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Homomorphisms of local algebras in positive characteristic[☆]

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ABSTRACT

We study some properties of regularity of homomorphisms of local rings of positive equicharacteristic. In particular we extend a theorem of Izumi about the linear equivalence of topologies in this case. The main tool we use is a monomialization theorem for homomorphisms of power series in positive characteristic.

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1. Introduction

The aim of this paper is to study some properties of regularity of homomorphisms of local \mathbb{k} -algebras, in particular when \mathbb{k} is a field of positive characteristic. In characteristic zero, the geometric rank of a homomorphism of local algebras $\varphi : A \longrightarrow B$ (denoted by $\text{grk}(\varphi)$) is a nice invariant that gives information about the structure of this homomorphism. In particular, a result due to P.M. Eakin and G.A. Harris [E-H] asserts that a homomorphism between rings of formal power series (or convergent power series) over a field of characteristic zero can be monomialized, and after monomialization the geometric rank is equal to the dimension of the image of the monomial homomorphism. Homomorphisms with maximal geometric rank have nice properties that we can summarize in the following theorem:

Theorem 1.1. (See [Ga2,E-H,Mi,Be-Za,Iz3].) *Let $\varphi : A \longrightarrow B$ be a homomorphism of analytic \mathbb{C} -algebras where B is an integral domain. Then the following properties are equivalent:*

- (i) $\text{grk}(\varphi) = \dim(\widehat{A}/\text{Ker}(\widehat{\varphi}))$.
- (ii) $\text{grk}(\varphi) = \dim(\widehat{A}/\text{Ker}(\widehat{\varphi})) = \dim(A/\text{Ker}(\varphi))$.

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- (iii) $\exists a \geq 1, b \geq 0$ such that $\widehat{\varphi}^{-1}(\mathfrak{m}_B^{an+b}) \subset \text{Ker}(\widehat{\varphi}) + \mathfrak{m}_A^n \ \forall n \in \mathbb{N}$.
- (iv) $\widehat{\varphi}(\widehat{A}) \cap B = \varphi(A)$.

Moreover S. Izumi proved (ii) \iff (iii) for any homomorphism of local rings of equicharacteristic zero [Iz4].

In characteristic zero, the geometric rank of $\varphi: A \longrightarrow B$ is equal to the rank of the B -module generated by $\Omega_{\mathbb{K}}^1(A)/\bigcap \mathfrak{m}_A^n$ in $\Omega_{\mathbb{K}}^1(B)/\bigcap \mathfrak{m}_B^n$. Unfortunately this definition does not extend well to positive characteristic for some obvious reasons (for instance look at the \mathbb{K} -homomorphism $\varphi: \mathbb{K}[[x]] \longrightarrow \mathbb{K}[[x]]$ defined by $\varphi(x) = x^p$ where $\text{char}(\mathbb{K}) = p$: this homomorphism obviously satisfies a linear Chevalley estimate).

In this paper we extend the definition of geometric rank in positive characteristic, using the transcendence degree of the \mathfrak{m}_B -adic valuation restricted to A (cf. Section 2). This last definition was first given by M. Spivakovsky in [Sp1]. In Section 3 we prove a result (cf. Theorem 3.4) about the structure of \mathbb{K} -homomorphisms between rings of power series over an infinite field of positive characteristic (similar to the result of P.M. Eakin and G.A. Harris [E-H] valid in characteristic zero). This result involves our definition of geometric rank and shows that it is the right analogue of the geometric rank defined usually in characteristic zero. This result is very close to a monomialization result of dominant homomorphisms in positive characteristic. Moreover the proof of it is algorithmic and allows us to compute the geometric rank.

In Section 4 we can deduce our first main result which is the positive characteristic analogue of the main result of [Iz3], i.e. (ii) \iff (iii) of Theorem 1.1 (Linear Chevalley's Lemma):

Theorem 4.2. *Let $\varphi: A \longrightarrow B$ be a homomorphism of local \mathbb{K} -algebras where \mathbb{K} is a field of positive characteristic. Assume that \widehat{A} is an integral domain and B is regular. Then the following conditions are equivalent:*

- (i) $\text{grk}(\varphi) = \dim(A)$.
- (ii) *There exist $a, b \in \mathbb{R}$ such that $a\nu_{\mathfrak{m}_A}(f) + b \geq \nu_{\mathfrak{m}_B}(\varphi(f))$ for any $f \in A$.*

Homomorphisms satisfying these equivalent conditions are called *regular* homomorphisms.

We would like to mention the work of R. Hübl [Hu] who gave sufficient conditions for general homomorphisms of local rings to satisfy condition (ii). He uses a deep result of S. Izumi and D. Rees about the so-called Rees valuations. Unfortunately these conditions are difficult to check in practice and we do not know if they are necessary conditions.

In characteristic zero, the result of Izumi is important in subanalytic geometry, since Bierstone and Milman showed the paramount importance of the dependence of linearity of the Chevalley function on parameters for the composite function property (cf. [B-M1, B-M2] or see [B-M3] for a general survey about the importance of the Gabrielov's Theorem and the Izumi's Theorem in subanalytic geometry).

The end of the paper is devoted to show how we can use the monomialization theorem (Theorem 3.4) in any characteristic in order to obtain new results or generalizations of known results about regular homomorphisms of local \mathbb{K} -algebras in any characteristic. For example in the second part of Section 4 we give an interpretation in terms of diophantine approximation of the fact that the Chevalley function of a homomorphism that is not regular is not bounded by an affine function.

In Section 5 we study homomorphisms of Henselian \mathbb{K} -algebras, where \mathbb{K} is a field of any characteristic (for definitions, see Section 5), which are generalizations of homomorphisms of convergent or formal power series rings, and we give some cases where such a homomorphism $\varphi: A \longrightarrow B$ satisfies $\widehat{\varphi}^{-1}(B) = A$ or $\text{Ker}(\widehat{\varphi}) = \text{Ker}(\varphi)\widehat{A}$. For this we first state a preparation theorem for Henselian \mathbb{K} -algebras (cf. Proposition 5.10). Indeed the Weierstrass Preparation Theorem is essentially the only tool that we need for this study. Then we give a Henselian version of Theorem 3.4 in any characteristic (cf. Theorem 5.19). We deduce from it a weak version of a theorem of A.M. Gabrielov [Ga2] (the analogue of (ii) \implies (iv) of Theorem 1.1) for good Henselian \mathbb{K} -algebras in any characteristic (cf. Definition 5.14 for the definition of a good Henselian \mathbb{K} -algebra). This is our second main result:

Theorem 5.21. *Let \mathbb{K} be a field of any characteristic. Let $\varphi: A \longrightarrow B$ be a homomorphism of good Henselian \mathbb{K} -algebras, where A and B are regular. If $\text{grk}(\varphi) = \dim(A)$ then φ is strongly injective (i.e. $\widehat{\varphi}^{-1}(B) = A$).*

In Section 6 we study two particular cases of homomorphisms of local rings that are regular. First, using the algorithmic proof of Theorem 5.19, we prove that φ is injective if and only $\text{grk}(\varphi) = 2$, when A is a two-dimensional integral \mathbb{k} -algebra with respect to a W -system (cf. Theorem 6.1). We deduce from this a generalization of a theorem due to S.S. Abhyankar and M. van der Put [Ab-vdP] (who studied the case when A is an analytic regular \mathbb{k} -algebra in any characteristic):

Theorem 6.3. *Let \mathbb{k} be a field of any characteristic. Let $\varphi : A \longrightarrow B$ be a homomorphism of good Henselian \mathbb{k} -algebras where A is regular and $\dim(A) = 2$. If φ is injective then it is strongly injective.*

The second particular case is the case of homomorphisms of analytic algebras defined by algebraic power series over any valued field. This case has been previously studied for $\mathbb{k} = \mathbb{C}$ in [To2, Be, Mi] using transcendental methods. We show here how to prove that such homomorphisms are regular using the monomialization theorem (cf. Corollary 6.8).

Finally, there are still remaining open problems. One of them is to know if the Gabrielov's Theorem (that asserts the following: if $\varphi : A \longrightarrow B$, a homomorphism of analytic \mathbb{C} -algebras, satisfies $\text{grk}(\varphi) = \dim(\hat{A}/\text{Ker}(\hat{\varphi}))$ then $\text{Ker}(\hat{\varphi}) = \text{Ker}(\varphi)\hat{A}$) extends to positive characteristic for analytic \mathbb{k} -algebras (and even for good Henselian \mathbb{k} -algebras in any characteristic). The proof of A. Gabrielov is quite difficult and the attempts to give a simpler proof (even over the field of complex number numbers \mathbb{C}) have not been successful. An other one is to extend these results in mixed characteristic.

1.1. Terminology

In this paper, rings are always assumed to be commutative Noetherian rings with unity. In any case \mathbb{k} denotes a field. A *local \mathbb{k} -algebra* will be a local ring A , with maximal ideal \mathfrak{m}_A , along with an injective homomorphism $\mathbb{k} \longrightarrow A$ such that the induced homomorphism $\mathbb{k} \longrightarrow A/\mathfrak{m}_A$ is a finite field extension. A *homomorphism of local rings* $\varphi : A \longrightarrow B$ means a ring homomorphism such that $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$ and the induced homomorphism $A/\mathfrak{m}_A \longrightarrow B/\mathfrak{m}_B$ is a finite extension of fields. The \mathfrak{m}_A -adic order $v_{\mathfrak{m}_A}$ is defined by $v_{\mathfrak{m}_A}(f) := \max\{n \in \mathbb{N} \mid f \in \mathfrak{m}_A^n\}$ for any $f \in A$. For any $f \in A$, where A is a local ring, $\text{in}(f)$ will denote the image of f in $\text{Gr}_{\mathfrak{m}_A} A$.

2. The geometric rank

Let $\varphi : A \longrightarrow B$ be a homomorphism of local \mathbb{k} -algebras and let us assume that A is an integral domain and B is regular. Consider the valuation $v = v_B \circ \varphi$ defined on $\text{Frac}(A/\text{Ker}(\varphi))$, the quotient field of the domain $A/\text{Ker}(\varphi)$. We denote by A_v the valuation ring associated to v and by \mathfrak{m}_v its maximal ideal. We denote by $\text{tr.deg}_{\mathbb{k}} v$ the transcendence degree of the field extension $\mathbb{k} \longrightarrow \frac{A_v}{\mathfrak{m}_v}$.

The Abhyankar's Inequality says in our context that

$$\text{tr.deg}_{\mathbb{k}} v + 1 \leq \dim(A/\text{Ker}(\varphi)) \quad (\leq \dim(A)).$$

Definition 2.1. (See [Sp1].) If $\text{Ker}(\varphi) \neq \mathfrak{m}_A$, the integer $\text{tr.deg}_{\mathbb{k}} v + 1$ is called the *geometric rank* of φ and denoted $\text{grk}(\varphi)$. If $\text{Ker}(\varphi) = \mathfrak{m}_A$, then $\text{grk}(\varphi) := 0$.

Lemma 2.2. *Let φ , A and B as above. Assume moreover that \hat{A} is an integral domain. Then $\text{grk}(\varphi) = \text{grk}(\hat{\varphi})$.*

Proof. We denote by A_v (resp. $\hat{A}_{\hat{v}}$) the valuation ring associated to $v = v_B \circ \varphi$ (resp. to $\hat{v} = v_B \circ \hat{\varphi}$) and \mathfrak{m}_v (resp. $\hat{\mathfrak{m}}_{\hat{v}}$) its maximal ideal. We have $\hat{\mathfrak{m}}_{\hat{v}} \cap A_v = \mathfrak{m}_v$ thus the quotient homomorphism $\frac{A_v}{\mathfrak{m}_v} \longrightarrow \frac{\hat{A}_{\hat{v}}}{\hat{\mathfrak{m}}_{\hat{v}}}$ is injective, hence $\text{grk}(\hat{\varphi}) \geq \text{grk}(\varphi)$.

On the other hand, if the images of $f_1, \dots, f_q \in \hat{A}_{\hat{v}}$ in the field $\mathbb{k}_{\hat{v}} = \frac{\hat{A}_{\hat{v}}}{\hat{\mathfrak{m}}_{\hat{v}}}$ are algebraically independent over \mathbb{k} , then we can consider elements $f'_1, \dots, f'_q \in A_v$ such that $f'_i - f_i \in \hat{\mathfrak{m}}_{\hat{v}}$. Thus the images of f'_1, \dots, f'_q in $\mathbb{k}_{\hat{v}}$ are algebraically independent over \mathbb{k} because their images coincide with the images of f_1, \dots, f_q . Hence $\text{grk}(\varphi) = \text{grk}(\hat{\varphi})$. \square

Lemma 2.3. Let $\varphi : A \longrightarrow B$ be a homomorphism of local \mathbb{k} -algebras where \widehat{A} is an integral domain and B is regular. If $\text{grk}(\varphi) = \dim(A)$ then φ and $\widehat{\varphi}$ are both injective.

Proof. Since $\dim(A) = \dim(\widehat{A})$ and $\text{Ker}(\varphi)$ and $\text{Ker}(\widehat{\varphi})$ are prime ideals, the assertion follows from the Abhyankar's Inequality. \square

Lemma 2.4. Let $\varphi : A \longrightarrow B$ and $\sigma : A' \longrightarrow A$ be homomorphisms of local \mathbb{k} -algebras, where A and A' are integral domains and B is regular. If σ is finite and injective then $\text{grk}(\varphi \circ \sigma) = \text{grk}(\varphi)$.

Proof. We may assume that φ is injective by replacing the local \mathbb{k} -algebras A and A' by $A/\text{Ker}(\varphi)$ and $A'/\sigma^{-1}(\text{Ker}(\varphi))$ respectively. We denote by ν and ν' the valuations induced by φ and $\varphi \circ \sigma$ respectively. Let $f \in A_\nu$. Then there are $a_i \in A'_{\nu'}$ and $k \in \mathbb{N}$ such that

$$a_0 f^k + a_1 f^{k-1} + \cdots + a_k = 0$$

because $\text{Frac}(A') \subset \text{Frac}(A)$ is finite. We can assume that at least one of the a_i 's satisfies $\nu'(a_i) = 0$ by dividing the last relation by an element a_{i_0} satisfying $\nu'(a_{i_0}) = \min_i \nu'(a_i)$. Then, if $\nu(f) = 0$, we see that the image of f in \mathbb{k}_ν satisfies a non-trivial integral equation over $\mathbb{k}_{\nu'}$. Then the field extension $\mathbb{k}_{\nu'} \longrightarrow \mathbb{k}_\nu$ is algebraic and $\text{grk}(\varphi) = \text{grk}(\varphi \circ \sigma)$. \square

Lemma 2.5. Let $\varphi : \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}'[[y_1, \dots, y_m]]$ be a homomorphism of formal power series rings where $\mathbb{k} \longrightarrow \mathbb{k}'$ is finite. Let $\varphi_{\mathbb{k}'}$ denote the induced homomorphism $\mathbb{k}'[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}'[[y_1, \dots, y_m]]$. Then $\text{grk}(\varphi) = \text{grk}(\varphi_{\mathbb{k}'})$.

Proof. The homomorphism $\mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}'[[x_1, \dots, x_n]]$ is finite and injective. Thus the result follows from Lemma 2.4. \square

Finally we give a combinatorial characterization of the geometric rank. For any $f \in \mathbb{k}[[y_1, \dots, y_m]]$, we denote by $\text{in}(f)$ the form of lowest degree in the power series expansion of f . We define a total ordering $<$ on \mathbb{N}^m in the following way: for any $\alpha, \beta \in \mathbb{N}^m$, we say that $\alpha < \beta$ if $(|\alpha|, \alpha_1, \dots, \alpha_m) < (|\beta|, \beta_1, \dots, \beta_m)$ for the left-lexicographic ordering, where $|\alpha| := \alpha_1 + \cdots + \alpha_m$. This ordering induces a monomial ordering on $\mathbb{k}[[y_1, \dots, y_m]]$. If $M = a_\alpha y^\alpha$ is a monomial, we define $\exp(M) := \alpha$. For any $f \in \mathbb{k}[[y_1, \dots, y_m]]$, we define $\text{in}_<(f)$ to be the monomial of least order in the power series expansion of f and $\exp(f) := \exp(\text{in}_<(f))$.

Proposition 2.6. Let $\varphi : A := \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow B := \mathbb{k}[[y_1, \dots, y_m]]$ be a homomorphism of formal power series rings. Let C be the minimal cone of \mathbb{R}^m containing $\exp(\varphi(f))$ for any $f \in \mathbb{k}[[x_1, \dots, x_n]]$. Then $\text{grk}(\varphi) = \dim(C)$.

Proof. Let us denote by ord the (y_1, \dots, y_m) -adic valuation on B and ν the valuation on A equal to $\text{ord} \circ \varphi$.

Let $\bar{f} \in \mathbb{k}_\nu$ being the image of $f \in A_\nu$. We may write $f = \frac{g}{h}$ where $g, h \in \mathbb{k}[[x_1, \dots, x_n]]$ and $\nu(g) = \nu(h)$. The homomorphism φ induces an injection $\mathbb{k}_\nu \longrightarrow \mathbb{k}_{\text{ord}} = \mathbb{k}(\frac{y_1}{y_m}, \dots, \frac{y_{m-1}}{y_m})$, and the image of \bar{f} under this injection is just $\frac{\text{in}(\varphi(g))}{\text{in}(\varphi(h))}$.

Let us denote $B' := \mathbb{k}[\text{in}(\varphi(f))]_{f \in \mathbb{k}[[x_1, \dots, x_n]]}$ and $\mathbb{K}' := \text{Frac}(B')$. First we will prove that $\text{grk}(\varphi) = \dim(B')$.

We have $\dim(B') = \dim(\text{Spec}(B'))$. But we can look at B' as a graded ring because any homogeneous component of any element of B' is in B' . If we consider $\text{Proj}(B')$, we see that $\dim(B') = \dim(\text{Proj}(B')) + 1$. So we have to prove that $\text{tr.deg}_{\mathbb{k}} \mathbb{k}_\nu$ is equal to the maximal number of algebraically independent elements of \mathbb{K}' of the form g/h where g and h are homogeneous of same degree.

Let us choose $f'_1, \dots, f'_r \in \mathbb{K}'$ algebraically independent over \mathbb{k} , such that $f'_i = \frac{g'_i}{h'_i}$ for any i , where $g'_i, h'_i \in B'$ are homogeneous of same degree. By definition there exist g_i and $h_i \in \mathbb{k}[[x_1, \dots, x_n]]$ such that $\text{in}(\varphi(g_i)) = g'_i$ and $\text{in}(\varphi(h_i)) = h'_i$ for any i . Let us denote $f_i := \frac{g_i}{h_i}$ for any i . Then $f_i \in A_v$ for all i and their images in \mathbb{k}_v are algebraically independent over \mathbb{k} . Then we see that $\text{grk}(\varphi) \geq \dim(B')$.

On the other hand, let $\bar{f}_1, \dots, \bar{f}_s \in \mathbb{k}_v$ be algebraically independent over \mathbb{k} . Let $f_i \in A_v$ be a lifting of \bar{f}_i for $1 \leq i \leq s$. For any i we may write $f_i = \frac{g_i}{h_i}$ where $g_i, h_i \in \mathbb{k}[[x_1, \dots, x_n]]$ and $v(g_i) = v(h_i)$. Let f'_i denote $\frac{\text{in}(\varphi(g_i))}{\text{in}(\varphi(h_i))}$ for $1 \leq i \leq s$. Then $f'_1, \dots, f'_s \in \mathbb{K}'$ are algebraically independent over \mathbb{k} . Thus $\text{grk}(\varphi) = \dim(B')$.

Now let us denote by B'' the sub- \mathbb{k} -algebra of $\mathbb{k}[y_1, \dots, y_m]$ generated by the $\text{in}_>(f)$ where $f \in B'$. Then, because B' is a \mathbb{k} -subalgebra of $\mathbb{k}[y_1, \dots, y_m]$ generated by homogeneous polynomials, the Hilbert function of B' is the same as the Hilbert function of B'' (for instance look at Proposition 6.6.1 of [K-R]). It implies that $\dim(B') = \dim(B'')$. But $\dim(B'')$ is exactly the dimension of C . Thus we have proved the proposition. \square

Corollary 2.7. Let $\varphi : A := \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow B := \mathbb{k}[[y_1, \dots, y_m]]$ be a homomorphism of formal power series rings. Let t be a variable over \mathbb{k} and let $\mathbb{K} := \mathbb{k}(t)$. Let $\varphi_{\mathbb{K}} : A' := \mathbb{K}[[x_1, \dots, x_n]] \longrightarrow B' := \mathbb{K}[[y_1, \dots, y_m]]$ be the homomorphism of formal power series rings induced by φ . Then $\text{grk}(\varphi_{\mathbb{K}}) = \text{grk}(\varphi)$.

Proof. According to the proof of Proposition 2.6, $\text{grk}(\varphi) = \dim(\mathbb{k}[\text{in}(f), f \in A])$ and $\text{grk}(\varphi_{\mathbb{K}}) = \dim(\mathbb{K}[\text{in}(f), f \in A'])$. If $g = \text{in}(f)$ with $f \in A'$ then $g = \text{in}(\lambda_1 f_1 + \dots + \lambda_s f_s)$ with $\lambda_i \in \mathbb{K}$ and $f_i \in A$ for $1 \leq i \leq s$. We may assume that $\lambda_i \in \mathbb{k}[t]$ for $1 \leq i \leq s$ by multiplying g by a non-zero element of \mathbb{K} . We write $\lambda_i = \sum_{j=0}^r \lambda_{i,j} t^j$ with $\lambda_{i,j} \in \mathbb{k}$ for $1 \leq i \leq s$ and $0 \leq j \leq r$. Then we get $g = \sum_{j=0}^r g_j t^j$ with $g_j = \text{in}(\sum_{i=1}^s \lambda_{i,j} f_i)$ for $0 \leq j \leq r$ because t is transcendent over \mathbb{k} . It follows that $\mathbb{K}[\text{in}(f), f \in A']$ and $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}[\text{in}(f), f \in A]$ are \mathbb{k} -isomorphic, hence $\text{grk}(\varphi) = \text{grk}(\varphi_{\mathbb{K}})$. \square

Proposition 2.8. Let $\varphi : A \longrightarrow B$, $\sigma_1 : A \longrightarrow A$ and $\sigma_2 : B \longrightarrow B$ be homomorphisms of local \mathbb{k} -algebras. Let us assume that there exist $a_1, a_2 > 0$ such that $\mathfrak{m}_A^{a_1 n} \subset \sigma_1(\mathfrak{m}_A^n)$ and $\mathfrak{m}_B^{a_2 n} \subset \sigma_2(\mathfrak{m}_B^n)$ for any $n \in \mathbb{N}$. Then $\text{grk}(\sigma_1) = \dim(A)$, $\text{grk}(\sigma_2) = \dim(B)$ and $\text{grk}(\varphi) = \text{grk}(\varphi \circ \sigma_1) = \text{grk}(\sigma_2 \circ \varphi)$.

Proof. We will prove the result for σ_2 , the proof for σ_1 being similar. Using the notation used in the proof of Proposition 2.6, $\text{grk}(\varphi) = \dim(B')$ is the degree of the Samuel polynomial $P(n)$ equal to $\dim_{\mathbb{k}}(\frac{\varphi(A)}{\varphi(A) \cap \mathfrak{m}_B^n})$ for $n \gg 0$. In the same way $\text{grk}(\sigma_2 \circ \varphi)$ is equal to the degree of the Samuel polynomial $Q(n)$ equal to $\dim_{\mathbb{k}}(\frac{\sigma_2(\varphi(A))}{\sigma_2(\varphi(A)) \cap \mathfrak{m}_B^n})$ for $n \gg 0$. By assumption we have $\mathfrak{m}_B^{a_2 n} \subset \sigma_2(\mathfrak{m}_B^n) \subset \mathfrak{m}_B^n$ for any $n \in \mathbb{N}$. Thus we get the following \mathbb{k} -linear maps:

$$\frac{\sigma_2(\varphi(A))}{\sigma_2(\varphi(A)) \cap \mathfrak{m}_B^{a_2 n}} \longrightarrow \frac{\sigma_2(\varphi(A))}{\sigma_2(\varphi(A)) \cap \sigma_2(\mathfrak{m}_B^n)} \longrightarrow \frac{\varphi(A)}{\varphi(A) \cap \mathfrak{m}_B^n} \longrightarrow \frac{\sigma_2(\varphi(A))}{\sigma_2(\varphi(A)) \cap \mathfrak{m}_B^n}$$

where the first and last arrows are obvious quotient homomorphisms (thus they are \mathbb{k} -linear) and where the second arrow is a surjective \mathbb{k} -linear map defined by choosing a lifting in $\frac{\varphi(A)}{\varphi(A) \cap \mathfrak{m}_B^n}$ of any element of $\frac{\sigma_2(\varphi(A))}{\sigma_2(\varphi(A)) \cap \sigma_2(\mathfrak{m}_B^n)}$.

Hence we have $Q(a_2 n) \geq P(n) \geq Q(n)$ for $n \gg 0$. Thus we see that $\deg(P) = \deg(Q)$, hence $\text{grk}(\sigma_2 \circ \varphi) = \text{grk}(\varphi)$. We get $\text{grk}(\sigma_2) = \dim(B)$ by choosing $\varphi = \text{id}_B$. \square

3. Algorithm for modifying a homomorphism of a given rank

We give here a positive characteristic version of a theorem proved by Eakin and Harris [E-H] in characteristic zero. This result is about the structure of homomorphisms of rings of formal power series over an infinite field of positive characteristic. First we give the following definition:

Definition 3.1. Let $\varphi: \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ be a homomorphism of formal power series rings. An *admissible transformation* of φ is a homomorphism $\bar{\varphi}$ related to φ in one of the following ways:

- (1) *Modification by automorphisms:* There exist a \mathbb{k} -automorphism τ of $\mathbb{k}[[x_1, \dots, x_n]]$ and a \mathbb{k} -automorphism σ of $\mathbb{k}[[y_1, \dots, y_m]]$ such that $\bar{\varphi} = \sigma \circ \varphi \circ \tau$.
- (2) *Modification by blowing-up:* There is $k \in \{1, \dots, m-1\}$ such that $\bar{\varphi} = \psi \circ \varphi$ where ψ is defined by

$$\begin{aligned}\psi(y_i) &= y_i & \text{for } i \leq k, \\ \psi(y_i) &= y_k y_i & \text{for } i > k.\end{aligned}$$

- (3) *Modification by ramification:* There is $d \in \mathbb{N}^*$ such that $\varphi = \bar{\varphi} \circ \psi_d$ where ψ_d is defined by

$$\psi_d(x_1) = x_1^d, \quad \text{and} \quad \psi_d(x_i) = x_i \quad \forall i \neq 1.$$

- (4) *Modification by contraction:* There is $k \in \{1, \dots, n-1\}$ such that $\varphi = \bar{\varphi} \circ \psi$ where ψ is defined by

$$\begin{aligned}\psi(x_i) &= x_i & \text{for } i \leq k, \\ \psi(x_i) &= x_i x_k & \text{for } i > k.\end{aligned}$$

Remark 3.2. We define the local \mathbb{k} -homomorphism $q: \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}[[x_1, \dots, x_n]]$ by $q(x_1) = x_1 x_2$ and $q(x_i) = x_i$ for $i > 1$. It is clear that the homomorphisms ψ defined in (4) of Definition 3.1 are compositions of q with permutations of the x_i 's. Thus we may use q instead of ψ in modification (4) of Definition 3.1. The same remark remains true for modifications by blowing-up.

Lemma 3.3. Let $\varphi: A := \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow B := \mathbb{k}[[y_1, \dots, y_m]]$ be a homomorphism of formal power series rings. Let us consider a modification $\bar{\varphi}$ of φ . Then $\text{grk}(\bar{\varphi}) = \text{grk}(\varphi)$. Moreover if there exist a and b such that

$$av_{\mathfrak{m}_A}(f) + b \geq v_{\mathfrak{m}_B}(\bar{\varphi}(f))$$

for any $f \in A$, then there exist a' and b' such that

$$a'v_{\mathfrak{m}_A}(f) + b' \geq v_{\mathfrak{m}_B}(\varphi(f))$$

for any $f \in A$.

Proof. The lemma is obvious for modifications of type (1).

The second statement is a consequence of the following inequalities:

$$v_B(f) \leq v_B(\psi(f)) \leq 2v_B(f) \quad \forall f \in B, \text{ for modifications of type (2),}$$

$$v_A(f) \leq v_A(\psi_d(f)) \leq dv_A(f) \quad \forall f \in A, \text{ for modifications of type (3),}$$

$$v_A(f) \leq v_A(\psi(f)) \leq 2v_A(f) \quad \forall f \in A, \text{ for modifications of type (4).}$$

Finally Proposition 2.8 gives us $\text{grk}(\bar{\varphi}) = \text{grk}(\varphi)$ in any cases. \square

Now we can state the key result of this article. The proof of this theorem is inspired by the proof of a similar result in characteristic zero proved by Eakin and Harris [E-H].

Theorem 3.4. Let \mathbb{k} be an infinite field of characteristic $p > 0$ and consider a homomorphism $\varphi: A := \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow B := \mathbb{k}[[y_1, \dots, y_m]]$ of power series rings. Then there exists a finite sequence of admissible homomorphisms of formal power series rings $(\varphi_i)_{i=0}^k: \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ such that $\varphi_0 = \varphi$ and $\varphi_k(x_i) = y_i^{p^{\alpha_i}} u_i$, for some units u_i , for $i \leq \text{grk}(\varphi)$, and $\varphi_k(x_i) = 0$ for $i > \text{grk}(\varphi)$. Moreover, for any i , $u_i = 1$ if $\alpha_i = 0$, and $\text{in}(u_i) = 1$ and $u_i \notin B^p$ if $\alpha_i > 0$.

Proof. If $\text{grk}(\varphi) = 0$, then $\varphi(f) = 0$ for all $f \in \mathfrak{m}_A$. So we have the result.

We assume now that $\text{grk}(\varphi) > 0$.

We will proceed by induction on the q -tuple $\mu = (\mu_1, \dots, \mu_q) \in (\mathbb{N} \cup \{+\infty\})^q$, defined later, ordered with the lexicographic order where $q \leq n$. At the beginning, $q = n$ and this q -tuple is $(+\infty, \dots, +\infty)$.

Step 0. If $\varphi(x_1) = 0$ we exchange x_n and x_1 . Then we define $q = n - 1$ and $\mu := (\mu_1, \dots, \mu_{n-1}) = (+\infty, \dots, +\infty)$.

Step 1. If $\varphi(x_1) \neq 0$, then we denote $d := \text{ord}(\varphi(x_1)) \in \mathbb{N}^*$. We denote by $g_d(y_1, \dots, y_m)$ the initial term of $\varphi(x_1)$. Let $(a_{i,j})_{i,j=1,\dots,m}$ be a non-singular matrix with entries in \mathbb{k} such that $g_d(a_{1,1}, \dots, a_{m,1}) \neq 0$ (\mathbb{k} is infinite). We define an automorphism ψ of $\mathbb{k}[[y_1, \dots, y_m]]$ by

$$\psi(y_j) := \sum_{k=1}^m a_{j,k} y_k \quad \forall j = 1, \dots, m.$$

So we get

$$\begin{aligned} \psi \circ \varphi(x_1) &= g_d(a_{1,1}, \dots, a_{m,1}) y_1^d + \{\text{terms of degree } d \text{ not divisible by } y_1^d\} \\ &\quad + \{\text{terms of degree } > d\}. \end{aligned}$$

By composing $\psi \circ \varphi$ on the right by the automorphism of $\mathbb{k}[[x_1, \dots, x_n]]$ consisting in dividing x_1 by $g_d(a_{1,1}, \dots, a_{m,1})$, we may assume that

$$\varphi(x_1) = y_1^d + \{\text{terms of degree } d \text{ not divisible by } y_1^d\} + \{\text{terms of degree } > d\}.$$

Now we define the homomorphism ψ by

$$\begin{aligned} \psi(y_1) &:= y_1, \\ \psi(y_i) &:= y_1 y_i, \quad \text{for } i > 1. \end{aligned}$$

We have $\psi \circ \varphi(x_1) = u y_1^d$, u being a unit of $\mathbb{k}[[y_1, \dots, y_m]]$ with $\text{in}(u) = 1$.

Step 2. If $d = ep^\alpha$ with $\gcd(e, p) = 1$, then we see that $\psi \circ \varphi = \varphi' \circ \tau'$ where $\tau'(x_1) = x_1^e$ and $\tau'(x_i) = x_i$ for $i \neq 1$, and $\varphi'(x_1) = u' y_1^{p^\alpha}$, $\varphi'(x_i) = \psi \circ \varphi(x_i)$ for $i \neq 1$ and $\text{in}(u') = 1$. So we can replace φ by φ' .

In particular, if $\gcd(d, p) = 1$, then we may assume $\varphi(x_1) = y_1$.

Then, if $u \in B^{p^\beta}$, with $\beta \leq \alpha$, then we have $\varphi(x_1) = u^{p^\beta} (y_1^{p^{\alpha-\beta}})^{p^\beta}$. So we see that $\psi \circ \varphi = \varphi' \circ \tau'$ where $\tau'(x_1) = x_1^{p^\beta}$ and $\tau'(x_i) = x_i$ for $i \neq 1$, and $\varphi'(x_1) = u' y_1^{p^{\alpha-\beta}}$, $\varphi'(x_i) = \psi \circ \varphi(x_i)$ for $i \neq 1$ and $\text{in}(u') = 1$.

So we may assume that $\varphi(x_1) = u y_1^{p^{\alpha_1}}$, $\text{in}(u) = 1$ and $u \notin B^p$ if $\alpha_1 \neq 0$. At this step, the q -tuple $(\mu_1, \dots, \mu_q) = (\alpha_1, +\infty, \dots, +\infty)$.

Step 3. We assume that $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$, for $i < j$, with $\text{in}(u_i) = 1$ and $u_i \notin B^p$ if $\alpha_i \neq 0$, and $\varphi(x_i) = 0$ for $i > q$. Moreover we assume that $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{j-1}$. We denote $\mu = (\alpha_1, \dots, \alpha_{j-1}, +\infty, \dots, +\infty) \in (\mathbb{N} \cup \{+\infty\})^q$.

We assume that $\text{in}(\varphi(x_j))$ contains a monomial of the form $cy_1^{k_1} \dots y_{j-1}^{k_{j-1}}$. If p^{α_i} divides k_i for all $i \leq j-1$, then we replace x_j by the element $x_j - cx_1^{k_1/p^{\alpha_1}} \dots x_{j-1}^{k_{j-1}/p^{\alpha_{j-1}}}$. We can go on and by induction, there are two cases. In the first case we can replace x_j by an element of the form $x_j - \sum_k c_k x_1^{k_1/p^{\alpha_1}} \dots x_{j-1}^{k_{j-1}/p^{\alpha_{j-1}}}$, where the sum is finite, and then we can assume that $\text{in}(\varphi(x_j))$ has no monomial of the form $cy_1^{k_1} \dots y_{j-1}^{k_{j-1}}$ where p^{α_i} divides k_i for all i . In the second case we can replace x_j by an element of the form $x_j - \sum_k c_k x_1^{k_1/p^{\alpha_1}} \dots x_{j-1}^{k_{j-1}/p^{\alpha_{j-1}}}$, where the sum is not necessarily finite, and then we have $\varphi(x_j) = 0$.

If $\varphi(x_j) = 0$, then we exchange x_q and x_j . Then we replace q by $q-1$ and $\mu := (\mu_1, \dots, \mu_q) = (\alpha_1, \dots, \alpha_{j-1}, +\infty, \dots, +\infty)$.

Step 4. We assume that $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$, for $i < j$, for some units u_i with $\text{in}(u_i) = 1$ and $u_i \notin B^p$ if $\alpha_i \neq 0$, and $\varphi(x_i) = 0$ for $i > q$. Moreover we assume that $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{j-1}$. As before we denote $\mu = (\alpha_1, \dots, \alpha_{j-1}, +\infty, \dots, +\infty) \in (\mathbb{N} \cup \{+\infty\})^q$.

Let us consider $cy_1^{k_1} \dots y_m^{k_m}$ a monomial of $\text{in}(\varphi(x_j))$. If one of k_j, \dots, k_m is different from zero, then after a permutation of the elements y_j, \dots, y_m , we can assume that $k_j \neq 0$. According to Step 3 we can assume that $\text{in}(\varphi(x_j))$ has no monomial of the form $cy_1^{k_1} \dots y_{j-1}^{k_{j-1}}$ where p^{α_i} divides k_i for all i , and we assume that $\varphi(x_j) \neq 0$.

Assume that for any non-zero monomial $M = cy_1^{k_1} \dots y_{j-1}^{k_{j-1}}$ of $\text{in}(\varphi(x_j))$, p^{α_i} divides k_i for any $i < l$, but for at least one such monomial p^{α_l} does not divide k_l . After a change of variables of the form $\sigma(y_i) = y_i$ for $i \leq l$ and $\sigma(y_i) = y_i + y_l^{\lambda_i}$ for $i > l$ and for some $\lambda_i \in \mathbb{N}$, we may assume that $\text{in}(\varphi(x_j))$ contains a non-zero monomial of the form $cy_1^{k_1} \dots y_l^{k_l}$ where p^{α_i} divides k_i for any $i < l$ and p^{α_l} does not divide k_l . Then after a composition with a homomorphism of the form $\psi(y_i) = y_i$ for $i \leq l$ and $y_i = y_1 y_i$ for $i > l$, we may assume that each monomial of $\text{in}(\varphi(x_j))$ depends only on y_1, \dots, y_l . And by Step 3, we may assume that for any monomial $cy_1^{k_1} \dots y_l^{k_l}$ of $\text{in}(\varphi(x_j))$, k_i is divisible by p^{α_i} for any $i < l$, and that k_l is not divisible by p^{α_l} . Finally we can exchange x_j and x_l and we can apply the following lemma with $\alpha = \alpha_l$:

Lemma 3.5. Under the hypothesis of Theorem 3.4, we assume that $\varphi(x_i) = y_i^{p^{\alpha_i}}$ for all $i < l$ and that the monomials of $\text{in}(\varphi(x_l))$ depend only on y_1, \dots, y_l . We assume moreover that for any monomial $cy_1^{k_1} \dots y_l^{k_l}$ of $\text{in}(\varphi(x_l))$, k_i is divisible by p^{α_i} for any $i < l$, and that k_l is not divisible by p^{α} . Then there exists a finite sequence of modifications of φ , such that $\bar{\varphi}$, the last homomorphism of the sequence, satisfies

$$\bar{\varphi}(x_i) = y_i^{p^{\alpha_i}} u_i \quad \text{for } i < l,$$

$$\bar{\varphi}(x_l) = y_l^{p^{\alpha'}} u_l$$

for some units u_j and with $\alpha' < \alpha$.

Proof. We have $\varphi(x_l) = M_1 v_1 + \dots + M_r v_r$ for some units v_i and some monomials M_i . We assume that this expression is minimal: it means that none of these monomials divides another one. The convex hull in \mathbb{N}^m of the set of elements (w_1, \dots, w_m) such that $\text{in}(\varphi(x_l))$ contains a non-zero monomial of the form $cy_1^{w_1} \dots y_m^{w_m}$ is a convex polyhedron P of dimension strictly less than l (because all such elements satisfy $w_{l+1} = \dots = w_m = 0$). Let $(w_1, \dots, w_l, 0, \dots, 0)$ be a vertex of this polyhedron. In particular w_l is not divisible by α . We may assume that M_1 corresponds to this vertex. We denote

by $(w_{1,k}, \dots, w_{m,k})$ the element of \mathbb{N}^m that corresponds to M_k for $k > 1$. Because M_1 is a vertex of P the cone defined by the following equations in the variables e_i :

$$\sum_{i=1}^l (w_{i,k} - w_i) e_i > 0$$

for all k such that the monomial M_k depends only on y_1, \dots, y_l , is a non-empty open set of $(\mathbb{R}_{\geq 0})^l$. Moreover for any $C > 0$, using modifications by blowing-up on the variables y_{l+1}, \dots, y_m , we may assume that the monomials M_k depending on at least one of y_{l+1}, \dots, y_m satisfy $w_{1,k} + \dots + w_{l,k} > C$. Hence the cone defined by the equations:

$$\sum_{i=1}^l (w_{i,k} - w_i) e_i > 0, \quad k = 2, \dots, r, \quad (1)$$

is a non-empty open set of $(\mathbb{R}_{\geq 0})^l$. Let (e_1, \dots, e_l) be l linearly independent vectors of this cone with coefficients in \mathbb{N} : we write $e_i = (e_{i,1}, \dots, e_{i,l})$ for each i . We can choose these vectors such that their determinant is not divisible by p and such that p does not divide $e_{i,l}$. Next we consider ψ defined by:

$$\begin{aligned} \psi(y_i) &= y_1^{e_{i,1}} \cdots y_l^{e_{i,l}} \quad \text{for } 1 \leq i \leq l, \\ \psi(y_i) &= y_i \quad \text{for } i > l. \end{aligned}$$

Hence, because the vectors e_i satisfy (1), $\psi \circ \varphi(x_l)$ is of the form $\psi(M_1)u_l$ for some unit u_l . More precisely we have:

$$\begin{aligned} \psi \circ \varphi(x_i) &= y_1^{p^{\alpha_i} e_{i,1}} \cdots y_l^{p^{\alpha_i} e_{i,l}} u_i \quad \text{for } i < l, \\ \psi \circ \varphi(x_l) &= y_1^{\sum_{i=1}^l w_i e_{i,1}} \cdots y_l^{\sum_{i=1}^l w_i e_{i,l}} u_l. \end{aligned}$$

Because the vectors are linearly independent and because their determinant is not divisible by p , we can reduce to the following case by using modifications of type (4):

$$\begin{aligned} \psi \circ \varphi(x_i) &= y_i^{p^{\alpha_i}} u_i \quad \text{for } i < l, \\ \psi \circ \varphi(x_l) &= y_l^{w_l e_{l,l} \det(e_{i,k})} u_l \end{aligned}$$

for some units u_i . And because $e_{i,l}$ and $\det(e_{i,k})$ are not divisible by p , according to the Cramer's rule and using Step 2, we may assume that:

$$\varphi(x_l) = y_l^{p^{\alpha'}} u'_l$$

where $\alpha' < \alpha$, because w_l is not divisible by p^α . \square

Finally, using Step 2, we may assume that

$$\varphi(x_i) = y_i^{p^{\alpha'_i}} u_i \quad \text{for } i \leq l,$$

for some units u_i , where $(\alpha'_1, \dots, \alpha'_l) <_{\text{lex}} (\alpha_1, \dots, \alpha_l)$ (this can be achieved by permuting the x'_i s and the y'_i s). Then, if we denote $\mu' = (\alpha'_1, \dots, \alpha'_l, +\infty, \dots, +\infty)$, we have $\mu >_{\text{lex}} \mu'$.

Step 5. We assume that we have $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$ for some units u_i with $\text{in}(u_i) = 1$, where $u_i \notin B^p$ whenever $\alpha_i > 0$, for any $i < j$, and $\varphi(x_i) = 0$ for $i > q$. We assume that $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{j-1}$ and we denote $\mu = (\alpha_1, \dots, \alpha_{j-1}, +\infty, \dots, +\infty) \in (\mathbb{N} \cup \{+\infty\})^q$.

According to Step 4, we may assume that none of the monomials of $\text{in}(\varphi(x_j))$ depends only on y_1, \dots, y_{j-1} . After a change of variables in y_j, \dots, y_m we may assume that one of the monomials of $\text{in}(\varphi(x_j))$ depends only on y_1, \dots, y_j . By composing with the homomorphism ψ defined by

$$\begin{aligned}\psi(y_i) &= y_i, & \text{for } i \leq j, \\ \psi(y_i) &= y_i y_j, & \text{for } i > j\end{aligned}$$

we can assume that $\text{in}(\varphi(x_j))$ depends only on y_1, \dots, y_j , but any of its monomials depends on y_j . So we have $\text{in}(\varphi(x_j)) = y_j^k P_{d-k}(y_1, \dots, y_j)$ where P_{d-k} is a homogeneous polynomial of degree k not divisible by y_j .

Thus, we may use Lemma 3.5 and assume that

$$\begin{aligned}\varphi(x_i) &= y_i^{\alpha_i} u_i & \text{for } i < j, \\ \varphi(x_j) &= y_j^{\alpha_j} u_j\end{aligned}$$

for some units u_i and some integer α . Finally, using Step 2, we may assume that $\varphi(x_i) = y_i^{p^{\alpha'_i}} u_i$ for some units u_i with $\text{in}(u_i) = 1$ and $u_i \notin B^p$ if $\alpha'_i \neq 0$. Moreover we see that $(\alpha'_1, \dots, \alpha'_{j-1}) \leq_{\text{lex}} (\alpha_1, \dots, \alpha_{j-1})$. Hence after permutation of the variables we may assume that $\alpha'_1 \leq \dots \leq \alpha'_j$ and $\mu' = (\alpha'_1, \dots, \alpha'_j, +\infty, \dots, +\infty) < \mu$.

Step 6. Eventually, we have $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$, for $i \leq q$ and $\varphi(x_i) = 0$ for $i > q$, where $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_q$ and u_i are units. In this case one checks that

$$\frac{A_v}{\mathfrak{m}_v} = \mathbb{k} \left(\frac{x_q}{x_{q-1}^{p^{\alpha_q - \alpha_{q-1}}}}, \dots, \frac{x_2}{x_1^{p^{\alpha_2 - \alpha_1}}} \right),$$

and we have $\text{grk}(\varphi) = q$. Because the geometric rank is invariant under modifications, we get the result. \square

Remark 3.6. If $\text{char}(\mathbb{k}) = 0$, the proof of the result of Eakin and Harris is similar. Namely, at Step 2 we get $\varphi(x_1) = y_1$ because any unit u with $\text{in}(u) = 1$ is a d -power for any $d \in \mathbb{N}^*$. Then we can skip Steps 3 and 4 because if $g(y_1, \dots, y_{j-1}) := \varphi(x_j)(y_1, \dots, y_{j-1}, 0, \dots, 0)$, then we replace x_j by $x'_j = x_j - g(x_1, \dots, x_{j-1})$ and $\varphi(x'_j)$ has no monomial of the form $c y_1^{k_1} \dots y_{j-1}^{k_{j-1}}$.

4. Linear Chevalley's Lemma

The aim of this section (and originally of the present paper) is to give an answer to a question that S. Izumi asked the author. This question is related to the following result of C. Chevalley on complete local rings:

Theorem 4.1. (See [Ch].) Let A be a complete local ring with maximal ideal \mathfrak{m} . Let (\mathfrak{a}_n) be a decreasing sequence of ideals of A such that $\bigcap_n \mathfrak{a}_n = \{0\}$. Then there exists a function $\beta: \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathfrak{a}_{\beta(n)} \subset \mathfrak{m}^n$ for any positive integer n .

In particular, if we consider an injective homomorphism of local rings $\varphi: A \longrightarrow B$ where A is complete, then there exists a function $\beta: \mathbb{N} \longrightarrow \mathbb{N}$ such that $\varphi^{-1}(\mathfrak{m}_B^{\beta(n)}) \subset \mathfrak{m}_A^n$ for any natural number n . This can be restated by saying that $\beta(v_{\mathfrak{m}_A}(f)) \geq v_{\mathfrak{m}_B}(\varphi(f))$ for any $f \in A$. The least function β satisfying this inequality is called the *Chevalley function* of φ . If β is bounded from above by a linear function we say that φ has a linear Chevalley estimate.

S. Izumi (in [Iz3] and [Iz5]), in the case of equicharacteristic zero local rings, proved that φ has a linear Chevalley estimate if and only if $\text{grk}(\varphi) = \dim(A)$. The question asked by S. Izumi was the following: is it possible to extend this result for local \mathbb{k} -algebras with $\text{char}(\mathbb{k}) > 0$? We can state such an analogue of the main result of [Iz3] in positive characteristic:

Theorem 4.2. *Let $\varphi: A \longrightarrow B$ be a homomorphism of local \mathbb{k} -algebras where \mathbb{k} is a field of positive characteristic. Assume that \widehat{A} is an integral domain and B is regular. Then the following conditions are equivalent:*

- (i) $\text{grk}(\varphi) = \dim(A)$.
- (ii) *There is $a, b \in \mathbb{R}$ such that $av_{\mathfrak{m}_A}(f) + b \geq v_{\mathfrak{m}_B}(\varphi(f))$ for any $f \in A$.*

Definition 4.3. A homomorphism of \mathbb{k} -algebras $\varphi: A \longrightarrow B$ such that $\text{grk}(\varphi) = \dim(A/\text{Ker}(\varphi))$ is called a *regular homomorphism* of \mathbb{k} -algebras.

4.1. Proof of Theorem 4.2

In order to give a proof of this theorem, we first state the following two lemmas:

Lemma 4.4. *Let $\sigma: A \longrightarrow B$ be a finite and injective homomorphism of complete local rings (we do not assume that the rings are local rings of equal characteristic). Then σ satisfies property (ii) of Theorem 4.2.*

Proof. By induction we only need to prove the lemma when B is generated by a single element over A . We denote by z this element which is integral over A . If $z \notin \mathfrak{m}_B$, then $\mathfrak{m}_B = \mathfrak{m}_A B$, thus, for any $n \in \mathbb{N}^*$, $\mathfrak{m}_B^n \cap A = \mathfrak{m}_A^n B \cap A \subset \mathfrak{m}_A^{n-C}$ for some $C \in \mathbb{N}$ not depending on n (by the Artin–Rees Lemma).

Let us assume from now on that $z \in \mathfrak{m}_B$. The \mathbb{k} -algebra $B/\mathfrak{m}_A B$ is a finite \mathbb{k} -module generated by $1, \dots, z^{d-1}$ modulo \mathfrak{m}_A for some $d \in \mathbb{N}^*$. Let us assume that $1, \dots, z^{d-1}$ is a \mathbb{k} -basis of this \mathbb{k} -algebra. Thus, by Theorem 30.6 [Na], we see that $1, z, \dots, z^{d-1}$ generate B as an A -module. It means that there exists an irreducible polynomial $P(Z) := Z^d + a_{d-1}Z^{d-1} + \dots + a_0 \in A[Z]$ such that B is isomorphic to $A[Z]/(P(Z))$. Moreover $a_i \in \mathfrak{m}_A$, for $0 \leq i \leq d-1$, because $1, \dots, z^{d-1}$ is a \mathbb{k} -basis of $B/\mathfrak{m}_A B$. Let $\alpha := \min_{0 \leq i \leq d-1} \{\text{ord}_A(a_i)\}$ (in particular $\alpha > 0$). Then $z^d \in \mathfrak{m}_A^\alpha B$. By induction, $z^{dn} \in \mathfrak{m}_A^{\alpha n} B$ for any $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$:

$$\begin{aligned} \mathfrak{m}_B^{(\alpha+d)n+1} &= (\mathfrak{m}_A B + (z))^{(\alpha+d)n+1} \subset \mathfrak{m}_A^{(\alpha+d)n+1} B + (z)\mathfrak{m}_A^{(\alpha+d)n} B + \dots + (z^{dn+1})\mathfrak{m}_A^{\alpha n} B + (z^{dn}) \\ &\subset \mathfrak{m}_A^{\alpha n} B + (z^{dn}) \subset \mathfrak{m}_A^{\alpha n} B. \end{aligned}$$

Thus $\mathfrak{m}_B^{(\alpha+d)n+1} \cap A \subset \mathfrak{m}_A^{\alpha n} B \cap A \subset \mathfrak{m}_A^{\alpha n-C}$ for some $C \in \mathbb{N}$ not depending on n . This proves the lemma because $\alpha > 0$. \square

Lemma 4.5. (See [Iz1].) *Let $\varphi: A \longrightarrow B$ and $\sigma: A' \longrightarrow A$ be two homomorphisms of local rings where σ is finite and injective and \widehat{A} is an integral domain. Then φ satisfies (ii) if and only if $\varphi \circ \sigma$ satisfies (ii).*

Proof. Because σ is finite and injective, if φ satisfies (ii) then $\varphi \circ \sigma$ satisfies (ii) by Lemma 4.4. In order to prove the “if”-part we follow the proof of Theorem 1.2 (1) \implies (2) of [Iz1] using the fact that there exist two positive constants c, d such that $v_A(fg) \leq c(v_A(f) + v_A(g)) + d \forall f, g \in A$ (cf. [Re]). \square

Now we can begin the proof of Theorem 4.2. We will first reduce to the case where A and B are complete. We remark that for any $f \in A$ we have $v_{m_A}(f) = v_{m_{\widehat{A}}}(f)$ because \widehat{A} is flat over A . So the order is invariant under completion. The Krull dimension and the geometric rank are also invariant under completion. Moreover, the inequality (ii) of Theorem 4.2 is equivalent to a similar estimate for $\widehat{\varphi}: \widehat{A} \longrightarrow \widehat{B}$ following remark 4.4 of [Lz1].

From now on we assume that A and B are complete and $\varphi = \widehat{\varphi}$. Then we show that (i) and (ii) are always true if $\dim(A) = 0$: $\text{grk}(\varphi) = \dim(A)$ is trivially true because $\text{grk}(\varphi) \leq \dim(A) = 0$. So (i) is true. In particular, using Lemma 2.3, φ is injective. On the other hand A is Artinian and so the following descending chain of ideals stabilizes: $A \supset \varphi^{-1}(m_B) \supset \cdots \supset \varphi^{-1}(m_B^n) \supset \cdots$. So there exists b such that $v_{m_B}(\varphi(f)) \leq b$ for any $f \in A \setminus \text{Ker} \varphi = A \setminus \{0\}$, and (ii) is true.

From now on we assume that A and B are complete, B is regular and $\dim(A) \geq 1$. In particular $B = \mathbb{k}'[[y_1, \dots, y_m]]$ where $\mathbb{k} \longrightarrow \mathbb{k}'$ is finite (it follows from the definition of a homomorphism of local \mathbb{k} -algebras).

Step 1. Assume that $\mathbb{k} = \mathbb{k}'$ and \mathbb{k} is an infinite field.

(I) Implication (ii) \implies (i). We may reduce to the case A is regular by using Lemma 2.4, i.e. $A = \mathbb{k}[[x_1, \dots, x_n]]$. Moreover we have $B = \mathbb{k}[[y_1, \dots, y_m]]$. We need to prove that $\text{grk}(\varphi) = n$.

Using Theorem 3.4, we get a commutative diagram as follows:

$$\begin{array}{ccc} \mathbb{k}[[x_1, \dots, x_n]] & \xrightarrow{\varphi} & \mathbb{k}[[y_1, \dots, y_m]] \\ \downarrow \sigma_1 & & \downarrow \sigma_2 \\ \mathbb{k}[[x_1, \dots, x_n]] & \xrightarrow{\bar{\varphi}} & \mathbb{k}[[y_1, \dots, y_m]] \end{array}$$

where σ_1 and σ_2 are compositions of homomorphisms defined in Definition 3.1 and $\bar{\varphi}(x_i) = y_i^{p^{\alpha_i}} u_i$ for $1 \leq i \leq \text{grk}(\varphi)$ and $\bar{\varphi}(x_i) = 0$ if $i > \text{grk}(\varphi)$. Let us denote $r := \text{grk}(\varphi)$. In particular we have $n \geq r$. By Lemma 3.3, we see that $\sigma_2 \circ \varphi$ satisfies property (ii). If $f \in \mathbb{k}[[x_1, \dots, x_n]]$ then $\sigma_2 \circ \varphi(f) = \bar{\varphi} \circ \sigma_1(f)$, thus the homogeneous component of minimal degree in the Taylor expansion of $\sigma_2 \circ \varphi(f)$ depends only on y_1, \dots, y_r . Thus, if we denote by $\pi: \mathbb{k}[[y_1, \dots, y_m]] \longrightarrow \mathbb{k}[[y_1, \dots, y_r]]$ the canonical projection, we see that the order of $\sigma_2 \circ \varphi(f)$ is the same as the order of $\pi \circ \sigma_2 \circ \varphi(f)$ for any $f \in \mathbb{k}[[x_1, \dots, x_n]]$. Thus $\pi \circ \sigma_2 \circ \varphi$ satisfies property (ii). Moreover $\text{grk}(\pi \circ \sigma_2 \circ \varphi) = \text{grk}(\pi \circ \bar{\varphi} \circ \sigma_1) = r = \text{grk}(\varphi)$. Thus we may assume that $A = \mathbb{k}[[x_1, \dots, x_n]]$, $B = \mathbb{k}[[y_1, \dots, y_m]]$ and $\text{grk}(\varphi) = m \leq n$.

By assumption there exist a and b such that $\varphi^{-1}(m_B^{ak+b}) \subset m_A^k$ for any $k \in \mathbb{N}$. So, for any $k \in \mathbb{N}$, we may define surjective \mathbb{k} -linear maps

$$\varphi(A)/(m_B^{ak+b} \cap \varphi(A)) \longrightarrow A/m_A^k$$

by choosing a lifting in A/m_A^k of any element of $\varphi(A)/(m_B^{ak+b} \cap \varphi(A))$.

Because $\varphi(A)/(m_B^{ak+b} \cap \varphi(A))$ is a \mathbb{k} -subspace of B/m_B^{ak+b} , we have the following equalities and inequalities for any $k \in \mathbb{N}$:

$$\begin{aligned} (ak + b + m - 1)! / ((ak + b - 1)!m!) &= \dim_{\mathbb{k}} B/m_B^{ak+b} \geq \dim_{\mathbb{k}} \varphi(A)/(m_B^{ak+b} \cap \varphi(A)) \\ &\geq \dim_{\mathbb{k}} A/m_A^k = (n + k - 1)! / ((k - 1)!n!). \end{aligned}$$

Hence, by comparing the degree in k of these two polynomials, we get $m \geq n$. Thus $n = \text{grk}(\varphi)$.

(II) Implication (i) \implies (ii). First of all we may assume that A is regular by using Lemmas 2.4 and 4.5.

Using Theorem 3.4, we may assume that $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$ for $1 \leq i \leq \text{grk}(\varphi)$ and $\varphi(x_i) = 0$ if $i > \text{grk}(\varphi)$. In this case (ii) is satisfied by taking $a = \max_i p^{\alpha_i}$ and $b = 0$.

Step 2. Assume that $\mathbb{k} = \mathbb{k}'$ and \mathbb{k} is a finite field.

As before we may reduce to the case $A = \mathbb{k}[[x_1, \dots, x_n]]$ and $B = \mathbb{k}[[y_1, \dots, y_m]]$. Let t be a variable over \mathbb{k} and let $\mathbb{K} := \mathbb{k}(t)$. Let $A' := \mathbb{K}[[x_1, \dots, x_n]]$ and $B' := \mathbb{K}[[y_1, \dots, y_m]]$. The homomorphism φ extends to a homomorphism $\varphi_{\mathbb{K}}: A' \rightarrow B'$ in an obvious way. According to Corollary 2.7 $\text{grk}(\varphi) = \text{grk}(\varphi_{\mathbb{K}})$.

On the other hand let us denote by $\tilde{\varphi}$ the homomorphism $\mathbb{k}[[x_1, \dots, x_n]][\mathbb{K}] \rightarrow \mathbb{k}[[y_1, \dots, y_m]][\mathbb{K}]$ induced by φ (where $\mathbb{k}[[x_1, \dots, x_n]][\mathbb{K}]$ is the image of $\mathbb{K} \otimes_{\mathbb{k}} \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{K}[[x_1, \dots, x_n]]$ defined by $\lambda \otimes f \mapsto \lambda f$). Let $f \in \mathbb{k}[[x_1, \dots, x_n]][\mathbb{K}]$. By multiplying f by an element of \mathbb{K} we may assume that $f = \sum_{i=0}^r f_i t^i$ with $f_i \in \mathbb{k}[[x_1, \dots, x_n]]$. Then $\tilde{\varphi}(f) \in (y)^k$ (resp. $f \in (x)^k$) if and only if $\varphi(f_i) \in (y)^k$ (resp. $f_i \in (x)^k$) for $1 \leq i \leq r$ and any $k \in \mathbb{N}$. Thus φ satisfies (ii) if and only if $\tilde{\varphi}$ satisfies (ii). It is clear that $\varphi_{\mathbb{K}}$ satisfies (ii) if and only if $\tilde{\varphi}$ satisfies (ii), $\varphi_{\mathbb{K}}$ being the completion of $\tilde{\varphi}$.

Thus we use Step 1 to conclude.

Step 3. Assume that $\mathbb{k} \neq \mathbb{k}'$. Using Lemma 2.4 and Cohen's Theorem (for example Corollary 31.6 of [Na]), we may find an injective finite homomorphism of \mathbb{k} -algebras $\sigma: A' \rightarrow A$ such that $\text{grk}(\varphi) = \text{grk}(\varphi \circ \sigma)$ and such that A' is regular. By Lemma 4.5 we can replace A by A' . So we assume that $A = \mathbb{k}[[x_1, \dots, x_n]]$ and $B = \mathbb{k}'[[y_1, \dots, y_m]]$. We denote by $A_{\mathbb{k}'}$ the \mathbb{k}' -algebra $A \hat{\otimes}_{\mathbb{k}} \mathbb{k}' = \mathbb{k}'[[x_1, \dots, x_n]]$. We denote by $\varphi_{\mathbb{k}'}$ the homomorphism $A_{\mathbb{k}'} \rightarrow B$ induced by φ . Because $\mathbb{k} \rightarrow \mathbb{k}'$ is finite, then $\text{grk}(\varphi) = \text{grk}(\varphi_{\mathbb{k}'})$ by Lemma 2.5. Using Lemma 5.4 [Iz3], we see that φ satisfies (ii) if and only if $\varphi_{\mathbb{k}'}$ satisfies (ii). Then the result follows from Step 2.

Finally, following W.F. Osgood [Os], S.S. Abhyankar [Ab1] and A.M. Gabrielov [Ga1], we give an example of injective homomorphisms of local rings for which the growth of the Chevalley function is greater than any given increasing function α :

Example 4.6. Let $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function and let \mathbb{k} be a field. Let $(n_i)_i$ be a sequence of natural numbers such that $n_{i+1} > \alpha(n_i)$ for any i and such that the element $\xi(Y) := \sum_{i \geq 1} Y^{n_i}$ is transcendental over $\mathbb{k}(Y)$ (such an element exists according to the constructive proof of Lemma 1 in [ML-S]). Let us define the homomorphism $\varphi: A := \mathbb{k}[[x_1, x_2, x_3]] \rightarrow B := \mathbb{k}[[y_1, y_2]]$ by

$$(\varphi(x_1), \varphi(x_2), \varphi(x_3)) = (y_1, y_1 y_2, y_1 \xi(y_2)).$$

Because $1, y_2, \xi(y_2)$ are algebraically independent over \mathbb{k} , φ is injective (cf. Part 1 of [Ab1]): indeed, let $f \in \text{Ker}(\varphi)$. We write $f = \sum_d f_d$, where f_d is a homogeneous polynomial of degree d . Then $\varphi(f) = \sum y_1^d f_d(1, y_2, \xi(y_2)) = 0$. Hence, we have $f_d(1, y_2, \xi(y_2)) = 0$ for all d . This implies that $f_d = 0$ for all d because $1, y_2, \xi(y_2)$ are algebraically independent.

For any positive natural number i we define:

$$f_i := x_1^{n_i-1} x_3 - (x_2^{n_1} x_1^{n_i-n_1} + \dots + x_2^{n_{i-1}} x_1^{n_i-n_{i-1}} + x_2^{n_i}).$$

Then we get:

$$\varphi(f_i) = y_1^{n_i} \xi(y_2) - y_1^{n_i} \sum_{k=1}^i y_2^{n_k} \in \mathfrak{m}_B^{n_i+n_{i+1}} \subset \mathfrak{m}_B^{\alpha(n_i)}.$$

But $f_i \notin \mathfrak{m}_A^{n_i+1}$ thus $\beta(n_i+1) > \alpha(n_i)$ where β is the Chevalley function associated to φ . Because $n_i \rightarrow +\infty$ when $i \rightarrow +\infty$, we get $\limsup \frac{\beta(n)}{\alpha(n)} \geq 1$.

4.2. Chevalley function and diophantine approximation

The aim of this section is to give an interpretation in terms of diophantine approximation of the fact that the Chevalley function of a homomorphism of complete local rings is not bounded by an affine function as soon as φ is not regular.

Let $\varphi: \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ be a homomorphism of local \mathbb{k} -algebras, where \mathbb{k} is an infinite field. Let us assume that $\text{grk}(\varphi) = n - 1$ and that φ is injective. Using Theorem 3.4, there exists a commutative diagram as follows:

$$\begin{array}{ccc} \mathbb{k}[[x]] & \xrightarrow{\varphi} & \mathbb{k}[[y]] \\ \downarrow \sigma_1 & & \downarrow \sigma_2 \\ \mathbb{k}[[x]] & \xrightarrow{\bar{\varphi}} & \mathbb{k}[[y]] \end{array}$$

such that the homomorphism σ_1 is a composition of homomorphisms of $\mathbb{k}[[x]]$ defined in Definition 3.1 and such that the homomorphism $\bar{\varphi}$ satisfies

$$\begin{aligned} \bar{\varphi}(x_i) &= y_i^{p^{\alpha_i}} u_i \quad \text{for some units } u_i \in \mathbb{k}[[y]] \text{ and } \alpha_i \in \mathbb{N}, \text{ for } i \leq n - 1, \text{ if } \text{char}(\mathbb{k}) = p > 0, \quad \text{or} \\ \bar{\varphi}(x_i) &= y_i \quad \text{for } i \leq n - 1, \text{ if } \text{char}(\mathbb{k}) = 0, \quad \text{and} \\ \bar{\varphi}(x_n) &= 0. \end{aligned}$$

Moreover, if $\text{char}(\mathbb{k}) = p > 0$, for any i , $u_i = 1$ whenever $\alpha_i = 0$, and $\text{in}(u_i) = 1$ and $u_i \notin \mathbb{k}[[y]]^p$ whenever $\alpha_i > 0$.

The homomorphism σ_2 is injective and $\text{grk}(\varphi) = \text{grk}(\sigma_2 \circ \varphi)$, thus $\text{grk}(\sigma_2 \circ \varphi) = n - 1$. From now on we will replace φ by $\sigma_2 \circ \varphi$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{k}[[x]] & \xrightarrow{\varphi_0 := \varphi} & \mathbb{k}[[y]] \\ \downarrow \psi_1 & \nearrow \varphi_1 & \\ \mathbb{k}[[x]] & & \\ \downarrow \psi_2 & \nearrow \varphi_l := \bar{\varphi} & \\ \vdots & & \\ \downarrow \psi_l & & \\ \mathbb{k}[[x]] & & \end{array}$$

where $\varphi_l := \bar{\varphi}$, and ψ_j , for $1 \leq j \leq l$, is one of the homomorphisms used in the modifications (1), (3) and (4) of Definition 3.1 (resp. called homomorphisms of types (1), (3) and (4)).

We can remark that if φ_{j+1} is not injective and ψ_{j+1} is a homomorphism of type (1) or (3), then φ_j is neither injective.

It is trivial for homomorphisms of type (1). If ψ_{j+1} is a homomorphism of type (3), let $f \in \text{Ker}(\varphi_{j+1})$ and let us write $d = p^r e$ with $e \wedge p = 1$. Then let us define $g := \prod_{\varepsilon \in \mathbb{U}_e} (f(\varepsilon x_1, x_2, \dots, x_d))^{p^r}$ where \mathbb{U}_e is the multiplicative group of the e -roots of unity in a finite extension of \mathbb{k} . Then $g \in \text{Im}(\psi_{j+1})$. Let $g' \in \mathbb{k}[[x]]$ such that $\psi_{j+1}(g') = g$. Then $\varphi_j(g') = \varphi_{j+1}(g) = 0$. Thus φ_j is not injective.

Nevertheless, if ψ_{j+1} is a homomorphism of type (4), φ_j may be injective while φ_{j+1} is not injective. Let us assume that φ_j , for $1 \leq j < k$, is injective and φ_k is not injective. In particular ψ_k is a homomorphism of type (4). Because φ_k is not injective, we have $\dim(\mathbb{k}[[x]]/\text{Ker}(\varphi_k)) = n - 1 = \text{grk}(\varphi_k)$ thus there exist a, b such that $\varphi_k^{-1}((y)^{an+b}) \subset \text{Ker}(\varphi_k) + (x)^n$ for any $n \in \mathbb{N}$ according to Theorem 4.2.

Let us remark that, for any j , there exist $a_j \geq 1$ and $b_j \geq 0$ such that $\psi_j^{-1}((x)^{a_j n + b_j}) \subset (x)^n$ for any $n \in \mathbb{N}$. Then there exist $a' \geq 1$ and $b' \geq 0$ such that $(\psi_{k-1} \circ \dots \circ \psi_1)^{-1}((x)^{a' n + b'}) \subset (x)^n$ for any $n \in \mathbb{N}$. If β_0 denotes the Chevalley function of φ and β_{k-1} denotes the Chevalley function of φ_{k-1} , then $\beta_0(n) \leq \beta_{k-1}(a'n + b')$ for any $n \in \mathbb{N}$. Thus β_0 is not bounded by an affine function because β_{k-1} is not bounded by an affine function. We will investigate the reason why β_{k-1} is not bounded by an affine function.

We have to consider the following situation (here φ represents φ_{k-1} , $\tilde{\varphi}$ represents φ_k and q represents ψ_k): we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{k}[[x]] & \xrightarrow{\varphi} & \mathbb{k}[[y]] \\ \downarrow q & \nearrow \tilde{\varphi} & \\ \mathbb{k}[[x]] & & \end{array}$$

where φ is injective and $r_1(\varphi) = n - 1$; q is the homomorphism defined by $q(x_i) = x_i$ for $i \neq 1$ and $q(x_1) = x_1 x_2$. Moreover $\tilde{\varphi}$ is not injective: $\text{grk}(\tilde{\varphi}) = n - 1 = \dim(\mathbb{k}[[x]]/\text{Ker}(\tilde{\varphi}))$. From Theorem 4.2, there exist $a \geq 1$, $b \geq 0$ such that $\tilde{\varphi}^{-1}((y)^{an+b}) \subset \text{Ker}(\tilde{\varphi}) + (x)^n$ for any $n \in \mathbb{N}$. Let $\tilde{z} \in \mathbb{k}[[x]]$ be a generator of $\text{Ker}(\tilde{\varphi})$. Let us denote, for any $g \in \mathbb{k}[[x]]$,

$$v_{\tilde{z}}(g) := \max\{k \in \mathbb{N} \mid g \in (\tilde{z}) + (x)^k\}$$

with the assumption $v_{\tilde{z}}(g) = +\infty$ if $g \in (\tilde{z})$. In particular $v_{\tilde{z}}(g) \leq \text{ord}(\tilde{\varphi}(g)) \leq av_{\tilde{z}}(g) + b$ for any $g \in \mathbb{k}[[x]]$. Then β is the Chevalley function of φ means exactly the following:

$$\begin{aligned} \forall f \in \mathbb{k}[[x]] \quad \text{ord}(\varphi(f)) &\leq \beta(\text{ord}(f)) \quad \text{and} \\ \forall n \in \mathbb{N} \exists f_n \in \mathbb{k}[[x]] \mid \text{ord}(f_n) &= n, \text{ord}(\varphi(f_n)) = \beta(\text{ord}(f_n)). \end{aligned}$$

This is equivalent to the fact that there is a function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\begin{aligned} \forall f \in \mathbb{k}[[x]] \quad v_{\tilde{z}}(q(f)) &\leq \gamma(\text{ord}(f)), \\ \forall n \in \mathbb{N} \exists f_n \in \mathbb{k}[[x]] \mid \text{ord}(f_n) &= n, v_{\tilde{z}}(f_n) = \gamma(\text{ord}(f_n)) \quad \text{and} \\ \forall n \in \mathbb{N} \quad \gamma(n) &\leq \beta(n) \leq a\gamma(n) + b. \end{aligned} \tag{2}$$

Let us consider the following three rings along with the canonical injections

$$A := \mathbb{k}[[x_1, \dots, x_n]] \xrightarrow{i_1} B := \frac{\mathbb{k}[[x_1, \dots, x_n]][t]}{(x_1 - tx_2)} \xrightarrow{i_2} C := \frac{\mathbb{k}[[x_1, \dots, x_n, t]]}{(x_1 - tx_2)}.$$

The homomorphism $\tau : C \rightarrow \mathbb{k}[[x_1, \dots, x_n]]$ defined by $\tau(x_1) = x_1 x_2$, $\tau(x_i) = x_i$ for $i > 1$ and $\tau(t) = x_1$ is an isomorphism and $\tau \circ i_2 \circ i_1 = q$. We will often omit the notations i_1 and i_2 in rest of the paper.

Let us remark the following fact:

The element $\tau^{-1}(\tilde{z}) \in C$ is not algebraic over A .

Indeed, if \tilde{z} was algebraic over A , then we would have a relation $a_0 + a_1 \tilde{z} + \dots + a_d \tilde{z}^d = 0$, such that $a_i \in A$ for $0 \leq i \leq d$ and $a_d \neq 0$. Because C is an integral domain, we may assume that $a_0 \neq 0$ by assuming that d is minimal. Thus we get $\varphi(a_0) = 0$, because $\tilde{\varphi}(\tilde{z}) = 0$, thus φ would not be injective which would contradicting the hypothesis. Hence \tilde{z} is not algebraic over A .

Let us denote by v_1 the valuation on A defined by its maximal ideal and let us denote by v_2 the valuation on C defined by its maximal ideal. We still denote by v_2 its restriction on A or B . Let us remark that $v_1(f) \leq v_2(f) \leq 2v_1(f)$ for any $f \in A$ and that $v_2(f) = v_1(q(f))$ for any $f \in A$.

Let us denote by \mathbb{K}_A (resp. \mathbb{K}_C) the field of fractions of A (resp. C). Let us remark that \mathbb{K}_A is also the field of fractions of B . Let us denote, for any $f \in \mathbb{K}_A$, $|f|_1 := e^{-v_1(f)}$, and for any $g \in \mathbb{K}_C$ let us denote $|g|_2 := e^{-v_2(g)}$. Then $|\cdot|_1$ and $|\cdot|_2$ are non-Archimedean norms on \mathbb{K}_A and \mathbb{K}_C respectively. Let us denote by $\widehat{\mathbb{K}}$ the completion of \mathbb{K}_A with respect to $|\cdot|_2$. We can remark that there is a natural injection $\mathbb{K}_C \hookrightarrow \widehat{\mathbb{K}}$.

Let us come back to \tilde{z} , the generator of $\text{Ker}(\tilde{\varphi})$. We have the following lemma:

Lemma 4.7. *The element $\tau^{-1}(\tilde{z})$ satisfies the following property: there exists a decreasing function $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that*

$$\left| \frac{f}{g} - \tau^{-1}(\tilde{z}) \right|_2 \geq \alpha(|g|_2) \quad \forall f \in A, g \in B.$$

Moreover, the Chevalley function of φ is not bounded by an affine function because of the following fact: If α is the greatest function satisfying the above inequality, then $\frac{\ln(\alpha(u))}{\ln(u)} \rightarrow 0$ as u goes to 0.

Proof. The fact (2) means that for any $f \in A$ and for any $g \in C$, we have $v_2(f - g\tau^{-1}(\tilde{z})) \leq \gamma(v_1(f))$ and this inequality is the best possible. This is equivalent to $v_2(f - g\tau^{-1}(\tilde{z})) \leq \gamma(v_1(f))$ for any $f \in A$ and any $g \in B$, where the inequality is the best possible, because C is the completion of B for v_2 . Thus for any $f \in A$ and for any $g \in B$, we have

$$v_2(f - g\tau^{-1}(\tilde{z})) \leq \gamma'(v_2(f)), \quad (3)$$

with $\gamma(\frac{n}{2}) \leq \gamma'(n) \leq \gamma(n)$ for any $n \in \mathbb{N}$, and this inequality is the best possible. We do not make any restriction if we assume that $v_2(f) = v_2(g\tau^{-1}(\tilde{z}))$: if it is not the case we have $v_2(f - g\tau^{-1}(\tilde{z})) \leq v_2(f)$, but clearly the least function γ' satisfying the inequality (3) for any f and g satisfies $\gamma'(n) \geq n$ for any $n \in \mathbb{N}$. Thus we get $|\frac{f}{g} - \tau^{-1}(\tilde{z})|_2 \geq \alpha(|g|_2)$ for any $f \in A$, $g \in B$ with $\alpha(u) := e^{-\gamma'(\ln(u))}$ for any $u > 0$. We get $\frac{\ln(\alpha(u))}{\ln(u)} \rightarrow 0$ as u goes to 0, because γ' is not bounded by an affine function, this following from the fact that β , thus γ , is neither bounded by an affine function. \square

Remark 4.8. Let us remark the following fact: if $z \in C$ is algebraic over A , then there does not exist any function $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $|\frac{f}{g} - z|_2 \geq \alpha(|g|_2) \quad \forall f \in A, g \in B$. Indeed since z is algebraic over A there exists a relation $a_d z^d + \dots + a_1 z + a_0$ such that $a_i \in A$ for $0 \leq i \leq d$ and $a_d \neq 0$. Because C is an integral domain, we may assume that $a_0 \neq 0$. Thus $z \cdot w = a_0$ with $w := -(a_d z^{d-1} + \dots + a_1) \in C$. For any $n \in \mathbb{N}$, let us denote by w_n an element of B such that $w_n - w \in \mathfrak{m}_C^n$ and $v_2(w_n) = v_2(w)$. Such a w_n exists because C is the completion of B . Thus we have $v_2(z - \frac{a_0}{w_n}) = n - v_2(w)$ for any $n \in \mathbb{N}$. Thus $|z - \frac{a_0}{w_n}|_2 \rightarrow 0$ as $n \rightarrow \infty$, but $|w_n|_2 = |w|_2 \neq 0$ for any $n \in \mathbb{N}$.

5. Homomorphisms of Henselian \mathbb{k} -algebras

In this section and the next one, we study a particular example of homomorphisms of local \mathbb{k} -algebras: namely the homomorphisms of W -system. Such homomorphisms generalize homomorphisms of analytic local rings in the sense that the local rings that we consider satisfy the Weierstrass Division Theorem. In particular we have been inspired by the work of S.S. Abhyankar and M. van der Put [Ab-vdP] on analytic \mathbb{k} -algebras.

5.1. Terminology

From now on we assume that \mathbb{k} is a field of any characteristic.

Definition 5.1. (See [D-L].) By a *Weierstrass System of local \mathbb{k} -algebras*, or a *W-system* over \mathbb{k} , we mean a family of \mathbb{k} -algebras $\mathbb{k}[[x_1, \dots, x_n]]$, $n \in \mathbb{N}$, such that:

- (i) For $n = 0$, the \mathbb{k} -algebra is \mathbb{k} .
For $n \geq 1$, $\mathbb{k}[[x_1, \dots, x_n]]_{(x_1, \dots, x_n)} \subset \mathbb{k}[[x_1, \dots, x_n]] \subset \mathbb{k}[[x_1, \dots, x_n]]$ and $\mathbb{k}[[x_1, \dots, x_{n+m}]] \cap \mathbb{k}[[x_1, \dots, x_n]] = \mathbb{k}[[x_1, \dots, x_n]]$ for $m \in \mathbb{N}$. For any permutation of $\{1, \dots, n\}$, denoted by σ , $\mathbb{k}[[x_{\sigma(1)}, \dots, x_{\sigma(n)}]] = \mathbb{k}[[x_1, \dots, x_n]]$.
- (ii) Any element of $\mathbb{k}[[x]]$, $x = (x_1, \dots, x_n)$, which is a unit in $\mathbb{k}[[x]]$, is a unit in $\mathbb{k}[[x]]$.
- (iii) Let $f \in (x)\mathbb{k}[[x]]$ such that $f(0, \dots, 0, x_n) \neq 0$. We denote $d := \text{ord}_n f(0, \dots, 0, x_n)$. Then for any $g \in \mathbb{k}[[x]]$ there exist a unique $q \in \mathbb{k}[[x]]$ and a unique $r \in \mathbb{k}[[x_1, \dots, x_{n-1}]][[x_n]]$ with $\deg_{x_n} r < d$ such that $g = qf + r$.
- (iv) (If $\text{char}(\mathbb{k}) > 0$.) If $h \in (y_1, \dots, y_m)\mathbb{k}[[y_1, \dots, y_m]]$ and $f \in \mathbb{k}[[x_1, \dots, x_n]]$ such that $f \neq 0$ and $f(h) = 0$, then there exists $g \in \mathbb{k}[[x]]$ irreducible in $\mathbb{k}[[x]]$ such that $g(h) = 0$ and such that there does not exist any unit $u(x) \in \mathbb{k}[[x]]$ with $u(x)g(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^{p\alpha}$ ($a_\alpha \in \mathbb{k}$).

Remark 5.2. Let $\mathbb{k}[[x]]$ be a W-system over \mathbb{k} .

- (i) From [D-L] the ring $\mathbb{k}[[x_1, \dots, x_n]]$ ($n \in \mathbb{N}$) is a Noetherian regular local ring with maximal ideal (x_1, \dots, x_n) and its completion at its maximal ideal is $\mathbb{k}[[x]]$.
- (ii) For any $f \in \mathbb{k}[[x_1, \dots, x_{n+m}]]$ and any $g_1, \dots, g_m \in (x)\mathbb{k}[[x_1, \dots, x_n]]$,

$$f(x_1, \dots, x_n, g_1(x), \dots, g_m(x)) \in \mathbb{k}[[x_1, \dots, x_n]]$$

[D-L].

- (iii) For any $f \in \mathbb{k}[[x]]$, if there is $g \in \mathbb{k}[[x]]$ such that $f = x_1 g$, then $g \in \mathbb{k}[[x]]$ [D-L].
- (iv) From Theorem 44.4 [Na], (iii) implies that $\mathbb{k}[[x]]$ is a Henselian local ring. In fact it is proven in [D-L] that $\mathbb{k}[[x]]$ has the Artin Approximation Property, and by [Po] and [Ro] (where it is proven that a local Noetherian ring has the Artin Approximation Property if and only if it is Henselian and excellent), we see that $\mathbb{k}[[x]]$ is excellent. In [D-L, Remark 10] it is said that if a family of excellent rings satisfies (i)–(iii), then it satisfies (iv).
- (v) Let $d > 1$, $d \wedge \text{char}(\mathbb{k}) = 1$, let $a \in \mathbb{k}^*$ be a d -th power in \mathbb{k} and let $f(x) \in (x)\mathbb{k}[[x]]$. It means that $P(T) = T^d - (a + f(x)) \in \mathbb{k}[[x]][T]$ has a non-zero solution modulo (x) . Thus, because $\mathbb{k}[[x]]$ is Henselian, $P(T)$ has a solution in $\mathbb{k}[[x]]$. Hence $a + f(x)$ has a d -th root in $\mathbb{k}[[x]]$.
If $d = \text{char}(\mathbb{k}) > 0$, $a \in \mathbb{k}^*$ is a d -th power in \mathbb{k} and $f(x) \in (x)\mathbb{k}[[x]]$ is a d -power in $\mathbb{k}[[x]]$, then $P(T) = T^d - (a + f(x))$ has d -root in $\mathbb{k}[[x]]$, thus it has a d -root in $\mathbb{k}[[x]]$ by the Artin Approximation Theorem [D-L].

In fact we can give a quick proof of the fact that W-systems satisfy the Artin Approximation Property if we assume that the rings of the family are excellent, using the Popescu's Smoothing Theorem (cf. [Po] or [Sp2]):

Theorem 5.3. (See [D-L].) Let $\mathbb{k}[[x]]$ be a W-system over \mathbb{k} and let us assume that $\mathbb{k}[[x_1, \dots, x_n]]$ is excellent for any $n \in \mathbb{N}$. Then for any $f = (f_1, \dots, f_p) \in \mathbb{k}[[x, y]]$ with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, for any $c \in \mathbb{N}$ and for any $\bar{y} \in (x)\mathbb{k}[[x]]^m$ such that $f(\bar{y}) = 0$ there exist $y_c \in (x)\mathbb{k}[[x]]^m$ such that $f(y_c) = 0$ and $\bar{y}_i - y_{c,i} \in (x)^c$.

Proof. We may assume that $p = 1$ by replacing (f_1, \dots, f_p) by

$$f := f_1^2 + x_1(f_2^2 + x_1(f_3^2 + x_1(\dots + x_1 f_p^2)^2 \dots)^2).$$

By assumption there exist $\bar{h}_i(x, y) \in \mathbb{k}[[x, y]]$, $1 \leq i \leq n$, such that

$$f(y) + \sum_{i=1}^n (y_i - \bar{y}_i(x)) \bar{h}_i(x, y) = 0.$$

Because the ring $\mathbb{k}[[y]](x)$ is Henselian and excellent, it satisfies the Artin Approximation Property for algebraic equations [Po] ($\mathbb{k}[[y]](x)$ is the Henselization of $\mathbb{k}[[y]][x]_{(y,x)}$). Thus there exist $f_i(x, y), h_i(x, y) \in \mathbb{k}[[y]](x)^n$, $1 \leq i \leq n$, such that

$$\begin{aligned} \bar{h}_i(x, y) - h_i(x, y), f_i(x, y) - \bar{y}_i(x) &\in (x, y)^c, \quad 1 \leq i \leq n, \quad \text{and} \\ f(y) + \sum_{i=1}^n (y_i - f_i(x, y)) h_i(x, y) &= 0. \end{aligned}$$

We may assume that $c \geq 2$. In this case the Jacobian matrix of $(y_i - f_i(x, y), 1 \leq i \leq n)$ with respect to y_1, \dots, y_n has determinant equal to 1 modulo (x, y) . The Henselian property asserts that there exist $y_{i,c}(x) \in \mathbb{k}[[x]]$ such that

$$y_{i,c}(x) - f_i(x, y_{1,c}(x), \dots, y_{n,c}(x)) = 0 \quad \text{for } 1 \leq i \leq n.$$

Then

$$\begin{aligned} f(y_{1,c}(x), \dots, y_{n,c}(x)) &= 0 \quad \text{and} \\ y_{i,c}(x) - \bar{y}_i &\in (x)^c, \quad 1 \leq i \leq n. \quad \square \end{aligned}$$

Remark 5.4. In the same way we may prove that the rings $\mathbb{k}[[x]]$ satisfy the Strong Artin Approximation Property (cf. Theorem 7.1 [D-L]) using the fact that a ring that satisfies the Artin Approximation Property satisfies also the Strong Artin Approximation Property [P-P].

Example 5.5.

- (i) The family $\mathbb{k}[[x_1, \dots, x_n]]$ is a W-system over \mathbb{k} .
- (ii) Let $\mathbb{k}\langle x_1, \dots, x_n \rangle$ be the Henselization of the localization of $\mathbb{k}[[x_1, \dots, x_n]]$ at the maximal ideal (x_1, \dots, x_n) . Then, for $n \geq 0$, the family $\mathbb{k}\langle x_1, \dots, x_n \rangle$ is a W-system over \mathbb{k} .
- (iii) The family $\mathbb{k}\{x_1, \dots, x_n\}$ (the ring of convergent power series in n variables over a valued field \mathbb{k}) is a W-system over \mathbb{k} .
- (iv) The family of Gevrey power series in n variables over a valued field is a W-system over \mathbb{k} [Br].

Definition 5.6. Let $\mathbb{k}[[x]]$ be a W-system over a field \mathbb{k} . A local ring A is a *local \mathbb{k} -algebra with respect to this W-system* if A is isomorphic to $\mathbb{k}[[x_1, \dots, x_n]][\mathbb{k}']/I$ for some $n > 0$, where \mathbb{k}' is a finite field extension of \mathbb{k} and I is an ideal of $\mathbb{k}[[x_1, \dots, x_n]][\mathbb{k}']$ ($\mathbb{k}[[x]][\mathbb{k}']$ is the image of the \mathbb{k} -homomorphism $\mathbb{k}[[x]][t_1, \dots, t_s] \rightarrow \mathbb{k}'[[x]]$ where x_i is sent on x_i and t_j is sent on ε_j , where $\varepsilon_1, \dots, \varepsilon_s$ is a \mathbb{k} -basis of \mathbb{k}').

A homomorphism of local \mathbb{k} -algebras $A \rightarrow B$ is called a *homomorphism of Henselian \mathbb{k} -algebras* if A and B are local \mathbb{k} -algebras with respect to the same W-system over \mathbb{k} and the homomorphism is a homomorphism of local \mathbb{k} -algebras.

Remark 5.7. If A is a local \mathbb{k} -algebra with respect to a W-system, then its residue field is a finite extension of \mathbb{k} . If $A \rightarrow B$ is a homomorphism of Henselian \mathbb{k} -algebras, then the residue field of B is a finite extension of the residue field of A .

Remark 5.8. Since an integral extension of a local Henselian ring is a Henselian ring [Na, 43.16], any local \mathbb{k} -algebra with respect to a W -system is a Henselian ring. Let $\mathbb{k}[[x]]$ be a W -system. Thus, from [La], $\mathbb{k}[[x]][\mathbb{k}']$ satisfies property (iii) of the definition of a W -system if $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite field extension. Moreover it is straightforward to show that $\mathbb{k}[[x]][\mathbb{k}']$ satisfies (i) and (ii) in the definition of a W -system. Finally, from Remark 5.2(iv), and since any finite extension of an excellent ring is an excellent ring, we see that $\mathbb{k}[[x]][\mathbb{k}']$ satisfies (iv) of the definition of a W -system. Hence $\mathbb{k}[[x]][\mathbb{k}']$ is a W -system with respect to \mathbb{k}' if $\mathbb{k}[[x]]$ is a W -system over \mathbb{k} and $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite field extension.

Definition 5.9. A homomorphism $\varphi: A \rightarrow B$ of Henselian \mathbb{k} -algebras is *strongly injective* if the map $\hat{A}/A \rightarrow \hat{B}/B$ induced by φ is injective (or equivalently if $\hat{\varphi}^{-1}(B) = A$).

Finally we give the following version of the Weierstrass Preparation Theorem:

Proposition 5.10 (Weierstrass Preparation Theorem). *Let A and B be local \mathbb{k} -algebras with respect to a W -system and $\varphi: A \rightarrow B$ be a homomorphism of Henselian \mathbb{k} -algebras. Then φ is finite if and only if φ is quasi-finite (i.e. $B/\mathfrak{m}_A B$ is finite over A/\mathfrak{m}_A).*

Proof. It is well known that (ii) of Definition 5.1 is equivalent to the proposition when there exist surjective homomorphisms $\mathbb{k}[[x_1, \dots, x_n]] \rightarrow A$ and $\mathbb{k}[[y_1, \dots, y_m]] \rightarrow B$ for some W -system $\mathbb{k}[[x]]$ ([To1] or [Ab2] for example). Because $\mathbb{k}[[x]][\mathbb{k}']$ is a W -system over \mathbb{k}' as soon as $\mathbb{k}[[x]]$ is a W -system over \mathbb{k} and $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite extension of fields, the proposition is proven. \square

Corollary 5.11. *Let A be a regular local \mathbb{k} -algebra with respect to a W -system $\mathbb{k}[[x]]$ and let (a_1, \dots, a_n) be a regular system of parameters of A . Let \mathbb{k}' be the coefficient field of A . Let $\varphi: \mathbb{k}[[x_1, \dots, x_n]][\mathbb{k}'] \rightarrow A$ be the unique homomorphism of local \mathbb{k}' -algebras such that $\varphi(x_i) = a_i$ for $1 \leq i \leq n$. Then φ is an isomorphism.*

Proof. Follows from Proposition 5.10. \square

5.2. Strongly injective homomorphisms

We state now the following results about homomorphisms of Henselian \mathbb{k} -algebras:

Lemma 5.12. (See [Ab-vdP, Lemma 2.1.2].) *Let φ be a homomorphism of Henselian \mathbb{k} -algebras. If φ is injective and finite, then $\hat{\varphi}$ is injective and finite and φ is strongly injective.*

Proof. Local \mathbb{k} -algebras are Zariski rings (cf. Theorem 9, Chapter VIII of [Z-S]). Then, using Theorems 5 and 11, Chapter VIII of [Z-S], we see that $\hat{\varphi}$ is finite and injective. Then using Theorem 15, Chapter VIII of [Z-S], we see that φ is strongly injective. \square

Let \mathbb{k} be a field of any characteristic and $\mathbb{k}[[x]]$ be a W -system with respect to \mathbb{k} . We define the local \mathbb{k} -homomorphism $q: \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{k}[[x_1, \dots, x_n]]$ by $q(x_1) = x_1 x_2$ and $q(x_i) = x_i$ for $i > 1$. For any $d \in \mathbb{N}^*$ we define the local \mathbb{k} -homomorphism $\psi_d: \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{k}[[x_1, \dots, x_n]]$ by $\psi_d(x_1) = x_1^d$ and $\psi_d(x_i) = x_i$ for $i \neq 1$.

Lemma 5.13. *Let A be a \mathbb{k} -algebra with respect to a W -system denoted by $\mathbb{k}[[x_1, \dots, x_n]]$. Any composition of \mathbb{k} -automorphisms of A and of homomorphisms of the form q and ψ_d is injective. Moreover \mathbb{k} -automorphisms and homomorphisms of the form ψ_d are strongly injective.*

Proof. It is clear that \mathbb{k} -automorphisms, homomorphisms ψ_d and q are injective. Moreover it is clear that \mathbb{k} -automorphisms are strongly injective.

Let $f(x_1, \dots, x_n) \in \mathbb{k}[[x_1, \dots, x_n]]$ such that $\psi_d(f) = f(x_1^d, x_2, \dots, x_n) \in \mathbb{k}[[x]]$. We have

$$f(x_1^d, x_2, \dots, x_n) = (x_1^d - y)q(x, y) + r(x, y)$$

with $r(x, y) \in \mathbb{k}[[x_2, \dots, x_n, y]][x_1]$ and $\deg_{x_1}(r) < d$ according to (iii) of Definition 5.1. On the other hand,

$$f(x_1, \dots, x_n) = (x_1 - y)\bar{q}(x, y) + \bar{r}$$

with $\bar{r} \in \mathbb{k}[[x_2, \dots, x_n, y]]$ according to the formal Weierstrass Division Theorem. Thus

$$f(x_1^d, x_2, \dots, x_n) = (x_1^d - y)\bar{q}(x_1^d, x_2, \dots, x_n, y) + \bar{r}$$

and because the division is unique in $\mathbb{k}[[x, y]]$, we see that $q(x, y) = \bar{q}(x_1^d, x_2, \dots, x_n, y)$ and $r = \bar{r}$ does not depend on x_1 . Since $r(x_2, \dots, x_n, x_1) = f(x_1, \dots, x_n)$, $f \in \mathbb{k}[[x]]$. \square

Definition 5.14. Let $\mathbb{k}[[x]]$ be a W -system over \mathbb{k} . We say that it is a *good W -system* if the homomorphism $q: \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{k}[[x_1, \dots, x_n]]$ defined by $q(x_1) = x_1 x_2$ and $q(x_i) = x_i$ for $i > 1$ is strongly injective. A homomorphism of local \mathbb{k} -algebras $A \rightarrow B$ is called a *homomorphism of good Henselian \mathbb{k} -algebras* if A and B are local \mathbb{k} -algebras with respect to some good W -system and the homomorphism is local. \square

Remark 5.15. It would be interesting to know if any W -system is a good W -system.

Lemma 5.16. *The W -systems $\mathbb{k}[[x]]$ presented in Example 5.5 are good W -systems over \mathbb{k} . The same is true for the respective W -systems over \mathbb{k}' , $\mathbb{k}[[x]][\mathbb{k}']$, where $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite field extension.*

Proof. It is clear for $\mathbb{k}[[x]]$. For the convergent power series, we have just to remark that for any $f = \sum_{\alpha} a_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{k}[[x]]$ such that $q(f)$ is convergent, there exist $R_1 > 0, \dots, R_n > 0$ such that $\sum_{\alpha} |a_{\alpha}| R_1^{\alpha_1} R_2^{\alpha_2} R_3^{\alpha_3} \cdots R_n^{\alpha_n} < +\infty$. Thus f is convergent because $\sum_{\alpha} |a_{\alpha}| (R_1 R_2)^{\alpha_1} R_2^{\alpha_2} R_3^{\alpha_3} \cdots R_n^{\alpha_n} < +\infty$. The proof is the same when we take a finite extension of the residue field (see [Ab-vdP, Lemma 2.2.1]). The proof is similar for Gevrey power series.

Let $f \in \mathbb{k}'[[x_1, \dots, x_n]]$. We have $f = \sum f_l \varepsilon_l$ where $f_l \in \mathbb{k}[[x_1, \dots, x_n]]$ for $1 \leq l \leq r$. Assume that $q(f) \in \mathbb{k}\langle x_1, \dots, x_n \rangle[\mathbb{k}']$. It is clear that $q(f_l) \in \mathbb{k}\langle x_1, \dots, x_n \rangle$ for $1 \leq l \leq r$. It is enough to prove that if $f \in \mathbb{k}[[x_1, \dots, x_n]]$ satisfies $q(f) \in \mathbb{k}\langle x_1, \dots, x_n \rangle$ then $f \in \mathbb{k}\langle x_1, \dots, x_n \rangle$. So let $f \in \mathbb{k}[[x_1, \dots, x_n]]$ such that

$$g := f(x_1 x_2, x_2, \dots, x_n) \in \mathbb{k}\langle x_1, \dots, x_n \rangle.$$

There exist $s \in \mathbb{N}$ and $a_i \in \mathbb{k}[x_1, \dots, x_n]_{(X)}$ for $0 \leq i \leq s$ such that

$$a_s g^s + \cdots + a_1 g + a_0 = 0. \quad (4)$$

We write $a_i = \sum_{\alpha} a_{i,\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for any i with $a_{i,\alpha} \in \mathbb{k}$. Multiplying relation (4) by some power of x_2 , we may assume that any $\alpha \in \mathbb{N}^n$ such that $a_{i,\alpha} \neq 0$ satisfies $\alpha_2 \geq \alpha_1$. Then there exist $b_i \in \mathbb{k}[x_1, \dots, x_n]_{(X)}$ such that $q(b_i) = a_i$ for $0 \leq i \leq s$. We have $b_s f^s + \cdots + b_1 f + b_0 = 0$, hence $f \in \mathbb{k}\langle x_1, \dots, x_n \rangle$ and q is strongly injective. \square

Lemma 5.17. (See [Ab-vdP, Lemma 2.1.3].) *Let $\varphi: A \rightarrow B$ and $\varphi': B \rightarrow C$ be homomorphisms of Henselian \mathbb{k} -algebras. If $\varphi' \circ \varphi$ is strongly injective then φ is strongly injective.*

Proof. Follows from the definitions. \square

Lemma 5.18. *Let $\mathbb{k}[[x]]$ be a W -system over \mathbb{k} . Let $\varphi: \mathbb{k}[[x_1, \dots, x_n]] \rightarrow \mathbb{k}[[y_1, \dots, y_m]][\mathbb{k}']$ where $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite field extension. Let $\varphi_{\mathbb{k}'}$ denote the induced homomorphism of Henselian \mathbb{k}' -algebras: $\mathbb{k}[[x_1, \dots, x_n]][\mathbb{k}'] \rightarrow \mathbb{k}[[y_1, \dots, y_m]][\mathbb{k}']$. Then $\text{grk}(\varphi) = \text{grk}(\varphi_{\mathbb{k}'})$.*

Proof. By Lemma 2.2, we may replace the Henselian algebras by their completions. Then the result comes from the fact that $\mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}'[[x_1, \dots, x_n]]$ is finite and Lemma 2.4. \square

Theorem 3.4 is still valid for homomorphisms of Henselian \mathbb{k} -algebras:

Theorem 5.19. *Let \mathbb{k} be an infinite field of any characteristic and let $\mathbb{k}[[x]]$ be a W-system over \mathbb{k} . Let us consider a homomorphism $\varphi: A \longrightarrow B$, where $A = \mathbb{k}[[x_1, \dots, x_n]]$ and $B = \mathbb{k}[[y_1, \dots, y_m]]$. Then there exists an admissible finite sequence of homomorphisms $(\varphi_i)_{i=0}^k: \mathbb{k}[[x_1, \dots, x_n]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ such that $\varphi_0 = \varphi$. The last homomorphism φ_k satisfies*

$$\begin{aligned}\varphi_k(x_i) &= y_i^{p^{\alpha_i}} u_i \quad \text{for some units } u_i \text{ if } \text{char}(\mathbb{k}) = p > 0, \quad \text{or} \\ \varphi_k(x_i) &= y_i \quad \text{if } \text{char}(\mathbb{k}) = 0, \text{ for } i \leq \text{grk}(\varphi), \quad \text{and} \\ \varphi_k(x_i) &= 0 \quad \text{for } i > \text{grk}(\varphi).\end{aligned}$$

Moreover, if $\text{char}(\mathbb{k}) = p > 0$, for any i , $u_i = 1$ whenever $\alpha_i = 0$, and $\text{in}(u_i) = 1$ and $u_i \notin B^p$ whenever $\alpha_i > 0$. \square

Proof. Modifications of types (2) and (4) are allowed according to (i) of the definition of W-systems and Remark 5.2(iii). Steps 0, 1, 4 and 5 involve only \mathbb{k} -automorphisms of $\mathbb{k}[[x_1, \dots, x_n]]$ and of $\mathbb{k}[[y_1, \dots, y_m]]$ that are defined by polynomials. For Steps 2, 4 and 5, using modifications of type (3), we take d -roots of elements of $\mathbb{k}[[x]]$ in $\mathbb{k}[[x]]$ and they are in $\mathbb{k}[[x]]$ from Remark 5.2(v). The only problem may occur at Step 3, where we replace x_j by an element of the form $x'_j := x_j - \sum_k c_k x_1^{k_1} \cdots x_{j-1}^{k_{j-1}}$ such that $\varphi(x'_j) = 0$ (and the sum is not finite) because we do not know if $x'_j \in \mathbb{k}[[x]]$. When $\text{char}(\mathbb{k}) = 0$ this is obvious because $\varphi(x_i) = y_i$ for $1 \leq i \leq j-1$ by assumption (see Remark 3.6).

From now on we assume that $\text{char}(\mathbb{k}) = p > 0$. We assume that $A = \mathbb{k}[[x_1, \dots, x_n]]$ and $B = \mathbb{k}[[y_1, \dots, y_m]]$ and we will prove that $x'_j \in A$. We will use the following lemma:

Lemma 5.20. *Assume that $\text{char}(\mathbb{k}) = p > 0$. Let us consider $\varphi: \mathbb{k}[[x_1, \dots, x_{j-1}]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ such that $\varphi(x_i) = y_i^{p^{\alpha_i}} u_i$ for some units u_i , for $1 \leq i \leq j-1$. Then φ is strongly injective.*

In particular, because $\varphi(x_j) = \varphi(\sum_k c_k x_1^{k_1} \cdots x_{j-1}^{k_{j-1}}) \in \mathbb{k}[[y_1, \dots, y_m]]$, we see that

$$\sum_k c_k x_1^{k_1} \cdots x_{j-1}^{k_{j-1}} \in \mathbb{k}[[y_1, \dots, y_m]],$$

and so $x'_j \in \mathbb{k}[[y_1, \dots, y_m]]$. \square

Now we give the proof of Lemma 5.20:

Proof. Let us denote by π the quotient homomorphism $\mathbb{k}[[y_1, \dots, y_m]] \longrightarrow \mathbb{k}[[y_1, \dots, y_{j-1}]]$. Then the homomorphism induced by $\pi \circ \varphi$:

$$\mathbb{k}[[x_1, \dots, x_{j-1}]]/(x_1, \dots, x_{j-1}) \longrightarrow \mathbb{k}[[y_1, \dots, y_{j-1}]]/(\pi \circ \varphi((x_1, \dots, x_{j-1})))$$

is finite. Using Proposition 5.10, we see that $\pi \circ \varphi$ is finite. Moreover, $\pi \circ \varphi$ is injective because $\text{grk}(\pi \circ \varphi) = j-1$. Using Lemma 5.12, we see that $\pi \circ \varphi$ is strongly injective, and from Lemma 5.17 we see that φ is strongly injective. \square

In particular we get the following result, which is a weak version of a theorem of A.M. Gabrielov [Ga2]:

Theorem 5.21. *Let \mathbb{k} be a field of any characteristic. Let $\varphi : A \longrightarrow B$ be a homomorphism of good Henselian \mathbb{k} -algebras, where A and B are regular. If $\text{grk}(\varphi) = \dim(A)$, then φ is strongly injective.*

Proof. Using Corollary 5.11, we may assume that $A = \mathbb{k}[\![x_1, \dots, x_n]\!]$ and $B = \mathbb{k}[\![y_1, \dots, y_m]\!][\mathbb{k}']$ where $\mathbb{k} \longrightarrow \mathbb{k}'$ is finite and $\mathbb{k}[\![x]\!]$ is a W-system over \mathbb{k} . If we replace A by $\mathbb{k}[\![x]\!][\mathbb{k}']$ then the geometric rank will not change by Lemma 5.18. Moreover if the induced homomorphism $\varphi_{\mathbb{k}'} : \mathbb{k}[\![x_1, \dots, x_n]\!][\mathbb{k}'] \longrightarrow \mathbb{k}[\![y_1, \dots, y_m]\!][\mathbb{k}']$ is strongly injective, then φ is strongly injective. So from now on we assume that $\mathbb{k} = \mathbb{k}'$.

From Corollary 2.7 we may assume that \mathbb{k} is an infinite field. Using Theorem 5.19, we see that $\sigma_1 \circ \varphi = \bar{\varphi} \circ \sigma_2$ where the homomorphisms σ_1 and σ_2 are compositions of \mathbb{k} -automorphisms of A and B , of homomorphisms of the form q and ψ_d , and $\bar{\varphi}$ is defined by $\bar{\varphi}(x_i) = y_i^{p^{\alpha_i}} u_i$, for some units u_i and some $\alpha_i \in \mathbb{N}$, for all i . Then using Lemmas 5.13, 5.17 and 5.20, we see that φ is strongly injective. \square

6. Two particular cases

6.1. The two-dimensional case

Example 4.6 shows that we can construct injective homomorphisms $\varphi : A \longrightarrow B$ with $\text{grk} \varphi < \dim(A)$ as soon as $\dim(A) \geq 3$. We prove here that it is not possible to find such examples when A is a Henselian \mathbb{k} -algebra and $\dim(A) \leq 2$.

In fact, it is obvious that if $\dim(A) = 1$ and φ is injective then $\text{grk}(\varphi) = 1$. Indeed, using Lemma 2.4, we can replace A by $\mathbb{k}[\![x]\!]$, where x is a single variable and B by $\mathbb{k}[\![y_1, \dots, y_m]\!][\mathbb{k}']$. Then the result is immediate.

When $\dim(A) = 2$ we have the following result that shows us that $\dim(A) = 2$ is a nice case as remarked by S.S. Abhyankar and M. van der Put in [Ab1] and [Ab-vdP]:

Theorem 6.1. *Let $\varphi : A \longrightarrow B$ be a homomorphism of Henselian \mathbb{k} -algebras where \hat{A} is an integral domain of dimension 2 and B is regular. Then φ is injective if and only if $\text{grk}(\varphi) = 2$.*

Proof. From Lemma 2.3, we see that $\text{grk}(\varphi) = 2$ implies that φ is injective. So from now on we assume that φ is injective.

By Theorem 2.1 [D-L] there exists an injective and finite homomorphism of Henselian \mathbb{k} -algebras $\pi : \mathbb{k}[\![x_1, x_2]\!] \longrightarrow A$ (where $\mathbb{k}[\![x]\!]$ is a W-system over \mathbb{k}), so using Lemma 2.4, we can replace A by $\mathbb{k}[\![x_1, x_2]\!]$. Because B is regular we assume that $B = \mathbb{k}[\![y_1, \dots, y_m]\!][\mathbb{k}']$ (Corollary 5.11) where \mathbb{k}' is a finite field extension of \mathbb{k} . Then, we can replace $\mathbb{k}[\![x_1, x_2]\!]$ by $\mathbb{k}[\![x_1, x_2]\!][\mathbb{k}']$ using Lemma 5.18.

Let t be a variable over \mathbb{k} and let $\mathbb{K} := \mathbb{k}(t)$. Let $\varphi_{\mathbb{K}} : \mathbb{K}[\![x_1, x_2]\!] \longrightarrow \mathbb{K}[\![y_1, \dots, y_m]\!]$ be the homomorphism induced by φ . If φ is injective then $\varphi_{\mathbb{K}}$ is also injective: otherwise there would exist a sequence $(f_n)_n \in \mathbb{k}[\![x_1, x_2]\!][\mathbb{K}]^{\mathbb{N}}$ such that $\varphi_{\mathbb{K}}(f_n) \in (y)^n$ and $f_n - f_{n+1} \in (x)^n$ for any $n \in \mathbb{N}$. Let $d := \text{ord}(f_n)$ for n large enough and let us denote by β the Chevalley function of φ . Let $N \in \mathbb{N}$ such that $\text{ord}(\varphi_{\mathbb{K}}(f_N)) > \beta(d)$. We may assume that $g := f_N \in \mathbb{k}[\![x_1, x_2]\!][\mathbb{k}(t)]$ by multiplying it by an element of \mathbb{K} . We write $g = \sum_{j=1}^r g_j t^j$ with $g_j \in \mathbb{k}[\![x_1, x_2]\!]$ for $0 \leq j \leq r$. Then $\varphi_{\mathbb{K}}(g) = \sum_{j=1}^r \varphi(g_j) t^j \in (y)^{\beta(d)+1}$ by assumption thus $\text{ord}(\varphi(g_j)) \geq \beta(d) + 1$ for $0 \leq j \leq r$, hence $\text{ord}(g_j) \geq d + 1$ by definition of β . This contradicts $\text{ord}(g) = d$. Hence $\varphi_{\mathbb{K}}$ is injective and we may assume that \mathbb{k} is infinite from Corollary 2.7.

To compute $\text{grk}(\varphi)$ we use the algorithmic proof of Theorem 3.4.

We first give the proof when $\text{char}(\mathbb{k}) = p > 0$. Using Step 1, we may assume that $\varphi(x_1) = y_1^d u$ for some unit u . Then we define

$$\Delta := \{ \alpha \in \mathbb{N} \mid \exists z \in \mathbb{k}[\![x_1, x_2]\!][\mathbb{k}'] \text{ with } \text{ord}(z) = 1, \text{ and } \varphi(z)(y_1, 0, \dots, 0) = y_1^{p^\alpha d} u \\ \text{such that } p \wedge d = 1 \text{ and } u \text{ is a unit} \}.$$

By assumption Δ is not empty. Let us denote by α the least integer of Δ . Let us choose an element $z_1 \in \mathbb{k}[[x_1, x_2]][\mathbb{k}']$ such that $\varphi(z_1) = y_1^{p^\alpha d} u$ with $p \wedge d = 1$ and u a unit. By using the following modification of type (2): $\psi(y_1) = y_1$ and $\psi(y_k) = y_k y_1$ for $k > 1$, we can replace φ by $\bar{\varphi} := \psi^k \circ \varphi$, for $k \geq 0$, such that $\bar{\varphi}(z_1) = y_1^{p^\alpha d} u$ for some unit u , with $p \wedge d = 1$. Let us choose $z_2 \in \mathbb{k}[[x_1, x_2]][\mathbb{k}']$ such that (z_1, z_2) is a regular system of parameters. Now, because α is the least integer of Δ , $\text{in}(\bar{\varphi}(z))$ has no monomial of the form cy_1^k such that p^α does not divide k . Then we can skip Step 4, and using Step 5, we can replace $\bar{\varphi}$ by $\bar{\bar{\varphi}}$ such that $\bar{\bar{\varphi}}(z_1) = y_1^{p^\alpha de_{1,1}} y_2^{p^\alpha de_{1,2}} u_1$, $\bar{\bar{\varphi}}(z_2) = y_1^{i_1 e_{1,1} + i_2 e_{2,1}} y_2^{i_1 e_{1,2} + i_2 e_{2,2}} u_2$ for some units u_1 and u_2 , and with $\bar{\varphi} = \psi' \circ \bar{\bar{\varphi}}$ where ψ' is a composition of blowing-ups and automorphisms of $\mathbb{k}[[y_1, \dots, y_m]][\mathbb{k}']$. Moreover the matrix $(e_{i,j})_{i,j}$ is invertible. Then using modifications of type (4) we transform $\bar{\bar{\varphi}}$ in $\bar{\bar{\bar{\varphi}}}$ such that $\bar{\bar{\bar{\varphi}}}(x) = y_1^{p^{\alpha_1}} u_1$ and $\bar{\bar{\bar{\varphi}}}(z') = y_2^{p^{\alpha_2}} u_2$ for some units u_1 and u_2 . Hence $\text{grk} \varphi = 2$.

Now, if $\text{char}(\mathbb{k}) = 0$ then we can do almost the same, but we do not need Δ . We just choose $z_1 = x_1$ and $z_2 = x_2$. After that the proof is the same as above. \square

Corollary 6.2. *Let $\varphi : \mathbb{k}[[x_1, x_2]] \longrightarrow \mathbb{k}[[y_1, \dots, y_m]]$ be an injective homomorphism of Henselian \mathbb{k} -algebras where \mathbb{k} is infinite. Let $\mathbb{k} \longrightarrow \mathbb{k}'$ be a finite field extension and let us assume that there is a W -system $\mathbb{k}'[[x]]$ over \mathbb{k}' such that $\mathbb{k}'[[x]] \cap \mathbb{k}[[x]] = \mathbb{k}[[x]]$. Then the induced homomorphism $\varphi_{\mathbb{k}'} : \mathbb{k}'[[x_1, x_2]] \longrightarrow \mathbb{k}'[[y_1, \dots, y_m]]$ is injective.*

Proof. By Theorem 6.1 $\text{grk}(\varphi) = 2$. Thus, if $\text{char}(\mathbb{k}) = p > 0$, φ can be transformed using modifications into a homomorphism $\bar{\varphi}$ such that $\bar{\varphi}(x_1) = y_1^{p^{\alpha_1}} u_1$ and $\bar{\varphi}(x_2) = y_2^{p^{\alpha_2}} u_2$ for some units. Then $\varphi_{\mathbb{k}'}$ can be transformed in the same way and $\text{grk}(\varphi_{\mathbb{k}'}) = 2$. Then $\varphi_{\mathbb{k}'}$ is injective by Lemma 2.3. The proof in characteristic zero is the same. \square

Using this result we deduce the following two results, the first being a generalization to the case of Henselian \mathbb{k} -algebras of a theorem of S.S. Abhyankar and M. van der Put (cf. Theorem 2.10 of [AbvdP]):

Theorem 6.3. *Let $\varphi : A \longrightarrow B$ be a homomorphism of good Henselian \mathbb{k} -algebras where A and B are regular and $\dim(A) = 2$. If φ is injective then it is strongly injective.*

Proof. We have $\text{grk}(\varphi) = 2$ from Theorem 6.1. Hence from Theorem 5.21 φ is strongly injective. \square

Corollary 6.4. *Let $\varphi : A \longrightarrow B$ denote a homomorphism of complete local \mathbb{k} -algebras where A is a two-dimensional integral domain and B is regular. Then φ is injective if and only if φ has a linear Chevalley estimate.*

Proof. It is obvious that φ is injective if it has a linear Chevalley estimate.

On the other hand the result follows from Theorems 6.1 and 4.2. \square

6.2. The algebraic case

Here we give a generalization of the main theorem of [To2,Be,Mi]. The result is the following: any homomorphism of analytic \mathbb{k} -algebras defined by algebraic power series has maximal geometric rank. This result has been proven for homomorphisms of analytic \mathbb{C} -algebras defined by polynomials in the three papers cited above.

Definition 6.5. Let $\varphi : A \longrightarrow B$ be a homomorphism of local \mathbb{k} -algebras. We define $r_2 := \dim(\frac{\hat{A}}{\text{Ker}(\hat{\varphi})})$ and $r_3 := \dim(\frac{A}{\text{Ker}(\varphi)})$. Moreover $r_1 := \text{grk}(\varphi)$.

It is clear that $r_2(\varphi) \leq r_3(\varphi)$. Moreover, from the definition, we see that $r_1(\hat{\varphi})$ is equal to the geometric rank of the homomorphism $\hat{A}/\text{Ker}(\hat{\varphi}) \longrightarrow \hat{B}$ induced by $\hat{\varphi}$, and using the Abhyankar's

Inequality [Ab1] and Lemma 2.2 we see that $r_1(\varphi) \leq r_2(\varphi)$. Thus we always have $r_1(\varphi) \leq r_2(\varphi) \leq r_3(\varphi)$. If $r_1(\varphi) = r_2(\varphi)$ we say that φ is *regular*. A difficult theorem of A. Gabrielov asserts that if $\varphi: A \rightarrow B$ is a regular homomorphism when A and B are quotients of convergent power series rings over \mathbb{C} then $r_2(\varphi) = r_3(\varphi)$, i.e. $\text{Ker}(\widehat{\varphi}) = \text{Ker}(\varphi)\widehat{A}$ (cf. [Ga2]).

Definition 6.6. A homomorphism $A \rightarrow B$ of Henselian \mathbb{k} -algebras is said to be a *homomorphism of algebraic \mathbb{k} -algebras* if A and B are local \mathbb{k} -algebras with respect to the W -system of algebraic power series (Example 5.5(ii)).

Theorem 6.7. Let $\varphi: A \rightarrow B$ a homomorphism of algebraic \mathbb{k} -algebras where B is regular if $\text{char}(\mathbb{k}) = p > 0$. Then $r_1(\varphi) = r_3(\varphi)$.

Proof. If $\text{char}(\mathbb{k}) = 0$ and B is not regular, by the existence of a resolution of singularities for $\text{Spec}(B)$, there exists a homomorphism of Henselian \mathbb{k} -algebras which is a composition of local blow-ups $\psi: B \rightarrow \mathbb{k}\langle y_1, \dots, y_m \rangle$. In particular $r_1(\psi \circ \varphi) = r_1(\varphi)$ by Proposition 2.8. Thus we may assume that B is regular.

Let us denote $A' := A/\text{Ker}(\varphi)$. Then $d := \dim(A') = r_3(\varphi)$. There exists a finite injective homomorphism $\mathbb{k}\langle x_1, \dots, x_d \rangle \rightarrow A'$ from the Weierstrass Preparation Theorem. Let us denote by τ the homomorphism induced by φ on $\mathbb{k}\langle x \rangle$. By Lemma 2.4, $r_1(\tau) = r_1(\varphi)$, and because τ is injective, $r_3(\tau) = d = r_3(\varphi)$.

Let t be a variable over \mathbb{k} . We may replace \mathbb{k} by $\mathbb{k}(t)$ since Corollary 2.7 and since the homomorphism induced by τ on $\mathbb{k}(t)\langle x \rangle$ is clearly injective. Now we apply Theorem 3.4 to τ . We get the following commutative diagram:

$$\begin{array}{ccc} \mathbb{k}\langle x \rangle & \xrightarrow{\tau} & \mathbb{k}\langle y \rangle \\ \downarrow \sigma_1 & & \downarrow \sigma_2 \\ \mathbb{k}\langle x \rangle & \xrightarrow{\bar{\tau}} & \mathbb{k}\langle y \rangle \end{array}$$

where $\bar{\tau}$ is as defined in (iii) of Theorem 3.4. In particular we see that $r_3(\bar{\tau}) = r_1(\bar{\tau})$ because $\text{Ker}(\bar{\tau}) = (x_{r_1(\tau)+1}, \dots, x_d)$. We have $r_1(\tau) = r_1(\sigma_2 \circ \tau)$ and $r_3(\tau) = r_3(\sigma_2 \circ \tau)$. Moreover $r_1(\tau) = r_1(\bar{\tau})$ according to Proposition 2.8. Thus we only have to prove that $r_3(\tau) = r_3(\bar{\tau})$.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{k}\langle x \rangle & \xrightarrow{\tau} & \mathbb{k}\langle y \rangle \\ \downarrow \sigma & \nearrow \psi & \\ \mathbb{k}\langle x \rangle & & \end{array}$$

where τ is injective and σ is one of the homomorphisms defined in (ii) of Theorem 3.4. We will prove that ψ is still injective. Thus this will prove by induction that $\bar{\tau}$ is injective and that $r_3(\tau) = r_3(\bar{\tau})$.

In order to prove that ψ is injective, we have to check the three following cases: If σ is an isomorphism, then it is clear that ψ is injective.

If $\sigma = \chi_d$ ($d \in \mathbb{N}^*$) is defined by $\chi_d(x_1) = x_1^d$, and $\chi_d(x_i) = x_i \ \forall i \neq 1$, we can write $d = p^f e$ with $e \wedge p = 1$. If $f \in \text{Ker}(\psi)$, then let us define $g := \prod_{\varepsilon \in \mathbb{U}_e} (f(\varepsilon x_1, x_2, \dots, x_d))^{p^f}$, where \mathbb{U}_e is the set of e -roots of unity in a finite field extension of \mathbb{k} . Then $g \in \mathbb{k}\langle x \rangle$ and $g \in \text{Im}(\sigma)$. Let $g' \in \mathbb{k}\langle x \rangle$ such that $\sigma(g') = g$. Then $\tau(g') = \psi(g) = 0$. Thus $g' = 0$ because τ is injective, hence $f = 0$ and ψ is injective.

Finally, let us assume that $\sigma = q$ defined by $q(x_i) = x_i$ for $i \neq 2$ and $q(x_2) = x_1 x_2$. Let $f \in \text{Ker}(\psi)$. Let $P(Y) \in \mathbb{k}[x][Y]$ be an irreducible polynomial having f as a root. Let us denote by $a_i \in \mathbb{k}[x]$, $0 \leq$

$i \leq r$, its coefficients (i.e. $a_r f^r + \cdots + a_1 f + a_0 = 0$). Then $\psi(a_0) = 0$ and $a_0 \neq 0$ if $f \neq 0$. If $d := \deg_{x_2}(a_0)$, then $g := x_1^d a_0 \in \text{Im}(\sigma)$ and $\psi(g) = 0$. Let $g' \in \mathbb{k}[x]$ such that $\sigma(g') = g$. Then $\tau(g') = \psi(g) = 0$. Thus $g' = 0$ because τ is injective, hence $a_0 = 0$, then $f = 0$ and ψ is injective. \square

Corollary 6.8. *Let $\varphi : \mathbb{k}\langle x \rangle \rightarrow \mathbb{k}\langle y \rangle / I\mathbb{k}\langle y \rangle$ be a homomorphism of analytic \mathbb{k} -algebras where k is a valued field, I is an ideal of $\mathbb{k}\langle y \rangle$ and such that $\varphi(x_i) \in \mathbb{k}\langle y \rangle / I$ for $1 \leq i \leq m$. Assume moreover that $\text{char}(\mathbb{k}) = 0$ or $I = (0)$. Then $r_1(\varphi) = r_3(\varphi)$.*

Proof. Let $\psi : \mathbb{k}\langle x \rangle \rightarrow \mathbb{k}\langle y \rangle / I$ be the homomorphism of Henselian \mathbb{k} -algebra defined by $\psi(x_i) := \varphi(x_i)$ for $1 \leq i \leq n$. Then we have $r_1(\psi) = r_3(\psi)$ by the preceding theorem. Moreover, by Lemma 2.2, we have $r_1(\psi) = r_1(\widehat{\psi}) = r_1(\widehat{\varphi}) = r_1(\widehat{\varphi})$ because $\widehat{\psi} = \widehat{\varphi}$. Clearly $\text{Ker}(\psi)\mathbb{k}\langle x \rangle \subset \text{Ker}(\varphi)$, thus $r_3(\varphi) \leq r_3(\psi)$. Thus $r_1(\varphi) \leq r_3(\varphi) \leq r_3(\psi) = r_1(\psi) = r_1(\varphi)$ and we get the conclusion. \square

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