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On the fractions of semi-Mackey and Tambara functors

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ABSTRACT

For a finite group G, a semi-Mackey (resp. Tambara) functor is regarded as a G-bivariant analog of a commutative monoid (resp. ring). As such, some naive algebraic constructions are generalized to this G-bivariant setting. In this article, as a G-bivariant analog of the fraction of a ring, we consider *fraction* of a Tambara (and a semi-Mackey) functor, by a multiplicative semi-Mackey subfunctor.

Contents

1. Introduction and preliminaries

For a finite group G, a semi-Mackey functor (resp. a Tambara functor) is regarded as a G-bivariant analog of a commutative monoid (resp. ring), as seen in [8]. As such, some naive algebraic

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constructions are generalized to this G-bivariant setting. For example an analog of ideal theory was considered in [5], and an analog of monoid-ring construction was considered in [4].

In the ordinary ring theory, *fraction* is another well-established construction. If we are given a multiplicatively closed subset S of a ring R, then there are associated a ring $S^{-1}R$ and a natural ring homomorphism $\ell_S : R \to S^{-1}R$ satisfying some universality. Similarly for monoids.

As a G-bivariant analog of this, we consider *fraction* of a Tambara (and a semi-Mackey) functor, by a multiplicative semi-Mackey subfunctor.

In this article, a monoid is always assumed to be unitary and commutative. Similarly a ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. We denote the category of monoids by *Mon*, the category of rings by *Ring*, and the category of abelian groups by *Ab*. A monoid homomorphism preserves units, and a ring homomorphism preserves 0 and 1. A *G-set* is a set equipped with a G-action $G \times X \to X$, and a *G-monoid* is a monoid equipped with a compatible G-action. Equivalently, a G-monoid is a pair (M, μ) of monoid M and group homomorphism $\mu : G \to Aut_{Mon}(M)$, where $Aut_{Mon}(M)$ denotes the group of monoid automorphisms of M.

We always assume that a multiplicatively closed subset $S \subseteq R$ contains 1. Thus a multiplicatively closed subset is nothing other than a submonoid of R^{μ}, where R^{μ} denotes the underlying multiplicative monoid of R. For any submonoid S of a monoid M, its *saturation* $\widetilde{S}$ is defined by

$$\tilde{s} = \{x \in M \mid ax = s \text{ for some } a \in M,\ s \in S\}.$$

Then $\tilde{s} \subseteq M$ is again a submonoid. S is called *saturated* if it satisfies $S = \tilde{s}$.

Remark also that if M is a G-monoid and $S \subseteq M$ is G-invariant, its saturation $\tilde{s}$ is also G-invariant.

Throughout this article, we use the same basic notation as in [5]. We fix a finite group G, whose unit element is denoted by e. Abbreviately we denote the trivial subgroup of G by e, instead of $\{e\}$. $H \leqslant G$ means H is a subgroup of G. ${}_G set$ denotes the category of finite G-sets and G-equivariant maps. The order of H is denoted by $|H|$, and the index of K in H is denoted by $|H : K|$, for any $K \leqslant H \leqslant G$.

For any category $\mathscr{C}$, we denote by $\mathrm{Ob}(\mathscr{C})$ the class of its objects, and for any pair of objects X and Y in $\mathscr{C}$, the set of morphisms from X to Y in $\mathscr{C}$ is denoted by $\mathscr{C}(X, Y)$.

2. Fraction of a semi-Mackey functor

Before constructing a fraction of a Tambara functor, we introduce the fraction of a semi-Mackey functor. First, we briefly recall the definition of a (semi-)Mackey functor. Although the notion of (semi-)Mackey functor seems to be well known, we add this section for the sake of self-containedness and to fix the notation.

Definition 2.1. A *semi-Mackey functor M on G* is a pair $M = (M^*, M_*)$ of a covariant functor

$$M_* : {}_G set \to Set,$$

and a contravariant functor

$$M^* : {}_G set \to Set$$

which satisfies the following. Here *Set* denotes the category of sets.

(1) For each object $X \in \mathrm{Ob}({}_G set)$, we have $M_*(X) = M^*(X)$. We denote this simply by $M(X)$.
(2) For any pair $X, Y \in \mathrm{Ob}({}_G set)$, if we denote the inclusions into $X \amalg Y$ by $\iota_X : X \hookrightarrow X \amalg Y$ and $\iota_Y : Y \hookrightarrow X \amalg Y$, then

$$\big(M^*(\iota_X), M^*(\iota_Y)\big) : M(X \amalg Y) \to M(X) \times M(Y)$$

becomes an isomorphism. $M(\emptyset) = \{*\}$ for the empty set $\emptyset$.

(3) (Mackey condition) If we are given a pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\xi"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\eta"'] & Y \end{tikzcd}$$

in ${}_G set$, then

$$\begin{tikzcd} M(X') \arrow[r, "M_*(\xi)"] & M(X) \\ M(Y') \arrow[u, "M^*(f')"] \arrow[r, "M_*(\eta)"'] \arrow[ur, phantom, "\circlearrowright"] & M(Y) \arrow[u, "M^*(f)"'] \end{tikzcd}$$

is commutative.

If M is a semi-Mackey functor, then $M(X)$ becomes a monoid for each $X \in \mathrm{Ob}({}_G set)$, and M^*, M_* become monoid-valued functors ${}_G set \to Mon$. In fact, a commutative multiplication on X is given by the folding map $\nabla : X \amalg X \to X$ as

$$M(X) \times M(X) \cong M(X \amalg X) \xrightarrow{M_*(\nabla)} M(X),$$

and the inclusion of the empty set $\iota : \emptyset \hookrightarrow X$ gives the unit $M_*(\iota) : M(\emptyset) \to M(X)$. Those $M^*(f), M_*(f)$ for morphisms f in ${}_G set$ are called *structure morphisms* of M. $M^*(f), M_*(f)$ are often abbreviated to f^*, f_*. Also remark that if H is a subgroup of G, then $M(G/H)$ is equipped with a natural $N_G(H)/H$-monoid structure. Here, $N_G(H) \leqslant G$ is the normalizer of H in G. Indeed for each $n \in N_G(H)/H$, the G-map

$$c_n : G/H \to G/H; \quad gH \mapsto gnH \quad (\forall g \in G)$$

induces a monoid automorphism $M^*(c_n)$ on $M(G/H)$.

A *morphism* of semi-Mackey functors $\vartheta : M \to N$ is a family of monoid homomorphisms

$$\vartheta = \{\vartheta_X \colon M(X) \to N(X)\}_{X \in \mathrm{Ob}({}_G set)},$$

natural with respect to the contravariant and the covariant parts. We denote the category of semi-Mackey functors by $SMack(G)$.

If M is a semi-Mackey functor on G, a *semi-Mackey subfunctor* $\mathscr{S} \subseteq M$ is a family of submonoids $\{\mathscr{S}(X) \subseteq M(X)\}_{X \in \mathrm{Ob}({}_G set)}$, satisfying

$$f^*(\mathscr{S}(Y)) \subseteq \mathscr{S}(X), \qquad f_*(\mathscr{S}(X)) \subseteq \mathscr{S}(Y)$$

for any $f \in {}_G set(X, Y)$. Then $\mathscr{S}$ itself becomes a semi-Mackey functor, and this is nothing other than a subobject in $SMack(G)$.

A semi-Mackey functor M on G is called a *Mackey functor* if it satisfies $M(X) \in \mathrm{Ob}(Ab)$ for any $X \in \mathrm{Ob}({}_G set)$. The full subcategory of Mackey functors is denoted by $Mack(G) \subseteq SMack(G)$. For the properties of Mackey functors, see for example [1].

Trivial example is the following.

Example 2.2. Let M be a semi-Mackey functor on G. If we define $M^{\times} \subseteq M$ by

$$M^{\times}(X) = \big(M(X)\big)^{\times} = \big\{\text{invertible elements in } M(X)\big\}$$

for each $X \in \mathrm{Ob}({}_G set)$, then $M^{\times} \subseteq M$ becomes a semi-Mackey subfunctor.

Proposition 2.3. *Let $\mathscr{S} \subseteq M$ be a semi-Mackey subfunctor.*

(1) $\mathscr{S}^{-1}M = \{\mathscr{S}(X)^{-1}M(X)\}_{X \in \mathrm{Ob}({}_G set)}$ *has a structure of a semi-Mackey functor induced from that on M. Here, $\mathscr{S}(X)^{-1}M(X)$ denotes the ordinary fraction of monoids.*

(2) *The natural monoid homomorphisms*

$$\ell_{\mathscr{S},X} : M(X) \to \mathscr{S}^{-1}M(X); \quad x \mapsto \frac{x}{1} \quad \big(\forall X \in \mathrm{Ob}({}_G set)\big)$$

form a morphism of semi-Mackey functors $\ell_{\mathscr{S}} : M \to \mathscr{S}^{-1}M$.

(3) *For any semi-Mackey functor M', the above $\ell_{\mathscr{S}}$ gives a bijection between the morphisms $\mathscr{S}^{-1}M \to M'$ and the morphisms $\vartheta : M \to M'$ satisfying $\vartheta(\mathscr{S}) \subseteq M'^{\times}$:*

$$SMack(G)\big(\mathscr{S}^{-1}M, M'\big) \xrightarrow{\cong} \big\{\vartheta \in SMack(G)\big(M, M'\big) \bigm| \vartheta(\mathscr{S}) \subseteq M'^{\times}\big\}.$$

Proof. By the universality of the fraction of monoids, for any $f \in {}_G set(X, Y)$, there exists a unique monoid homomorphism

$$f^* : \mathscr{S}^{-1}M(Y) \to \mathscr{S}^{-1}M(X)$$

compatible with f^* for M, given by

$$f^*\left(\frac{y}{t}\right) = \frac{f^*(y)}{f^*(t)} \quad \left(\forall \frac{y}{t} \in \mathscr{S}^{-1}M(Y)\right).$$

Similarly f_* for $\mathscr{S}^{-1}M$ is obtained uniquely by

$$f_*\left(\frac{x}{s}\right) = \frac{f_*(x)}{f_*(s)} \quad \left(\forall \frac{x}{s} \in \mathscr{S}^{-1}M(X)\right),$$

compatibly with f_* for M. Obviously $\mathscr{S}^{-1}M$ becomes a semi-Mackey functor, with these structure morphisms.

The rest also immediately follows from the properties of ordinary fraction of monoids. Since we discuss this again for Tambara functors in Proposition 4.6, we omit the details here. We remark that analogs of Corollary 4.8 and Corollary 4.9 also hold, which will be left to the reader. □

In particular, we can take the fraction $M^{-1}M$ of a semi-Mackey functor M by itself. This can be understood in a more functorial way as follows.

Remark 2.4. If $F : Mon \to Ab$ is a functor preserving finite products, then from any semi-Mackey functor M, we obtain a Mackey functor

$$F(M) = \big\{F\big(M(X)\big)\big\}_{X \in \mathrm{Ob}({}_G set)}.$$

This gives a functor, which we also abbreviate to F

$$F : \mathit{SMack}(G) \to \mathit{Mack}(G).$$

(Similarly for functors $\mathit{Mon} \to \mathit{Mon}$, $\mathit{Ab} \to \mathit{Ab}$, and $\mathit{Ab} \to \mathit{Mon}$.)

Example 2.5. The group-completion functor

$$K_0 : \mathit{Mon} \to \mathit{Ab}$$

and the functor taking the group of invertible elements

$$(\ )^{\times} : \mathit{Mon} \to \mathit{Ab}$$

yield functors

$$K_0 : \mathit{SMack}(G) \to \mathit{Mack}(G),$$
$$(\ )^{\times} : \mathit{SMack}(G) \to \mathit{Mack}(G).$$

Moreover the adjoint properties of the original functors are enhanced to this Mackey-functorial level. In fact, it can be easily shown that K_0 is left adjoint to the inclusion functor $\mathit{Mack}(G) \hookrightarrow \mathit{SMack}(G)$, and $(\ )^{\times}$ is right adjoint to the same functor.

Thus for any pair of semi-Mackey functors M and M', we have a natural isomorphism

$$\begin{aligned} \mathit{SMack}(G)\big(K_0(M), M'\big) &\cong \mathit{Mack}(G)\big(K_0(M), M'^{\times}\big) \\ &\cong \mathit{SMack}(G)\big(M, M'^{\times}\big) \\ &= \big\{\vartheta \in \mathit{SMack}(G)\big(M, M'\big) \bigm| \vartheta(M) \subseteq M'^{\times}\big\}, \end{aligned}$$

which re-creates the adjoint isomorphism in Proposition 2.3, in the case of $\mathscr{S} = M$.

Definition 2.6. For any semi-Mackey subfunctor $\mathscr{S} \subseteq M$, we define its *saturation* $\widetilde{\mathscr{S}}$ by

$$\widetilde{\mathscr{S}}(X) = \big(\mathscr{S}(X)\big)^{\sim}.$$

$\widetilde{\mathscr{S}} \subseteq M$ becomes again a semi-Mackey subfunctor. We say $\mathscr{S}$ is *saturated* if it satisfies $\mathscr{S} = \widetilde{\mathscr{S}}$.

Remark 2.7. Let M be a semi-Mackey functor on G.

(1) If a semi-Mackey subfunctor $\mathscr{S} \subseteq M$ satisfies $\mathscr{S} \subseteq M^{\times}$, then $\ell_{\mathscr{S}}$ becomes an isomorphism. In particular if $\mathscr{S}$ belongs to $\mathit{Mack}(G)$, then we have $\mathscr{S} \subseteq M^{\times}$ and thus $\ell_{\mathscr{S}}$ is an isomorphism.
(2) For any semi-Mackey subfunctor $\mathscr{S} \subseteq M$, we have a natural isomorphism $\mathscr{S}^{-1}M \xrightarrow{\cong} \widetilde{\mathscr{S}}^{-1}M$ compatible with $\ell_{\mathscr{S}}$ and $\ell_{\widetilde{\mathscr{S}}}$.

Proof. These can be confirmed on each object $X \in \mathrm{Ob}({}_G\mathit{set})$. See also Remark 4.5. □

3. Semi-Mackey subfunctors generated by $S \subseteq M(G/e)$

In this section, we state the construction of semi-Mackey subfunctors $\mathscr{S} \subseteq M$ from a saturated G-invariant submonoid $S \subseteq M(G/e)$.

The following proposition is also used critically in the next section.

Proposition 3.1. *Let $\mathscr{S} \subseteq M$ be a semi-Mackey subfunctor. Then, for any $f \in {}_G set(X, Y)$,*

$$\mathscr{S}(X) \subseteq \big(f^*\mathscr{S}(Y)\big)^{\sim}$$

is satisfied. Namely, for any $s \in \mathscr{S}(X)$, there exist some $a \in M(X)$ and $\bar{s} \in \mathscr{S}(Y)$ satisfying $f^*(\bar{s}) = as$. Indeed, $\bar{s}$ can be chosen as $\bar{s} = f_*(s)$.

Proof. Let

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a pullback diagram, and let $\Delta : X \to X \times_Y X$ be the diagonal map. If we put

$$Z = \Big(X \underset{Y}{\times} X\Big) - \Delta(X),$$
$$q_1 = p_1|_Z, \qquad q_2 = p_2|_Z,$$

then

$$\begin{tikzcd} X \amalg Z \arrow[r, "\mathrm{id}_X \cup q_2"] \arrow[d, "\mathrm{id}_X \cup q_1"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

also becomes a pullback diagram. Thus by Mackey condition, we obtain

$$f^* f_*(s) = (\mathrm{id}_X \cup q_1)_*(\mathrm{id}_X \cup q_2)^*(s) = s \cdot \big(q_{1*}q_2^*(s)\big)$$

for any $s \in M(X)$.

In particular when $s \in \mathscr{S}(X)$, if we put $a = q_{1*}q_2^*(s)$ and $\bar{s} = f_*(s)$, then it follows $f^*(\bar{s}) = as$ and $\bar{s} \in \mathscr{S}(Y)$. $\square$

Corollary 3.2. *If $\mathscr{S} \subseteq M$ is a saturated semi-Mackey subfunctor, then for any $X \in \mathrm{Ob}({}_G set)$, we have*

$$\mathscr{S}(X) = \big(\mathrm{pt}_X^*\big(\mathscr{S}(G/G)\big)\big)^{\sim},$$

where $\mathrm{pt}_X : X \to G/G$ is the constant map. Thus $\mathscr{S}$ is determined by $\mathscr{S}(G/G)$.

Proof. This immediately follows from $\mathrm{pt}_X^*(\mathscr{S}(G/G)) \subseteq \mathscr{S}(X) \subseteq (\mathrm{pt}_X^*(\mathscr{S}(G/G)))^{\sim}$ and $\mathscr{S}(X) = (\mathscr{S}(X))^{\sim}$. $\square$

Remark 3.3. Let M be a semi-Mackey functor on G. To give a semi-Mackey subfunctor $\mathscr{S} \subseteq M$ is equivalent to give a submonoid $\mathscr{S}(X) \subseteq M(X)$ for each transitive $X \in \mathrm{Ob}({}_G set)$, in such a way that

(i) $f_*(\mathscr{S}(X)) \subseteq \mathscr{S}(Y)$,
(ii) $f^*(\mathscr{S}(Y)) \subseteq \mathscr{S}(X)$

are satisfied for any $f \in {}_G set(X, Y)$ between transitive $X, Y \in \mathrm{Ob}({}_G set)$.

In fact, if we define $\mathscr{S}(X)$ for any (not necessarily transitive) $X \in \mathrm{Ob}({}_G set)$ by

$$\mathscr{S}(X) = \bigl\{(s_1, \ldots, s_n) \in M(X) \bigm| s_i \in \mathscr{S}(X_i)\ (1 \leqslant \forall i \leqslant n)\bigr\}$$

using the orbit decomposition $X = X_1 \amalg \cdots \amalg X_n$, then $\mathscr{S} \subseteq M$ becomes a semi-Mackey subfunctor. (Here, we are identifying $M(X)$ with $M(X_1) \times \cdots \times M(X_n)$ by the isomorphism $(\iota_i^*)_{1 \leqslant i \leqslant n} : M(X) \xrightarrow{\cong} \prod_{1 \leqslant i \leqslant n} M(X_i)$ induced from the inclusions $\iota_i : X_i \hookrightarrow X$.)

Starting from a G-invariant submonoid $S \subseteq M(G/e)$, we can construct semi-Mackey subfunctors of M in the following way.

Proposition 3.4. *Let $S \subseteq M(G/e)$ be a saturated G-invariant submonoid. For each transitive $X \in \mathrm{Ob}({}_G set)$, define $\mathscr{L}_S(X)$ by*

$$\mathscr{L}_S(X) = \gamma_{X*}(S)$$

for some $\gamma_X \in {}_G set(G/e, X)$. Then $\mathscr{L}_S \subseteq M$ becomes a semi-Mackey subfunctor.

Obviously we have $\mathscr{L}_S(G/e) = S$, and $\mathscr{L}_S$ is the minimum one among the semi-Mackey subfunctors $\mathscr{S}$ satisfying $\mathscr{S}(G/e) \supseteq S$.

Proof. First remark that the definition of $\mathscr{L}_S(X)$ does not depend on the choice of γ_X, since X is transitive and S is G-invariant. We show the conditions in Remark 3.3 are satisfied.

Let $f \in {}_G set(X, Y)$ be any morphism between transitive $X, Y \in \mathrm{Ob}({}_G set)$. Obviously we have $f_*(\mathscr{L}_S(X)) = (f \circ \gamma_X)_*(S) = \mathscr{L}_S(Y)$.

For a morphism $\gamma_Y \in {}_G set(G/e, Y)$, the fiber product of f and γ_Y can be written in the form

$$\begin{tikzcd} \coprod_{1 \leqslant i \leqslant n} G/e \arrow[r, "\nabla"] \arrow[d, "\bigcup_{1 \leqslant i \leqslant n} \gamma_i"'] \arrow[dr, phantom, "\square"] & G/e \arrow[d, "\gamma_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with some $\gamma_1, \ldots, \gamma_n \in {}_G set(G/e, X)$. Thus for any $s \in S$ we have

$$f^* \gamma_{Y*}(s) = \prod_{1 \leqslant i \leqslant n} \gamma_{i*}(s) \in \mathscr{L}_S(X).$$

Namely, we have $f^*(\mathscr{L}_S(Y)) \subseteq \mathscr{L}_S(X)$. $\square$

Proposition 3.5. *Let $S \subseteq M(G/e)$ be a saturated G-invariant submonoid. Put $S_0 = (\mathrm{pt}^*_{G/e})^{-1}(S) \subseteq M(G/G)$. For each transitive $X \in \mathrm{Ob}({}_G set)$, define $\mathscr{U}_S(X)$ by*

$$\mathscr{U}_S(X) = \bigl((\mathrm{pt}_X)^*(S_0)\bigr)^{\sim}.$$

Then $\mathscr{U}_S \subseteq M$ becomes a semi-Mackey subfunctor. Obviously we have $\mathscr{U}_S(G/e) \subseteq S$.

Proof. We show the conditions in Remark 3.3 are satisfied.

Let $f \in {}_G set(X, Y)$ be any morphism between transitive $X, Y \in \mathrm{Ob}({}_G set)$. For any $s \in \mathscr{U}_S(Y)$, by definition, there exist $a \in M(Y)$ and $t \in S_0$ such that $as = \mathrm{pt}_Y^*(t)$ holds. Then we have

$$f^*(a) f^*(s) = f^* \mathrm{pt}_Y^*(t) = \mathrm{pt}_X^*(t),$$

which means $f^*(\mathscr{U}_S(Y)) \subseteq \mathscr{U}_S(X)$.

It remains to show (i). We use the following lemma.

Lemma 3.6. *For any transitive $X \in \mathrm{Ob}({}_G set)$ and any $t \in S_0$, we have*

$$(\mathrm{pt}_X)_* \mathrm{pt}_X^*(t) \in S_0.$$

Proof. If we take a pullback diagram

$$\begin{tikzcd} \amalg_n G/e \arrow[r, "\exists \zeta = \bigcup_{1 \leqslant i \leqslant n} \zeta_i"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\mathrm{pt}_X"] \\ G/e \arrow[r, "\mathrm{pt}_{G/e}"'] & G/G \end{tikzcd}$$

then we have

$$\mathrm{pt}_{G/e}^*(\mathrm{pt}_X)_* \mathrm{pt}_X^*(t) = \nabla_* \zeta^* \mathrm{pt}_X^*(t) = \prod_{1 \leqslant i \leqslant n} \zeta_i^* \mathrm{pt}_X^*(t) = \big(\mathrm{pt}_{G/e}^*(t)\big)^n \in S. \qquad \square$$

For any $s \in \mathscr{U}_S(X)$, by definition, there exist $a \in M(X)$ and $t \in S_0$ such that $as = \mathrm{pt}_X^*(t)$. Thus we have $f_*(a) f_*(s) = f_* \mathrm{pt}_X^*(t)$. By Proposition 3.1, there exists $b \in M(Y)$ satisfying

$$b \cdot f_* \mathrm{pt}_X^*(t) = \mathrm{pt}_Y^*(\mathrm{pt}_Y)_* f_* \mathrm{pt}_X^*(t) = \mathrm{pt}_Y^*(\mathrm{pt}_X)_* \mathrm{pt}_X^*(t).$$

Thus we obtain

$$b f_*(a) f_*(s) = \mathrm{pt}_Y^*(\mathrm{pt}_X)_* \mathrm{pt}_X^*(t). \tag{3.1}$$

By Lemma 3.6, we have $(\mathrm{pt}_X)_* \mathrm{pt}_X^*(t) \in S_0$, and thus (3.1) implies

$$f_*(s) \in \big(\mathrm{pt}_Y^*(S_0)\big)^{\sim} = \mathscr{U}_S(Y),$$

and condition (i) follows. $\square$

Proposition 3.7. *Let $S \subseteq M(G/e)$ be a saturated G-invariant submonoid. Then $\mathscr{U}_S$ is the maximum one among semi-Mackey subfunctors $\mathscr{S}$ satisfying $\mathscr{S}(G/e) \subseteq S$.*

Proof. Let $\mathscr{S}$ be any semi-Mackey subfunctor satisfying $\mathscr{S}(G/e) \subseteq S$. We have $\mathscr{S} \subseteq \widetilde{\mathscr{S}}$. Since S is saturated, $\widetilde{\mathscr{S}}$ also satisfies $\widetilde{\mathscr{S}}(G/e) \subseteq S$. Since $\widetilde{\mathscr{S}} \subseteq M$ is a semi-Mackey subfunctor, we have

$$\mathrm{pt}_{G/e}^*\big(\widetilde{\mathscr{S}}(G/G)\big) \subseteq \widetilde{\mathscr{S}}(G/e) \subseteq S.$$

Thus it follows

$$\widetilde{\mathscr{S}}(G/G) \subseteq \left(\mathrm{pt}^*_{G/e}\right)^{-1}(S) \ (= S_0).$$

Since $\widetilde{\mathscr{S}}$ is saturated, for any $X \in \mathrm{Ob}({}_G set)$ we have

$$\widetilde{\mathscr{S}}(X) = \left(\mathrm{pt}^*_X\left(\widetilde{\mathscr{S}}(G/G)\right)\right)^{\sim} \subseteq \left(\mathrm{pt}^*_X(S_0)\right)^{\sim} = \mathscr{U}_S(X)$$

by Corollary 3.2. Thus we obtain $\mathscr{S} \subseteq \widetilde{\mathscr{S}} \subseteq \mathscr{U}_S$. $\square$

Corollary 3.8. *For any saturated G-invariant submonoid $S \subseteq M(G/e)$, we have $\mathscr{L}_S \subseteq \mathscr{U}_S$. In particular we have $\mathscr{U}_S(G/e) = S$.*

Moreover, for any semi-Mackey subfunctor $\mathscr{S} \subseteq M$ satisfying $\mathscr{S}(G/e) = S$, we have $\mathscr{L}_S \subseteq \mathscr{S} \subseteq \mathscr{U}_S$.

Proof. This follows from Proposition 3.4 and Proposition 3.7. $\square$

4. Fraction of a Tambara functor

First we briefly recall the definition of exponential diagrams and Tambara functors.

Remark 4.1. (See [7].) For each $X \in \mathrm{Ob}({}_G set)$, let ${}_G set/X$ denote the slice category of ${}_G set$ over X. (Namely, its objects are G-maps to X.) For any G-map $f \in {}_G set(X,Y)$ and any object $(A \xrightarrow{p} X) \in \mathrm{Ob}({}_G set/X)$, we define $\Pi_f(A \xrightarrow{p} X) = (\Pi_f(A) \xrightarrow{\pi} Y)$ by

$$\Pi_f(A) = \left\{ (y,\sigma) \;\middle|\; \begin{array}{c} y \in Y, \\ \sigma : f^{-1}(y) \to A \text{ a map of sets}, \\ p \circ \sigma = \mathrm{id}_{f^{-1}(y)} \end{array} \right\},$$

$$\pi(y,\sigma) = y.$$

G acts on $\Pi_f(A)$ by $g \cdot (y,\sigma) = (gy, {}^g\sigma)$, where ${}^g\sigma$ is the map defined by

$${}^g\sigma(x) = g\sigma\left(g^{-1}x\right) \quad \left(\forall x \in f^{-1}(gy)\right).$$

If $a \in ({}_G set/X)((A \xrightarrow{p} X), (A' \xrightarrow{p'} X))$ is a morphism (namely, $a \in {}_G set(A,A')$ satisfying $p' \circ a = p$), then we define $(\Pi_f(a)) \in ({}_G set/Y)(\Pi_f(A \xrightarrow{p} X), \Pi_f(A' \xrightarrow{p'} X))$ by

$$\Pi_f(a)(y,\sigma) = (y, a \circ \sigma).$$

Then Π_f gives a functor $\Pi_f : {}_G set/X \to {}_G set/Y$, which is right adjoint to the functor taking pullback along f

$$X \underset{Y}{\times} - : {}_G set/Y \to {}_G set/X.$$

By the adjoint property, for any $p \in {}_G set(A,X)$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] & X \times_Y \Pi_f(A) \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & \circlearrowright & \Pi_f(A) \arrow[ll, "\pi"] \end{tikzcd} \tag{4.1}$$

where ρ is the pullback of f by π, and λ is the morphism corresponding to $\mathrm{id}_{\Pi_f(A)}$ under the adjoint isomorphism and $p \circ \lambda$ becomes the pullback of π by f.

Any commutative diagram in ${}_G set$

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] \arrow[d, phantom, "exp"] & Z \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & & B \arrow[ll, "q"] \end{tikzcd}$$

isomorphic to (4.1) is called an *exponential diagram*. For the properties of exponential diagrams, see [7].

Definition 4.2. A *Tambara functor T on G* is a triplet $T = (T^*, T_+, T_\bullet)$ of two covariant functors

$$T_+ : {}_G set \to Set, \qquad T_\bullet : {}_G set \to Set$$

and one contravariant functor

$$T^* : {}_G set \to Set$$

which satisfies the following.

(1) $T^\alpha = (T^*, T_+)$ is a Mackey functor on G.
(2) $T^\mu = (T^*, T_\bullet)$ is a semi-Mackey functor on G.

Since T^α, T^μ are semi-Mackey functors, we have $T^*(X) = T_+(X) = T_\bullet(X)$ for each $X \in \mathrm{Ob}({}_G set)$. We denote this by $T(X)$.

(3) (Distributive law) If we are given an exponential diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] \arrow[d, phantom, "exp"] & Z \arrow[l, "\lambda"'] \arrow[d, "\rho"] \\ Y & & B \arrow[ll, "q"] \end{tikzcd}$$

in ${}_G set$, then

$$\begin{tikzcd} T(X) \arrow[d, "T_\bullet(f)"'] & T(A) \arrow[l, "T_+(p)"'] \arrow[r, "T^*(\lambda)"] \arrow[d, phantom, "\circlearrowright"] & T(Z) \arrow[d, "T_\bullet(\rho)"] \\ T(Y) & & T(B) \arrow[ll, "T_+(q)"] \end{tikzcd}$$

is commutative.

If $T = (T^*, T_+, T_\bullet)$ is a Tambara functor, then $T(X)$ becomes a ring for each $X \in \mathrm{Ob}({}_G set)$, whose additive (resp. multiplicative) structure is induced from that on $T^\alpha(X)$ (resp. $T^\mu(X)$). Those $T^*(f), T_+(f), T_\bullet(f)$ for morphisms f in ${}_G set$ are called *structure morphisms* of T. For each $f \in {}_G set(X, Y)$,

- $T^*(f) : T(Y) \to T(X)$ is a ring homomorphism, called the *restriction* along f.
- $T_+(f) : T(X) \to T(Y)$ is an additive homomorphism, called the *additive transfer* along f.
- $T_\bullet(f) : T(X) \to T(Y)$ is a multiplicative homomorphism, called the *multiplicative transfer* along f.

$T^*(f), T_+(f), T_\bullet(f)$ are often abbreviated to f^*, f_+, $f_\bullet$.

A *morphism* of Tambara functors $\varphi : T \to S$ is a family of ring homomorphisms

$$\varphi = \{\varphi_X \colon T(X) \to S(X)\}_{X \in \mathrm{Ob}({}_G set)},$$

natural with respect to all of the contravariant and the covariant parts. We denote the category of Tambara functors by *Tam*(G).

Example 4.3.

(1) If we define Ω by

$$\Omega(X) = K_0({}_G set/X)$$

for each $X \in \mathrm{Ob}({}_G set)$, where the right hand side is the Grothendieck ring of the category of finite G-sets over X, then Ω becomes a Tambara functor on G. This is called the *Burnside Tambara functor* ([7] or [5]).

(2) Let R be a G-ring. If we define $\mathcal{P}_R$ by

$$\mathcal{P}_R(X) = \{G\text{-maps from } X \text{ to } R\}$$

for each $X \in \mathrm{Ob}({}_G set)$, then $\mathcal{P}_R$ becomes a Tambara functor on G. This is called the *fixed point functor* associated to R ([7] or [5]). For each $f \in {}_G set(X, Y)$, the multiplicative transfer $f_\bullet : \mathcal{P}_R(X) \to \mathcal{P}_R(Y)$ is given by

$$\big(f_\bullet(\alpha)\big)(y) = \prod_{x \in f^{-1}(y)} \alpha(x) \quad \big(\forall \alpha \in \mathcal{P}(X)\big).$$

In this section, we construct a fraction of a Tambara functor by a semi-Mackey subfunctor $\mathscr{S} \subseteq T^\mu$. As in Example 2.2, we have a trivial semi-Mackey subfunctor $(T^\mu)^\times$, which we also denote simply by $T^\times$.

Proposition 4.4. *Let T be a Tambara functor on G and let $\mathscr{S} \subseteq T^\mu$ be a semi-Mackey subfunctor. Then $\mathscr{S}^{-1}T = \{\mathscr{S}(X)^{-1}T(X)\}_{X \in \mathrm{Ob}({}_G set)}$ has a structure of a Tambara functor induced from that on T.*

Moreover, the natural ring homomorphisms

$$\ell_{\mathscr{S},X} : T(X) \to \mathscr{S}^{-1}T(X); \quad x \mapsto \frac{x}{1} \quad \big(\forall X \in \mathrm{Ob}({}_G set)\big)$$

form a morphism of Tambara functors $\ell_{\mathscr{S}} : T \to \mathscr{S}^{-1}T$.

Proof. As shown in Proposition 2.3, $\mathscr{S}^{-1}T^\mu$ has a structure of a semi-Mackey functor, with structure morphisms defined by

$$f^*\left(\frac{y}{t}\right) = \frac{f^*(y)}{f^*(t)} \quad \left(\forall \frac{y}{t} \in \mathscr{S}^{-1}T(Y)\right),$$
$$f_\bullet\left(\frac{x}{s}\right) = \frac{f_\bullet(x)}{f_\bullet(s)} \quad \left(\forall \frac{x}{s} \in \mathscr{S}^{-1}T(X)\right),$$

for each $f \in {}_G set(X, Y)$.

Thus it suffices to give additive transfers for $\mathscr{S}^{-1}T$, compatibly with the structure on $\mathscr{S}^{-1}T^{\mu}$. Let $f \in {}_G set(X,Y)$ be any morphism.

Let $\frac{x}{s} \in \mathscr{S}^{-1}T(X)$ be any element. If we put $\bar{s} = f_{\bullet}(s)$, then by Proposition 3.1, we have $f^*(\bar{s}) = as$ for some $a \in T(X)$. We define the additive transfer of $\mathscr{S}^{-1}T$ along f by

$$f_+ : \mathscr{S}^{-1}T(X) \to \mathscr{S}^{-1}T(Y); \quad \frac{x}{s} \mapsto \frac{f_+(ax)}{\bar{s}}. \tag{4.2}$$

To show the well-definedness, suppose we have $\frac{x}{s} = \frac{x'}{s'}$ in $\mathscr{S}^{-1}T(X)$. Namely, there exists $t \in \mathscr{S}(X)$ such that $ts'x = tsx'$. Let $a, a', b \in T(X)$ and $\bar{s}, \bar{s}', \bar{t} \in \mathscr{S}(Y)$ be elements satisfying

$$f^*(\bar{s}) = as, \qquad f^*(\bar{s}') = a's' \tag{4.3}$$

and

$$f^*(\bar{t}) = bt.$$

Then, by the projection formula, we have

$$\begin{aligned}\bar{t}\bar{s}' f_+(ax) &= f_+\big(axf^*(\bar{t}\bar{s}')\big) = f_+\big(axbta's'\big) = f_+\big(aa'bts'x\big),\\ \bar{t}\bar{s} f_+\big(a'x'\big) &= f_+\big(a'x'f^*(\bar{t}\bar{s})\big) = f_+\big(a'x'btas\big) = f_+\big(aa'bts'x\big).\end{aligned}$$

This means we have $\frac{f_+(ax)}{\bar{s}} = \frac{f_+(a'x')}{\bar{s}'}$ in $\mathscr{S}^{-1}T(Y)$, and f_+ is well defined. Also, this argument shows that we can use arbitrary $a \in T(X)$ and $\bar{s} \in \mathscr{S}(Y)$ instead of $f_{\bullet}(s)$ to define $f_+(\frac{x}{s})$ by (4.2), as long as they satisfy $f^*(\bar{s}) = as$.

To show the additivity of f_+, let $\frac{x}{s}$ and $\frac{x'}{s'}$ be arbitrary elements in $\mathscr{S}^{-1}T(X)$, and take $a, a', \bar{s}, \bar{s}'$ satisfying (4.3). Then we have $f^*(\bar{s}\bar{s}') = aa'ss'$, and thus

$$f_+\left(\frac{x}{s} + \frac{x'}{s'}\right) = f_+\left(\frac{s'x + sx'}{ss'}\right) = \frac{f_+(aa'(s'x + sx'))}{\bar{s}\bar{s}'}.$$

On the other hand, we have

$$f_+\left(\frac{x}{s}\right) + f_+\left(\frac{x'}{s'}\right) = \frac{f_+(ax)}{\bar{s}} + \frac{f_+(a'x')}{\bar{s}'} = \frac{\bar{s}' f_+(ax) + \bar{s} f_+(a'x')}{\bar{s}\bar{s}'}.$$

By the projection formula, we have

$$\begin{aligned}\bar{s}' f_+(ax) + \bar{s} f_+\big(a'x'\big) &= f_+\big(axf^*(\bar{s}')\big) + f_+\big(a'x'f^*(\bar{s})\big)\\ &= f_+\big(aa'\big(s'x + sx'\big)\big),\end{aligned}$$

and thus

$$f_+\left(\frac{x}{s} + \frac{x'}{s'}\right) = f_+\left(\frac{x}{s}\right) + f_+\left(\frac{x'}{s'}\right).$$

With these definitions, we can easily confirm $\ell_{\mathscr{S},X} \circ f^* = f^* \circ \ell_{\mathscr{S},Y}$, $\ell_{\mathscr{S},Y} \circ f_{\bullet} = f_{\bullet} \circ \ell_{\mathscr{S},X}$ and $\ell_{\mathscr{S},Y} \circ f_+ = f_+ \circ \ell_{\mathscr{S},X}$ for each $f \in {}_G set(X,Y)$.

It remains to show the compatibilities between these (to-be) structure morphisms.

(i) (**Functoriality of $()_+$**)

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms in ${}_G set$. For any $\frac{x}{s} \in \mathscr{S}^{-1}T(X)$, there exist $a \in T(X)$ and $b \in T(Y)$ satisfying

$$f^* f_\bullet(s) = as, \tag{4.4}$$

$$g^* g_\bullet\big(f_\bullet(s)\big) = bf_\bullet(s), \tag{4.5}$$

by Proposition 3.1. Thus we have

$$g_+ f_+\left(\frac{x}{s}\right) = g_+\left(\frac{f_+(ax)}{f_\bullet(s)}\right) = \frac{g_+(bf_+(ax))}{g_\bullet f_\bullet(s)}.$$

On the other hand by (4.4) and (4.5), we have

$$(g \circ f)^*(g \circ f)_\bullet(s) = f^*(b) f^* f_\bullet(s) = f^*(b)as,$$

and thus

$$(g \circ f)_+\left(\frac{x}{s}\right) = \frac{(g \circ f)_+(f^*(b)ax)}{(g \circ f)_\bullet(s)} = \frac{g_+(bf_+(ax))}{g_\bullet f_\bullet(s)}.$$

(ii) (**Mackey condition for $(\mathscr{S}^{-1}T)^\alpha$**)

Let

$$\begin{tikzcd} X' \arrow[r, "\xi"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\eta"'] & Y \end{tikzcd}$$

be any pullback diagram in ${}_G set$. For any $\frac{y}{t} \in \mathscr{S}^{-1}T(Y')$, there exists $b' \in T(Y')$ satisfying

$$\eta^* \eta_\bullet(t) = b't, \tag{4.6}$$

and

$$f^* \eta_+\left(\frac{y}{t}\right) = f^*\left(\frac{\eta_+(b'y)}{\eta_\bullet(t)}\right) = \frac{f^*\eta_+(b'y)}{f^*\eta_\bullet(t)} = \frac{\xi_+ f'^*(b'y)}{\xi_\bullet f'^*(t)}.$$

On the other hand by (4.6), we have

$$\xi^* \xi_\bullet f'^*(t) = \xi^* f^* \eta_\bullet(t) = f'^* \eta^* \eta_\bullet(t) = f'^*\big(b'\big) f'^*(t),$$

and thus

$$\xi_+ f'^*\left(\frac{y}{t}\right) = \xi_+\left(\frac{f'^*(y)}{f'^*(t)}\right) = \frac{\xi_+(f'^*(b')f'^*(y))}{\xi_\bullet f'^*(t)}.$$

(iii) (**Distributive law for $\mathscr{S}^{-1}T$**)
Let

$$\begin{tikzcd} X & A \arrow[l, "p"'] & Z \arrow[l, "\lambda"'] \\ Y \arrow[from=u, "f"'] \arrow[ru, phantom, "exp"] & & B \arrow[ll, "q"] \arrow[from=u, "\rho"] \end{tikzcd}$$

be any exponential diagram in ${}_G set$. For any $\frac{x}{s} \in \mathscr{S}^{-1}T(A)$, there exists $a \in T(A)$ satisfying

$$p^* p_\bullet(s) = as, \tag{4.7}$$

and

$$f_\bullet p_+\left(\frac{x}{s}\right) = f_\bullet\left(\frac{p_+(ax)}{p_\bullet(s)}\right) = \frac{f_\bullet p_+(ax)}{f_\bullet p_\bullet(s)} = \frac{q_+\rho_\bullet\lambda^*(ax)}{f_\bullet p_\bullet(s)}.$$

On the other hand, if we put $\bar{s} = f_\bullet p_\bullet(s)$ and $b = \rho_\bullet\lambda^*(a)$, then by (4.7), we have

$$q^*(\bar{s}) = q^* f_\bullet p_\bullet(s) = \rho_\bullet\lambda^* p^* p_\bullet(s) = \rho_\bullet\lambda^*(as) = b\rho_\bullet\lambda^*(s),$$

and thus

$$q_+\rho_\bullet\lambda^*\left(\frac{x}{s}\right) = q_+\left(\frac{\rho_\bullet\lambda^*(x)}{\rho_\bullet\lambda^*(s)}\right) = \frac{q_+(b\rho_\bullet\lambda^*(x))}{\bar{s}} = \frac{q_+(\rho_\bullet\lambda^*(ax))}{f_\bullet p_\bullet(s)}.$$

Thus $\mathscr{S}^{-1}T$ becomes a Tambara functor, and Proposition 4.4 is shown. □

Remark 4.5. Let T be a Tambara functor on G.

(1) If a semi-Mackey subfunctor $\mathscr{S} \subseteq T^\mu$ satisfies $\mathscr{S} \subseteq T^\times$, then $\ell_{\mathscr{S}}$ becomes an isomorphism of Tambara functors. In particular if $\mathscr{S}$ belongs to $Mack(G)$, then we have $\mathscr{S} \subseteq T^\times$ and thus $\ell_{\mathscr{S}}$ is an isomorphism.
(2) For any semi-Mackey subfunctor $\mathscr{S} \subseteq T^\mu$, we have a natural isomorphism $\mathscr{S}^{-1}T \xrightarrow{\cong} \widetilde{\mathscr{S}}^{-1}T$ of Tambara functors compatible with $\ell_{\mathscr{S}}$ and $\ell_{\widetilde{\mathscr{S}}}$.

Proof. These can be confirmed on each object $X \in \mathrm{Ob}({}_G set)$, by the ordinary commutative ring theory. □

Naturally, the morphism $\ell_{\mathscr{S}} : T \to \mathscr{S}^{-1}T$ satisfies the expected universality.

Proposition 4.6. *Let $\varphi : T \to T'$ be a morphism of Tambara functors, and let $\mathscr{S} \subseteq T^\mu$, $\mathscr{S}' \subseteq T'^\mu$ be semi-Mackey subfunctors. If φ satisfies $\varphi(\mathscr{S}) \subseteq \mathscr{S}'$, then there exists unique morphism*

$$\tilde{\varphi} : \mathscr{S}^{-1}T \to \mathscr{S}'^{-1}T'$$

compatible with φ

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\ell_{\mathscr{S}}"'] \arrow[dr, phantom, "\circlearrowright"] & T' \arrow[d, "\ell_{\mathscr{S}'}"] \\ \mathscr{S}^{-1}T \arrow[r, "\tilde{\varphi}"'] & \mathscr{S}'^{-1}T' \end{tikzcd}$$

Proof. By the ordinary commutative ring theory, there exists a unique ring homomorphism

$$\tilde{\varphi}_X : \mathscr{S}^{-1}T(X) \to \mathscr{S}'^{-1}T'(X)$$

for each $X \in \mathrm{Ob}({}_G\mathit{set})$, satisfying $\tilde{\varphi}_X \circ \ell_{\mathscr{S},X} = \ell_{\mathscr{S}',X} \circ \varphi_X$. This is given by

$$\tilde{\varphi}_X\left(\frac{x}{s}\right) = \frac{\varphi_X(x)}{\varphi_X(s)}$$

for any $\frac{x}{s} \in \mathscr{S}^{-1}T(X)$.

It suffices to show $\tilde{\varphi} = \{\tilde{\varphi}_X\}_{X\in \mathrm{Ob}({}_G\mathit{set})}$ is compatible with $f^*, f_+, f_\bullet$ for any morphism $f \in {}_G\mathit{set}(X,Y)$. Compatibility with f^* and $f_\bullet$ immediately follows from the definitions.

We show the compatibility with f_+. For any $\frac{x}{s} \in \mathscr{S}^{-1}T(X)$, there exists $a \in T(X)$ satisfying $f^*f_\bullet(s) = as$. It follows

$$f^*f_\bullet\varphi_X(s) = \varphi_X f^* f_\bullet(s) = \varphi_X(a)\varphi_X(s),$$

and thus we obtain

$$\begin{aligned} \tilde{\varphi}_Y f_+\left(\frac{x}{s}\right) &= \tilde{\varphi}_Y\left(\frac{f_+(ax)}{f_\bullet(s)}\right) = \frac{\varphi_Y f_+(ax)}{\varphi_Y f_\bullet(s)} \\ &= \frac{f_+\varphi_X(ax)}{f_\bullet\varphi_X(s)} = f_+\left(\frac{\varphi_X(x)}{\varphi_X(s)}\right) = f_+\tilde{\varphi}_X\left(\frac{x}{s}\right). \end{aligned}$$

$$\begin{tikzcd} \mathscr{S}^{-1}T(X) \arrow[r, "\tilde{\varphi}_X"] \arrow[d, "f_+"'] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{S}'^{-1}T'(X) \arrow[d, "f_+"] \\ \mathscr{S}^{-1}T(Y) \arrow[r, "\tilde{\varphi}_Y"'] & \mathscr{S}'^{-1}T'(Y) \end{tikzcd} \qquad \square$$

Corollary 4.7. *Let T be a Tambara functor, and let $\mathscr{S} \subseteq T^\mu$ be a semi-Mackey subfunctor. Then $\ell_{\mathscr{S}}$ gives a bijection between the morphisms $\mathscr{S}^{-1}T \to T'$ and the morphisms $\varphi : T \to T'$ satisfying $\varphi(\mathscr{S}) \subseteq T'^\times$:*

$$- \circ \ell_{\mathscr{S}} : \mathit{Tam}(G)\big(\mathscr{S}^{-1}T, T'\big) \xrightarrow{\cong} \big\{\varphi \in \mathit{Tam}(G)\big(T,T'\big) \bigm| \varphi(\mathscr{S}) \subseteq T'^\times\big\}.$$

Proof. This immediately follows from Remark 4.5 and Proposition 4.6. $\square$

Once Proposition 4.4 is shown, some natural compatibilities immediately follow from Proposition 4.6.

Corollary 4.8. *Let $\varphi : T \to T'$ be a morphism of Tambara functors.*

(1) *If $\mathscr{S} \subseteq T^{\mu}$ is a semi-Mackey subfunctor, then*

$$\varphi(\mathscr{S}) = \{\varphi_X(\mathscr{S}(X))\}_{X \in \mathrm{Ob}({}_G set)}$$

gives a semi-Mackey subfunctor $\varphi(\mathscr{S}) \subseteq T'^{\mu}$, and we obtain a morphism $\mathscr{S}^{-1}T \to (\varphi(\mathscr{S}))^{-1}T'$ compatible with φ.

(2) *If $\mathscr{S}' \subseteq T'^{\mu}$ is a semi-Mackey subfunctor, then*

$$\varphi^{-1}(\mathscr{S}') = \{\varphi_X^{-1}(\mathscr{S}'(X))\}_{X \in \mathrm{Ob}({}_G set)}$$

gives a semi-Mackey subfunctor $\varphi^{-1}(\mathscr{S}') \subseteq T$, and we obtain a morphism $(\varphi^{-1}(\mathscr{S}'))^{-1}T \to \mathscr{S}'^{-1}T'$ compatible with φ.

Proof. This immediately follows from Proposition 4.6. □

Corollary 4.9. *Let T be a Tambara functor, and let $\mathscr{S} \subseteq \mathscr{S}' \subseteq T$ be semi-Mackey subfunctors. Then the image $\bar{\mathscr{S}'} = \ell_{\mathscr{S}}(\mathscr{S}')$ of $\mathscr{S}'$ under the morphism $\ell_{\mathscr{S}} : T \to \mathscr{S}^{-1}T$ becomes a semi-Mackey subfunctor $\bar{\mathscr{S}'} \subseteq \mathscr{S}^{-1}T$, and there is a natural isomorphism*

$$\mathscr{S}'^{-1}T \xrightarrow{\cong} \bar{\mathscr{S}'}^{-1}(\mathscr{S}^{-1}T)$$

compatible with $\ell_{\mathscr{S}'}$ and $\ell_{\bar{\mathscr{S}'}} \circ \ell_{\mathscr{S}}$.

Proof. This immediately follows from Corollary 4.8 and an objectwise argument from ordinary commutative ring theory. □

5. Compatibility with the Tambarization

In [4], we constructed a functor (*Tambarization*)

$$\mathcal{T} : SMack(G) \to Tam(G),$$

which is left adjoint to the functor taking multiplicative parts

$$(-)^{\mu} : Tam(G) \to SMack(G).$$

$\mathcal{T}$ is regarded as a G-bivariant analog of the monoid-ring functor

$$Mon \to Ring; \quad Q \mapsto \mathbb{Z}[Q].$$

In this view, we denote $\mathcal{T}(M)$ by $\Omega[M]$ for any $M \in \mathrm{Ob}(SMack(G))$.

For each $X \in \mathrm{Ob}({}_G set)$, by definition $(\Omega[M])(X)$ is the Grothendieck ring

$$\Omega[M](X) = K_0(M\text{-}{}_G set/X),$$

where $M\text{-}{}_G set/X$ is the category defined as follows.

- An object in $M\text{-}{}_G set/X$ is a pair $(A \xrightarrow{p} X, m)$ of $(A \xrightarrow{p} X) \in \mathrm{Ob}({}_G set/X)$ and $m \in M(A)$.
- A morphism from $(A_1 \xrightarrow{p_1} X, m_1)$ to $(A_2 \xrightarrow{p_2} X, m_2)$ is a morphism $f \in {}_G set(A_1, A_2)$ satisfying $p_2 \circ f = p_1$ and $M^*(f)(m_2) = m_1$.

– The sum of $(A_1 \xrightarrow{p_1} X, m_1)$ and $(A_2 \xrightarrow{p_2} X, m_2)$ is

$$\big(A_1 \amalg A_2 \xrightarrow{p_1 \cup p_2} X, m_1 \amalg m_2\big),$$

where $m_1 \amalg m_2$ is the element in $M(A_1 \amalg A_2)$ corresponding to (m_1, m_2) under the natural isomorphism $M(A_1 \amalg A_2) \cong M(A_1) \times M(A_2)$.

– The product of $(A_1 \xrightarrow{p_1} X, m_1)$ and $(A_2 \xrightarrow{p_2} X, m_2)$ is $(A \xrightarrow{p} X, m_1 \star m_2)$, where

$$\begin{tikzcd} A \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, "\square"] & A_2 \arrow[d, "p_2"] \\ A_1 \arrow[r, "p_1"'] & X \end{tikzcd}$$

is a pullback diagram, and

$$p = p_1 \circ \varpi_1 = p_2 \circ \varpi_2,$$
$$m_1 \star m_2 = \varpi^*(m_1) \cdot \varpi_2^*(m_2).$$

We denote the equivalence class of $(A \xrightarrow{p} X, m)$ in $\Omega[M](X)$ by $[A \xrightarrow{p} X, m]$. Any element in $\Omega[M](X)$ can be written in the form of

$$\big[A_1 \xrightarrow{p_1} X, m_1\big] - \big[A_2 \xrightarrow{p_2} X, m_2\big]$$

for some $(A_1 \xrightarrow{p_1} X, m_1), (A_2 \xrightarrow{p_2} X, m_2) \in \mathrm{Ob}(M\text{-}{}_G set/X)$.

Remark 5.1. This kind of construction seems to be firstly done by Jacobson in [3], and later by Hartmann and Yalçın in [2], to obtain a Green functor from a monoid-valued additive contravariant functor.

Recently this construction was utilized to obtain a Tambara functor from a semi-Mackey functor in [4]. This can be also regarded as a generalization of crossed Burnside Tambara functors considered in [6].

For the later use, we briefly recall the construction of the adjoint isomorphism

$$Tam(G)\big(\Omega[M], T\big) \cong SMack(G)\big(M, T^{\mu}\big),$$
$$\varphi \leftrightarrow \vartheta,$$

for each $M \in \mathrm{Ob}(SMack(G))$, $T \in \mathrm{Ob}(Tam(G))$ (Theorem 2.15 in [4]).

For any $\varphi \in Tam(G)(\Omega[M], T)$, the corresponding ϑ is given by

$$\vartheta_X : M(X) \to T^{\mu}(X),$$
$$m \mapsto \varphi_X\big(\big[X \xrightarrow{\mathrm{id}_X} X, m\big]\big)$$

for each $X \in \mathrm{Ob}({}_G set)$.

For any $\vartheta \in SMack(G)(M, T^{\mu})$, the corresponding φ is given by

$$\varphi_X : \Omega[M](X) \to T(X),$$
$$\big[A_1 \xrightarrow{p_1} X, m_1\big] - \big[A_2 \xrightarrow{p_2} X, m_2\big] \mapsto T_+(p_1) \circ \vartheta_{A_1}(m_1) - T_+(p_2) \circ \vartheta_{A_2}(m_2)$$

for each $X \in \mathrm{Ob}({}_G set)$.

From this, for any semi-Mackey functor $M \in \mathrm{Ob}(\mathit{SMack}(G))$, the adjunction morphism

$$\varepsilon : M \to \Omega[M]^{\mu}$$

corresponding to $\mathrm{id}_{\Omega[M]}$ is given by

$$\begin{aligned} \varepsilon_X : M(X) &\to \Omega[M](X), \\ m &\mapsto \left[X \xrightarrow{\mathrm{id}_X} X, m\right] \end{aligned} \tag{5.1}$$

for each $X \in \mathrm{Ob}({}_G\mathit{set})$. Remark that, for any $\varphi \in \mathit{Tam}(G)(\Omega[M], T)$ and corresponding $\vartheta \in \mathit{SMack}(G)(M, T^{\mu})$, we have

$$\varphi \circ \varepsilon = \vartheta. \tag{5.2}$$

By (5.1), it is shown that ε_X is monomorphic for any X, and thus M can be regarded as a semi-Mackey subfunctor $M \subseteq \Omega[M]^{\mu}$ through ε. Thus if we are given a semi-Mackey subfunctor $\mathscr{S} \subseteq M$, we can localize $\Omega[M]$ by $\varepsilon(\mathscr{S})$. We denote the fraction $\varepsilon(\mathscr{S})^{-1}(\Omega[M])$ simply by $\mathscr{S}^{-1}(\Omega[M])$.

Proposition 5.2. *Let M be a semi-Mackey functor on G, and let $\mathscr{S} \subseteq M$ be a semi-Mackey subfunctor. We have a natural isomorphism of Tambara functors*

$$\mathscr{S}^{-1}\big(\Omega[M]\big) \cong \Omega\big[\mathscr{S}^{-1}M\big].$$

Proof. It suffices to construct a natural bijection

$$\mathit{Tam}(G)\big(\mathscr{S}^{-1}\big(\Omega[M]\big), T\big) \cong \mathit{Tam}(G)\big(\Omega\big[\mathscr{S}^{-1}M\big], T\big)$$

for each $T \in \mathrm{Ob}(\mathit{Tam}(G))$. This is obtained from

$$\begin{aligned} \mathit{Tam}(G)\big(\Omega\big[\mathscr{S}^{-1}M\big], T\big) &\cong \mathit{SMack}(G)\big(\mathscr{S}^{-1}M, T^{\mu}\big) \\ &\cong \big\{\vartheta \in \mathit{SMack}(G)\big(M, T^{\mu}\big) \bigm| \vartheta(\mathscr{S}) \subseteq \big(T^{\mu}\big)^{\times} = T^{\times}\big\} \end{aligned}$$

and

$$\begin{aligned} \mathit{Tam}(G)\big(\mathscr{S}^{-1}\big(\Omega[M]\big), T\big) &= \mathit{Tam}(G)\big(\varepsilon(\mathscr{S})^{-1}\big(\Omega[M]\big), T\big) \\ &\cong \big\{\varphi \in \mathit{Tam}(G)\big(\Omega[M], T\big) \bigm| \varphi\big(\varepsilon(\mathscr{S})\big) \subseteq T^{\times}\big\} \\ &\cong \big\{\vartheta \in \mathit{SMack}(G)\big(M, T^{\mu}\big) \bigm| \vartheta(\mathscr{S}) \subseteq T^{\times}\big\}. \qquad \square \end{aligned}$$

6. Compatibility with ideal quotients

In [5], an ideal of a Tambara functor T was defined as follows.

Definition 6.1. Let T be a Tambara functor. An *ideal* $\mathscr{I}$ of T is a family of ideals $\mathscr{I}(X) \subseteq T(X)$ $(\forall X \in \mathrm{Ob}({}_G\mathit{set}))$ satisfying

(i) $f^*(\mathscr{I}(Y)) \subseteq \mathscr{I}(X)$,
(ii) $f_+(\mathscr{I}(X)) \subseteq \mathscr{I}(Y)$,
(iii) $f_\bullet(\mathscr{I}(X)) \subseteq f_\bullet(0) + \mathscr{I}(Y)$

for any $f \in {}_G\mathit{set}(X, Y)$.

As shown in [5], for any ideal $\mathscr{I} \subseteq T$, the quotients

$$(T/\mathscr{I})(X) = T(X)/\mathscr{I}(X) \quad \big(X \in \mathrm{Ob}({}_G set)\big)$$

form a Tambara functor $T/\mathscr{I}$, and the projections

$$p_X : T(X) \to T(X)/\mathscr{I}(X) \quad \big(X \in \mathrm{Ob}({}_G set)\big)$$

form a morphism of Tambara functors $p : T \to T/\mathscr{I}$.

The following gives some examples of ideals [5].

Example 6.2. Let T be a Tambara functor, and $I \subseteq T(G/e)$ be a G-invariant ideal of $T(G/e)$. For each $X \in \mathrm{Ob}({}_G set)$, define $\mathscr{I}_I(X)$ by

$$\mathscr{I}_I(X) = \bigcap_{\gamma \in {}_G set(G/e, X)} \big(\gamma^*\big)^{-1}(I). \tag{6.1}$$

Then $\mathscr{I}_I \subseteq T$ becomes an ideal of T, which is the maximum one among ideals $\mathscr{I}$ satisfying $\mathscr{I}(G/e) = I$.

Remark 6.3. Let T be a Tambara functor. Let $I \subseteq T(G/e)$ be a G-invariant ideal and let $S \subseteq T(G/e)^{\mu}$ be a saturated G-invariant submonoid. For any ideal $\mathscr{I} \subseteq T$ satisfying $\mathscr{I}(G/e) = I$ and any semi-Mackey subfunctor $\mathscr{S} \subseteq T$ satisfying $\mathscr{S}(G/e) = S$, the following are equivalent.

(1) $I \cap S = \emptyset$.
(2) $\mathscr{I} \cap \mathscr{S} = \emptyset$. Namely, $\mathscr{I}(X) \cap \mathscr{S}(X) = \emptyset$ for any non-empty $X \in \mathrm{Ob}({}_G set)$.

Proof. Obviously (2) implies (1). Conversely, assume (1) holds. Then, for any $X \in \mathrm{Ob}({}_G set)$ and $\gamma \in {}_G set(G/e, X)$, since

$$\gamma^*\big(\mathscr{I}(X)\big) \subseteq I \quad \text{and} \quad \gamma^*\big(\mathscr{S}(X)\big) \subseteq S,$$

we obtain

$$\mathscr{I}(X) \cap \mathscr{S}(X) \subseteq \big(\gamma^*\big)^{-1}(I \cap S) = \emptyset. \qquad \square$$

Proposition 6.4. *Let T be a Tambara functor. Let $\mathscr{I} \subseteq T$ be an ideal and $\mathscr{S} \subseteq T^{\mu}$ be a semi-Mackey subfunctor, satisfying $\mathscr{I} \cap \mathscr{S} = \emptyset$.*

(1) *If we define $\mathscr{S}^{-1}\mathscr{I} \subseteq \mathscr{S}^{-1}T$ by*

$$\mathscr{S}^{-1}\mathscr{I}(X) = \left\{ \alpha \in \mathscr{S}^{-1}T(X) \;\middle|\; \alpha = \frac{x}{s} \text{ for some } x \in \mathscr{I}(X),\ s \in \mathscr{S}(X) \right\}$$

for each $X \in \mathrm{Ob}({}_G set)$, then $\mathscr{S}^{-1}\mathscr{I}$ becomes an ideal of $\mathscr{S}^{-1}T$.

(2) *Let $p : T \to T/\mathscr{I}$ be the projection, and put $\bar{\mathscr{S}} = p(\mathscr{S})$. Then we have a natural isomorphism of Tambara functors*

$$\upsilon : \mathscr{S}^{-1}T/\mathscr{S}^{-1}\mathscr{I} \xrightarrow{\cong} \bar{\mathscr{S}}^{-1}(T/\mathscr{I}),$$

compatible with projections

$$
\begin{tikzcd}
T \arrow[r, "p"] \arrow[dd, "\ell_{\mathscr{S}}"'] & T/\mathscr{I} \arrow[dr, "\ell_{\bar{\mathscr{S}}}"] & \\
 & \circlearrowleft & \bar{\mathscr{S}}^{-1}(T/\mathscr{I}) \\
\mathscr{S}^{-1}T \arrow[rr] & & \mathscr{S}^{-1}T/\mathscr{S}^{-1}\mathscr{I} \arrow[u, "\cong"', "\upsilon"]
\end{tikzcd}
\tag{6.2}
$$

Proof. By the ordinary ideal theory for rings, $\mathscr{S}^{-1}\mathscr{I}(X) \subseteq \mathscr{S}^{-1}T(X)$ becomes an ideal for each $X \in \mathrm{Ob}({}_G\mathit{set})$. Thus it suffices to show

$$f^*\big(\mathscr{S}^{-1}\mathscr{I}(Y)\big) \subseteq \mathscr{S}^{-1}\mathscr{I}(X), \qquad f_+\big(\mathscr{S}^{-1}\mathscr{I}(X)\big) \subseteq \mathscr{S}^{-1}\mathscr{I}(Y),$$

and

$$f_\bullet\big(\mathscr{S}^{-1}\mathscr{I}(X)\big) \subseteq f_\bullet(0) + \mathscr{S}^{-1}\mathscr{I}(Y).$$

Let $f \in {}_G\mathit{set}(X,Y)$ be any morphism. For any $y \in \mathscr{I}(Y)$ and $t \in \mathscr{S}(Y)$, we have $f^*(\frac{y}{t}) = \frac{f^*(y)}{f^*(t)} \in \mathscr{S}^{-1}\mathscr{I}(X)$, and thus

$$f^*\big(\mathscr{S}^{-1}\mathscr{I}(Y)\big) \subseteq \mathscr{S}^{-1}\mathscr{I}(X).$$

For any $x \in \mathscr{I}(X)$ and $s \in \mathscr{S}(X)$, if we take $a \in T(X)$ satisfying $as = f^*f_\bullet(s)$, then we have $f_+(\frac{x}{s}) = \frac{f_+(ax)}{f_\bullet(s)} \in \mathscr{S}^{-1}\mathscr{I}(Y)$, and thus

$$f_+\big(\mathscr{S}^{-1}\mathscr{I}(X)\big) \subseteq \mathscr{S}^{-1}\mathscr{I}(Y).$$

Besides, by $f_\bullet(\frac{x}{s}) - f_\bullet(0) = \frac{f_\bullet(x) - f_\bullet(s)f_\bullet(0)}{f_\bullet(s)} = \frac{f_\bullet(x) - f_\bullet(0)}{f_\bullet(s)} \in \mathscr{S}^{-1}\mathscr{I}(Y)$, we obtain

$$f_\bullet\big(\mathscr{S}^{-1}\mathscr{I}(X)\big) \subseteq f_\bullet(0) + \mathscr{S}^{-1}\mathscr{I}(Y)$$

for any $f \in {}_G\mathit{set}(X,Y)$. Thus $\mathscr{S}^{-1}\mathscr{I} \subseteq \mathscr{S}^{-1}T$ becomes an ideal.

By the ordinary ideal theory for rings, for any $X \in \mathrm{Ob}({}_G\mathit{set})$, there is a ring isomorphism

$$\upsilon_X : \mathscr{S}^{-1}T/\mathscr{S}^{-1}\mathscr{I}(X) \xrightarrow{\cong} \bar{\mathscr{S}}^{-1}(T/\mathscr{I})(X),$$
$$\frac{x}{s} + \mathscr{S}^{-1}\mathscr{I}(X) \mapsto \frac{p(x)}{p(s)} \quad \left(\forall \frac{x}{s} \in \mathscr{S}^{-1}T(X)\right),$$

which makes (6.2) commutative at X. Since the structure morphisms of $\bar{\mathscr{S}}^{-1}(T/\mathscr{I})$ and $\mathscr{S}^{-1}T/\mathscr{S}^{-1}\mathscr{I}$ are those induced from T, we can check that $\upsilon = \{\upsilon_X\}_{X \in \mathrm{Ob}({}_G\mathit{set})}$ becomes an isomorphism of Tambara functors. $\square$

7. Fraction and field-like Tambara functors

As in [5], we say a Tambara functor T is *field-like* if the zero ideal $(0) \subsetneq T$ is maximal with respect to the inclusion. In this section, we consider fractions by the following semi-Mackey subfunctor, and investigate the relations between field-like Tambara functors.

Example 7.1. Let T be a Tambara functor. If we put

$$\mathfrak{Z} = \{ s \in T(G/e) \mid s \text{ is not a zero divisor} \},$$

then we obtain two semi-Mackey subfunctors $\mathscr{L}_{\mathfrak{Z}} \subseteq T^{\mu}$ and $\mathscr{U}_{\mathfrak{Z}} \subseteq T^{\mu}$.

We introduce the following condition from [5].

Definition 7.2. A Tambara functor T is said to satisfy (MRC) if, for any $f \in {}_{G}set(X, Y)$ between transitive $X, Y \in \mathrm{Ob}({}_{G}set)$, the restriction f^* is monomorphic. Remark that we may assume $X = G/e$.

Remark 7.3. Let T be a Tambara functor and $\mathscr{I}_{(0)} \subseteq T$ be the ideal corresponding to $(0) \subseteq T(G/e)$ as in Example 6.2. If we define T_{MRC} by $T_{\mathrm{MRC}} = T/\mathscr{I}_{(0)}$, then T_{MRC} satisfies (MRC). Besides, T satisfies (MRC) if and only if $T = T_{\mathrm{MRC}}$.

Fact 7.4. (See Theorem 4.21 in [5].) A Tambara functor satisfies (MRC) if and only if T is a Tambara subfunctor of $\mathcal{P}_{T(G/e)}$.

Fact 7.5. (See Theorem 4.32 in [5].) For any Tambara functor $T \neq 0$, the following are equivalent.

(1) T is field-like.
(2) T satisfies (MRC), and $T(G/e)$ has no non-trivial G-invariant ideal.

First, we show that if T is field-like itself, then nothing is changed under the fraction by $\mathscr{U}_{\mathfrak{Z}}$.

Proposition 7.6. *If T is a field-like Tambara functor, then we have*

$$T^{\times}(G/e) = \mathfrak{Z} = \Big\{ s \in T(G/e) \;\Big|\; \prod_{g \in G} gs \neq 0 \Big\}.$$

Proof. For any $s \in T(G/e)$, put $\tilde{s} = \prod_{g \in G} gs$. Since we have

$$T^{\times}(G/e) \subseteq \mathfrak{Z} \subseteq \{ s \in T(G/e) \mid \tilde{s} \neq 0 \},$$

it suffices to show

$$\{ s \in T(G/e) \mid \tilde{s} \neq 0 \} \subseteq T^{\times}(G/e).$$

Take any $s \in \{ s \in T(G/e) \mid \tilde{s} \neq 0 \}$. Since $\tilde{s} \neq 0$ and T contains no non-trivial ideal, we have $\langle \tilde{s} \rangle = T$. In particular we have $\langle \tilde{s} \rangle(G/e) \ni 1$.

On the other hand, since $\tilde{s}$ is G-invariant, it can be easily shown that we have

$$\langle \tilde{s} \rangle(G/e) = \{ r\tilde{s} \mid r \in T(G/e) \}.$$

Thus there exists some $r \in T(G/e)$ such that $r\tilde{s} = 1$, which means $\tilde{s} \in T^{\times}(G/e)$. Consequently we obtain $s \in T^{\times}(G/e)$. $\square$

Proposition 7.7. *Let T be a field-like Tambara functor. If $S \subseteq T(G/e)$ is a saturated G-invariant submonoid contained in $\mathfrak{Z}$, then we have $\mathscr{L}_S \subseteq \mathscr{U}_S \subseteq T^{\times}$.*

Proof. Remark that T is a Tambara subfunctor of a fixed point functor. Especially we have $(\mathrm{pt}_{G/e})_{\bullet}\mathrm{pt}^*_{G/e}(x) = x^{|G|}$ for any $x \in T(G/G)$. Thus if $x \in T(G/G)$ satisfies $\mathrm{pt}^*_{G/e}(x) \in S$ $(\subseteq T^{\times}(G/e))$, then it satisfies $x \in T^{\times}(G/G)$. Namely we have

$$\left(\mathrm{pt}^*_{G/e}\right)^{-1}(S) \subseteq T^{\times}(G/G).$$

Thus it follows

$$\mathscr{U}_S(X) \subseteq \left(\mathrm{pt}^*_X\left(T^{\times}(G/G)\right)\right)^{\sim} \subseteq T^{\times}(X)$$

for any transitive $X \in \mathrm{Ob}({}_G\mathit{set})$. Thus it follows $\mathscr{U}_S \subseteq T^{\times}$. $\square$

Corollary 7.8. *For any field-like Tambara functor T, we have*

$$\mathscr{U}_{\mathfrak{Z}}^{-1}T \cong \mathscr{L}_{\mathfrak{Z}}^{-1}T \cong T.$$

Proof. This follows from Remark 4.5 and Proposition 7.7. $\square$

In the following, we investigate when $\mathscr{U}_{\mathfrak{Z}}^{-1}T$ becomes field-like.

Remark 7.9. Let T be a Tambara functor, and let $\mathscr{S} \subseteq T^{\mu}$ be a semi-Mackey subfunctor. Then the following are equivalent.

(1) $\mathscr{S}^{-1}T$ satisfies (MRC).
(2) For any transitive $X \in \mathrm{Ob}({}_G\mathit{set})$ and any $x \in T(X)$ admitting some $s \in \mathscr{S}(G/e)$ satisfying $s \cdot \gamma_X^*(x) = 0$ for $\gamma_X \in {}_G\mathit{set}(G/e, X)$, there exists some $t \in \mathscr{S}(X)$ such that $tx = 0$.

Especially, if $\mathscr{S}$ satisfies $\mathscr{S}(G/e) \subseteq \mathfrak{Z}$, then these are also equivalent to:

(2)′ For any transitive $X \in \mathrm{Ob}({}_G\mathit{set})$ and any $x \in T(X)$ satisfying $\gamma_X^*(x) = 0$ for $\gamma_X \in {}_G\mathit{set}(G/e, X)$, there exists some $t \in \mathscr{S}(X)$ such that $tx = 0$.

(Conditions (2) and (2)′ do not depend on the choice of $\gamma_X \in {}_G\mathit{set}(G/e, X)$.)

Proposition 7.10. *Let T be a Tambara functor, and let $\mathscr{S} \subseteq T^{\mu}$ be a semi-Mackey subfunctor satisfying $\mathscr{S}(G/e) \subseteq \mathfrak{Z}$. Let $p : T \to T/\mathscr{I}_{(0)} = T_{\mathrm{MRC}}$ be the projection, and $\bar{\mathscr{S}} \subseteq T_{\mathrm{MRC}}$ be the image of $\mathscr{S}$ under p. Then we have the following.*

(1) $\mathscr{S}^{-1}T/\mathscr{S}^{-1}\mathscr{I}_{(0)} \cong \bar{\mathscr{S}}^{-1}T_{\mathrm{MRC}}$.
(2) *$\mathscr{S}^{-1}T$ satisfies* (MRC) *if and only if the ideal $\mathscr{S}^{-1}\mathscr{I}_{(0)} \subseteq \mathscr{S}^{-1}T$ is equal to* (0).
(3) *If $\mathscr{S}^{-1}T$ satisfies* (MRC), *then $\mathscr{S}^{-1}T \cong \bar{\mathscr{S}}^{-1}T_{\mathrm{MRC}}$.*

Proof. (1) follows from Proposition 6.4, since $\mathscr{S}$ satisfies $\mathscr{I}_{(0)} \cap \mathscr{S} = \emptyset$. (2) follows from Remark 7.9. In fact, for any transitive $X \in \mathrm{Ob}({}_G\mathit{set})$, the following are equivalent.

(i) $\mathscr{S}^{-1}\mathscr{I}_{(0)}(X) = 0$.
(ii) For any $x \in \mathscr{I}_{(0)}(X)$, there exists $t \in \mathscr{S}(X)$ satisfying $tx = 0$.

(iii) For any $x \in T(X)$ satisfying $\gamma_X^*(x) = 0$ for $\gamma_X \in {}_G set(G/e, X)$, there exists $t \in \mathscr{S}(X)$ satisfying $tx = 0$.

(3) follows from (1) and (2). □

Lemma 7.11. *Let T be a Tambara functor. If T satisfies one of the following conditions, then $\mathscr{U}_3^{-1}T$ satisfies* (MRC).

(i) *T satisfies* (MRC).
(ii) *For any transitive $X \in \mathrm{Ob}({}_G set)$, if we let $\gamma_X \in {}_G set(G/e, X)$ be a G-map, then*

$$(\gamma_X)_+(1) \in \mathscr{U}_3(X)$$

holds. (*Remark that this does not depend on γ_X.*)

Proof. We use the criterion of Remark 7.9.

(i) This is obvious, since $\gamma_X^*(x) = 0$ implies $x = 0$.
(ii) By the projection formula, $\gamma_X^*(x) = 0$ implies

$$\gamma_{X+}(1) \cdot x = \gamma_{X+}\gamma_X^*(x) = 0. \qquad \square$$

As an immediate consequence of Lemma 7.11, we have:

Proposition 7.12. *Let T be a Tambara functor. If $T(G/e)$ is an integral domain and if T satisfies one of the conditions* (i), (ii) *in Lemma* 7.11, *then $\mathscr{U}_3^{-1}T$ becomes a field-like Tambara functor.*

Proof. By Lemma 7.11, $\mathscr{U}_3^{-1}T$ satisfies (MRC). Since $(\mathscr{U}_3^{-1}T)(G/e)$ is a field, $\mathscr{U}_3^{-1}T$ becomes field-like by Fact 7.5. □

Example 7.13.

(1) For any G-ring R, the fixed point functor $\mathcal{P}_R$ satisfies condition (i) in Lemma 7.11. Especially if R is an integral domain, then $\mathscr{U}_3^{-1}\mathcal{P}_R$ becomes a field-like Tambara functor by Proposition 7.12.
(2) If $T(G/e)$ has no $|G|$-torsion, then the Tambara functor T satisfies condition (ii) in Lemma 7.11.

Proof. (1) follows immediately from the definition of $\mathcal{P}_R$. We show (2). Let $X \in \mathrm{Ob}({}_G set)$ be transitive. We may assume $X = G/H$, for some $H \leqslant G$. It suffices to show $(p_e^H)_+(1) \in \mathscr{U}_3(X)$.

By the existence of a pullback diagram

$$\begin{tikzcd} \coprod_{|G:H|} G/e \arrow[r, "\exists\zeta"] \arrow[d, "\bigcup_{|G:H|} p_e^H"'] \arrow[dr, phantom, "\square"] & G/e \arrow[d, "p_e^G"] \\ G/H \arrow[r, "p_H^G"'] & G/G \end{tikzcd}$$

we have

$$\left(p_H^G\right)^*\left(p_e^G\right)_+(1) = |G:H| \cdot \left(p_e^H\right)_+(1). \tag{7.1}$$

In particular if $H = e$, we obtain

$$\left(p_e^G\right)^*\left(p_e^G\right)_+(1) = |G| \cdot 1. \tag{7.2}$$

Since $T(G/e)$ has no $|G|$-torsion, we have $|G| \cdot 1 \in \mathfrak{Z}$, and thus (7.2) implies

$$\left(p_e^G\right)_+(1) \in \left(\left(p_e^G\right)^*\right)^{-1}\left(|G| \cdot 1\right) \subseteq \left(\left(p_e^G\right)^*\right)^{-1}(\mathfrak{Z}),$$

namely

$$\left(p_e^G\right)_+(1) \in \left(\mathrm{pt}_{G/e}^*\right)^{-1}(\mathfrak{Z}).$$

From (7.1), we obtain

$$|G:H| \cdot \left(p_e^H\right)_+(1) = \left(p_H^G\right)^*\left(p_e^G\right)_+(1) = \mathrm{pt}_X^*\left(\left(p_e^G\right)_+(1)\right) \in \mathrm{pt}_X^*\left(\left(\mathrm{pt}_{G/e}^*\right)^{-1}(\mathfrak{Z})\right),$$

and thus $\gamma_{X+}(1) \in \mathscr{U}_{\mathfrak{Z}}(X)$. $\square$

Corollary 7.14. *Let $\Omega \in \mathrm{Ob}(Tam(G))$ be the Burnside Tambara functor. Then $\mathscr{U}_{\mathfrak{Z}}^{-1}\Omega$ becomes a field-like Tambara functor.*

Proof. Since $\Omega(G/e)$ is an integral domain with no $|G|$-torsion, this immediately follows from Proposition 7.12 and Example 7.13. $\square$

Caution 7.15. In [5], we also considered an analogous notion of an integral domain, as a '*domain-like*' Tambara functor. In [5], a Tambara functor T is called *domain-like* if the zero ideal $(0) \subseteq T$ is *prime*. Typical examples of domain-like Tambara functors are $T = \Omega$ and the fixed point functor $T = \mathcal{P}_R$ associated to an integral domain R (with a G-action). For this Tambara functor T, the associated fraction $\mathscr{U}_{\mathfrak{Z}}^{-1}T$ becomes field-like as shown in Example 7.13 and Corollary 7.14. However in general, we will have to assume some more conditions on a domain-like Tambara functor, if we expect $\mathscr{U}_{\mathfrak{Z}}^{-1}T$ to be field-like.

By using Proposition 7.10, we can calculate $\mathscr{U}_{\mathfrak{Z}}^{-1}\Omega$. First we remark the following.

Remark 7.16. For each $H \leqslant G$, let $\mathcal{O}(H)$ denote a set of representatives of conjugacy classes of subgroups of H.

Then $\Omega(G/H)$ is a free module over

$$\left\{G/K = \left[G/K \xrightarrow{p_K^H} G/H\right] \,\middle|\, K \in \mathcal{O}(H)\right\},$$

where $p_K^H : G/K \to G/H$ is the canonical projection.

Especially, for any transitive $X \cong G/H \in \mathrm{Ob}({}_G set)$, any $\alpha \in \Omega(X)$ can be decomposed uniquely as

$$\alpha = \sum_{K \in \mathcal{O}(H)} m_K\left[G/K \xrightarrow{p_K^H} G/H\right] \quad (m_K \in \mathbb{Z}). \tag{7.3}$$

Proposition 7.17. *We have an isomorphism of Tambara functors*

$$\mathscr{U}_{\mathfrak{Z}}^{-1}\Omega \cong \mathcal{P}_{\mathbb{Q}}.$$

Proof. As in Corollary 7.14, $\mathscr{U}_3^{-1}\Omega$ satisfies (MRC). Thus by Proposition 7.10, it suffices to show

$$\bar{\mathscr{U}}_3^{-1}\Omega_{\mathrm{MRC}} \cong \mathcal{P}_{\mathbb{Q}},$$

where $\bar{\mathscr{U}}_3 \subseteq \Omega_{\mathrm{MRC}}$ is the image of $\mathscr{U}_3$ under the projection $\Omega \to \Omega_{\mathrm{MRC}}$.

As shown in [5], the family of ring isomorphisms $\{\wp_H\}_{H\leqslant G}$

$$\wp_H : (\Omega/\mathscr{I}_{(0)})(G/H) \to \mathcal{P}_{\mathbb{Z}}(G/H) = \mathbb{Z},$$

$$\sum_{K\in\mathcal{O}(H)} m_K\big[G/K \xrightarrow{p_K^H} G/H\big] \mapsto \sum_{K\in\mathcal{O}(H)} m_K|H:K|$$

gives an isomorphism of Tambara functors $\wp : \Omega/\mathscr{I}_{(0)} \xrightarrow{\cong} \mathcal{P}_{\mathbb{Z}}$.

Remark that, for $m \in \mathbb{Z}$, we have $\wp_H(m[G/H \xrightarrow{\mathrm{id}} G/H]) = 0$ if and only if $m = 0$. Additionally, since $m[G/H \xrightarrow{\mathrm{id}} G/H] \in \mathscr{U}_3(G/H)$ for any $0 \neq m \in \mathbb{Z}$, we have

$$\wp(\bar{\mathscr{U}}_3)(G/H) = \mathbb{Z} \setminus \{0\}$$

for any $H \leqslant G$. Thus it follows

$$\bar{\mathscr{U}}_3^{-1}(\Omega/\mathscr{I}_{(0)}) \cong \wp(\bar{\mathscr{U}}_3)^{-1}\mathcal{P}_{\mathbb{Z}} \cong \mathcal{P}_{\mathbb{Q}}. \qquad \square$$

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