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On defining characteristic representations of finite reductive groups

Olivier Brunat [a], Frank Lübeck [b,*]

[a] UFR de Mathématiques, Université Denis Diderot – Paris 7, 175, rue du Chevaleret, F-75013 Paris, France
[b] Lehrstuhl D für Mathematik, RWTH Aachen, Templergraben 64, D-52062 Aachen, Germany

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ABSTRACT

We give parameterizations of the irreducible representations of finite groups of Lie type in their defining characteristic.

1. Introduction

We consider series of finite groups of Lie type which are specified by a root datum and a finite order automorphism of that root datum. For each power q of a prime p this determines a connected reductive algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ (an algebraic closure of the field with p elements) and a group of fixed points $\mathbf{G}^F$ of a Frobenius morphism $F:\mathbf{G}\to\mathbf{G}$, up to isomorphism.

We are interested in a parameterization of the irreducible modules of $\mathbf{G}^F$ over $\bar{\mathbb{F}}_p$.

A known solution to this task is to use that the groups $\mathbf{G}^F$ are groups with a split (B,N)-pair of characteristic p. There exists a parameterization of the absolutely irreducible representations over

* Corresponding author.
E-mail addresses: brunat@math.jussieu.fr (O. Brunat), Frank.Luebeck@Math.RWTH-Aachen.De (F. Lübeck).

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a field of characteristic p of such groups. For details we refer to the description by Curtis in [2, B, Thm. 5.7]. In our setup it would be very technical to construct the data for this parameterization from the given root datum with Frobenius action. That parameterization looks very different for different Frobenius actions on the same algebraic group.

In the literature on representations of connected reductive algebraic groups and finite groups of Lie type in their defining characteristic most authors restrict their descriptions to the case of simply-connected algebraic groups and the finite groups of Lie type arising from these.

In this case there is a nice combinatorial parameterization of the absolutely irreducible modules of the algebraic group by the set of dominant weights. The irreducible modules of the finite groups are restrictions of those of the algebraic group and Steinberg [19] described a nice subset of dominant weights which yields representatives of the isomorphism classes of these modules. A generalization to connected reductive groups with simply-connected derived group can be found in [11, App. 1.3].

Jantzen considers in [14] general connected reductive algebraic groups but does not consider the finite groups of Lie type. In the general case it is no longer true that all irreducible representations of the finite groups are restrictions from the algebraic group.

In this paper we give a parameterization of the irreducible representations in defining characteristic for arbitrary finite groups of Lie type. It is very concrete and computable starting from the given root datum for the algebraic group and Frobenius action on the root datum. The description will not become more complicated for twisted Frobenius actions.

Here is an overview of the content of the other sections of this paper. Section 2 contains a description of our setup. We describe how root data and Frobenius actions on root data can be represented and how to compute certain related data. Some of the results in this section may be of independent interest. For example, we describe a construction of a certain covering group of an arbitrary connected reductive group, which generalizes the well-known simply-connected coverings of semisimple groups (see Proposition 2.5).

In Section 3 we first recall the results about defining characteristic representations of the algebraic groups and the finite groups of Lie type arising from simply-connected semisimple groups which we have mentioned above. Then we state our main result in Theorem 3.5 where we consider arbitrary finite groups of Lie type. In the end of that section we work out an example in some detail (certain centralizers of semisimple elements in exceptional groups of type E_8).

In Section 4 we give a more detailed description of the parameter sets in our main theorem for finite groups of Lie type arising from any simple connected reductive group. As an application of these results we work out the number of semisimple conjugacy classes for all of these finite groups. The results of this application were obtained before by the first named author with a completely different proof. The new proof given here is more elementary.

2. Root data for finite groups of Lie type

2.1. Connected reductive algebraic groups

Let $\mathbf{G}$ be a connected reductive group over an algebraically closed field $\bar{k}$. We recall how $\mathbf{G}$ is determined by a root datum, for more details we refer to [18, 7.4, 9.6].

For each maximal torus $\mathbf{T}$ of $\mathbf{G}$ there is an associated root datum $\Psi = (X, R, Y, R^\vee)$ which together with $\bar{k}$ determines $\mathbf{G}$ up to isomorphism. Here, X and Y are the character and cocharacter groups of $\mathbf{T}$, respectively, both isomorphic to $\mathbb{Z}^r$ for some r called the rank of $\mathbf{G}$ (or of $\mathbf{T}$ or of Ψ). These are in duality via a natural pairing $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z}$. Here R is a finite subset of X, called the roots. There is a bijection ${}^\vee : R \to R^\vee \subset Y$, $\alpha \mapsto \alpha^\vee$, to the set $R^\vee$ of coroots, such that $\langle\alpha, \alpha^\vee\rangle = 2$ for all $\alpha \in R$.

Each $\alpha \in R$ defines reflections $s_\alpha : X \to X$, $x \mapsto x - \langle x, \alpha^\vee\rangle\alpha$, and $s_\alpha^\vee : Y \to Y$, $y \mapsto y - \langle\alpha, y\rangle\alpha^\vee$. The group W generated by all s_α is called the Weyl group of $\mathbf{G}$ or Ψ, it is isomorphic to the group $W^\vee$ generated by the $s_\alpha^\vee$. We have $RW = R$ and $R^\vee W^\vee = R^\vee$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset R$ be a set of simple roots, that is each root is a linear combination of simple ones with either non-negative or non-positive coefficients. The integer l is called the semisimple rank of $\mathbf{G}$ and Ψ. The set $\{s_\alpha \mid \alpha \in \Delta\}$ is a set of Coxeter generators of W and Δ is linearly independent as subset of $X \otimes_{\mathbb{Z}} \mathbb{Q}$. The matrix $C \in \mathbb{Z}^{l\times l}$, $C_{ij} = \langle\alpha_j, \alpha_i^\vee\rangle$ for $1 \leqslant i, j \leqslant l$ is called the Cartan matrix of Ψ.

We have $\Delta W = R$. The matrix C is the Cartan matrix of a crystallographic root system, the set Δ can be reordered such that C has a block diagonal form whose diagonal blocks are in the list given in [5, 3.6]. Cartan matrices can be encoded in a compact way by Dynkin diagrams, this is also explained in [5, 3.6].

We now introduce a compact description of a root datum which is useful to specify a root datum and for computations. This is for example used in the GAP [17] programs of the CHEVIE [10] project.

Given $\Psi = (X, R, Y, R^\vee)$ we can choose $\mathbb{Z}$-bases of X and Y which are dual to each other and represent elements of $x \in X$ and $y \in Y$ by their coordinate row vectors with respect to these bases (so, we have $\langle x, y\rangle = yx^{\mathrm{tr}}$, where $^{\mathrm{tr}}$ means the transpose).

For Δ as above we define matrices $A, A^\vee \in \mathbb{Z}^{l\times r}$ where the i-th row of A contains the coordinates of α_i and the i-th row of $A^\vee$ those of $\alpha_i^\vee$.

From A and $A^\vee$ we can compute the whole root datum: the i-th rows of the two matrices determine the generators s_{α_i} and $s_{\alpha_i}^\vee$ of W and $W^\vee$, and the orbits of the rows of A under W yield R (and similarly for $R^\vee$). The product $C = A^\vee A^{\mathrm{tr}} \in \mathbb{Z}^{l\times l}$ is the Cartan matrix of Ψ.

Vice versa, let $A, A^\vee \in \mathbb{Z}^{l\times r}$ be two matrices such that $C = A^\vee A^{\mathrm{tr}} \in \mathbb{Z}^{l\times l}$ is the Cartan matrix of a crystallographic root system, and let $\bar{k}$ be an algebraically closed field. Then there exists a connected reductive algebraic group over $\bar{k}$ which yields $(A, A^\vee)$ as described above (use [18, 7.4.1, 9.5.1, 10.1]).

Definition 2.1. We call a pair of matrices $(A, A^\vee) \in (\mathbb{Z}^{l\times r})^2$ *root datum matrices* if $C = A^\vee A^{\mathrm{tr}} \in \mathbb{Z}^{l\times l}$ is the Cartan matrix of a crystallographic root system.

Remark 2.2.

(a) Fixing the type of a root datum via a Cartan matrix $C \in \mathbb{Z}^{l\times l}$ (or, equivalently, a Dynkin diagram), the corresponding connected reductive groups of adjoint type are described by the root datum matrices (Id_l, C) (the simple roots are a basis of X), and the corresponding groups of simply-connected type are described by $(C^{\mathrm{tr}}, \mathrm{Id}_l)$ (the simple coroots are a basis of Y).

(b) For $i = 1, 2$ let $\mathbf{G}_i$ be a connected reductive group over $\bar{k}$ with a maximal torus $\mathbf{T}_i$. Let $(A_i, A_i^\vee)$ be corresponding root datum matrices. Then the direct product $\mathbf{G}_1 \times \mathbf{G}_2$ can be described with respect to the maximal torus $\mathbf{T}_1 \times \mathbf{T}_2$ by root datum matrices $(A, A^\vee)$ where A and $A^\vee$ are block diagonal with diagonal blocks A_1, A_2 and $A_1^\vee, A_2^\vee$, respectively.

The following observation will be useful later. We formulate it with roots, there is a similar statement for the coroots.

Lemma 2.3. *Let $\Psi = (X, R, Y, R^\vee)$ be a root datum and $\Delta \subset R$ a set of simple roots.*

(a) *The Cartan matrix $C = (\langle \alpha_j, \alpha_i^\vee\rangle)_{i,j}$ or, equivalently, the Dynkin diagram of Ψ labeled by Δ, determines the set R of roots as linear combinations of those in Δ.*

(b) *The set of roots R as linear combinations of Δ determines the Cartan matrix C or, equivalently, the Dynkin diagram of Ψ.*

Proof. (a) The set R is the union of orbits of Δ under the Weyl group W which is generated by the s_α with $\alpha \in \Delta$. For $\beta \in R$ (which is a $\mathbb{Z}$-linear combination of Δ) we have $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee\rangle\alpha$, so the action of s_α on the $\mathbb{Z}$-lattice spanned by Δ is completely determined by the Cartan matrix.

(b) For any two simple roots $\alpha_i, \alpha_j \in \Delta$ the subset of R consisting of linear combinations of α_i and α_j is the same as the sub-root system spanned by these two roots (see [12, Prop. in 1.10]).

So, to find the bond between α_i and α_j in the Dynkin diagram labeled by Δ we look at the subset of positive roots in R which are linear combinations of α_i and α_j. There are 2, 3, 4 or 6 such roots, corresponding to no, a single, a double or a triple bond (types $2A_1$, A_2, B_2, G_2), respectively. In the last two cases an arrow must be added pointing to the shorter root, this is the one occurring with the largest coefficient in the linear combinations. □

2.2. Homomorphisms of root data

We recall some information from [14, II 1.13–1.15]. For $i = 1, 2$ let $\mathbf{G}_i$ be connected reductive groups over the same algebraically closed field $\bar{k}$, with maximal tori $\mathbf{T}_i$ and corresponding root data $\Psi_i = (X_i, R_i, Y_i, R_i^\vee)$.

A homomorphism from Ψ_1 to Ψ_2 is given by a $\mathbb{Z}$-linear map $f : X_2 \to X_1$ such that f induces a bijection $R_2 \to R_1$ and its dual map $f^\vee : Y_1 \to Y_2$ induces a bijection $R_1^\vee \to R_2^\vee$.

For each such homomorphism of root data there is a homomorphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$ that maps $\mathbf{T}_1 \to \mathbf{T}_2$ such that $\phi|_{\mathbf{T}_1}$ induces $f^\vee : Y_1 \to Y_2$ and $\ker\phi \leqslant Z(\mathbf{G}_1) \leqslant T_1$, where $Z(\mathbf{G}_1)$ denotes the center of $\mathbf{G}_1$. More precisely, $\ker\phi = \{t \in T_1 \mid f(x)(t) = 1 \text{ for all } x \in X_2\}$.

The map ϕ is surjective if and only if $Y_2/f^\vee(Y_1)$ is finite, and ϕ is an isomorphism if and only if $f^\vee$ (or f) is invertible.

Moreover, ϕ is called an isogeny if it is surjective and has a finite kernel, that is $f^\vee$ maps Y_1 injectively onto a finite index subgroup of Y_2.

The root datum associated to a connected reductive group is unique up to isomorphism.

To construct homomorphisms of root data we will use the following lemma.

Lemma 2.4. *A homomorphism of root data is determined by a $\mathbb{Z}$-linear map $f^\vee : Y_1 \to Y_2$ which induces a bijection $R_1^\vee \to R_2^\vee$ and which maps $R_1^\perp$ to $R_2^\perp$, where $R_i^\perp = \{y \in Y_i \mid \langle \alpha, y\rangle = 0 \text{ for } \alpha \in R_i\}$.*

Proof. The given map $f^\vee : Y_1 \to Y_2$ induces its dual map $f : X_2 \to X_1$ as follows. For $x_2 \in X_2$ the image $f(x_2) \in X_1$ is the unique element such that $\langle f(x_2), y_1\rangle = \langle x_2, f^\vee(y_1)\rangle$ for all $y_1 \in Y_1$.

We need to show that the map f induces a bijection from $R_2 \to R_1$.

Let $\Delta_1 = \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots in R_1 and $\Delta_1^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ be the corresponding coroots. Since $f^\vee$ is $\mathbb{Z}$-linear and induces a bijection $R_1^\vee \to R_2^\vee$, it must map $\Delta_1^\vee$ to a set of simple coroots of $R_2^\vee$. So, there is a set of simple roots $\Delta_2 = \{\beta_1, \ldots, \beta_l\}$ of R_2 such that $f^\vee(\alpha_j^\vee) = \beta_j^\vee$ for $1 \leqslant j \leqslant l$.

Now we use Lemma 2.3(b) to conclude that the Cartan matrices of Ψ_1 and Ψ_2 are the same, more precisely $\langle \alpha_j, \alpha_i^\vee\rangle = \langle \beta_j, \beta_i^\vee\rangle$ for all $1 \leqslant i, j \leqslant l$.

We show the lemma by checking that $f(\beta_i) = \alpha_i$ for $1 \leqslant i \leqslant l$.

Note that $\mathbb{Q}Y_1 = \mathbb{Q}\Delta_1^\vee \oplus \mathbb{Q}R_1^\perp$ because the Cartan matrix of Ψ_1 has full rank l. So, we can show that $f(\beta_i) = \alpha_i$ by showing that $\langle f(\beta_i), \alpha_j^\vee\rangle = \langle \alpha_i, \alpha_j^\vee\rangle$ for $1 \leqslant j \leqslant l$ and that $\langle f(\beta_i), y\rangle = \langle \alpha_i, y\rangle$ for all $y \in R_1^\perp$.

The first follows because the Cartan matrices of Ψ_1 and Ψ_2 are the same: $\langle f(\beta_i), \alpha_j^\vee\rangle = \langle \beta_i, f^\vee(\alpha_j^\vee)\rangle = \langle \beta_i, \beta_j^\vee\rangle = \langle \alpha_i, \alpha_j^\vee\rangle$.

The second follows because $f^\vee(y) \in R_2^\perp$ for $y \in R_1^\perp$: $\langle f(\beta_i), y\rangle = \langle \beta_i, f^\vee(y)\rangle = 0 = \langle \alpha_i, y\rangle$. $\square$

Of course, there is also a similar version of the lemma where the roles of X_i and Y_i are interchanged.

2.3. Frobenius morphisms

From now we assume that our field $\bar{k} = \bar{\mathbb{F}}_p$ is an algebraic closure of the finite prime field with p elements, and that $\mathbf{G}$ is defined over the finite subfield $\mathbb{F}_q \leqslant \bar{k}$ with q elements. We refer to [8, Chapter 3] for an explanation of this notion. There is a corresponding Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. We consider the root datum of $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$ that is contained in a Borel subgroup $\mathbf{B}$ with $F(\mathbf{B}) = \mathbf{B}$ and $F(\mathbf{T}) = \mathbf{T}$. Then F induces a map on X which is of the form qF_0 where F_0 defines an automorphism of root data of finite order which permutes the set of simple roots Δ that is determined by $\mathbf{B}$. This follows from [8, 3.17] (the τ in that theorem is our F_0^{-1}) and [8, 3.6(ii)] (which shows that F_0 has finite order).

Vice versa, each qF_0 with F_0 of finite order is induced by some Frobenius morphism F of $\mathbf{G}$ as above; F is uniquely determined by F_0 and q up to conjugation by an element in $\mathbf{T}$. See [18, 9.6] for more details.

The finite groups of fixed points $G(q) = \mathbf{G}^F$ are called finite groups of Lie type. The group $G(q)$ is determined up to isomorphism by the root datum Ψ of $\mathbf{G}$, F_0 and q. (But various such tuples of data can yield isomorphic groups $G(q)$.)

If the root datum is described by root datum matrices $(A, A^\vee)$ and the elements of X and Y are considered as row vectors then F_0 can be described by an invertible matrix in $\mathbb{Z}^{r\times r}$ of finite order.

We remark that in this setup we do not cover the Suzuki and Ree groups. These are fixed points of simple reductive groups of types B_2, F_4 and G_2 under generalized Frobenius morphisms whose square is a Frobenius morphism as considered above (for q an odd power of 3 in case G_2 and an odd power of 2 in the other two cases). But in these cases parameterizations of the irreducible defining characteristic representations are known, see Theorem 3.2 and Remark 3.7.

2.4. A covering group

A semisimple group $\mathbf{G}$ has a covering by a simply-connected group. In this subsection we explicitly construct such a covering $\tilde{\mathbf{G}}$ for general connected reductive $\mathbf{G}$. If F is a Frobenius morphism on $\mathbf{G}$ we also construct a Frobenius morphism $\tilde{F}$ on $\tilde{\mathbf{G}}$ which induces F on $\mathbf{G}$.

Proposition 2.5. *Let $\mathbf{G}$ be a connected reductive group, defined over $\mathbb{F}_q$ with Frobenius morphism F. Let the root datum $\Psi = (X, R, Y, R^\vee)$ of $\mathbf{G}$ and F be described by root datum matrices $(A, A^\vee)$ and F_0 as above.*

There are root datum matrices $(\tilde{A}, \tilde{A}^\vee)$ and an automorphism $\tilde{F}_0$ of finite order of the corresponding root datum $\tilde{\Psi} = (\tilde{X}, \tilde{R}, \tilde{Y}, \tilde{R}^\vee)$, and a homomorphism $\tilde{\Psi} \to \Psi$ with the following properties.

- (a) *The connected reductive group $\tilde{\mathbf{G}}$ over $\bar{\mathbb{F}}_p$ determined by $\tilde{\Psi}$ is a direct product of simple simply-connected groups and a central torus $\tilde{\mathbf{Z}}^0$.*
- (b) *The homomorphism $\tilde{\Psi} \to \Psi$ induces an isogeny $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$.*
- (c) *π induces an isomorphism from $\tilde{\mathbf{Z}}^0$ to the connected center $\mathbf{Z}^0$ of $\mathbf{G}$.*
- (d) *There is a Frobenius morphism $\tilde{F} : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ corresponding to $\tilde{F}_0$ and q which induces F on $\mathbf{G}$.*

Proof. We first construct $\tilde{A}$ and $\tilde{A}^\vee$. Let $C = A^\vee A^{\mathrm{tr}} \in \mathbb{Z}^{l\times l}$ be the Cartan matrix of Ψ and r be the rank of Ψ. Let $\tilde{A} = (C^{\mathrm{tr}} \mid 0) \in \mathbb{Z}^{l\times r}$ be the matrix with C as the first l columns and $r - l$ zero columns, and similarly let $\tilde{A}^\vee = (\mathrm{Id}_l \mid 0) \in \mathbb{Z}^{l\times r}$. Then $(\tilde{A}, \tilde{A}^\vee)$ are root datum matrices because $\tilde{A}^\vee \tilde{A}^{\mathrm{tr}} = C$, so determine a root datum $\tilde{\Psi} = (\tilde{X}, \tilde{R}, \tilde{Y}, \tilde{R}^\vee)$ and a connected reductive group $\tilde{\mathbf{G}}$ over $\bar{\mathbb{F}}_p$. After reordering of the simple roots $\tilde{A}$ and $\tilde{A}^\vee$ have block diagonal form, the blocks corresponding to the simple components of $\tilde{\mathbf{G}}$. So, it is the root datum of a direct product of the simple components and a torus. The simple components are simply-connected, see Remark 2.2.

Let $B \in \mathbb{Z}^{(r-l)\times r}$ be a matrix whose rows describe a $\mathbb{Z}$-basis of $R^\perp \leqslant Y$. Then the matrix $M^{\mathrm{tr}} = \binom{A^\vee}{B} \in \mathbb{Z}^{r\times r}$ describes a $\mathbb{Z}$-linear map $f^\vee : \tilde{Y} \to Y$ which defines a homomorphism of root data: It maps the simple coroots to simple coroots and so by Lemma 2.3(a) the coroots $\tilde{R}^\vee$ to $R^\vee$ (the root data have the same Cartan matrix). And it induces an isomorphism $\tilde{R}^\perp \to R^\perp$. Hence we can use Lemma 2.4.

We can compute B as follows: Its rows are a $\mathbb{Z}$-basis of the set of solutions $y \in \mathbb{Z}^r$ of $yA^{\mathrm{tr}} = 0$. With the Smith normal form algorithm we can compute invertible integer matrices P and Q such that PAQ has diagonal form, so the last $r - l$ columns of AQ are zero (and the first l columns are $\mathbb{Q}$-linearly independent). We can take the last $r - l$ rows of Q^{tr} as matrix B.

The map $f^\vee : \tilde{Y} \to Y$ is injective, its image is generated by $R^\vee$ and $R^\perp$. So, the image is invariant under F_0^{tr} and we can define $\tilde{F}_0^{\mathrm{tr}} : \tilde{Y} \to \tilde{Y}$ by $\tilde{y}\tilde{F}_0^{\mathrm{tr}} := f^{\vee-1}(f^\vee(\tilde{y})F_0^{\mathrm{tr}}) = \tilde{y}M^{\mathrm{tr}}F_0^{\mathrm{tr}}M^{-\mathrm{tr}}$. This defines an automorphism of finite order of $\tilde{\Psi}$. Now, $\tilde{\Psi}$, $\tilde{F}_0^{\mathrm{tr}}$ and a prime power q determine a reductive $\tilde{\mathbf{G}}$, defined over $\mathbb{F}_q$ with Frobenius morphism $\tilde{F}$. We have a surjective homomorphism $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$, and $\tilde{F}$ induces a Frobenius morphism F' on $\mathbf{G}$, which induces F_0^{tr} on Y. So, modifying $\tilde{F}$ by a conjugation with an appropriate torus element we can assume that $\tilde{F}$ induces F on $\mathbf{G}$.

The kernel $K := \ker(\pi)$ of the covering $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is finite because M (and M^{tr}) have full $\mathbb{Q}$-rank r, so $XM \leqslant \tilde{X}$ is of finite index. We have $R^\perp = Y(\mathbf{Z}^0)$ and $\tilde{R}^\perp = Y(\tilde{\mathbf{Z}}^0)$ and since $f^\vee$ induces an isomorphism between these two lattices, the homomorphism π induces an isomorphism $\tilde{\mathbf{Z}}^0 \to \mathbf{Z}^0$. In Section 2.5 we show how to compute the kernel of π explicitly. $\square$

Lemma 2.6. *Let $\tilde{\mathbf{G}}$, $\mathbf{G}$ be connected reductive groups with a surjective homomorphism $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ and central kernel K. Let $\tilde{F}$ be a Frobenius morphism of $\tilde{\mathbf{G}}$ with $\tilde{F}(K) = K$ and F be the induced Frobenius morphism on $\mathbf{G}$. The induced map $\pi : \tilde{\mathbf{G}}^{\tilde{F}} \to \mathbf{G}^F$ is in general not surjective. We define $\mathcal{L}(K) = \{z^{-1}\tilde{F}(z) \mid z \in K\}$. Then $\pi(\tilde{\mathbf{G}}^{\tilde{F}})$ is a normal subgroup of $\mathbf{G}^F$ and there is a natural isomorphism*

$$\mathbf{G}^F/\pi\big(\tilde{\mathbf{G}}^{\tilde{F}}\big) \xrightarrow{\sim} K/\mathcal{L}(K).$$

Proof. We first show that $\pi(\tilde{\mathbf{G}}^{\tilde{F}})$ is normal. Let $\tilde{h} \in \tilde{\mathbf{G}}^{\tilde{F}}$, $g \in \mathbf{G}^F$ and $\tilde{g} \in \tilde{\mathbf{G}}$ with $\pi(\tilde{g}) = g$. Then $\tilde{F}(\tilde{g}) = \tilde{g}z$ for some $z \in K$. It follows $\tilde{F}(\tilde{g}^{-1}) = z^{-1}\tilde{g}^{-1}$. Hence $\tilde{F}(\tilde{g}^{-1}\tilde{h}\tilde{g}) = z^{-1}\tilde{g}^{-1}\tilde{h}\tilde{g}z = \tilde{g}^{-1}\tilde{h}\tilde{g}$ because K and so z is central. This shows that $g^{-1}\pi(\tilde{h})g = \pi(\tilde{g}^{-1}\tilde{h}\tilde{g}) \in \pi(\tilde{\mathbf{G}}^{\tilde{F}})$.

Since the group K as subgroup of the center of $\tilde{\mathbf{G}}$ is abelian if follows that $\mathcal{L}(K)$ is a subgroup of K.

We have $\mathbf{G} \cong \tilde{\mathbf{G}}/K$ and for $g \in \tilde{\mathbf{G}}$ we have $gK \in (\tilde{\mathbf{G}}/K)^F \cong \mathbf{G}^F$ if and only if $g^{-1}\tilde{F}(g) \in K$. We consider the map

$$\mathbf{G}^F \cong (\tilde{\mathbf{G}}/K)^F \to K/\mathcal{L}(K), \qquad gK \mapsto g^{-1}\tilde{F}(g)\mathcal{L}(K).$$

Since K is central, it is easy to check that this is a well-defined homomorphism. The Lang–Steinberg theorem (for $\tilde{\mathbf{G}}$) shows that this map is surjective. An element gK is in the kernel of this map if and only if gK contains an element of $\tilde{\mathbf{G}}^{\tilde{F}}$. $\square$

2.5. Torus elements

Given a root datum $\Psi = (X, R, Y, R^\vee)$ for $\mathbf{G}$ and $\mathbf{T}$, we can recover $\mathbf{T}$ by the isomorphism $\mathbf{T} \cong Y \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p^\times$. Via some fixed isomorphism we identify the multiplicative group $\bar{\mathbb{F}}_p^\times$ with the additive group $\mathbb{Q}_{p'}/\mathbb{Z}$ of elements of p'-order in $\mathbb{Q}/\mathbb{Z}$. See [6, 3.1] for more details.

Choosing dual bases of X and Y, we can describe Ψ by root datum matrices $(A, A^\vee)$ and identify $\mathbf{T} \cong Y \otimes_{\mathbb{Z}} (\mathbb{Q}_{p'}/\mathbb{Z})$ with r-tuples of elements in $\mathbb{Q}_{p'}/\mathbb{Z}$. In this setup we can compute $y(c)$ for $y \in Y$ and $c \in \mathbb{Q}_{p'}/\mathbb{Z}$, and apply $x \in X$ and F to $t \in \mathbf{T} = (\mathbb{Q}_{p'}/\mathbb{Z})^r$ as follows:

$$y(c) = c \cdot y, \qquad x(t) = tx^{\mathrm{tr}} \in \mathbb{Q}_{p'}/\mathbb{Z}, \qquad F(t) = qtF_0^{\mathrm{tr}} \in \mathbf{T}.$$

The center of $\mathbf{G}$ is the intersection of the kernels of all (simple) roots in $\mathbf{T}$. We can compute it as the solutions $t \in \mathbf{T}$ of the system of equations $tA^{\mathrm{tr}} = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^l$. The F-fixed points $\mathbf{T}^F$ of $\mathbf{T}$ are the solutions $t \in \mathbf{T}$ of the system of equations $t(qF_0^{\mathrm{tr}} - \mathrm{Id}_r) = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^r$.

We consider the isogeny from Proposition 2.5, $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$. In the proof of the proposition we have computed a matrix M describing the map $f : X \to \tilde{X}$ for the corresponding homomorphism of root data.

We can compute the kernel K of π as set of solutions $t \in \mathbf{T}$ of the system of equations

$$tM^{\mathrm{tr}} = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^r.$$

(The $\mathbb{Z}$-span of the rows of M is the image $f(X) \leqslant \tilde{X}$.) And we can compute the $\tilde{F}$-action on the elements $t \in K$ by

$$\tilde{F}(t) = qt\tilde{F}_0^{\mathrm{tr}}.$$

This yields an explicit description of the elements in K, $K^{\tilde{F}}$ and $\mathcal{L}(K)$.

2.6. The derived subgroup

We will also need to consider the derived group $\mathbf{G}'$ of $\mathbf{G}$ and the quotient torus $\mathbf{G}/\mathbf{G}'$. We use the description in [18, 8.1.9] or [14, 1.18].

The images of all coroots generate $\mathbf{T} \cap \mathbf{G}'$ and a character $x \in X$ has $\mathbf{T} \cap \mathbf{G}'$ in its kernel if and only if $x \in (R^\vee)^\perp$.

We compute a matrix $D \in \mathrm{GL}_r(\mathbb{Z})$ such that the last $l-r$ columns of $A^\vee D$ are zero (for example by the Smith normal form algorithm). Instead of $(A, A^\vee)$ and F_0 we then consider the isomorphic data $(AD^{-\mathrm{tr}}, A^\vee D)$ and $D^{\mathrm{tr}} F_0 D^{-\mathrm{tr}}$.

Now we get root datum matrices for $\mathbf{G}'$ by taking the first l columns of $AD^{-\mathrm{tr}}$ and $A^\vee D$, and the restriction of F to $\mathbf{G}'$ is described by the upper left $l \times l$ corner of $D^{\mathrm{tr}} F_0 D^{-\mathrm{tr}}$.

Furthermore, the lower right $(r-l) \times (r-l)$ corner of $D^{\mathrm{tr}} F_0 D^{-\mathrm{tr}}$ describes the Frobenius action induced on the torus $\mathbf{G}/\mathbf{G}'$.

Lemma 2.7. *Let $\pi : \tilde{\mathbf{G}} = \tilde{\mathbf{G}}' \times Z^0 \to \mathbf{G}$ be the covering and $\tilde{F}$ be the Frobenius morphism of $\tilde{\mathbf{G}}$ as constructed in Proposition 2.5.*

Then $\pi(\tilde{\mathbf{G}}'^{\tilde{F}}) \leqslant \mathbf{G}^F$ is a normal subgroup, and the quotient $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ is an abelian group of order prime to p.

Proof. That $\pi(\tilde{\mathbf{G}}'^{\tilde{F}}) \leqslant \mathbf{G}^F$ is normal can be shown as in the proof of Lemma 2.6, using that $\tilde{\mathbf{G}}'$ is normal in $\tilde{\mathbf{G}}$.

In [8, proof of 13.20] it is shown that $\mathbf{G}^F = \mathbf{T}^F \cdot \pi(\tilde{\mathbf{G}}'^{\tilde{F}})$. So, the quotient $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ is isomorphic to $\mathbf{T}^F/(\mathbf{T}^F \cap \pi(\tilde{\mathbf{G}}'^{\tilde{F}}))$, hence it is abelian and of order prime to p. $\square$

3. Irreducible representations in defining characteristic

In this section we consider (finite dimensional rational) irreducible representations of our connected reductive algebraic groups $\mathbf{G}$ and the finite groups of Lie type $\mathbf{G}^F$ over the defining field $\bar{k} = \bar{\mathbb{F}}_p$ of $\mathbf{G}$. As before, let $\Psi = (X, R, Y, R^\vee)$ be the root datum of $\mathbf{G}$ and $F_0 : X \to X$ and q be the finite order automorphism and the prime power determined by F.

3.1. Representations of connected reductive groups

In this subsection $\bar{k}$ can be any algebraically closed field. We fix a set $\Delta \subset R$ of simple roots. The set

$$X_+ = \{x \in X \mid \langle x, \alpha^\vee \rangle \geqslant 0 \text{ for } \alpha \in \Delta\}$$

is called the set of dominant weights of $\mathbf{G}$ (or Ψ).

One can associate to each irreducible representation of $\mathbf{G}$ over $\bar{k}$ a highest weight $\lambda \in X_+$. Chevalley proved the following basic theorem, see [14, 2.7]:

Theorem 3.1. *Associating the highest weight induces a bijection from the isomorphism classes of irreducible representations of $\mathbf{G}$ over $\bar{k}$ to the set X_+ of dominant weights.*

For $\lambda \in X_+$ we denote by $L(\lambda)$ the corresponding irreducible module, and by ρ_λ the corresponding representation.

3.2. Finite groups of Lie type, simply-connected case

In this subsection we assume that $\mathbf{G}$ is semisimple and of simply-connected type. In this case the simple coroots are a $\mathbb{Z}$-basis of Y. The elements of the dual basis $\{\omega_1, \ldots, \omega_l\} \subset X$ are called the fundamental weights. For a positive integer b we call the subset

$$X_b = \{x \in X_+ \mid \langle x, \alpha^\vee \rangle < b \text{ for all } \alpha \in \Delta\}$$
$$= \{a_1\omega_1 + \cdots + a_l\omega_l \mid 0 \leqslant a_i < b \text{ for } 1 \leqslant i \leqslant l\}$$

of dominant weights the set of b-restricted weights.

Steinberg proved the following theorem, see [19, 13.3]. Although we have excluded the cases of Suzuki and Ree groups from our general setup we include them in this theorem.

Theorem 3.2.

(a) *The restrictions of ρ_λ with $\lambda \in X_q$ to $\mathbf{G}^F$ remain irreducible. This induces a bijection from X_q to the isomorphism classes of irreducible representations of $\mathbf{G}^F$ over $\bar{\mathbb{F}}_q$.*

(b) *Let $\mathbf{G}$ be of type B_2, F_4 or G_2, q^2 be an odd power of $p = 2$, 2 or 3, respectively, and F_0 be of order 2. We consider the set X'_q of dominant weights $\sum_{i=1}^{l} a_i\omega_i$ with $0 \leqslant a_i < q\sqrt{p}$ if α_i is a short simple root and $0 \leqslant a_i < q/\sqrt{p}$ otherwise. Then the restrictions of ρ_λ with $\lambda \in X'_q$ induce a bijection from X'_q to the isomorphism classes of irreducible representations of $\mathbf{G}^F$ over $\bar{\mathbb{F}}_q$.*

3.3. Finite tori

Let $\mathbf{G} = \mathbf{T}$ be a torus. The irreducible representations of $\mathbf{T}$ are the characters $X(T)$. Let $F_0 : X \to X$ be the finite order automorphism induced by F, and let $m \in \mathbb{N}$ such that $F_0^m = \mathrm{Id}$. We have $\mathbf{T}^F \leqslant \mathbf{T}^{F^m}$. Since these groups are abelian, each irreducible representation of $\mathbf{T}^F$ can be extended to one of $\mathbf{T}^{F^m}$, see [13, 5.5]. Using Section 2.5 we see that the group $\mathbf{T}^{F^m}$ is isomorphic to a direct product of r cyclic groups of order $q^m - 1$. And restriction yields a bijection from the set of characters $\{\rho_\lambda \mid \lambda \in X_{q^m-1}\}$ of $\mathbf{T}$ to the set of irreducible characters over $\bar{\mathbb{F}}_q$ of the finite group $\mathbf{T}^{F^m}$.

Remark 3.3. All irreducible representations of a finite torus $\mathbf{T}^F$ over $\bar{\mathbb{F}}_q$ are restrictions of irreducible representations (characters) of the torus $\mathbf{T}$.

In the general case there seems to be no nice description of a subset of X which yields the pairwise different characters of $\mathbf{T}^F$, for this one has to compute an explicit parameterization of $\mathbf{T}^F$. This can be done by solving the system of equations $t(qF_0^{\mathrm{tr}} - \mathrm{Id}_r) = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^r$, as explained in Section 2.5 (we see that the order of $\mathbf{T}^F$ is just the characteristic polynomial of F_0 evaluated at q).

3.4. Extending representations

Proposition 3.4. *Let $\mathbf{G}$ be a connected reductive group over $\bar{k}$ $(= \bar{\mathbb{F}}_p)$ with Frobenius morphism F. Let $H \leqslant \mathbf{G}^F$ be a normal subgroup such that $\mathbf{G}^F/H$ is an abelian group of order prime to p. Then each irreducible representation of H over $\bar{k}$ can be extended to a representation of $\mathbf{G}^F$. And each irreducible representation of $\mathbf{G}^F$ over $\bar{k}$ restricts irreducibly to H.*

Proof. We use Clifford theory, see for example [1, 9.18]. It follows that the two statements in the proposition are equivalent. We show the latter: the restriction of every irreducible $\bar{k}\mathbf{G}^F$-module V to H is irreducible.

The restriction is a direct sum of irreducible $\bar{k}H$-modules W_i,

$$V_H = \bigoplus_{i=1}^{r} W_i,$$

we show $r = 1$. Let U be a Sylow-p-subgroup of $\mathbf{G}^F$. Then $U \leqslant H$ because H has p'-index. The only simple $\bar{k}U$-module is the trivial module. Therefore, each W_i must have at least a one-dimensional subspace on which U acts trivially. So, V contains at least an r-dimensional subspace on which U acts trivially.

Now we use that the group $\mathbf{G}^F$ is a finite group with split (B,N)-pair in characteristic p, see [9, Cor. 4.2.5]. Thus we can apply a result by Richen and Curtis that says that the subspace of V fixed by U is one-dimensional, see [7, 4.3(c)]. Hence $r=1$ and V_H is an irreducible kH-module. $\square$

3.5. Parameterization of irreducible representations of finite groups of Lie type

We can now describe the main result of this paper.

As before, let $\mathbf{G}$ be a connected reductive group over $\bar{k}$, defined over $\mathbb{F}_q$ with corresponding Frobenius morphism F, given by root datum matrices $(A, A^\vee)$ and a finite order matrix F_0, as explained in Section 2.

In Proposition 2.5 we have constructed a covering $\pi : \tilde{\mathbf{G}} = \tilde{\mathbf{G}}' \times Z^0 \to \mathbf{G}$ and a Frobenius morphism $\tilde{F}$ of $\tilde{\mathbf{G}}$ inducing F on $\mathbf{G}$. We write K for the kernel of π and $\mathbf{G}'$ for the derived subgroup of $\mathbf{G}$.

Theorem 3.5. *The irreducible representations of $\mathbf{G}^F$ over $\bar{k}$ can be parameterized by the direct product of the following three sets*:

(A) *the q-restricted weights of $\tilde{\mathbf{G}}'$ which have $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ in their kernel,*
(B) *the group $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$,*
(C) *and the group $(\mathbf{G}/\mathbf{G}')^F$.*

Proof. This follows from Steinberg's Theorem 3.2 applied to $\tilde{\mathbf{G}}'^{\tilde{F}}$ and Clifford theory, see for example [1, 9.18]. We give more details.

We know from Lemma 2.7 that $\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ is a normal subgroup of $\mathbf{G}^F$ with abelian quotient of order prime to p. Thus we can apply Proposition 3.4 to see that all irreducible $\bar{k}\mathbf{G}^F$-modules are extensions of irreducible $\bar{k}\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$-modules. By Clifford theory the extensions of a fixed $\bar{k}\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$-module to $\mathbf{G}^F$ are parameterized by the group of linear characters of the quotient group $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ which is isomorphic to the quotient group itself.

The irreducible representations of $\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ can be interpreted as the irreducible representations of $\tilde{\mathbf{G}}'^{\tilde{F}}$ which have $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ in their kernel. And, since $\tilde{\mathbf{G}}'$ is simply-connected, the irreducible representations of $\tilde{\mathbf{G}}'^{\tilde{F}}$ are by Theorem 3.2 parameterized by the q-restricted weights of $\tilde{\mathbf{G}}'$. An element $z \in Z(\tilde{\mathbf{G}}') \leqslant \tilde{T} \cap \tilde{\mathbf{G}}'$ lies in the kernel of an irreducible representation with highest weight $\lambda \in \tilde{X}(\tilde{T} \cap \tilde{\mathbf{G}}')$ if its (only) eigenvalue is 1. This eigenvalue can be read off at the weight space of the highest weight by evaluating λ at z. This shows that the irreducible representations of $\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$ can be parameterized by the set (A).

We can parameterize the characters $\mathrm{Hom}(\mathbf{G}^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}}), \bar{k}^\times)$ in two steps, first by the restriction to $\mathrm{Hom}(\mathbf{G}'^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}}), \bar{k}^\times)$ and then by the characters $\mathrm{Hom}(\mathbf{G}^F/\mathbf{G}'^F, \bar{k}^\times)$ (again by Clifford theory because all characters in $\mathrm{Hom}(\mathbf{G}'^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}}), \bar{k}^\times)$ extend to $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}})$). The latter yields our parameter set (C) using $\mathbf{G}^F/\mathbf{G}'^F \cong (\mathbf{G}/\mathbf{G}')^F$ which follows from the Lang–Steinberg theorem. The set (B) we get from the isomorphism $\mathbf{G}'^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}}) \cong K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$. This follows from Lemma 2.6 applied to the induced map $\pi : \tilde{\mathbf{G}}' \to \mathbf{G}'$ which has the finite kernel $K' = K \cap \tilde{\mathbf{G}}'$. The lemma shows $\mathbf{G}'^F/\pi(\tilde{\mathbf{G}}'^{\tilde{F}}) \cong K'/\mathcal{L}(K')$. This last group is isomorphic to $K'^{\tilde{F}} = K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ which follows from dualizing the exact sequence $1 \to K'^{\tilde{F}} \to K' \to \mathcal{L}(K') \to 1$. $\square$

We now indicate how to compute the parameter sets (A), (B) and (C). In Proposition 2.5 we have constructed the root datum of $\tilde{\mathbf{G}}$ such that the first l coordinates and the last $r-l$ coordinates of $\tilde{X}$ and $\tilde{Y}$ correspond to the factors of the direct product $\tilde{T} = (\tilde{T} \cap \tilde{\mathbf{G}}') \times Z^0$. Thus it is easy to decide which elements of $K^{\tilde{F}}$, computed as in Section 2.5, are contained in $\tilde{\mathbf{G}}'$, this yields the set (B).

In Section 2.5 we have also shown how to evaluate a $\lambda \in X(\tilde{T} \cap \tilde{\mathbf{G}}')$ at a torus element. This way we can decide which q-restricted weights of $\tilde{\mathbf{G}}'$ have $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ in their kernel. This determines the set (A).

For the set (C) we need to compute the structure of the abelian group $\mathbf{G}^F/\mathbf{G}'^F \cong (\mathbf{G}/\mathbf{G}')^F$. In subsection 2.6 we have described how to compute the F-action on the torus $\mathbf{G}/\mathbf{G}'$. We can use Section 2.5 again to compute the F-fixed points of $\mathbf{G}/\mathbf{G}'$.

Remark 3.6.

(a) Assume that the derived group $\mathbf{G}'$ of $\mathbf{G}$ is simply-connected. Then each irreducible $\bar{k}\mathbf{G}^F$-module is the restriction of an irreducible $\bar{k}\mathbf{G}$-module. In [11, App. 1.3] Herzig gives another parameterization in this case: Namely by all q-restricted weights of $\mathbf{G}$ (these are infinitely many if $\mathbf{G}$ is not semisimple) and showing that two q-restricted weights λ_1, λ_2 yield the same restriction to $\mathbf{G}^F$ if and only if $\lambda_1 - \lambda_2 \in (q \cdot \mathrm{id} - F_0)(R^\vee)^\perp$.

(b) In general, not all irreducible $\bar{k}\mathbf{G}^F$-modules are restrictions of modules of the algebraic group $\mathbf{G}$. As an example consider $\mathbf{G} = \mathrm{PGL}_{l+1}(\bar{k})$, the adjoint groups of type A_l, with Frobenius map F such that $\mathbf{G}^F = \mathrm{PGL}_{l+1}(q)$. For some prime powers q the finite group $\mathbf{G}^F$ has non-trivial $\bar{k}$-representations of dimension 1. Such representations are not restrictions from $\mathbf{G}$ because $\mathbf{G}$ is perfect.

Proof. We show the first statement of (a) using our setup. If $\mathbf{G}'$ is simply-connected our parameterization of irreducible $\bar{k}\mathbf{G}^F$ modules in Theorem 3.5 is particularly simple: the set (A) consists of all q-restricted weights of $\mathbf{G}'$, the group (B) is trivial and (C) is the finite torus $(\mathbf{G}/\mathbf{G}')^F$.

Since $\mathbf{G}'$ is simply-connected, X contains $\tilde{\omega}_i$ with $\langle \tilde{\omega}_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant l$. So, for each q-restricted weight λ' of $\mathbf{G}'$ there is a $\lambda \in X$ such that the module $L(\lambda)$ of $\mathbf{G}$ restricts to $\mathbf{G}'$ as $L(\lambda')$. Together with Steinberg's Theorem 3.2 this shows that each irreducible representation $\tilde{\rho}$ of $\mathbf{G}'^F$ can be extended to a representation ρ of the algebraic group $\mathbf{G}$. All the other extensions of $\tilde{\rho}$ to $\mathbf{G}^F$ are obtained by tensoring $\rho|_{\mathbf{G}^F}$ with the linear characters of $\mathbf{G}^F/\mathbf{G}'^F$. But these are also obtained as restrictions of linear characters of the algebraic group $\mathbf{G}/\mathbf{G}'$ as we have seen in Remark 3.3. $\square$

3.6. A variant

As a variant of Theorem 3.5 we could have first given a parameterization of the irreducible representations of $\tilde{\mathbf{G}}^{\tilde{F}}$ and use Clifford theory only for the quotient $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}^{\tilde{F}})$. But our description in Theorem 3.5 often leads to a more natural parameterization.

For example, let $\mathbf{G} = \mathrm{GL}_{l+1}(\bar{k})$ and $q \equiv 1 \pmod{l+1}$. Then $\tilde{\mathbf{G}} = \mathrm{SL}_{l+1}(\bar{k}) \times Z^0$ and the kernel of π, $K = K^{\tilde{F}}$, is cyclic of order $l+1$ and is isomorphic to $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}^{\tilde{F}})$. The irreducible representations of $\tilde{\mathbf{G}}^{\tilde{F}} \cong \mathrm{SL}_{l+1}(q) \times (Z^0)^{\tilde{F}}$ (the second factor is cyclic of order $q-1$) are easy to describe. But it is a bit complicated to describe the subset which has K in its kernel. The quotient $\mathbf{G}^F/\pi(\tilde{\mathbf{G}}^{\tilde{F}})$ is cyclic of order $l+1$, so its irreducible representations are also easy to describe.

Our parameterization in Theorem 3.5 is more natural in this example: The derived subgroup of $\mathbf{G}$ and of $\tilde{\mathbf{G}}$ is $\mathrm{SL}_{l+1}(\bar{k})$ and so is simply-connected. Hence we are in the situation of Remark 3.6(a), our set (A) consists of all q-restricted dominant weights of $\mathrm{SL}_{l+1}(\bar{k})$ and our set (B) is trivial. The set (C) corresponds to the $q-1$ linear characters of $\mathrm{GL}_{l+1}(q)/\mathrm{SL}_{l+1}(q)$.

Remark 3.7. A variant of the main Theorem 3.5 is also true if $\mathbf{G}^F$ has Suzuki or Ree groups as components. Since the Suzuki and Ree groups have trivial center, we can assume that $\mathbf{G}^F$ arises from an algebraic group such that the Suzuki and Ree components are coming from direct factors of $\mathbf{G}$ of simply-connected type. We can then deal with these components using Theorem 3.2(b).

3.7. An example

Let us consider as an example a reductive group $\mathbf{G}$ which occurs as the centralizer of a semisimple element in the simple algebraic group of type E_8, equipped with a Frobenius morphism F. It is given by root datum matrices A, $A^\vee$, and a matrix F_0, as explained in Sections 2.1 and 2.3:

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 1 \end{pmatrix}, \qquad A^\vee := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$F_0 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 1 \\ -2 & -3 & -4 & -6 & -5 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We do not fix the q, but we want to investigate all finite groups of Lie type for any prime power q which are determined by these data.

The group G has rank 8 and semisimple rank 7. Looking at the Cartan matrix $A^\vee A^{\mathrm{tr}}$ we see that the pairs of simple roots number 1 and 3, number 4 and 5 and number 6 and 7 each span a sub-root system of type A_2, and root number 2 spans a subsystem of type A_1. The matrix AF_0 yields a permutation of the rows of A, the permutation is $(1,3)(4,6)(5,7)$. Thus, the data describe groups $\mathbf{G}^F$ which are central products of components of type ${}^2A_2(q)$, $A_2(q^2)$, $A_1(q)$ and a finite torus of rank 1.

Now we look at the covering group $\tilde{\mathbf{G}}$ of $\mathbf{G}$ constructed in Proposition 2.5. We do not need the matrices $\tilde{A}$ and $\tilde{A}^\vee$, but the matrix M^{tr} is essential which describes the homomorphism $\tilde{Y} \to Y$ that determines the covering $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$. As described in the proof of Proposition 2.5 we can compute a $\mathbb{Z}$-basis of $R^\perp$ by applying the Smith normal form algorithm to A. This yields invertible integer matrices P and Q such that PAQ is of diagonal form (the diagonal entries are six times 1 and one 3). Then M^{tr} is given by the rows of $A^\vee$ and the last row of Q^{tr}, the latter spans $R^\perp$. We furthermore need $\tilde{F}_0$ which defines the Frobenius morphism on the covering group $\tilde{\mathbf{G}}$. We can compute it with M^{tr} as $\tilde{F}_0^{\mathrm{tr}} = M^{\mathrm{tr}} F_0^{\mathrm{tr}} M^{-\mathrm{tr}}$. We get

$$M^{\mathrm{tr}} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \end{pmatrix}, \qquad \tilde{F}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using these two matrices we can determine the finite kernel K of the covering $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ and its $\tilde{F}$-fixed points $K^{\tilde{F}}$, as explained in Section 2.5. To find K we solve the system of equations

$$tM^{\mathrm{tr}} = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^r.$$

To do so, we use again the Smith normal form algorithm to find matrices $P, Q \in \mathrm{GL}_r(\mathbb{Z})$ such that $PM^{\mathrm{tr}}Q$ is diagonal. The diagonal entries in our example are six times 1, 3 and 6. It is easy to write down the solutions of

$$t_1\big(PM^{\mathrm{tr}}Q\big) = 0 \in (\mathbb{Q}_{p'}/\mathbb{Z})^r.$$

If the i-th diagonal entry of the diagonal matrix is an integer n then the i-th entry of any solution t_1 has the form $i/n_{p'}$ for one $0 \leqslant i < n_{p'}$ (where $n_{p'}$ is the largest divisor of n prime to p). Having found all solutions t_1 of this last equation we get the solutions of the original equation as $t = t_1 P$. In practice we first compute all solutions $t \in (\mathbb{Q}/\mathbb{Z})^r$, because we have not yet said anything about the q and so the p. In our example we have 18 solutions for t_1 over $\mathbb{Q}/\mathbb{Z}$, they have the form $(0, 0, 0, 0, 0, 0, \frac{i}{3}, \frac{j}{6})$. And multiplying with P we get for t the 18 $\mathbb{Z}$-linear combinations of the two elements $(\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0, \frac{2}{3})$ and $(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{2})$.

We find $K^{\tilde{F}}$ by applying the Frobenius $\tilde{F}$ to the elements just found. This action is for any $t \in \tilde{\mathbf{T}}$ given by

$$\tilde{F}(t) = t\big(q\tilde{F}_0^{\mathrm{tr}}\big).$$

To be able to evaluate this on $t \in K$ we need to know the residue of q modulo all denominators of the coordinates of $t \in K$. A common denominator of all these entries is

$$\begin{aligned} m &:= \text{the largest elementary divisor of } M^{\mathrm{tr}} \\ &= \mathrm{lcm}\big(\text{entries of Smith normal form of } M^{\mathrm{tr}}\big) = 6. \end{aligned}$$

We still do not fix q, but the remaining computations are done for any congruence class c of a prime power modulo m separately, assuming that $q \equiv c \pmod{m}$. In our example we have to distinguish the cases of $q \equiv 1, 2, 3, 4, 5 \pmod 6$.

In cases $c = 2$ or 4 we have $p = 2$ (the prime dividing c and m), and in this case the kernel K only contains the 9 elements given above which are of order 1 or 3. Similarly, in case $c = 3$ we have $p = 3$ and K only contains the two elements of order 1 and 2. In the other cases K contains all 18 elements given above.

For the computation of $K^{\tilde{F}}$ we comment on the case $c = 2$. Multiplying the elements of K by $q\tilde{F}_0^{\mathrm{tr}}$ and using that $q \equiv 2 \pmod 6$ we find that only the three multiples of $(\frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0)$ are mapped to themselves.

We need to decide which of these $\tilde{F}$-fixed elements lie in $\tilde{\mathbf{G}}'$. This is easy to see, because the first l basis elements of $\tilde{X}$ and $\tilde{Y}$ correspond to the maximal torus of the semisimple factor and derived subgroup $\tilde{\mathbf{G}}'$. So, here the $\tilde{F}$-fixed elements of K all lie in $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ because their last coordinate is 0.

Considering also the other cases for c we find that $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ is cyclic of order 3 if $q \equiv 2$ or $5 \pmod 6$ and it is trivial in the other cases. We have found the group (B) for our parameterization of the irreducible representations of $\mathbf{G}^F$.

The elements of $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ are also needed to find our parameter set (A). This consists of all q-restricted weights $\lambda \in \tilde{X}^+$ of $\tilde{\mathbf{G}}'$ which are trivial on $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$. This means

$$t\lambda^{\mathrm{tr}} = 0 \in \mathbb{Q}_{p'}/\mathbb{Z} \quad \text{for all } t \in K^{\tilde{F}} \cap \tilde{\mathbf{G}}'.$$

These equations can be reformulated in terms of integers by multiplying with a common multiple m' of all denominators in $t \in K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ (a divisor of m). In our example we can multiply with $m' = 3$ and then consider the equations modulo m':

$$\big(m't\big)\lambda^{\mathrm{tr}} \equiv 0 \pmod{m'} \quad \text{for all } t \in K^{\tilde{F}} \cap \tilde{\mathbf{G}}'.$$

Writing all $(m't)$ for a set of generators t of $K^{\tilde{F}} \cap \tilde{\mathbf{G}}'$ in one matrix we can further simplify the system of equations by computing the Hermite normal form (mod m') of this matrix. In our example we get no non-trivial equation if $c \notin \{2, 5\}$. So, in these cases all q^l q-restricted weights λ lie in our parameter set (A). If $c = 2$ or 5, the set (A) contains only those q-restricted λ which fulfill the single equation

$$(1, 0, 2, 1, 2, 2, 1)\lambda^{\mathrm{tr}} = 0 \pmod 3.$$

Using this equation it is easy to check for a concrete q and q-restricted weight if it is in the parameter set (A).

For general q we can also count the number of parameters in the set (A). For this we use the following trivial lemma.

Lemma 3.8. *Let $q, c, i, m \in \mathbb{N}$ with $0 \leqslant i, c < m$ and $q \equiv c \pmod m$. Then the number of integers j with $0 \leqslant j < q-1$ and $j \equiv i \pmod m$ is $(q-c)/m$ if $i \geqslant c$ and $(q-c)/m+1$ for $i < c$.*

This lemma can be applied recursively to count the sets (A). For example in the case $q \equiv 2 \pmod 3$ above we need to count the $\lambda = (\lambda_1, \dots, \lambda_7) \in \mathbb{Z}^7$ with $0 \leqslant \lambda_i < q$ for $i = 1, \dots 7$ and $1 \cdot \lambda_1 + 0 \cdot \lambda_2 + \dots + 1 \cdot \lambda_7 \equiv 0 \pmod 3$.

From the lemma we can easily deduce how often each congruence class (mod 3) is hit by $1 \cdot \lambda_1$, $0 \cdot \lambda_2$, and so on. Combining this it is easy to count how often each congruence class (mod 3) is hit by $1 \cdot \lambda_1 + 0 \cdot \lambda_2$. In the next step we find the numbers for the expressions $1 \cdot \lambda_1 + 0 \cdot \lambda_2 + 2 \cdot \lambda_3$. Going on recursively, we find for each congruence class (mod 3) the number of q-restricted λ with $(1, 0, 2, 1, 2, 2, 1)\lambda^{\mathrm{tr}}$ in that class. In particular, we find for the 0-class the number of q-restricted weights in (A), it is $(q^7 + 2q)/3$ for $q = 2, 5 \pmod 6$.

Finally, we need the set (C), the structure of $(\mathbf{G}/\mathbf{G}')^F$. Using subsection 2.6 we can find the matrix of F_0 acting on the characters of this torus via the transformation of the matrix A to Smith normal form. In our example we find the 1×1 identity matrix. So the group of F-fixed points in this torus is cyclic of order $q-1$. In general the order of a finite torus is the characteristic polynomial of F_0 evaluated at q. The precise structure of the finite abelian group for a specific q is found by the Smith normal form of the characteristic matrix at q. See [6, Chapter 3] for more details.

To summarize: The parameter group (C) is for any q cyclic of order $q-1$. For $q \equiv 2 \pmod 3$ the parameter group (B) is of order 3 and the set (A) contains $(q^7 + 2q)/3$ weights. For $q \equiv 0, 1 \pmod 3$ the group (B) is trivial and the set (A) contains all q^7 q-restricted weights.

4. The case when G is simple

In this last section of the paper we want to apply our main Theorem 3.5 to all finite groups of Lie type arising from simple algebraic groups $\mathbf{G}$. As an application we determine the number of semisimple classes of these groups.

As before, we exclude here the Suzuki and Ree groups, in these cases the q^2, respectively q^4, irreducible representations were already described in Theorem 3.2(b).

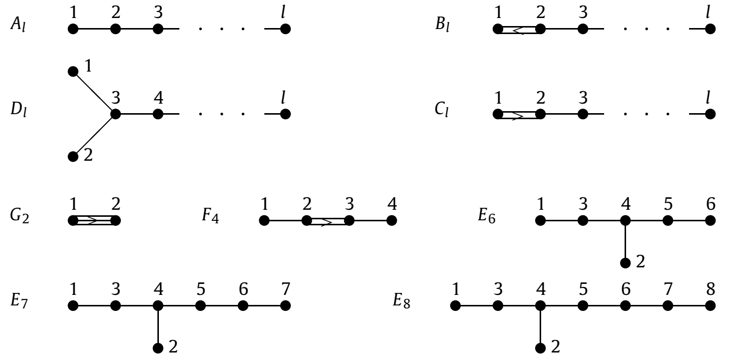
For each type of irreducible root system R, we choose a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_l\} \subseteq R$. We fix a numbering of the simple roots via the Dynkin diagrams given in Table 1. The node labeled by i corresponds to the simple root α_i of Δ. This is the labeling used in Chevie; see [10] (the often used Bourbaki labeling is different for types B, C, D, where it starts to count from the right side of the shown diagrams).

For a Frobenius morphism F of $\mathbf{G}$ we consider a root datum of $\mathbf{G}$ with respect to a maximally split maximal torus. Then F_0 permutes the set of simple roots and induces a graph automorphism of the Dynkin diagram. This graph automorphism can be non-trivial in cases A_l with $l \geqslant 2$, D_l with $l \geqslant 4$ and E_6. We also write F_ϵ instead of F in these cases with $\epsilon = 1$ in case of the trivial graph automorphism, $\epsilon = -1$ in case of the graph automorphism of order 2 (permuting nodes 1 and 2 in case D_l) and $\epsilon = 3$ in the case of a graph automorphism of order 3 of the Dynkin diagram of type D_4 (permuting the nodes with cycle $(1, 2, 4)$).

Let $\mathbf{G}_{sc}$ be the simply-connected simple group of the same type as $\mathbf{G}$. As explained in Proposition 2.5 we have an isogeny $\mathbf{G}_{sc} \to \mathbf{G}$ with a central kernel K and $\mathbf{G}_{sc}$ has a Frobenius morphism that induces F on $\mathbf{G}$, we denote that also by F or F_ϵ.

As in the proof of Proposition 2.5 we choose as root datum matrices for $\mathbf{G}_{sc}$ the pair $(C^{\mathrm{tr}}, \mathrm{Id})$, where C is the Cartan matrix corresponding to the chosen numbering of Δ. This means that in the root datum $(\tilde{X}, \tilde{R}, \tilde{Y}, \tilde{R}^\vee)$ of $\mathbf{G}_{sc}$ we use the simple coroots as basis of $\tilde{Y}$ and the fundamental weights as basis of $\tilde{X}$. The matrix for F_0 is the permutation matrix for the graph automorphism induced by F. As before, we identify a maximal torus of $\mathbf{G}_{sc}$ with $\tilde{Y} \otimes_{\mathbb{Z}} (\mathbb{Q}_{p'}/\mathbb{Z}) \cong (\mathbb{Q}_{p'}/\mathbb{Z})^l$.

Table 1
Dynkin diagram of irreducible root systems.



4.1. A parameterization of the irreducible representations in defining characteristic

Let $\mathbf{G}$, $\mathbf{G}_{sc}$ and K be as above. We want to give a parameterization of the irreducible defining characteristic representations of $\mathbf{G}^F$ by describing the parameter sets (A), (B) and (C) of Theorem 3.5.

Since $\mathbf{G}$ is semisimple we have $\mathbf{G} = \mathbf{G}'$ and so the group (C) is trivial in all cases considered here.

The parameter sets (A) and (B) only depend on K^F, the F-fixed elements of the kernel of the isogeny $\mathbf{G}_{sc} \to \mathbf{G}$.

We now consider the possibilities for $\mathbf{G}$, K and K^F for the various types of root systems separately.

We make use of the result in [16, §6.2] which describes explicitly the elements of the center Z of $\mathbf{G}_{sc}$ in all cases (as elements of $(\mathbb{Q}_{p'}/\mathbb{Z})^l$ as explained above).

For any positive integers l and q, we will write $\mathcal{E}_{l,q}$ for the set of tuples $(\lambda_1, \dots, \lambda_l) \in \mathbb{Z}^l$ such that $0 \leqslant \lambda_i < q$ for all $1 \leqslant i \leqslant l$.

4.1.1. Type A_l

The group Z is cyclic of order $m = (l+1)_{p'}$, generated by

$$z = \left(\frac{1}{m}, \frac{2}{m}, \dots, \frac{l}{m}\right) \in (\mathbb{Q}_{p'}/\mathbb{Z})^l.$$

For each divisor e of $l+1$ there is an algebraic group $\mathbf{G}$ such that the index of $\mathbb{Z}R \leqslant X$ is e, we denote its type by $(A_l)_e$. So, $e = l+1$ yields the simply-connected groups, isomorphic to $\mathrm{SL}_{l+1}(\bar{k})$, and $e = 1$ yields the adjoint groups, isomorphic to $\mathrm{PGL}_{l+1}(\bar{k})$.

Assume that $\mathbf{G}$ is of type $(A_l)_e$.

Then K is the subgroup of Z of order $((l+1)/e)_{p'} = m/e_{p'}$ (the group generated by $e_{p'}z$).

For the Frobenius morphism F_ϵ on $\mathbf{G}_{sc}$ and $i \in \mathbb{Z}$ we have $F_\epsilon(iz) = iz$ if and only if $(q-\epsilon)i \in m\mathbb{Z}$ if and only if $(m/\gcd(m, q-\epsilon)) \mid i$.

Combining, we find that the group K^{F_ϵ} is the subgroup of Z of order

$$d := \gcd\bigl(m/e_{p'}, \gcd(m, q-\epsilon)\bigr) = \gcd(m/e_{p'}, q-\epsilon) = \gcd\bigl((l+1)/e, q-\epsilon\bigr)$$

(the last equation because $q-\epsilon$ is prime to p).

We have found that the parameter group (B) is cyclic of order d. The set (A) consists of the q-restricted weights which are trivial on the generator $(m/d)z$ of K^{F_ϵ}. These are the $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\sum_{i=1}^{l} i\lambda_i \equiv 0 \mod d. \tag{1}$$

4.1.2. Types B_l and C_l

In these two cases, the center Z of $\mathbf{G}_{sc}$ has order $m = \gcd(2, p+1)$ and has the following generators:

Type	Generator
B_l	$(\frac{1}{m}, 0, \dots, 0)$
C_l	$(\frac{l}{m}, \frac{l-1}{m}, \dots, \frac{2}{m}, 0, \frac{1}{m})$

There are two possibilities for $\mathbf{G}$, the simply-connected type where K and so K^F are trivial, and the adjoint type where $K = Z$ and clearly $K^F = K$ (since K is of order 1 or 2).

So, when $p = 2$ or $\mathbf{G}$ is simply-connected then the parameter group (B) is trivial and the parameter set (A) consists of all q-restricted weights. Otherwise, for odd q and $\mathbf{G}$ of adjoint type, the group (B) is of order 2 and the parameter set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ satisfying the following equation. In the case of type B_l, the equation is

$$\lambda_1 \equiv 0 \mod 2, \tag{2}$$

and in case of type C_l it depends on the parity of l. This is

$$\sum_{1 \leqslant i \leqslant l,\ i \text{ even}} \lambda_i \equiv 0 \mod 2 \quad \text{or} \quad \sum_{1 \leqslant i \leqslant l,\ i \text{ odd}} \lambda_i \equiv 0 \mod 2, \tag{3}$$

according to l being odd or even.

4.1.3. Type D_l, $l \geqslant 4$

Assume that $l = 2k+1$ is odd. Then the center Z is cyclic of order $m = 4$ for odd p and $m = 1$ for $p = 2$, and is generated by

$$z = \left(\frac{1}{m}, \frac{3}{m}, \frac{2}{m}, 0, \frac{2}{m}, 0, \frac{2}{m}, \dots, 0, \frac{2}{m}\right).$$

There are three possibilities for $\mathbf{G}$, the simply-connected type for which K is trivial, or $\mathbf{G}$ is isomorphic to $\mathrm{SO}_{2l}(\bar{k})$ for which $\mathbb{Z}R$ is of index 2 in X and K is generated by $2z$, or the group of adjoint type where $K = Z$.

So, if $p = 2$ or if $\mathbf{G}$ is simply-connected then the parameter group (B) is trivial and (A) consists of all q-restricted weights.

If p is odd and $\mathbf{G}$ of type SO, then $K^{F_\epsilon} = K$ (since $2z$ is the only element of order 2 in Z), so the group (B) is of order 2. In this case, $\mathbf{G}^F = \mathrm{SO}_{2l}^{\epsilon}(q)$, the parameter set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\lambda_1 + \lambda_2 \equiv 0 \mod 2. \tag{4}$$

Let p be odd and $\mathbf{G}$ be of adjoint type, then $K = Z$. If $q \equiv \epsilon \mod 4$ then $K^{F_\epsilon} = K$, so the parameter group (B) is cyclic of order 4 and the parameter set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\sum_{i=1}^{k} 2\lambda_{2i+1} \equiv \lambda_2 - \lambda_1 \mod 4. \tag{5}$$

Otherwise, if $q \equiv -\epsilon \mod 4$ then K^{F_ϵ} is of order 2 and the parameter sets (B) and (A) are the same as in the SO-case.

4.1.4. Type D_l, $l \geqslant 4$

Assume now that $l = 2k$ is even. Then Z is elementary abelian of order 4 if p is odd and trivial if $p = 2$. If p is odd then Z is generated by

$$z_1 = \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \dots, 0, \frac{1}{2}\right) \quad \text{and} \quad z_2 = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \dots, 0, \frac{1}{2}\right),$$

if $p = 2$ we set $z_1 = z_2 = 1$.

Here, for any q, F_1 is the identity on Z, F_{-1} permutes z_1 and z_2, and in case $l = 4$ the Frobenius F_3 permutes z_1, z_2 and $z_1 + z_2$ cyclically.

If $\mathbf{G}$ is simply-connected or $p = 2$, then $K = K^F = 1$, the parameter group (B) is trivial and the set (A) consists of all q-restricted weights.

If the index of $\mathbb{Z}R$ in X is 2, there are two possibilities for $\mathbf{G}$. Either $\mathbf{G}$ has only Frobenius morphisms of type F_1, then $\mathbf{G}$ is isomorphic to a half spin group $\mathrm{HSpin}_{2l}(\bar{k})$ and $K = K^F$ is generated by z_1 (or by z_2). In this case, for odd p, the parameter group (B) is of order 2 and the set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\lambda_2 + \lambda_4 + \dots + \lambda_{2k} \equiv 0 \mod 2. \tag{6}$$

Otherwise, K is generated by $z_1 + z_2$ and $K = K^{F_\epsilon}$ for $\epsilon \in \{\pm 1\}$. Then $\mathbf{G}$ is isomorphic to a special orthogonal group $\mathrm{SO}_{2l}(\bar{k})$ and $\mathbf{G}^{F_\epsilon}$ is isomorphic to $\mathrm{SO}^{\epsilon}_{2l}(q)$. For odd p the parameter group (B) is also of order 2 and the set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\lambda_1 + \lambda_2 \equiv 0 \mod 2. \tag{7}$$

The final possibility is that $\mathbf{G}$ is of adjoint type and $K = Z$. Then $K^{F_{-1}}$ is generated by $z_1 + z_2$ and for odd p and $\epsilon = -1$ we get the same parameter sets (B) and (A) as in the SO-case. Furthermore, we have $K^{F_1} = K$, so for odd p and $\epsilon = 1$ the parameter group (B) is elementary abelian of order 4 and the parameter set (A) consists of the weights $(\lambda_1, \dots, \lambda_l) \in \mathcal{E}_{l,q}$ such that

$$\begin{cases} \displaystyle\sum_{i=2}^{k} \lambda_{2i} \equiv \lambda_1 \mod 2, \\ \lambda_1 = \lambda_2 \mod 2. \end{cases} \tag{8}$$

If $l = 4$ then $K^{F_3} = 1$ and we get the same parameterization as in the simply-connected case for $F = F_3$.

4.1.5. Types G_2, F_4 and E_8

In these cases Z and so $K = K^F$ and the parameter group (B) are trivial. The set (A) consists of all q-restricted weights.

4.1.6. Type E_6

The group Z is cyclic of order $m = 3$ if $p \neq 3$ and $m = 1$ if $p = 3$, it is generated by $z = (\frac{1}{m}, 0, \frac{2}{m}, 0, \frac{1}{m}, \frac{2}{m})$. The group $\mathbf{G}$ can either be simply-connected or of adjoint type.

If $\mathbf{G}$ is simply-connected or $p = 3$ then $K = K^{F_\epsilon} = 1$, the parameter group (B) is trivial and (A) consists of all q-restricted weights.

If $\mathbf{G}$ is of adjoint type then $K = Z$. We have $K^{F_\epsilon} = K$ if $q \equiv \epsilon \mod 3$. In that case the parameter group (B) is cyclic of order 3 and the parameter set (A) consists of the weights $(\lambda_1, \dots, \lambda_6) \in \mathcal{E}_{6,q}$ such that

$$\lambda_1 - \lambda_3 + \lambda_5 - \lambda_6 \equiv 0 \mod 3. \tag{9}$$

For $q \equiv -\epsilon \mod 3$ we have $K^{F_\epsilon} = 1$ and the parameter sets are as in the simply-connected case.

4.1.7. Type E_7

The group Z is cyclic of order $m = 2$ if $p \neq 2$ and $m = 1$ if $p = 2$, it is generated by $z = (0, \frac{1}{m}, 0, 0, \frac{1}{m}, 0, \frac{1}{m})$. The group $\mathbf{G}$ is either simply-connected or of adjoint type.

If $\mathbf{G}$ is simply-connected or if $p = 2$ then $K = K^F$ is trivial, the parameter group (B) is trivial and the set (A) consists of all q-restricted weights.

If $\mathbf{G}$ is of adjoint type then $K = K^F = Z$. For odd p the parameter group (B) is of order 2 and the set (A) consists of the weights $(\lambda_1, \ldots, \lambda_7) \in \mathcal{E}_{7,q}$ such that

$$\lambda_2 + \lambda_5 + \lambda_7 \equiv 0 \mod 2. \tag{10}$$

4.2. Application: number of semisimple classes

In this section, we will compute the number of isomorphism classes of irreducible $\bar{\mathbb{F}}_p$-modules (or, equivalently, the number of semisimple classes) of the finite groups $\mathbf{G}^F$ for all simple algebraic groups $\mathbf{G}$ defined over $\mathbb{F}_q$.

Theorem 4.1.

(a) *Let $\mathbf{G}$ be a connected reductive group of semisimple rank l, such that its derived group $\mathbf{G}'$ is simply-connected. Let $Z(\mathbf{G})^0$ be the connected component of the center of $\mathbf{G}$. We assume that $\mathbf{G}$ is defined over $\mathbb{F}_q$ and denote $F: \mathbf{G} \to \mathbf{G}$ the corresponding Frobenius morphism. Then the number of semisimple conjugacy classes of $\mathbf{G}^F$ is $q^l|(Z(\mathbf{G})^0)^F|$. In particular, if $\mathbf{G}$ is semisimple of simply-connected type this number is q^l.*

(b) *Now let $\mathbf{G}$ be a simple connected reductive group of rank l, defined over $\mathbb{F}_q$ with corresponding Frobenius morphism F. Then the number of semisimple conjugacy classes of $\mathbf{G}^F$ is either q^l, or it is given in Table 2.*

Proof. (a) This follows from Theorem 3.5. Under the given assumptions the parameter set (B) is trivial, and the set (A) contains all q-restricted weights. The parameter set (C) contains $|(\mathbf{G}/\mathbf{G}')^F| = |(Z(\mathbf{G})^0)^F|$ elements. See also [6, 3.7.6(ii)] for a completely different proof of this result.

(b) This will be shown in the rest of this section. Here the parameter group (C) is always trivial. We need to go through all the cases of subsection 4.1. Whenever the group (B) is trivial, the set (A) consists of the q^l elements in $\mathcal{E}_{l,q}$. In Table 2 we collect the cases with non-trivial (B) and find the cardinalities of the sets (A) by counting the solutions of certain modular equations. □

We denote by Λ the set of parameters (A) for the group $\mathbf{G}^F = (\mathbf{G}_{sc}/K)^F$. By Theorem 3.5, the number of isomorphism classes of irreducible $\bar{\mathbb{F}}_p$-modules of $\mathbf{G}^F$ is $|K^F| \cdot |\Lambda|$.

The following lemma will be useful in several cases.

Lemma 4.2. *Assume that q is odd, and for any positive integers n and $\nu \in \{0, 1\}$, define*

$$E_{n,\nu} = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \,\middle|\, 0 \leqslant \lambda_i \leqslant q-1,\ \sum_{i=1}^{n} \lambda_i \equiv \nu \bmod 2 \right\}.$$

Then, we have

$$|E_{n,\nu}| = \frac{q^n + 1 - 2\nu}{2}.$$

Proof. This follows easily by induction on n. □

Table 2
Number of semisimple classes (φ is the Euler φ-function).

Type	K^F	F	$\mathbf{G}$	Condition	|Semisimple classes|
A_l	$\mathbb{Z}_d$	F_ϵ	$(A_l)_e$	$d=\gcd(\frac{l+1}{e}, q-\epsilon)$	$\sum_{d'\mid d}\varphi(d')q^{(l+1)/d'-1}$
B_l	$\mathbb{Z}_2$		adjoint	$p\neq 2$	q^l+q^{l-1}
C_l	$\mathbb{Z}_2$		adjoint	$p\neq 2$	$q^l+q^{\lfloor l/2\rfloor}$
D_l, l even	$\mathbb{Z}_2^2$	F_1	adjoint	$p\neq 2$	$q^l+q^{l-2}+2q^{l/2}$
	$\mathbb{Z}_2$	F_{-1}	adjoint	$p\neq 2$	q^l+q^{l-2}
	$\mathbb{Z}_2$	F_ϵ	SO	$p\neq 2$	q^l+q^{l-2}
	$\mathbb{Z}_2$	F_1	HSpin	$p\neq 2$	$q^l+q^{l/2}$
D_l, l odd	$\mathbb{Z}_4$	F_ϵ	adjoint	$q\equiv\epsilon \bmod 4$	$q^l+q^{l-2}+2q^{(l-3)/2}$
	$\mathbb{Z}_2$	F_ϵ	adjoint	$q\equiv-\epsilon \bmod 4$	q^l+q^{l-2}
	$\mathbb{Z}_2$	F_ϵ	SO	$p\neq 2$	q^l+q^{l-2}
E_6	$\mathbb{Z}_3$	F_ϵ	adjoint	$q\equiv\epsilon \bmod 3$	q^6+2q^2
E_7	$\mathbb{Z}_2$		adjoint	$p\neq 2$	q^7+q^4

Types B_l and C_l. We must consider the case that $p\neq 2$ and $K=Z$. Then $\mathbf{G}$ is of adjoint type. If $\mathbf{G}$ is of type B_l we use Eq. (2) and obtain

$$\Lambda=\{(\lambda_1,\dots,\lambda_l)\in\mathcal{E}_{l,q} \mid \lambda_l\in 2\mathbb{Z}\}.$$

Thus, Lemma 4.2 gives $|\Lambda|=q^{l-1}\cdot\frac{q+1}{2}$. Now, since $|K^F|=2$, the entry for case B_l in Table 2 follows.

If $\mathbf{G}$ is of type C_l with $l=2k$ (resp. $l=2k+1$), then in Eq. (3) there are k summands (resp. $k+1$ summands) in the sum. Hence, Lemma 4.2 gives

$$|\Lambda|=q^k\cdot\frac{q^k+1}{2}\quad\left(\text{resp. } q^k\cdot\frac{q^{k+1}+1}{2}\right).$$

Since $k=\lfloor l/2\rfloor$ and $|K^F|=2$, the entry for type C_l in Table 2 follows.

Type D_l. We only need to consider the case $p\neq 2$. First assume that $l=2k$. We compute the number of elements in the set (A) for $\mathbf{G}_{\mathrm{ad}}^{F_1}$ using Eq. (8) as follows. If λ_1 is odd, then λ_2 is odd. This implies that $\lambda_4+\lambda_6+\dots+\lambda_{2k}\in 2\mathbb{Z}+1$. By Lemma 4.2, there are $q^{k-1}(\frac{q-1}{2})^2\cdot\frac{q^{k-1}-1}{2}$ such solutions. Similarly, there are $q^{k-1}(\frac{q+1}{2})^2\cdot\frac{q^{k-1}+1}{2}$ solutions such that λ_1 is even. Therefore, we deduce that

$$|\Lambda|=q^{k-1}\cdot\frac{q^{k+1}+2q+q^{k-1}}{4}.$$

In the same way, using Lemma 4.2, we count the number of solutions of Eqs. (7) and (6), giving $|\Lambda|$ for $\mathrm{SO}_{2l}^\epsilon(q)$ and $\mathrm{HSpin}_{2l}(q)$, respectively.

Suppose now that $l=2k+1$. Assume that $K=Z$ and that $q\equiv\epsilon \bmod 4$. By Eq. (5) we have to find the number of solutions $(\lambda_1,\dots,\lambda_{2k})\in\mathcal{E}_{2k,q}$ of $2(x_3+x_5+\dots+x_{2k+1})+x_1-x_2\in 4\mathbb{Z}$. There are $q^{k-1}\cdot\frac{q^k+1}{2}$ tuples $(\lambda_1,\dots,\lambda_{2k-1})$ with $0\leqslant\lambda_i\leqslant q-1$, such that $(\lambda_3+\lambda_5+\dots+\lambda_{2k+1})$ is even. For each tuple, we have to find the number of solutions of $\lambda_1-\lambda_2\in 4\mathbb{Z}$. There are $(\frac{q+3}{4})^2+3\cdot(\frac{q-1}{4})^2$ such solutions. Thus, there are $n_0=\frac{1}{8}\cdot q^{k-1}\cdot(q^{k+2}+3q^k+q^2+3)$ solutions $(\lambda_1,\dots,\lambda_{2k+1})\in\mathcal{E}_{2k+1,q}$, such that $\lambda_3+\lambda_5+\dots+\lambda_{2k+1}$ is even. Similarly, there are $n_1=\frac{1}{8}\cdot q^{k-1}\cdot(q^{k+2}-q^k-q^2+1)$ solutions $(\lambda_1,\dots,\lambda_{2k+1})\in\mathcal{E}_{2k+1,q}$, such that $\lambda_1+\lambda_3+\dots+\lambda_{2k-1}$ is odd. Adding up we find for the case $|K^{F_\epsilon}|=4$ that

$$|\Lambda|=n_0+n_1=\frac{q^{k-1}}{4}(q^{k+2}+q^k+2).$$

Finally, again with Lemma 4.2, we count the solutions of Eq. (4) and obtain $|\Lambda|$ for the cases $\mathrm{SO}_{2l}^{\epsilon}(q)$, and for the cases with $\mathbf{G}$ of adjoint type and $q \equiv -\epsilon \bmod 4$.

Types E_6 and E_7. We only need to consider $\mathbf{G}$ of adjoint type. In type E_6 with $q \equiv \epsilon \bmod 3$ we compute $|\Lambda|$ using Eq. (9) and Lemma 3.8. For type E_7 and odd p we conclude as above using Eq. (10) and Lemma 4.2.

Type A_l. We have postponed this case because it is a bit trickier to derive a closed formula for the cardinality $|\Lambda|$. We need to count the solutions of Eq. (1) to find the first line of Table 2. Instead of counting the solutions of Eq. (1) we can introduce another coordinate λ_0, count the solutions of the equation

$$\sum_{i=0}^{l} i\lambda_i \equiv 0 \mod d$$

with $0 \leqslant \lambda_i < q$ for $0 \leqslant i \leqslant l$, and divide the result by q.

The number of solutions of this modified equation follows from the following lemma applied with $n = l+1$ and $m = d$.

Lemma 4.3. *Let $n \geqslant 2$ be an integer, and $m \mid n$. Let $\epsilon \in \{-1, 1\}$. For any positive integer q with $m \mid (q-\epsilon)$ define $E = \{0, \dots, q-1\}$ and write $X = \{(\lambda_0, \lambda_1, \dots, \lambda_{n-1}) \in E^n \mid \sum_{i=0}^{n-1} ix_i \equiv 0 \bmod m\}$. Then*

$$|X| = \frac{1}{m} \sum_{d|m} \varphi(d) q^{n/d}.$$

Proof. The following proof was shown to us by Darij Grinberg, who generously allowed us to include it in this article.

For a non-negative integer t let

$$X_t = \left\{ (\lambda_0, \dots, \lambda_{n-1}) \in E^n \;\middle|\; \sum_{i=0}^{n-1} i\lambda_i = t \right\}.$$

We want to investigate $|X| = \sum_{t \in m\mathbb{Z}} |X_t|$.

For integers j with $0 \leqslant j \leqslant n-1$ we consider the polynomials

$$P_j(z) = 1 + z^j + z^{2j} + \dots + z^{(q-1)j} \in \mathbb{C}[z].$$

It is clear that $|X_t|$ is the coefficient of z^t in the product of the $P_j(z)$:

$$P(z) = \prod_{j=0}^{n-1} P_j(z) = \sum_{t=0}^{\infty} |X_t| z^t.$$

Now let $\zeta \in \mathbb{C}$ be a primitive m-th root of unity. We will use repeatedly the fact that for $t \in \mathbb{Z}$ the sum $\sum_{k=0}^{m-1} \zeta^{tk}$ equals m if $m \mid t$ and equals 0 otherwise (use the formula for geometric sums).

We evaluate

$$\sum_{k=0}^{m-1} P\left(\zeta^k\right) = \sum_{k=0}^{m-1} \sum_{t=0}^{\infty} |X_t| \zeta^{kt} = \sum_{t=0}^{\infty} |X_t| \sum_{k=0}^{m-1} \left(\zeta^t\right)^k = \sum_{\substack{t=0 \\ m|t}}^{\infty} |X_t| \cdot m = m|X|.$$

From now we fix a $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant m-1$ and set $d = \frac{m}{\gcd(m,k)}$.

We will show that

$$P\big(\zeta^k\big) = q^{n/d}.$$

This proves the lemma, because for $d \mid m$ we have

$$\big|\big\{0 \leqslant k < m \;\big|\; d = m/\gcd(m,k)\big\}\big| = \left|\frac{m}{d} \cdot \big\{0 \leqslant i < d \;\big|\; \gcd(i,d) = 1\big\}\right| = \varphi(d).$$

It is easy to evaluate each $P_j(z)$ at ζ^k for $0 \leqslant j \leqslant n-1$:

$$P_j\big(\zeta^k\big) = 1 + \zeta^{jk} + \big(\zeta^{jk}\big)^2 + \cdots + \big(\zeta^{jk}\big)^{q-1} = \begin{cases} q, & \text{if } m \mid jk \\ 1, & \text{if } m \nmid jk \text{ and } \epsilon = 1 \\ -\zeta^{-kj}, & \text{if } m \nmid jk \text{ and } \epsilon = -1 \end{cases}$$

because for $m \nmid jk$ every m consecutive summands sum up to 0. In case $m \mid (q-1)$ it remains the last summand which is 1. And in case $m \mid (q+1)$ an additional summand $(\zeta^{jk})^q = \zeta^{-jk}$ cancels all the previous ones.

For an integer k, $d = m/\gcd(m,k)$ and $j \in \mathbb{Z}$ we have that $m \mid jk$ if and only if $d \mid j$.

Since $m \mid n$ we also have $d \mid n$ and so there are n/d indices j with $0 \leqslant j \leqslant n-1$ and $P_j(\zeta^k) = q$.

In case $m \mid (q-1)$ we have $P_j(\zeta^k) = 1$ for the remaining j with $d \nmid j$. Taking the product we get

$$P\big(\zeta^k\big) = q^{n/d}.$$

To see that the same is true in case $m \mid (q+1)$ we must show that

$$\prod_{\substack{j=0 \\ d \nmid j}}^{n-1} \big(-\zeta^{-jk}\big) = 1.$$

The root of unity ζ^k and so also ζ^{-k} has order d $(= m/\gcd(m,k))$. So, if $d \mid j$ we have $(\zeta^{-k})^j = 1$ and we get

$$\prod_{\substack{j=0 \\ d \nmid j}}^{n-1} \big(-\zeta^{-jk}\big) = (-1)^{n-n/d} \cdot \prod_{j=0}^{n-1} \big(\zeta^{-k}\big)^j = (-1)^{n-n/d} \cdot \big(\zeta^{-k}\big)^{n(n-1)/2}.$$

Since $d \mid n$, we have: $d \nmid n(n-1)/2$ iff (d is even and (n/d) is odd) iff $(n/d)(d-1) = n - n/d$ is odd, and in this case $(d/2) \mid n(n-1)/2$ so that $(\zeta^{-k})^{n(n-1)/2} = -1$. This shows that the right hand side in the last displayed equation is always 1.

This proves the lemma. $\square$

Remark 4.4. The results given in Table 2 are not new. They were worked out in [3] using sophisticated results from the ordinary representation theory of the groups $\mathbf{G}^F$ in good characteristic. The completely different approach in this section is more elementary (and it works for arbitrary root data and characteristics).

For small rank groups, in particular the exceptional types, detailed parameterizations of all conjugacy classes were computed, this also yields the number of semisimple conjugacy classes, see [15].

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