as soc_i, $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(2(p-1)\rho))$ as $\mathrm{soc}_{i,\lambda}$ and $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(2(p-1)\rho))^1$ as $\mathrm{soc}^1_{i,\lambda}$:

$$\mathrm{soc}_1 = \mathrm{soc}_{1,(p-2)\rho},$$

$$\mathrm{soc}_2 = \mathrm{soc}_{2,(p-2)\omega_1+\omega_2} \oplus \mathrm{soc}_{2,(p-3)\omega_2} \oplus \mathrm{soc}_{2,(p-4)\omega_1},$$

$$\mathrm{soc}_3 = \mathrm{soc}_{3,2\omega_1+(p-3)\omega_2} \oplus \mathrm{soc}_{3,0} \oplus \mathrm{soc}_{3,(p-4)\omega_1+\omega_2},$$

$$\mathrm{soc}_4 = \mathrm{soc}_{4,2\omega_1+(p-2)\omega_2} \oplus \mathrm{soc}_{4,(p-2)\omega_1+\omega_2},$$

$$\mathrm{soc}_5 = \mathrm{soc}_{5,0}.$$

Lemma. *Assume $p \geqslant 5$.*

(i) *There are isomorphisms B-modules*

$$\mathrm{soc}^1_{1,(p-2)\rho} \simeq \rho \simeq \mathrm{soc}^1_{2,(p-2)\omega_1+\omega_2},$$

$$\mathrm{soc}^1_{2,(p-3)\omega_2} \simeq \mathrm{soc}^1_{3,0},$$

$$\mathrm{soc}^1_{3,2\omega_1+(p-3)\omega_2} \simeq \omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big),$$

$$\mathrm{soc}^1_{4,2\omega_1+(p-2)\omega_2} \simeq \omega_2, \qquad \mathrm{soc}^1_{4,(p-2)\omega_1+\omega_2} \simeq \omega_1,$$

$$\mathrm{soc}^1_{5,0} \simeq \mathbb{k}.$$

(ii) *There is an isomorphism* $\mathrm{soc}^1_{3,0} \simeq \mathrm{soc}^1_{2,(p-3)\omega_2}$ *of B-modules, which is a unique B-extension of $\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$ by $2\omega_2$, and also is a unique B-extension of ω_2 by $\omega_2 \otimes \mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(\omega_2))$.*

(iii) $\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}$ *is a unique B-extension of $\omega_2 \otimes \mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2))$ by ρ, and is a unique B-extension of ω_1 by $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\rho))$.*

(iv) $\mathrm{soc}^1_{2,(p-4)\omega_1}$ *is a unique B-extension of $\omega_2 \otimes \mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2))$ by $2\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$, is a unique B-extension of ω_1 by $\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1))$, and also is a unique B-extension of* $\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}$ *by $3\omega_1$.*

Proof. As the arguments are all similar, let us just explain (iv), assuming the rest.

To see the unicities, it suffices to show

$$\begin{aligned}\Bbbk &\simeq \mathrm{Ext}^1_B\big(2\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big), \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\ &\simeq \mathrm{Ext}^1_B\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)\big), \omega_1\big)\\ &\simeq \mathrm{Ext}^1_B\big(3\omega_1, \mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big).\end{aligned} \tag{2}$$

Consider three long exact sequences

$$\begin{aligned}&\mathrm{Ext}^1_B\big(3\omega_1, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\ &\quad\to \mathrm{Ext}^1_B\big(2\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big), \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\ &\quad\to \mathrm{Ext}^1_B\big(\rho, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\ &\quad\to \mathrm{Ext}^2_B\big(3\omega_1, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big),\end{aligned} \tag{3}$$

$$\mathrm{Ext}^i_B(3\omega_1, \omega_1) \to \mathrm{Ext}^i_B\big(3\omega_1, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big) \to \mathrm{Ext}^i_B(3\omega_1, -\omega_1+2\omega_2), \tag{4}$$

$i \in \mathbb{N}$, and

$$\begin{aligned}&\mathrm{Ext}^1_B(\rho, \omega_1) \to \mathrm{Ext}^1_B\big(\rho, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\ &\quad\to \mathrm{Ext}^1_B(\rho, -\omega_1+2\omega_2) \to \mathrm{Ext}^2_B(\rho, \omega_1).\end{aligned} \tag{5}$$

As $p \geqslant 5$ by the standing hypothesis, $\mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(-2\omega_1)) = 0$ by Andersen, 3.2 (2), and hence the spectral sequence

$$\mathrm{E}_2^{ij} = \mathrm{Ext}^i_G\big(\Bbbk, \mathrm{H}^j\big(\mathcal{B}, \mathcal{L}(-2\omega_1)\big)\big) \simeq \mathrm{Ext}^i_G\big(\Bbbk, \mathrm{R}^j\mathrm{ind}_B^G(-2\omega_1)\big) \quad\Rightarrow\quad \mathrm{Ext}^{i+j}_B(\Bbbk, -2\omega_1)$$

degenerates to yield

$$0 = \mathrm{Ext}^\bullet_B(\Bbbk, -2\omega_1) \simeq \mathrm{Ext}^\bullet_B(3\omega_1, \omega_1).$$

Likewise

$$\mathrm{Ext}^\bullet_B(3\omega_1, -\omega_1+2\omega_2) = 0 = \mathrm{Ext}^\bullet_B(\rho, \omega_1).$$

It follows from (4) that

$$\mathrm{Ext}^\bullet_B\big(3\omega_1, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big) = 0,$$

and from (5)

$$\begin{aligned}\mathrm{Ext}^1_B\big(\rho, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big) &\simeq \mathrm{Ext}^1_B(\rho, -\omega_1+2\omega_2)\\ &\simeq \mathrm{H}^1(B, -\alpha_1) \simeq \Bbbk.\end{aligned}$$

Then by (3)

$$\begin{aligned}
&\mathrm{Ext}^1_B\big(2\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big), \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\
&\quad\simeq \mathrm{Ext}^1_B\big(\rho, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big)\\
&\quad\simeq \Bbbk,
\end{aligned}$$

that is the first isomorphism of (2). Likewise the rest.

Now $\mathrm{soc}^1_{2,(p-4)\omega_1}$ has a filtration of B-modules

$$\mathrm{soc}^1_{2,(p-4)\omega_1} > \mathrm{soc}^{1,1}_{2,(p-4)\omega_1} > \mathrm{soc}^{1,2}_{2,(p-4)\omega_1} > \mathrm{soc}^{1,3}_{2,(p-4)\omega_1} > 0$$

such that

$$\begin{gathered}
\mathrm{soc}^1_{2,(p-4)\omega_1}/\mathrm{soc}^{1,1}_{2,(p-4)\omega_1} \simeq 3\omega_1,\\
\mathrm{soc}^{1,1}_{2,(p-4)\omega_1}/\mathrm{soc}^{1,2}_{2,(p-4)\omega_1} \simeq \rho,\\
\mathrm{soc}^{1,2}_{2,(p-4)\omega_1}/\mathrm{soc}^{1,3}_{2,(p-4)\omega_1} \simeq -\omega_1+2\omega_2,\\
\mathrm{soc}^{1,3}_{2,(p-4)\omega_1} \simeq \omega_1.
\end{gathered}$$

Just suppose that the short exact sequence of B-modules

$$0 \to \mathrm{soc}^{1,3}_{2,(p-4)\omega_1} \to \mathrm{soc}^{1,2}_{2,(p-4)\omega_1} \to (-\omega_1+2\omega_2) \to 0$$

split. Then there would be a G_1B-submodule M of soc^2 containing soc^1 such that $M/\mathrm{soc}^1 \simeq L((p-4)\omega_1) \otimes p(-\omega_1+2\omega_2)$. That would induce an exact sequence of G-modules

$$\mathrm{ind}^G_{G_1B}\big(M \otimes p(\omega_1-2\omega_2)\big) \to L\big((p-4)\omega_1\big) \to \mathrm{R}^1\mathrm{ind}^G_{G_1B}\big(\mathrm{soc}^1 \otimes p(\omega_1-2\omega_2)\big)$$

with

$$\begin{aligned}
\mathrm{ind}^G_{G_1B}\big(M \otimes p(\omega_1-2\omega_2)\big) &\leqslant \mathrm{ind}^G_{G_1B}\big(\hat{\nabla}\big(2(p-1)\rho\big) \otimes p(\omega_1-2\omega_2)\big)\\
&\simeq \mathrm{H}^0\big(\mathcal{B}, \mathcal{L}\big((3p-2)\omega_1-2\omega_2\big)\big) = 0.
\end{aligned}$$

But $\mathrm{R}^1\mathrm{ind}^G_{G_1B}(\mathrm{soc}^1 \otimes p(\omega_1-2\omega_2))$ has no G-composition factor whose G_1-irreducible part is $L((p-4)\omega_1)$, absurd. It follows by the unicity 3.2 (i) that

$$\mathrm{soc}^{1,2}_{2,(p-4)\omega_1} \simeq \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)$$

as B-modules. Likewise the short exact sequence of B-modules

$$0 \to \mathrm{soc}^{1,2}_{2,(p-4)\omega_1} \to \mathrm{soc}^{1,1}_{2,(p-4)\omega_1} \to \rho \to 0$$

does not split; $L((p-4)\omega_1)$ is not a G-composition factor of $\mathrm{H}^0(\mathcal{B}, \mathcal{L}((p-2)\rho))$ [2]. Then by (iii) one obtains an isomorphism of B-modules

$$\mathrm{soc}^{1,1}_{2,(p-4)\omega_1} \simeq \mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}.$$

Also the exact sequence of B-modules

$$0 \to \mathrm{soc}^{1,1}_{2,(p-4)\omega_1} \to \mathrm{soc}^1_{2,(p-4)\omega_1} \to 3\omega_1 \to 0$$

does not split, and by (2)

$$\mathrm{Ext}^1_B\big(3\omega_1, \mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big) \simeq \Bbbk.$$

It follows that $\mathrm{soc}^1_{2,(p-4)\omega_1}$ is a unique B-extension of $\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}$ by $3\omega_1$.

Now just suppose that the exact sequence of B-modules

$$0 \to \rho \to \mathrm{soc}^1_{2,(p-4)\omega_1}/\big\{\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big\} \to 3\omega_1 \to 0$$

split. Then, as $\mathrm{Ext}^1_B(3\omega_1, -\omega_1+2\omega_2) = 0 = \mathrm{Ext}^1_B(3\omega_1, \omega_1)$,

$$\mathrm{Ext}^1_B\big(3\omega_1, \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big) = 0,$$

and hence $3\omega_1 \leqslant \mathrm{soc}^1_{2,(p-4)\omega_1}$, which would force the epi $\mathrm{soc}^1_{2,(p-4)\omega_1} \to 3\omega_1$ to split as $\mathrm{soc}^1_{2,(p-4)\omega_1}$ is multiplicity-free, absurd. Thus one obtains an isomorphism of B-modules

$$\mathrm{soc}^1_{2,(p-4)\omega_1}/\big\{\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big\} \simeq 2\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)$$

by 3.2 (i). Moreover, if the exact sequence

$$0 \to \omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big) \to \mathrm{soc}^1_{2,(p-4)\omega_1} \to 2\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big) \to 0$$

were split, then

$$\big\{\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2)\big)\big\} \oplus \rho \simeq \ker\big(\mathrm{soc}^1_{2,(p-4)\omega_1} \to 3\omega_1\big) \simeq \mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2},$$

contradicting (iii), and hence the sequence does not split. It follows from (2) that $\mathrm{soc}^1_{2,(p-4)\omega_1}$ is a unique B-extension of $\omega_2 \otimes \mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(-\omega_1+\omega_2))$ by $2\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$.

Finally, just suppose that the exact sequence of B-modules

$$0 \to \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\rho)\big) \to \mathrm{soc}^1_{2,(p-4)\omega_1}/\omega_1 \to 3\omega_1 \to 0$$

split. Then, as $\mathrm{Ext}^1_B(3\omega_1, \omega_1) = 0$, one would have $3\omega_1 \leqslant \mathrm{soc}^1_{2,(p-4)\omega_1}$, splitting the epi $\mathrm{soc}^1_{2,(p-4)\omega_1} \to 3\omega_1$, absurd. Then by 3.2 (ii)

$$\mathrm{soc}^1_{2,(p-4)\omega_1}/\omega_1 \simeq \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)\big) \otimes \omega_1$$

as B-modules. If $\mathrm{soc}^1_{2,(p-4)\omega_1} \simeq \omega_1 \oplus \{\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)) \otimes \omega_1\}$, then $-\omega_1 + 2\omega_2 \leqslant \mathrm{soc}^1_{2,(p-4)\omega_1}$, and hence

$$\begin{aligned}\omega_1 \oplus (-\omega_1 + 2\omega_2) &\simeq \ker\bigl(\mathrm{soc}^1_{2,(p-4)\omega_1} \to \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)\bigr) \otimes \omega_1\bigr)\\ &\simeq \omega_2 \otimes \mathrm{H}^0\bigl(P_{\alpha_2}/B, \mathcal{L}(-\omega_1 + \omega_2)\bigr),\end{aligned}$$

contradicting 3.2 (i). As $\mathrm{Ext}^1_B(\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)) \otimes \omega_1, \omega_1) \simeq \Bbbk$ by (2) again, $\mathrm{soc}^1_{2,(p-4)\omega_1}$ is also a unique B-extension of ω_1 by $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)) \otimes \omega_1$, as desired. □

3.5. We now finish the verification of 3.4 (1) for $\mu = 2(p-1)\rho$. As each argument is straightforward, however, let us just explain a most complicated one to show

$$\mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}\bigl(\mathrm{soc}^1_{2,(p-3)\omega_2}\bigr)\bigr) = 0.$$

There is an exact sequence

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \to \mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}\bigl(\mathrm{soc}^1_{2,(p-3)\omega_2}\bigr)\bigr) \to \mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}(2\omega_2)\bigr).\end{aligned} \tag{1}$$

On the other hand, one has an exact sequence of G-modules

$$\begin{aligned}&\mathbf{Mod}_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \to \mathbf{Mod}_{\mathcal{B}}\bigl(\mathcal{L}(\omega_1), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \to \mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\rho)\bigr)\bigr), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \to \mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\bigr), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \to \mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}(\omega_1), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\end{aligned} \tag{2}$$

with for each $j \in \mathbb{N}$

$$\begin{aligned}\mathrm{Ext}^j_{\mathcal{B}}\bigl(\mathcal{L}(\omega_1), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr) &\simeq \mathrm{H}^j\bigl(\mathcal{B}, \mathcal{L}\bigl(\mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\simeq \delta_{0j}\mathrm{H}^0\bigl(\mathcal{B}, \mathcal{L}(\omega_1)\bigr)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{B}}\bigl(\mathcal{L}\bigl(\mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\rho)\bigr)\bigr), \mathcal{L}\bigl(\omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \simeq \mathrm{H}^1\bigl(\mathcal{B}, \mathcal{L}\bigl(\mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\rho)\bigr)^* \otimes \omega_1 \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr)\\ &\quad \simeq \mathrm{H}^1\bigl(G/P_{\alpha_1}, \mathcal{L}\bigl(\mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\bigr) \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr) \otimes \mathrm{H}^0\bigl(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\bigr)\bigr)\bigr).\end{aligned}$$

As $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1)) \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$ has a P_{α_1}-filtration with the subquotients $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1))$ atop of ω_2,

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)$$

admits a P_{α_1}-filtration of the subquotients

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)\big)$$

atop of

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\big) \otimes \omega_2 \simeq \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - \omega_2)\big).$$

In turn,

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(2\omega_1)\big)$$

has a P_{α_1}-filtration of the subquotients $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(3\omega_1 - 2\omega_2))$ atop of $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - \omega_2))$, and hence

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - 2\omega_2)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)$$

has a P_{α_1}-filtration of the subquotients

$$\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(3\omega_1 - 2\omega_2)\big), \quad \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - \omega_2)\big) \quad \text{and} \quad \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - \omega_2)\big)$$

from the top in order. As $\mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(\omega_1 - \omega_2)) = 0$, one obtains

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\rho)\big)\big), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) &\simeq \mathrm{H}^1\big(\mathcal{B}, \mathcal{L}(3\omega_1 - 2\omega_2)\big) \\ &\simeq \mathrm{H}^0\big(\mathcal{B}, \mathcal{L}(\omega_1)\big).\end{aligned}$$

There is also an exact sequence

$$\begin{aligned}&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \\ &\quad \to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \\ &\quad \to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1 + \omega_2)\big)\big), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \\ &\quad \to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big)\end{aligned}$$

with

$$\mathrm{Ext}^\bullet_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \simeq \mathrm{H}^\bullet\big(\mathcal{B}, \mathcal{L}(\omega_1 - \omega_2)\big) = 0,$$

and hence

$$\begin{aligned}&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \\ &\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(-\omega_1 + \omega_2)\big)\big), \mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big),\end{aligned}$$

which fits in an exact sequence

$$\begin{aligned}
&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big),\mathcal{L}(\omega_2)\big)\\
&\quad\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big),\mathcal{L}\big(\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big)\big)\big)\\
&\quad\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big),\mathcal{L}(2\omega_1)\big)
\end{aligned}$$

with

$$\begin{aligned}
&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big),\mathcal{L}(\omega_2)\big)\\
&\quad\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)^*\big)\big)\\
&\quad\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big)\big)\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(-\omega_1+\omega_2)\big)=0
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_2\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2)\big)\big),\mathcal{L}(2\omega_1)\big)\\
&\quad\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}\big((-\omega_2)\otimes \mathrm{H}^0\big(P_{\alpha_2}/B,\mathcal{L}(\rho)\big)\big)\big)\\
&\quad=0\quad \text{as } \mathrm{H}^\bullet\big(P_{\alpha_2}/B,\mathcal{L}(-\omega_2)\big)=0.
\end{aligned}$$

Thus the exact sequence (2) reads as

$$\begin{aligned}
&0\to \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(\omega_1)\big)\to \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(\omega_1)\big)\\
&\quad\to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big),\mathcal{L}\big(\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big)\big)\big)\to 0,
\end{aligned}$$

and hence

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big),\mathcal{L}\big(\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big)\big)\big)=0$$

by the irreducibility of $\mathrm{H}^0(\mathcal{B},\mathcal{L}(\omega_1))$. Finally, there is an exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\rho)\big)\big),\mathcal{L}(2\omega_2)\big)\to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big),\mathcal{L}(2\omega_2)\big)\\
&\quad\to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}(\omega_1),\mathcal{L}(2\omega_2)\big)
\end{aligned}$$

with

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\rho)\big)\big),\mathcal{L}(2\omega_2)\big)\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big)\big)\big)\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}(\omega_1)\big)=0$$

and

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}(\omega_1),\mathcal{L}(2\omega_2)\big)\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}(-\omega_1+2\omega_2)\big)=0,$$

and hence $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{L}(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}),\mathcal{L}(2\omega_2))=0$. It now follows from (1) that

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}\big),\mathcal{L}\big(\mathrm{soc}^1_{2,(p-3)\omega_2}\big)\big)=0,$$

as desired.

3.6. Dualizing $\hat{\nabla}(2(p-1)\rho)$ one obtains

$$\begin{gathered}
\mathrm{soc}_1\big(\hat{\nabla}(0)\big)=\mathrm{soc}_{1,0}\big(\hat{\nabla}(0)\big),\\
\mathrm{soc}_2\big(\hat{\nabla}(0)\big)=\mathrm{soc}_{2,2\omega_1+(p-2)\omega_2}\big(\hat{\nabla}(0)\big)\oplus\mathrm{soc}_{2,(p-2)\omega_1+\omega_2}\big(\hat{\nabla}(0)\big),\\
\mathrm{soc}_3\big(\hat{\nabla}(0)\big)=\mathrm{soc}_{3,2\omega_1+(p-3)\omega_2}\big(\hat{\nabla}(0)\big)\oplus\mathrm{soc}_{3,0}\big(\hat{\nabla}(0)\big)\oplus\mathrm{soc}_{3,(p-4)\omega_1+\omega_2}\big(\hat{\nabla}(0)\big),\\
\mathrm{soc}_4\big(\hat{\nabla}(0)\big)=\mathrm{soc}_{4,(p-2)\omega_1+\omega_2}\big(\hat{\nabla}(0)\big)\oplus\mathrm{soc}_{4,(p-3)\omega_2}\big(\hat{\nabla}(0)\big)\oplus\mathrm{soc}_{4,(p-4)\omega_1}\big(\hat{\nabla}(0)\big),\\
\mathrm{soc}_5\big(\hat{\nabla}(0)\big)=\mathrm{soc}_{5,(p-2)\rho}\big(\hat{\nabla}(0)\big),
\end{gathered}$$

where as B-modules

$$\begin{gathered}
\mathrm{soc}_{1,0}\big(\hat{\nabla}(0)\big)^1\simeq\Bbbk,\\
\mathrm{soc}_{2,2\omega_1+(p-2)\omega_2}\big(\hat{\nabla}(0)\big)^1\simeq-\omega_2,\\
\mathrm{soc}_{2,(p-2)\omega_1+\omega_2}\big(\hat{\nabla}(0)\big)^1\simeq-\omega_1,\\
\mathrm{soc}_{3,2\omega_1+(p-3)\omega_2}\big(\hat{\nabla}(0)\big)^1\simeq-(\omega_1)\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1-\omega_2)\big),\\
\mathrm{soc}_{4,(p-2)\omega_1+\omega_2}\big(\hat{\nabla}(0)\big)^1\simeq\mathrm{soc}_{5,(p-2)\rho}\big(\hat{\nabla}(0)\big)^1\simeq-\rho,
\end{gathered}$$

$\mathrm{soc}_{3,0}(\hat{\nabla}(0))^1\simeq\mathrm{soc}_{4,(p-3)\omega_2}(\hat{\nabla}(0))^1$ is a unique nonsplit B-extension of $-2\omega_2$ by $(-\omega_1)\otimes\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(\omega_1-\omega_2))$ and is a unique nonsplit B-extension of $(-\omega_2)\otimes\mathrm{H}^0(P_{\alpha_2}/B,\mathcal{L}(-2\omega_1+\omega_2))$ by $-\omega_2$, $\mathrm{soc}_{3,(p-4)\omega_1+\omega_2}(\hat{\nabla}(0))^1$ is a unique nonsplit B-extension of $\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(\omega_1-2\omega_2))$ by $-\omega_1$ and is a unique nonsplit B-extension of $-\rho$ by $(-\omega_2)\otimes\mathrm{H}^0(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2))$, and $\mathrm{soc}_{4,(p-4)\omega_1}(\hat{\nabla}(0))^1$ is a unique nonsplit B-extension of $(-2\omega_1)\otimes\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(\omega_1-\omega_2))$ by $(-\omega_2)\otimes\mathrm{H}^0(P_{\alpha_2}/B,\mathcal{L}(-\omega_1+\omega_2))$, is a unique nonsplit B-extension of $(-\omega_1)\otimes\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-2\omega_2))$ by $-\omega_1$, and is a unique nonsplit B-extension of $-3\omega_1$ by $\mathrm{soc}_{3,(p-4)\omega_1+\omega_2}(\hat{\nabla}(0))^1$.

3.7. For $\mu=2(p-1)\omega_1+(p+1)\omega_2$ we omit the computations entirely similar to those in the case of $2(p-1)\rho$. One has, abbreviating $\mathrm{soc}_i(\hat{\nabla}(2(p-1)\omega_1+(p+1)\omega_2))$ as soc_i,

$$\begin{gathered}
\mathrm{soc}_1=\mathrm{soc}_{1,(p-2)\omega_1+\omega_2},\\
\mathrm{soc}_2=\mathrm{soc}_{2,0}\oplus\mathrm{soc}_{2,(p-4)\omega_1+\omega_2}\oplus\mathrm{soc}_{2,2\omega_1+(p-3)\omega_2},\\
\mathrm{soc}_3=\mathrm{soc}_{3,2\omega_1+(p-2)\omega_2}\oplus\mathrm{soc}_{3,(p-3)\omega_2}\oplus\mathrm{soc}_{3,(p-4)\omega_1}\oplus\mathrm{soc}_{3,(p-2)\omega_1+\omega_2},\\
\mathrm{soc}_4=\mathrm{soc}_{4,0}\oplus\mathrm{soc}_{4,(p-4)\omega_1+\omega_2}\oplus\mathrm{soc}_{4,(p-2)\rho},\\
\mathrm{soc}_5=\mathrm{soc}_{5,(p-3)\omega_2},
\end{gathered}$$

where as B-modules

$$\begin{gathered}
\mathrm{soc}^1_{1,(p-2)\omega_1+\omega_2}\simeq\mathrm{soc}^1_{2,(p-4)\omega_1+\omega_2}\simeq\rho,\\
\mathrm{soc}^1_{2,2\omega_1+(p-3)\omega_2}\simeq\mathrm{soc}^1_{3,(p-3)\omega_2}\simeq\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big),
\end{gathered}$$

$$\mathrm{soc}^1_{3,2\omega_1+(p-2)\omega_2} \simeq \omega_2,$$

$$\mathrm{soc}^1_{3,(p-2)\omega_1+\omega_2} \simeq \mathrm{soc}^1_{4,(p-2)\rho} \simeq \omega_1,$$

$$\mathrm{soc}^1_{4,0} \simeq \mathrm{soc}^1_{5,(p-3)\omega_2} \simeq \Bbbk,$$

$$\mathrm{soc}^1_{4,(p-4)\omega_1+\omega_2} \simeq \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big),$$

$\mathrm{soc}^1_{2,0}$ is a unique nonsplit B-extension of $\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$ by $2\omega_2$ and is a unique nonsplit B-extension of ω_2 by $\omega_2 \otimes \mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(\omega_2))$, and $\mathrm{soc}^1_{3,(p-4)\omega_1}$ is a unique nonsplit B-extension of $\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(2\omega_1 - \omega_2))$ by ρ and is a unique nonsplit B-extension of $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$ by $\mathrm{H}^0(P_{\alpha_2}/B, \mathcal{L}(\rho))$.

Dualizing $\hat{\nabla}(2(p-1)\omega_1 + (p+1)\omega_2)$ yields the G_1T-socle series of $\hat{\nabla}((p-3)\omega_2)$. One thus obtains for all p-regular μ the decomposition

$$F_*\mathcal{L}_{\mathcal{B}}(\mu) \simeq \coprod_{i\in\mathbb{N}} \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(\mu)\big)\big) \simeq \coprod_i \coprod_{\lambda\in\Lambda_1} \mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big) \otimes_{\Bbbk} L(\lambda).$$

Thus, denoting an alcove by A, one may symbolically express the decomposition as

$$\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(A)\big) \simeq \coprod_i \mathcal{L}_{\bar{\mathcal{B}}}\big(\mathrm{soc}_i\big(\hat{\nabla}(A)\big)\big). \tag{1}$$

We postpone verification of the assertion that

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big), \mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big)\big) \simeq \Bbbk$$

till we see in Section 4

$$\mathrm{Ext}^k_{\mathcal{B}}\big(\mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big), \mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{j,\nu}\big(\hat{\nabla}(\mu)\big)^1\big)\big) = 0 \quad \forall k \geqslant 1,\ \forall \lambda, \nu \in \Lambda_1,$$

which helps the computation of the endomorphisms. Then we will obtain the unicity of the decomposition of $F_*\mathcal{L}(\mu)$ into indecomposables by Krull–Schmidt–Azumaya [22, 1.6.1].

4. Localization of $\bar{D}$-modules

In this section we will show for $G \in \{\mathrm{SL}_2(\Bbbk), \mathrm{SL}_3(\Bbbk), \mathrm{Sp}_4(\Bbbk)\}$ that $F_*\mathcal{L}_{\mathcal{B}}(\mu)$ for $\mu \in \Lambda$ is tilting on the flag variety $\mathcal{B} = G/B$ only if $p \geqslant h$ the Coxeter number of G, and that if $p \geqslant h$, then all $F_*\mathcal{L}_{\mathcal{B}}(\mu)$ for p-regular $\mu \in \Lambda$ are indeed tilting. It will follow from the Beilinson–Baer that for such p and μ the derived localization theorem holds on $\mathcal{B}$ for $\bar{D}^{(0)}_{\mathcal{B}}(\mu) = \Gamma(\mathcal{B}, \bar{\mathcal{D}}^{(0)}_{\mathcal{B}}(\mu))$, $\bar{\mathcal{D}}^{(0)}_{\mathcal{B}}(\mu) = \mathcal{M}od_{\mathcal{B}^{(1)}}(\mathcal{L}_{\mathcal{B}}(\mu), \mathcal{L}_{\mathcal{B}}(\mu))$; $F_*\mathcal{L}_{\mathcal{B}}(2(p-1)\rho - \mu) \simeq (F_*\mathcal{L}_{\mathcal{B}}(\mu))^\vee$ induces triangulated equivalences

$$\mathrm{D}^b\big(\bar{D}^{(0)}_{\mathcal{B}}(\mu)\mathbf{mod}\big) \underset{\mathbb{R}\Gamma(\mathcal{B},?)}{\overset{\bar{\mathcal{D}}^{(0)}_{\mathcal{B}}(\mu)\otimes^{\mathbb{L}}_{\bar{D}^{(0)}_{\mathcal{B}}(\mu)}?}{\rightleftarrows}} \mathrm{D}^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}^{(0)}_{\mathcal{B}}(\mu)\big)\big)$$

quasi-inverse to each other between the derived category of $\bar{D}_{\mathcal{B}}^{(0)}(\mu)$-modules of finite type and the derived category of coherent $\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\mu)$-modules. In case $\mu = 0$ by what we have formulated in Section 2 the equivalence is G-equivariant:

$$\mathrm{D}_G^b\big(\bar{D}_{\mathcal{B}}^{(0)}\mathbf{mod}\big) \underset{\mathbb{R}\Gamma(\mathcal{B},?)}{\overset{\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}\otimes^{\mathbb{L}}_{\bar{D}_{\mathcal{B}}^{(0)}}?}{\rightleftarrows}} \mathrm{D}_G^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}\big)\big).$$

4.1. Let us first consider in some general framework. Let thus X be any smooth projective variety over $\Bbbk$. We say a coherent $\mathcal{O}_X$-module $\mathcal{M}$ is tilting iff:

(T1) $\mathrm{Ext}_X^i(\mathcal{M}, \mathcal{M}) = 0\ \forall i \geqslant 1$,
(T2) $\mathcal{M}$ Karoubian generates $\mathrm{D}^b(\mathrm{coh}(X))$, i.e., the smallest triangulated subcategory of $\mathrm{D}^b(\mathrm{coh}(X))$ containing $\mathcal{M}$ and closed under taking a direct summand is the whole of $\mathrm{D}^b(\mathrm{coh}(X))$,
(T3) $\mathrm{gl\,dim}\,\mathbf{Mod}_X(\mathcal{M}, \mathcal{M}) < \infty$.

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and set $\bar{\mathcal{D}}_X^{(m)}(\mathcal{L}) = \mathcal{M}od_{X^{(m+1)}}(\mathcal{L}, \mathcal{L})$, $\bar{D}_X^{(m)}(\mathcal{L}) = \Gamma(X, \bar{\mathcal{D}}_X^{(m)}(\mathcal{L})) = \mathbf{Mod}_{X^{(m+1)}}(\mathcal{L}, \mathcal{L})$, $m \in \mathbb{N}$. Let $\mathrm{D}^b(\mathrm{coh}(\bar{\mathcal{D}}_X^{(m)}(\mathcal{L})))$ be the bounded derived category of the category $\mathrm{coh}(\bar{\mathcal{D}}_X^{(m)}(\mathcal{L}))$ of coherent $\bar{\mathcal{D}}_X^{(m)}(\mathcal{L})$-modules, and $\mathrm{D}^b(\bar{D}_X^{(m)}(\mathcal{L})\mathbf{mod})$ the bounded derived category of the category $\bar{D}_X^{(m)}(\mathcal{L})\mathbf{mod}$ of $\bar{D}_X^{(m)}(\mathcal{L})$-modules of finite type. Let $F_X : X \to F$ be the absolute Frobenius endomorphism of X. One has from Beilinson's lemma [5,6].

Lemma. *If $(F_{X*}^{m+1}\mathcal{L})^\vee$ is tilting on X, there arise triangulated equivalences*

$$\mathrm{D}^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}_X^{(m)}(\mathcal{L})\big)\big) \underset{\bar{\mathcal{D}}_X^{(m)}(\mathcal{L})\otimes^{\mathbb{L}}_{\bar{D}_X^{(m)}(\mathcal{L})}?}{\overset{\mathbb{R}\Gamma(X,?)}{\rightleftarrows}} \mathrm{D}^b\big(\bar{D}_X^{(m)}(\mathcal{L})\mathbf{mod}\big)$$

quasi-inverse to each other.

Proof. To ease the notation, we will suppress X when appropriate.

Assume $(F_*^{m+1}\mathcal{L})^\vee$ is tilting. Put $\hat{D}_X^{(m)}(\mathcal{L}) = \mathbf{Mod}_X(F_*^{m+1}\mathcal{L}, F_*^{m+1}\mathcal{L})$. There are natural $\Bbbk$-algebra isomorphisms

$$\hat{D}_X^{(m)}(\mathcal{L})^{\mathrm{op}} \simeq \mathbf{Mod}_X\big(\big(F_*^{m+1}\mathcal{L}\big)^\vee, \big(F_*^{m+1}\mathcal{L}\big)^\vee\big) \quad \text{via } f \mapsto f^\vee \tag{1}$$

and

$$\hat{D}_X^{(m)}(\mathcal{L}) \simeq \mathbf{Mod}_{X^{(m+1)}}(\mathcal{L}, \mathcal{L})^{(-m-1)} = \bar{D}^{(m)}(\mathcal{L})^{(-m-1)} \quad \text{via } \delta \mapsto \delta, \tag{2}$$

where $\bar{D}^{(m)}(\mathcal{L})^{(r)}$, $r \in \mathbb{Z}$, is the ring $\bar{D}^{(m)}(\mathcal{L})$ with the $\Bbbk$-linear structure twisted by $\xi \mapsto \xi^{p^{-r}}$, $\xi \in \Bbbk$. By Beilinson–Baer [5,6] one has triangulated equivalences

$$\mathrm{D}^b\big(\mathrm{coh}(X)\big) \underset{?\otimes^{\mathbb{L}}_{\hat{D}_X^{(m)}(\mathcal{L})^{\mathrm{op}}}(F_*^{m+1}\mathcal{L})^\vee}{\overset{\mathbb{R}\mathbf{Mod}_X((F_*^{m+1}\mathcal{L})^\vee,?)}{\rightleftarrows}} \mathrm{D}^b\big(\mathbf{mod}\,\hat{D}_X^{(m)}(\mathcal{L})^{\mathrm{op}}\big) \tag{3}$$

quasi-inverse to each other. On the other hand, recall from [14, 3.4.2] Morita-equivalence

$$\mathrm{coh}\big(X^{(m+1)}\big) \underset{\mathcal{M}od_{X^{(m+1)}}(\mathcal{L},\mathcal{O}_{X^{(m+1)}})\otimes_{\bar{\mathcal{D}}^{(m+1)}(\mathcal{L})}?}{\overset{\mathcal{L}\otimes_{X^{(m+1)}}?}{\rightleftarrows}} \mathrm{coh}\big(\bar{\mathcal{D}}_X^{(m)}(\mathcal{L})\big). \tag{4}$$

One has also an isomorphism of sheaves of rings on X

$$\mathcal{O}_{X^{(m+1)}} \simeq \mathcal{O}_X \quad \text{via } a \leftarrow\!\shortmid a. \tag{5}$$

Thus putting (1)–(5) together yields a triangulated equivalence

$$\begin{array}{ccc}
\mathrm{D}^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}^{(m)}(\mathcal{L})\big)\big) & & \mathcal{V} \\
\sim\downarrow & & \downarrow \\
\mathrm{D}^b\big(\mathrm{coh}\big(X^{(m+1)}\big)\big) & & \mathcal{M}od_{X^{(m+1)}}(\mathcal{L},\mathcal{O}_{X^{(m+1)}})\otimes^{\mathbb{L}}_{\bar{\mathcal{D}}^{(m)}(\mathcal{L})}\mathcal{V} \\
\sim\downarrow & & \downarrow \\
\mathrm{D}^b\big(\mathrm{coh}(X)\big) & & \mathcal{M}od_X\big(F_*^{m+1}\mathcal{L},\mathcal{O}_X\big)\otimes^{\mathbb{L}}_{\bar{\mathcal{D}}^{(m)}(\mathcal{L})}\mathcal{V} \\
\sim\downarrow & & \downarrow \\
\mathrm{D}^b\big(\mathbf{mod}\,\hat{D}^{(m)}(\mathcal{L})^{\mathrm{op}}\big) & & \mathbb{R}\mathbf{Mod}_X\big(\big(F_*^{m+1}\mathcal{L}\big)^\vee,\mathcal{M}od_X\big(F_*^{m+1}\mathcal{L},\mathcal{O}_X\big)\otimes^{\mathbb{L}}_{\bar{\mathcal{D}}^{(m)}(\mathcal{L})}\mathcal{V}\big). \\
\sim\downarrow & & \\
\mathrm{D}^b\big(\hat{D}^{(m)}(\mathcal{L})\mathbf{mod}\big) & & \\
\sim\downarrow & & \\
\mathrm{D}^b\big(\bar{D}^{(m)}(\mathcal{L})\mathbf{mod}\big) & &
\end{array} \tag{6}$$

Now if $\mathcal{V} \to \mathcal{E}$ is an injective resolution of $\bar{\mathcal{D}}^{(m+1)}(\mathcal{L})$-modules,

$$\begin{aligned}
&\mathbb{R}\mathbf{Mod}_X\big(\big(F_*^{m+1}\mathcal{L}\big)^\vee,\mathcal{M}od_X\big(F_*^{m+1}\mathcal{L},\mathcal{O}_X\big)\otimes_{\bar{\mathcal{D}}_{\mathcal{B}}^{(m)}(\mathcal{L})}\mathcal{V}\big) \\
&\quad\simeq \mathbf{Mod}_X\big(\big(F_*^{m+1}\mathcal{L}\big)^\vee,\mathcal{M}od_X\big(F_*^{m+1}\mathcal{L},\mathcal{O}_X\big)\otimes_{\bar{\mathcal{D}}_{\mathcal{B}}^{(m)}(\mathcal{L})}\mathcal{E}\big)
\end{aligned}$$

$$\begin{aligned}
&\simeq \mathbf{Mod}_X\big(\mathcal{O}_X, \big(F_*^{m+1}\mathcal{L}\big) \otimes_X \mathcal{M}od_X\big(F_*^{m+1}\mathcal{L}, \mathcal{O}_X\big) \otimes_{\bar{\mathcal{D}}^{(m)}(\mathcal{L})} \mathcal{E}\big) \\
&\simeq \Gamma\big(X, \mathcal{M}od_X\big(F_*^{m+1}\mathcal{L}, F_*^{m+1}\mathcal{L}\big) \otimes_{\bar{\mathcal{D}}^{(m)}(\mathcal{L})} \mathcal{E}\big) \simeq \Gamma\big(X, F_*^{m+1}\mathcal{E}\big) \simeq \Gamma(X, \mathcal{E})^{(-m-1)} \\
&\simeq \mathbb{R}\Gamma(X, \mathcal{V})^{(-m-1)} \quad \text{as } \mathcal{E} \text{ is flasque by [21, 2.4.6(vii), p. 99]},
\end{aligned}$$

where $\Gamma(X, \mathcal{E})^{(-m-1)}$ is the abelian group $\Gamma(X, \mathcal{E})$ with the $\Bbbk$-linear structure twisted by $\xi \mapsto \xi^{p^{m+1}}$ and likewise $\mathbb{R}\Gamma(X, \mathcal{V})^{(-m-1)}$. Thus, coupled with the twist in (2), the equivalence (6) reads as $\mathcal{V} \mapsto \mathbb{R}\Gamma(X, \mathcal{V})$.

Likewise in the opposite direction. □

4.2. Let G be any reductive algebraic group over $\Bbbk$. If $\nu \in \Lambda$, as $F_{\mathcal{B}/\Bbbk*}\mathcal{L}(\mu + p\nu) \simeq (F_{\mathcal{B}/\Bbbk*}\mathcal{L}(\mu)) \otimes_{\mathcal{B}^{(1)}} \mathcal{L}(\nu)^{(1)}$ by the projection formula, one has an isomorphism of $\mathcal{O}_{\mathcal{B}}$-rings

$$\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\mu + p\nu) \simeq \bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\mu). \tag{1}$$

It also follows, as $F_*\mathcal{L}(\mu + p\nu) \simeq (F_*\mathcal{L}(\mu)) \otimes_{\mathcal{B}} \mathcal{L}(\nu)$, that

$$F_*\mathcal{L}(\mu + p\nu) \text{ is tilting iff } F_*\mathcal{L}(\mu) \text{ is tilting.} \tag{2}$$

Note also that

$$\begin{aligned}
\mathrm{Ext}_{\mathcal{B}}^{\bullet}\big(F_*\mathcal{L}\big(2(p-1)\rho - \mu\big), F_*\mathcal{L}\big(2(p-1)\rho - \mu\big)\big) &\simeq \mathrm{Ext}_{\mathcal{B}}^{\bullet}\big(\big(F_*\mathcal{L}(\mu)\big)^{\vee}, \big(F_*\mathcal{L}(\mu)\big)^{\vee}\big) \\
&\simeq \mathrm{Ext}_{\mathcal{B}}^{\bullet}\big(F_*\mathcal{L}(\mu), F_*\mathcal{L}(\mu)\big)
\end{aligned} \tag{3}$$

and that

$$\begin{aligned}
&\bar{D}_{\mathcal{B}}^{(0)}\big(2(p-1)\rho - \mu\big)^{\mathrm{op}} \\
&\quad = \mathbf{Mod}_{\mathcal{B}^{(1)}}\big(\mathcal{L}\big(2(p-1)\rho - \mu\big), \mathcal{L}\big(2(p-1)\rho - \mu\big)\big)^{\mathrm{op}} \\
&\quad \simeq \big\{\mathbf{Mod}_{\mathcal{B}}\big(F_*\mathcal{L}\big(2(p-1)\rho - \mu\big), F_*\mathcal{L}\big(2(p-1)\rho - \mu\big)\big)^{(1)}\big\}^{\mathrm{op}} \\
&\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(\big(F_*\mathcal{L}\big(2(p-1)\rho - \mu\big)\big)^{\vee}, \big(F_*\mathcal{L}\big(2(p-1)\rho - \mu\big)\big)^{\vee}\big)^{(1)} \\
&\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(\phi_*\big(\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho - \mu\big)\big)^{\vee}\big), \phi_*\big(\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho - \mu\big)\big)^{\vee}\big)\big)^{(1)} \\
&\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(\phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho - \mu\big)^*\big), \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho - \mu\big)^*\big)\big)^{(1)} \\
&\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(\phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big), \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big)\big)^{(1)} \\
&\quad \simeq \mathbf{Mod}_{\mathcal{B}}\big(F_*\mathcal{L}(\mu), F_*\mathcal{L}(\mu)\big)^{(1)} \simeq \bar{D}_{\mathcal{B}}^{(0)}(\mu).
\end{aligned} \tag{4}$$

If $F_*\mathcal{L}_{\mathcal{B}}(2(p-1)\rho - \mu) \simeq (F_*\mathcal{L}_{\mathcal{B}}(\mu))^{\vee}$ is tilting, then by Beilinson–Baer 4.1 we will have triangulated equivalences

$$\mathrm{D}^b\big(\mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\mu)\big)\big) \underset{\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\mu) \otimes^{\mathbb{L}}_{\bar{D}_{\mathcal{B}}^{(0)}(\mu)} ?}{\overset{\mathbb{R}\Gamma(\mathcal{B}, ?)}{\rightleftarrows}} \mathrm{D}^b\big(\bar{D}_{\mathcal{B}}^{(0)}(\mu)\mathbf{mod}\big) \tag{5}$$

quasi-inverse to each other.

Not all $F_*\mathcal{L}(\mu)$ are tilting, however. For example, if $\mu = (p-1)\rho$, $\hat{\nabla}((p-1)\rho)$ is the Steinberg G-module St, and hence

$$F_*\mathcal{L}\big((p-1)\rho\big) \simeq \phi_*(\mathrm{St} \otimes_{\Bbbk} \mathcal{O}_{\bar{\mathcal{B}}}) \simeq \mathrm{St} \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}}.$$

If $F_*\mathcal{L}((p-1)\rho)$ were tilting, so would be $\mathcal{O}_{\mathcal{B}}$, and hence $\mathbb{R}\mathbf{Mod}_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}, ?) \simeq \mathbb{R}\Gamma(\mathcal{B}, ?)$ would yield a triangulated equivalence $\mathrm{D}^b(\mathrm{coh}(\mathcal{B})) \to \mathrm{D}^b(\mathbf{mod}\,\Bbbk)$; $\mathbf{Mod}_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}}) \simeq \Bbbk$. But $\mathcal{L}(-\rho) \in \mathrm{coh}(\mathcal{B})$ with $\mathbb{R}\Gamma(\mathcal{B}, \mathcal{L}(-\rho)) = 0$, absurd.

There is also a restriction on the characteristic of $\Bbbk$.

Proposition. *Let $G \in \{\mathrm{SL}_2(\Bbbk), \mathrm{SL}_3(\Bbbk), \mathrm{Sp}_4(\Bbbk)\}$ and $\mu \in \Lambda$. If $F_*\mathcal{L}(\mu)$ is tilting on $\mathcal{B}$, then $p \geqslant h$.*

Proof. We may assume G is either $\mathrm{SL}_3(\Bbbk)$ or $\mathrm{Sp}_4(\Bbbk)$. If $F_*\mathcal{L}(\mu)$ is tilting, we will have by (5) the derived localization theorem holding for $\bar{D}_{\mathcal{B}}^{(0)}(2(p-1)\rho - \mu)$. By (2) we may assume $2(p-1)\rho - \mu \in \Lambda_1$.

Assume first $G = \mathrm{SL}_3(\Bbbk)$ in characteristic 2. Then $2(p-1)\rho - \mu \in \{0, \omega_1, \omega_2, \rho\}$. One has

$$\mathcal{L}(-2\omega_1) \simeq \mathcal{O}_{\mathcal{B}} \otimes_{\mathcal{B}^{(1)}} \mathcal{L}(-\omega_1)^{(1)} \in \mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}\big),$$

$$\mathcal{L}(-\omega_1) \simeq \mathcal{L}_{\mathcal{B}}(\omega_1) \otimes_{\mathcal{B}^{(1)}} \mathcal{L}(-\omega_1)^{(1)} \in \mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\omega_1)\big),$$

$$\mathcal{L}(-\omega_2) \simeq \mathcal{L}_{\mathcal{B}}(\omega_2) \otimes_{\mathcal{B}^{(1)}} \mathcal{L}(-\omega_2)^{(1)} \in \mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\omega_2)\big),$$

$$\mathcal{L}(-\omega_1 + \omega_2) \simeq \mathcal{L}_{\mathcal{B}}(\rho) \otimes_{\mathcal{B}^{(1)}} \mathcal{L}(-\omega_1)^{(1)} \in \mathrm{coh}\big(\bar{\mathcal{D}}_{\mathcal{B}}^{(0)}(\rho)\big)$$

with

$$\mathbb{R}\Gamma\big(\mathcal{B}, \mathcal{L}(-2\omega_1)\big) = \mathbb{R}\Gamma\big(\mathcal{B}, \mathcal{L}(-\omega_1)\big) = \mathbb{R}\Gamma\big(\mathcal{B}, \mathcal{L}(-\omega_2)\big) = \mathbb{R}\Gamma\big(\mathcal{B}, \mathcal{L}(-\omega_1 + \omega_2)\big) = 0$$

by [19, II.5.5 and 5.4], 3.2 (2) and (3). It follows that if $p = 2$, no $F_*\mathcal{L}(\mu)$ can be tilting. Likewise for $\mathrm{Sp}_4(\Bbbk)$ in small characteristic. □

4.3. Assume $p \geqslant h$ and that $\mu \in \Lambda$ is p-regular. We now verify (T1)–(T3) for $F_*\mathcal{L}(\mu)$. By 4.2 (2) and by our decomposition of $F_*\mathcal{L}(\mu)$ in 3.1 we have only to check for a single μ in each alcove contained in a box. Thus the assertion holds for $G = \mathrm{SL}_2(\Bbbk)$ by [16, 4.2]. Also for $G = \mathrm{SL}_3(\Bbbk)$ we know that $F_*\mathcal{L}(2(p-1)\rho) \simeq (F_*\mathcal{O}_{\mathcal{B}})^{\vee}$ is tilting from [16]. Then (T1) and (T3) holds for $F_*\mathcal{O}_{\mathcal{B}}$ by 4.2 (3) and (5), and the verification of (T2) for $F_*\mathcal{O}_{\mathcal{B}}$ is entirely analogous to the case of $(F_*\mathcal{O}_{\mathcal{B}})^{\vee}$. Thus we may assume $G = \mathrm{Sp}_4(\Bbbk)$ and have only to check (T1)–(T3) for $F_*\mathcal{L}(2(p-1)\rho)$ and $F_*\mathcal{L}(2(p-1)\omega_1 + (p+1)\omega_2)$, and to check (T2) for $(F_*\mathcal{L}(2(p-1)\rho))^{\vee}$ and $(F_*\mathcal{L}(2(p-1)\omega_1 + (p+1)\omega_2))^{\vee}$.

4.4. Verification of (T1)

Let $G = \mathrm{Sp}_4(\Bbbk)$ and retain the notations from Section 3. One has

$$\begin{aligned}
\operatorname{Ext}^\bullet_{\mathcal{B}}\big(F_*\mathcal{L}(\mu), F_*\mathcal{L}(\mu)\big) &\simeq \operatorname{Ext}^\bullet_{\mathcal{B}}\big(\phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big), \phi_*\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big)\big)\\
&\simeq \operatorname{Ext}^\bullet_{\bar{\mathcal{B}}}\big(\mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big), \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)\big)\big) \quad \text{as } \phi \text{ is an isomorphism of schemes}\\
&\simeq \mathrm{H}^\bullet\big(\bar{\mathcal{B}}, \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}(\mu)^* \otimes_{\Bbbk} \hat{\nabla}(\mu)\big)\big) \quad \text{by [15, III.6.3, 6.7]}\\
&\simeq \mathrm{H}^\bullet\big(\bar{\mathcal{B}}, \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(2(p-1)\rho-\mu\big) \otimes_{\Bbbk} \hat{\nabla}(\mu)\big)\big) \quad \text{by [19, II.9.2]}\\
&\simeq \mathrm{H}^\bullet\big(\bar{\mathcal{B}}, \mathcal{L}_{\bar{\mathcal{B}}}\big(\hat{\nabla}\big(\big(2(p-1)\rho-\mu\big) \otimes_{\Bbbk} \hat{\nabla}(\mu)\big)\big)\big) \quad \text{by the tensor identity}\\
&\simeq \mathrm{H}^\bullet\big(\bar{\mathcal{B}}, q_*\mathcal{L}_{\mathcal{B}}\big(\big(2(p-1)\rho-\mu\big) \otimes_{\Bbbk} \hat{\nabla}(\mu)\big)\big) \quad \text{by [19, I.5.18]}\\
&\simeq \mathrm{H}^\bullet\big(\mathcal{B}, \mathcal{L}_{\mathcal{B}}\big(\big(2(p-1)\rho-\mu\big) \otimes_{\Bbbk} \hat{\nabla}(\mu)\big)\big).
\end{aligned} \tag{1}$$

Consider first the case $\mu = 2(p-1)\rho$; we are to show $\mathrm{H}^i(\mathcal{B}, \mathcal{L}(\hat{\nabla}(2(p-1)\rho))) = 0\ \forall i \geqslant 1$. This has been done in [4]. Let us, however, write down a simpler proof readily applicable to other μ's. Recall from 3.4 the structure of G_1B-modules on the G_1T-socle layers of $\hat{\nabla}(2(p-1)\rho)$. As the weights of $\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))$ are all dominant, it suffices to show

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\rho)\big)^{[1]}\big)\big) = 0 \quad \forall i \geqslant 1.$$

The P_{α_1}-irreducible $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\rho))^{[1]}$ of highest weight $p\rho$ fits in an exact sequence of P_{α_1}-modules

$$0 \to \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(s_{1,1}\cdot p\rho)\big) \to \mathrm{H}^1\big(P_{\alpha_1}/B, \mathcal{L}(s_1\cdot p\rho)\big) \to \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\rho)\big)^{[1]} \to 0$$

with $s_{1,1}\cdot$ being the reflection on the wall $\{x \in \lambda \mid \langle x+\rho, \alpha_1^\vee\rangle = p\}$. For each $i \geqslant 1$

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(s_{1,1}\cdot p\rho)\big)\big)\big) \simeq \mathrm{H}^i\big(\mathcal{B}, \mathcal{L}(s_{1,1}\cdot p\rho)\big) = 0,$$

and hence $\mathrm{H}^i(\mathcal{B}, \mathcal{L}(\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\rho))^{[1]})) \simeq \mathrm{H}^{i+1}(\mathcal{B}, \mathcal{L}(s_1\cdot p\rho)) = 0$ by the standard vanishing for $s_1\cdot p\rho$ [1], as desired.

Consider next the case $\mu = 2(p-1)\omega_1 + (p+1)\omega_2$. We must show

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \hat{\nabla}\big(2(p-1)\omega_1 + (p+1)\omega_2\big)\big)\big) = 0 \quad \forall i \geqslant 1.$$

In view of the G_1T-socle series of $\nabla(2(p-1)\omega_1 + (p+1)\omega_2)$ in 3.7 it suffices to show for each $i \geqslant 1$

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \big(\mathrm{soc}^1_{4,(p-4)\omega_1+\omega_2}\big)^{[1]}\big)\big) = 0 = \mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)^{[1]}\big)\big).$$

Moreover, by the first equality the second equality will follow from

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]}\big)\big) = 0 \quad \forall i \geqslant 1. \tag{2}$$

For the first we must show

$$\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)^{[1]}\big)\big) = 0 \quad \forall i \geqslant 1. \tag{3}$$

Now $\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1))^{[1]}$ is a simple P_{α_1}-module of highest weight $p\omega_1$, which fits in a short exact sequence of P_{α_1}-modules

$$
\begin{aligned}
0 &\to \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}\big((p-3)\omega_2 + p\omega_1 - \alpha_1\big)\big) \to \mathrm{H}^1\big(P_{\alpha_1}/B, \mathcal{L}\big((p-3)\omega_2 + s_1 \cdot p\omega_1\big)\big) \\
&\to (p-3)\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)^{[1]} \to 0.
\end{aligned}
$$

As

$$
\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}\big((p-3)\omega_2 + p\omega_1 - \alpha_1\big)\big)\big)\big) \simeq \mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-2)\rho\big)\big) = 0 \quad \forall i \geqslant 1
$$

by Kempf, it follows that

$$
\begin{aligned}
&\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)^{[1]}\big)\big) \\
&\quad \simeq \mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big(\mathrm{H}^1\big(P_{\alpha_1}/B, \mathcal{L}\big((p-3)\omega_2 + s_1 \cdot p\omega_1\big)\big)\big)\big) \\
&\quad \simeq \mathrm{H}^{i+1}\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 + s_1 \cdot p\omega_1\big)\big) \\
&\quad \simeq \mathrm{H}^{i+1}\big(\mathcal{B}, \mathcal{L}\big(s_1 \cdot \big(p\omega_1 + (p-3)\omega_2\big)\big)\big) \\
&\quad = 0 \quad \text{by [1, Fig. 1, p. 255] again.}
\end{aligned}
$$

Turning to (2),

$$
\begin{aligned}
&\mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((p-3)\omega_2 \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]}\big)\big) \\
&\quad \simeq \mathrm{H}^i\big(G/P_{\alpha_2}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((p-3)\omega_2\big)\big) \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]}\big)\big)
\end{aligned}
$$

with

$$
\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((p-3)\omega_2\big)\big) \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]}
$$

an irreducible P_{α_2}-module of highest weight $p\omega_1 + (2p-3)\omega_2$, which therefore fits in a short exact sequence of P_{α_2}-modules

$$
\begin{aligned}
0 &\to \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((3p-4)\omega_1 + \omega_2\big)\big) \to \mathrm{H}^1\big(P_{\alpha_2}/B, \mathcal{L}\big(s_2 \cdot \big(p\omega_1 + (2p-3)\omega_2\big)\big)\big) \\
&\to \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((p-3)\omega_2\big)\big) \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]} \to 0.
\end{aligned}
$$

As

$$
\mathrm{H}^i\big(G/P_{\alpha_2}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((3p-4)\omega_1 + \omega_2\big)\big)\big)\big) \simeq \mathrm{H}^i\big(\mathcal{B}, \mathcal{L}\big((3p-4)\omega_1 + \omega_2\big)\big) = 0 \quad \forall i \geqslant 1,
$$

by Kempf, it follows that

$$\begin{aligned}
&\mathrm{H}^i\big(G/P_{\alpha_2}, \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}\big((p-3)\omega_2\big)\big) \otimes_{\Bbbk} \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)^{[1]}\big)\big)\\
&\quad\simeq \mathrm{H}^i\big(G/P_{\alpha_2}, \mathcal{L}\big(\mathrm{H}^1\big(P_{\alpha_2}/B, \mathcal{L}\big(s_2 \cdot \big(p\omega_1 + (2p-3)\omega_2\big)\big)\big)\big)\big)\\
&\quad\simeq \mathrm{H}^{i+1}\big(\mathcal{B}, \mathcal{L}\big(s_2 \cdot \big(p\omega_1 + (2p-3)\omega_2\big)\big)\big)\\
&\quad= 0 \quad \text{by [1, Fig. 1, p. 255]}.
\end{aligned}$$

Thus (T1) holds for $F_*\mathcal{L}(2(p-1)\omega_1 + (p+1)\omega_2)$.

Let us now complete the proof of 3.1 by verifying

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big), \mathcal{L}_{\mathcal{B}}\big(\mathrm{soc}_{i,\lambda}\big(\hat{\nabla}(\mu)\big)^1\big)\big) \simeq \Bbbk.$$

Again, we will only write down an argument for a most complicated case to show

$$\begin{aligned}
&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}_{3,(p-4)\omega_1}\big(\hat{\nabla}\big(2(p-1)\omega_1 + (p+1)\omega_2\big)\big)^1\big),\\
&\mathcal{L}\big(\mathrm{soc}_{3,(p-4)\omega_1}\big(\hat{\nabla}\big(2(p-1)\omega_1 + (p+1)\omega_2\big)\big)^1\big)\big) \simeq \Bbbk. \qquad (4)
\end{aligned}$$

We will suppress $\hat{\nabla}(2(p-1)\omega_1 + (p+1)\omega_2)$ for simplicity. One has by (T1) a long exact sequence

$$\begin{aligned}
&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\quad\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big), \mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\quad\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1 \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(2\omega_1 - \omega_2)\big)\big), \mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big) \to 0.
\end{aligned}$$

There is an exact sequence

$$\begin{aligned}
\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) &\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\to \mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)\big)\big)
\end{aligned}$$

with

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1)\big)\big)\big) \simeq \mathrm{H}^0\big(\mathcal{B}, \mathcal{L}\big((-\omega_1) \otimes \mathrm{H}^0\big(P_{\alpha_1}/B, \mathcal{L}(\omega_1 - \omega_2)\big)\big)\big) = 0$$

as $\mathrm{H}^\bullet(P_{\alpha_1}/B, \mathcal{L}(-\omega_1)) = 0$, and

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\rho)\big)\big)\big) \simeq \mathrm{H}^0\big(\mathcal{B}, \mathcal{L}\big((-\omega_2) \otimes \mathrm{H}^0\big(P_{\alpha_2}/B, \mathcal{L}(\omega_2)\big)\big)\big) = 0$$

as $\mathrm{H}^\bullet(P_{\alpha_2}/B, \mathcal{L}(-\omega_2)) = 0$. It follows that

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}(\rho), \mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big) = 0 \qquad (5)$$

and

$$\begin{aligned}&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\&\quad\simeq\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big),\end{aligned}$$

which fits in an exact sequence

$$\begin{aligned}0&\to\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\to\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\&\to\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}(\rho)\big)\\&\to\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\to\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\end{aligned}\tag{6}$$

with

$$\begin{aligned}&\mathrm{Ext}^i_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\quad\simeq\mathrm{H}^i\big(\mathcal{B},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\quad\simeq\mathrm{H}^i\big(G/P_{\alpha_1},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\end{aligned}$$

and

$$\begin{aligned}&\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}(\rho)\big)\\&\quad\simeq\mathrm{H}^0\big(\mathcal{B},\mathcal{L}\big(\omega_2\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\quad\simeq\mathrm{H}^0\big(\mathcal{B},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1)\big)\big)\big)\simeq\mathrm{H}^0\big(\mathcal{B},\mathcal{L}(2\omega_1)\big).\end{aligned}$$

As $\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2))\otimes\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2))$ has a P_{α_1}-filtration of the subquotients $\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(4\omega_1-2\omega_2))$, $\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2))$ and $\Bbbk$ from the top in order,

$$\begin{aligned}&\mathrm{H}^i\big(G/P_{\alpha_1},\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\&\quad\simeq\begin{cases}\mathrm{H}^0(G/P_{\alpha_1},\mathcal{L}(\Bbbk))\simeq\Bbbk & \text{if } i=0,\\ \mathrm{H}^1(\mathcal{B},\mathcal{L}(4\omega_1-2\omega_2))\simeq\mathrm{H}^0(\mathcal{B},\mathcal{L}(2\omega_1)) & \text{if } i=1,\\ 0 & \text{elsewhere.}\end{cases}\end{aligned}$$

Thus sequence (6) reads as an exact sequence

$$\begin{aligned}&0\to\Bbbk\to\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\to\mathrm{H}^0\big(\mathcal{B},\mathcal{L}(2\omega_1)\big)\\&\quad\to\mathrm{H}^0\big(\mathcal{B},\mathcal{L}(2\omega_1)\big)\to\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big).\end{aligned}\tag{7}$$

As $\mathrm{H}^0(\mathcal{B},\mathcal{L}(2\omega_1))$ is G-irreducible, it suffices to show that $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{L}(\omega_1\otimes\mathrm{H}^0(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2))),\mathcal{L}(\mathrm{soc}^1_{3,(p-4)\omega_1}))$ does not have $\mathrm{H}^0(\mathcal{B},\mathcal{L}(2\omega_1))$ as its composition factor. There is by (T1) an exact sequence

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}(3\omega_1-\omega_2),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\quad\to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\quad\to \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(\omega_1)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)=0.
\end{aligned}$$

Moreover,

$$\mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}(3\omega_1-\omega_2),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes \mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)$$

fits in an exact sequence

$$\begin{aligned}
&\mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\
&\quad\to \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes \mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\\
&\quad\to \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes\rho\big)\big)
\end{aligned}$$

with

$$\mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes\rho\big)\big)\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-2\omega_1+2\omega_2)\big)\big)\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(\omega_2)\big)$$

while

$$\begin{aligned}
&\mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big)\big)\\
&\quad\simeq \mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-2\omega_1)\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1)\big)\big)\big)\\
&\quad\simeq \mathrm{H}^0\big(G/P_{\alpha_1},\mathcal{L}\big(\mathrm{H}^1\big(G/P_{\alpha_1},\mathcal{L}(-2\omega_1)\big)\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1)\big)\big)\big)\\
&\quad\simeq \mathrm{H}^0\big(G/P_{\alpha_1},\mathcal{L}\big(\mathrm{H}^0\big(G/P_{\alpha_1},\mathcal{L}(2\omega_1-\alpha_1)\big)^*\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1)\big)\big)\big)\quad\text{by the Serre duality}\\
&\quad\simeq \mathrm{H}^0\big(G/P_{\alpha_1},\mathcal{L}\big((-\omega_2)\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1)\big)\big)\big)\simeq \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(2\omega_1-\omega_2)\big)=0.
\end{aligned}$$

Thus

$$\mathrm{H}^1\big(\mathcal{B},\mathcal{L}\big((-3\omega_1+\omega_2)\otimes \mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\leqslant \mathrm{H}^0\big(\mathcal{B},\mathcal{L}(\omega_2)\big).$$

It follows that there is an epi

$$\mathrm{H}^0\big(\mathcal{B},\mathcal{L}(\omega_2)\big)\twoheadrightarrow \mathrm{Ext}^1_{\mathcal{B}}\big(\mathcal{L}\big(\omega_1\otimes \mathrm{H}^0\big(P_{\alpha_1}/B,\mathcal{L}(2\omega_1-\omega_2)\big)\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big),$$

and hence

$$\mathbf{Mod}_{\mathcal{B}}\big(\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big),\mathcal{L}\big(\mathrm{soc}^1_{3,(p-4)\omega_1}\big)\big)\simeq \Bbbk,$$

as desired.

4.5.

Remark. (T1) holds for $(F_*\mathcal{O}_{\mathcal{B}})^\vee$ in characteristic 2 and 3 also.

4.6. Verification of (T2)

Let $G = \mathrm{Sp}_4(\Bbbk)$. Consider first the case $\mu = 2(p-1)\rho$. Let $\Lambda_0 = \{0, \omega_1, 2\omega_1, 3\omega_1, \omega_2, 2\omega_2, -\omega_1 + 2\omega_2, \rho\}$ be the set of weights of all $\mathrm{soc}_{i,\lambda}(\hat{\nabla}(2(p-1)\rho))^1$, $i \in [1,5]$, $\lambda \in \Lambda_1$. By our decomposition of $F_*\mathcal{L}(2(p-1)\rho)$ we have all $\mathcal{L}(\nu)$, $\nu \in \Lambda_0$, contained in the triangulated subcategory of $\mathrm{D}^b(\mathrm{coh}(\mathcal{B}))$ Karoubian generated by $F_*\mathcal{L}(2(p-1)\rho)$. It thus suffices to show that $\{\mathcal{L}(\nu) \mid \nu \in \Lambda_0\}$ Karoubian generates $\mathrm{D}^b(\mathrm{coh}(\mathcal{B}))$.

Let $\langle\Lambda_0\rangle$ be the triangulated subcategory of $\mathrm{D}^b(\mathrm{coh}(\mathcal{B}))$ Karoubian generated by $\mathcal{L}(\nu)$, $\nu \in \Lambda_0$. By [16, 5.1] one has only to check $\mathcal{L}(\xi) \in \langle\Lambda_0\rangle$ $\forall \xi \in \Lambda$. $\forall \lambda \in \Lambda^+$, put $\nabla(\lambda) = \Gamma(\mathcal{B}, \mathcal{L}(\lambda))$. Recall

$$\mathrm{ch}\,\nabla(\omega_1) = e^{\omega_1} + e^{\omega_1-\omega_2} + e^{-\omega_1} + e^{-\omega_1+\omega_2}. \tag{1}$$

Then there is an exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules [16, 5.1.1]

$$0 \to \mathcal{O}_{\mathcal{B}} \to \mathcal{L}(\omega_1)^{\oplus\binom{4}{3}} \to \mathcal{L}(2\omega_1)^{\oplus\binom{4}{2}} \to \mathcal{L}(3\omega_1)^{\oplus\binom{4}{1}} \to \mathcal{L}(4\omega_1) \to 0. \tag{2}$$

As all but the last term belong to $\langle\Lambda_0\rangle$,

$$\mathcal{L}(n\omega_1) \in \langle\Lambda_0\rangle \quad \forall n \in \mathbb{Z}. \tag{3}$$

Next, $\langle\Lambda_0\rangle \ni \mathcal{L}(\omega_2) \otimes \nabla(\omega_1) \simeq \mathcal{L}(\omega_2 \otimes \nabla(\omega_1))$ with $\mathcal{L}(\omega_2 \otimes \nabla(\omega_1))$ having by (1) a filtration of the subquotients $\mathcal{L}(\rho)$, $\mathcal{L}(\omega_1)$, $\mathcal{L}(-\omega_1+\omega_2)$, and $\mathcal{L}(-\omega_1+2\omega_2)$. As all but $\mathcal{L}(-\omega_1+\omega_2)$ belong to $\langle\Lambda_0\rangle$, so does $\mathcal{L}(-\omega_1+\omega_2)$. Likewise $\langle\Lambda_0\rangle \ni \mathcal{L}(\rho) \otimes \nabla(\omega_1) \simeq \mathcal{L}(\rho \otimes \nabla(\omega_1))$ with $\mathcal{L}(\rho \otimes \nabla(\omega_1))$ having a filtration of the subquotients $\mathcal{L}(2\omega_1+\omega_2)$, $\mathcal{L}(2\omega_1)$, $\mathcal{L}(\omega_2)$, and $\mathcal{L}(2\omega_2)$, and hence $\mathcal{L}(2\omega_1+\omega_2) \in \langle\Lambda_0\rangle$ also. Thus $\mathcal{L}(k\omega_1+\omega_2) \in \langle\Lambda_0\rangle$ $\forall k \in [-1,2]$. Then by (2)

$$\mathcal{L}(n\omega_1+\omega_2) \in \langle\Lambda_0\rangle \quad \forall n \in \mathbb{Z}. \tag{4}$$

In turn, $\langle\Lambda_0\rangle \ni \mathcal{L}(n\omega_1+\omega_2) \otimes \nabla(\omega_1) \simeq \mathcal{L}((n\omega_1+\omega_2) \otimes \nabla(\omega_1))$ with $\mathcal{L}((n\omega_1+\omega_2) \otimes \nabla(\omega_1))$ having a filtration of the subquotients $\mathcal{L}((n+1)\omega_1+\omega_2)$, $\mathcal{L}((n+1)\omega_1)$, $\mathcal{L}((n-1)\omega_1+\omega_2)$, and $\mathcal{L}((n-1)\omega_1+2\omega_2)$. It follows from (3) and (4) that $\mathcal{L}((n-1)\omega_1+2\omega_2) \in \langle\Lambda_0\rangle$, and hence

$$\mathcal{L}(n\omega_1+2\omega_2) \in \langle\Lambda_0\rangle \quad \forall n \in \mathbb{Z}. \tag{5}$$

Repeat the argument with $n\omega_1+\omega_2$ replaced by $n\omega_1+2\omega_2$ to obtain all $\mathcal{L}(n\omega_1+3\omega_2) \in \langle\Lambda_0\rangle$, $n \in \mathbb{Z}$, and then all $\mathcal{L}(n\omega_1+m\omega_2) \in \langle\Lambda_0\rangle$, $m \in \mathbb{N}$.

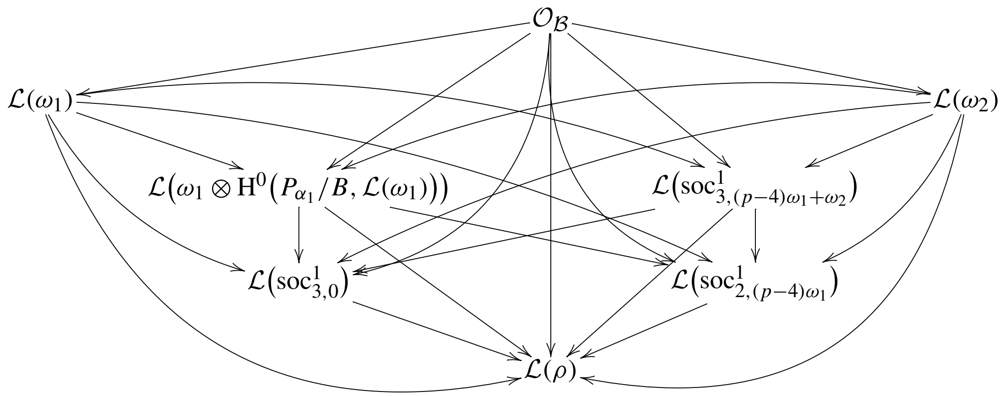
Finally, $\langle\Lambda_0\rangle \ni \mathcal{L}(n\omega_1) \otimes \nabla(\omega_1) \simeq \mathcal{L}(n\omega_1 \otimes \nabla(\omega_1))$ with $\mathcal{L}(n\omega_1 \otimes \nabla(\omega_1))$ having a filtration of the subquotients $\mathcal{L}((n+1)\omega_1)$, $\mathcal{L}((n+1)\omega_1-\omega_2)$, $\mathcal{L}((n-1)\omega_1)$, and $\mathcal{L}((n-1)\omega_1+\omega_2)$. It follows that $\mathcal{L}((n+1)\omega_1-\omega_2) \in \langle\Lambda_0\rangle$, and hence $\mathcal{L}(n\omega_1-\omega_2) \in \langle\Lambda_0\rangle$ $\forall n \in \mathbb{Z}$. Repeat the argument replacing $n\omega_1$ by $n\omega_1-\omega_2$ to obtain all $\mathcal{L}(n\omega_1-2\omega_2) \in \langle\Lambda_0\rangle$, and then all $\mathcal{L}(n\omega_1-m\omega_2) \in \langle\Lambda_0\rangle$, $m \in \mathbb{Z}$. Thus $\mathcal{L}(n\omega_1+m\omega_2) \in \langle\Lambda_0\rangle$, $\forall n,m \in \mathbb{Z}$, as desired.

Likewise $\{\mathcal{L}(-\nu) \mid \nu \in \Lambda_0\}$ Karoubian generates $\mathrm{D}^b(\mathrm{coh}(\mathcal{B}))$, and hence $F_*\mathcal{O}_\mathcal{B} \simeq (F_*\mathcal{L}(2(p-1)\rho))^\vee$ Karoubian generates $\mathrm{D}^b(\mathrm{coh}(\mathcal{B}))$. The same arguments apply to $F_*\mathcal{L}(2(p-1)\omega_1+(p+1)\omega_2)$ and to $F_*\mathcal{L}((p-3)\omega_2) \simeq (F_*\mathcal{L}(2(p-1)\omega_1+(p+1)\omega_2))^\vee$, and hence (T2) holds for all $F_*\mathcal{L}(\mu)$, $\mu \in \Lambda$ p-regular, by 4.3.

4.7. Verification of (T3)

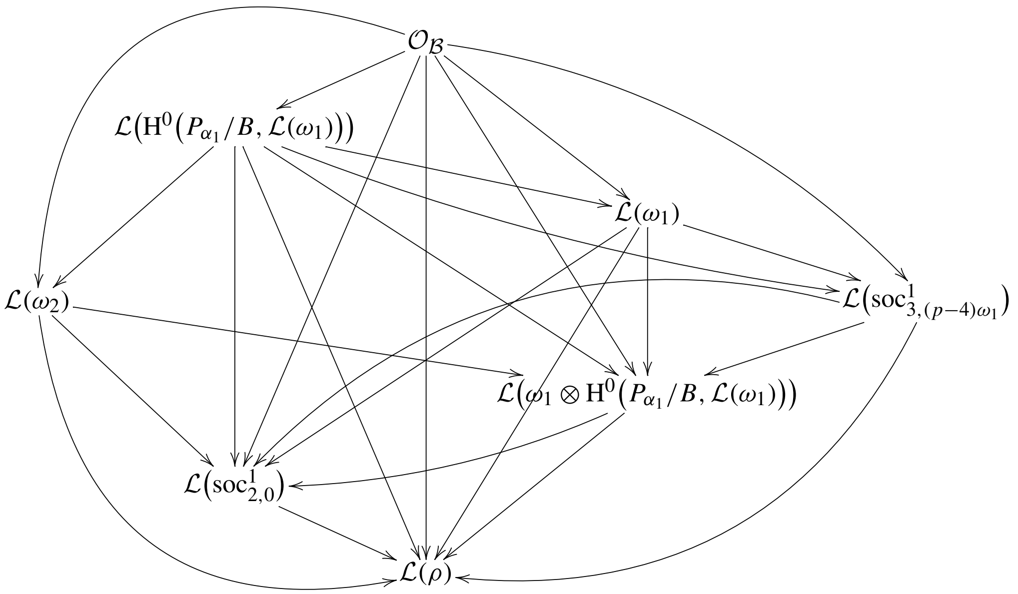
Let $G = \mathrm{Sp}_4(\Bbbk)$. To see that (T3) is holding for $F_*\mathcal{L}(\mu)$, we have only to show that its distinct indecomposable direct summands form a strong exceptional collection [10], [16, 3.3], i.e., that the $\mathcal{O}_\mathcal{B}$-endomorphism ring of each of these sheaves is isomorphic to $\Bbbk$, and that the graph of these sheaves as the vertices with an arrow from one to another iff there is a nonzero morphism of $\mathcal{O}_\mathcal{B}$-modules from the one to the other does not contain a circuit. We will abbreviate $(\mathrm{soc}_{i,\lambda}\hat{\nabla}(\mu))^1$ as $\mathrm{soc}^1_{i,\lambda}$.

Consider first the case $\mu = 2(p-1)\rho$. One first checks that all nonzero summands $\mathcal{L}(\mathrm{soc}_{i,\lambda})^1$ have 1-dimensional endomorphism algebra over $\mathcal{O}_\mathcal{B}$. Thus the distinct indecomposable direct summands of $F_*\mathcal{L}(2(p-1)\rho)$ are $\mathcal{O}_\mathcal{B}$, $\mathcal{L}(\omega_1)$, $\mathcal{L}(\omega_2)$, $\mathcal{L}(\rho)$, $\mathcal{L}(\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1)))$, $\mathcal{L}(\mathrm{soc}^1_{3,0})$, $\mathcal{L}(\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2})$, and $\mathcal{L}(\mathrm{soc}^1_{2,(p-4)\omega_1})$. Using the characterizations of $\mathrm{soc}^1_{3,0}$, $\mathrm{soc}^1_{3,(p-4)\omega_1+\omega_2}$, and $\mathrm{soc}^1_{2,(p-4)\omega_1}$ in 3.5 and using (T1) that those sheaves have no mutual $\mathcal{O}_\mathcal{B}$-extensions, one finds the graph to be



verifying (T3).

Likewise if $\mu = 2(p-1)\omega_1 + (p+1)\omega_2$, the distinct indecomposable direct summands of $F_*\mathcal{L}(2(p-1)\omega_1+(p+1)\omega_2)$ are $\mathcal{O}_\mathcal{B}$, $\mathcal{L}(\omega_1)$, $\mathcal{L}(\omega_2)$, $\mathcal{L}(\rho)$, $\mathcal{L}(\mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1)))$, $\mathcal{L}(\omega_1 \otimes \mathrm{H}^0(P_{\alpha_1}/B, \mathcal{L}(\omega_1)))$, $\mathcal{L}(\mathrm{soc}^1_{2,0})$, and $\mathcal{L}(\mathrm{soc}^1_{3,(p-4)\omega_1})$, the graph of which is given by



without circuits.

4.8. This completes the proof of the first statement of

Proposition. *Let $G \in \{\mathrm{SL}_2(\Bbbk), \mathrm{SL}_3(\Bbbk), \mathrm{Sp}_4(\Bbbk)\}$ and $\mu \in \Lambda$. If $F_*\mathcal{L}(\mu)$ is tilting on $\mathcal{B}$, then $p \geqslant h$, in which case for μ p-regular, $F_*\mathcal{L}(\mu)$ is tilting and hence the derived localization theorem holds for $\bar{D}^{(0)}_{\mathcal{B}}(\mu)$. In particular, for $p \geqslant h$ and μ p-regular all $\bar{D}^{(0)}_{\mathcal{B}}(\mu)$ are derived Morita equivalent.*

Proof. The last assertion follows from the Morita equivalence of $\bar{\mathcal{D}}^{(0)}_{\mathcal{B}}(\mu)$ and $\mathcal{O}_{\mathcal{B}^{(1)}}$. □

4.9. By what we have formulated in Section 2 we obtain

Corollary. *For $G \in \{\mathrm{SL}_2(\Bbbk), \mathrm{SL}_3(\Bbbk), \mathrm{Sp}_4(\Bbbk)\}$ if $p \geqslant h$, the derived localization theorem for $\bar{D}^{(0)}_{\mathcal{B}}$ is G-equivalent*; *there are equivalences*

$$\mathrm{D}^b_G\big(\bar{D}^{(0)}_{\mathcal{B}}\mathbf{mod}\big) \underset{\mathbb{R}\Gamma(\mathcal{B},?)}{\overset{\bar{\mathcal{D}}^{(0)}_{\mathcal{B}} \otimes^{\mathbb{L}}_{\bar{D}^{(0)}_{\mathcal{B}}} ?}{\rightleftarrows}} \mathrm{D}^b_G\big(\mathrm{coh}\big(\bar{\mathcal{D}}^{(0)}_{\mathcal{B}}\big)\big)$$

quasi-inverse to each other.

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References

[1] H.H. Andersen, On the structure of the cohomology of line bundles on G/B, J. Algebra 71 (1981) 245–258.
[2] H.H. Andersen, An inversion formula for the Kazhdan–Lusztig polynomials for affine Weyl groups, Adv. Math. 60 (2) (1986) 125–153.
[3] H.H. Andersen, M. Kaneda, Loewy series of modules for the first Frobenius kernel in a reductive algebraic group, Proc. London Math. Soc. (3) 59 (1989) 74–98.
[4] H.H. Andersen, M. Kaneda, On the D-affinity of the flag variety in type B_2, Manuscripta Math. 103 (3) (2000) 393–399.
[5] D. Baer, Tilting sheaves in representation theory of algebras, Manuscripta Math. 60 (1988) 323–347.
[6] A.A. Beilinson, Coherent sheaves on $\mathbb{P}_n$ and problems of linear algebra, Funct. Anal. Appl. 12 (1979) 214–216.
[7] P. Berthelot, $\mathcal{D}$-modules arithmétiques I. Opérateurs différentiels de niveau fini, Ann. Sci. École Norm. Sup. (4) 29 (1996) 185–272.
[8] P. Berthelot, $\mathcal{D}$-modules arithmétiques II. Descente par Frobenius, Mem. Soc. Math. Fr. 81 (2000).
[9] R. Bezrukavnikov, I. Mirkovic, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, Ann. of Math., in press.
[10] A.I. Bondal, Representation of associative algebras and coherent sheaves, Math. USSR Izv. 34 (1) (1990) 23–42.
[11] A. Borel, Operations on algebraic $\mathcal{D}$-modules, in: A. Borel (Ed.), Algebraic D-Modules, Academic Press, New York, 1987, pp. 207–270.
[12] M. Demazure, P. Gabriel, Groupes algébriques I, Masson, Paris, 1970.
[13] A. Grothendieck, J. Dieudonné, Eléments de Géométrie Algébrique I, Springer, Berlin, 1971.
[14] B. Haastert, Über Differentialoperatoren und $\mathbb{D}$-Moduln in positiver Charakteristik, Dissertation, Univ. Hamburg, 1986.
[15] R. Hartshorne, Algebraic Geometry, Springer, New York, 1977.
[16] Y. Hashimoto, M. Kaneda, D. Rumynin, On localization of $\bar{D}$-modules, in: Z. Lin et al. (Eds.), Proc. Joint Summer Research Conf. on Representations of Algebraic Groups, Quantum Groups, and Lie Algebras, Amer. Math. Soc., in press.
[17] R. Hotta, T. Tanisaki, Dkagun to daisûgun, Springer, Tokyo, 1995 (in Japanese).
[18] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., vol. 9, Springer, 1972.
[19] J.C. Jantzen, Representations of Algebraic Groups, Academic Press, Orlando, 1987.
[20] M. Kaneda, Direct images of $\mathcal{D}$-modules in prime characteristic, Res. Inst. Math. Sci. kôkyûroku 1382 (2004) 154–170.
[21] M. Kashiwara, P. Schapira, Sheaves on Manifolds, Springer, Berlin, 1990.
[22] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Orlando, 1989.
[23] T.A. Springer, Linear Algebraic Groups, second ed., Progr. Math., vol. 9, Birkhäuser, Basel, 1998.