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Large and symmetric: The Khukhro–Makarenko theorem on laws — without laws

Anton A. Klyachko ^{*,1}, Maria V. Milentyeva

*Faculty of Mechanics and Mathematics, Moscow State University, Leninskie gory,
Moscow 119991, Russian Federation*

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ABSTRACT

We prove a generalisation of the Khukhro–Makarenko theorem on large characteristic subgroups with laws. This general fact implies new results on groups, algebras, and even graphs and other structures. Concerning groups, we obtain, e.g., a fact in a sense dual to the Khukhro–Makarenko theorem. A graph-theoretic corollary is an analogue of this theorem in which planarity plays the role of a multilinear identity. Also, we answer a question of Makarenko and Shumyatsky.

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* Corresponding author.

E-mail addresses: klyachko@mech.math.msu.su (A.A. Klyachko), mariamil@yandex.ru (M.V. Milentyeva).

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0. Introduction

The following short theorem generalised and strengthened various results scattered through the literature (see, e.g., [1,7], and Section 21.1.4 of [2]).

Khukhro–Makarenko theorem. (See [10].) *If a group has a finite-index subgroup satisfying an outer commutator identity, then this group also has a characteristic finite-index subgroup satisfying the same identity.*

An *outer* (or *multilinear*) *commutator identity* is an identity of the form $[\dots[x_1, \dots, x_t] \dots] = 1$ with some meaningful arrangement of brackets, where all letters $x_1, \dots, x_t$ are different. Examples of such identities are solvability, nilpotency, centre-by-metabelianity, etc. A formal definition looks as follows. Let $F(x_1, x_2, \dots)$ be a free group of countable rank. An *outer commutator of weight 1* is just a letter x_i . An *outer commutator of weight $t > 1$* is a word of the form $w(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)]$, where u and v are outer commutators of weights r and $t - r$, respectively. An *outer commutator identity* is an identity of the form $w = 1$, where w is an outer commutator.

The Khukhro–Makarenko theorem has various applications (see, e.g., [6,11,13] and the references therein). Paper [3] contains a significantly simpler proof (than the original one) and a better estimate of the index of the characteristic subgroup. At the same time, in [11] and [12], there were established some results (about groups and algebras) similar to but not following from the Khukhro–Makarenko theorem.

Paper [13] was an attempt to clean up the mess; it contains a very general proposition about groups with operators (in the sense of [9], see also [4]) including as special cases all known and several new results of this type. However, later, several facts that do not fit into the general scheme of [13] were discovered. Some of these facts are rather delicate [14], and others are quite simple (an impatient reader may glance into the last section). Generally, a Khukhro–Makarenko-like theorem looks as follows.

A theorem. *If somewhere there is something (in the classical case, a subgroup in a group) large (of finite index) and good (satisfying a multilinear identity), then there is also something large, good, and symmetric (automorphism-invariant).*

In this paper, we make another try to catch everything. In Section 1, we prove the Main theorem containing as special cases all known and several new results similar to above (on groups, algebras, graphs, and other objects). The main idea is considering “multilinear properties” instead of multilinear identities.

One of the new corollaries of the Main theorem (Section 2) shows that, in the Khukhro–Makarenko theorem, an identity, i.e. the triviality of a verbal subgroup, can be replaced by something similar to the triviality. For example, it is true that a group containing a finite-index subgroup whose 2014th derived subgroup is amenable (or periodic, or locally finite, etc.) contains a characteristic finite-index subgroup with the same property.

Another corollary (Section 3) can be considered as a result dual to the Khukhro–Makarenko theorem. It shows, e.g., that, *for any finite normal subgroup N of any group G of bounded exponent, there exists a characteristic finite subgroup $H \triangleleft G$ such that the spectrum (i.e. the set of orders of all elements) of the quotient group G/H is contained in the spectrum of G/N .*

Section 4 contains similar results for algebras over fields.

Another proposition from Section 2 gives a positive answer to a question of Makarenko and Shumyatsky and strengthens the main theorem of [14].

In Section 5, we show that some properties of graphs behave similarly to multilinear commutator identities from the Khukhro–Makarenko theorem. For example, it is true that, *if some graph can be made planar by removing a finite number of edges, then this finite set of edges can be chosen invariant with respect to all automorphisms of the initial graph.*

As a reward to readers who reached the last section, we present two elementary-school problems on the subject.

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1. The main theorem

Recall that a *semilattice* is a partially ordered set $\mathcal{L}$ in which any finite subset $\mathcal{N} \subseteq \mathcal{L}$ has a least upper bound $\sup \mathcal{N} \in \mathcal{L}$. A *directed semilattice* is a semilattice which is a downward directed partially ordered set; this means that, for any finite set $\mathcal{N} \subseteq \mathcal{L}$, there exists an element $\inf \mathcal{N} \in \mathcal{L}$ such that $\inf \mathcal{N} \leq N$ for any $N \in \mathcal{N}$. Note that, in our notation, $\sup$ is the least upper bound and $\inf$ is *some* lower bound; hopefully, this will not lead to a mess (see also the remark below, after the definition of codimension). A semilattice $\mathcal{L}$ is called *Noetherian* if all increasing chains in $\mathcal{L}$ terminate, i.e. there are no infinite chains of the form $N_1 < N_2 < \dots$, where $N_i \in \mathcal{L}$. A semilattice $\mathcal{L}$ is called a *lattice* if each finite subset $\mathcal{N} \subseteq \mathcal{L}$ has a greatest lower bound (which will be denoted $\inf \mathcal{N}$).

We call a t -ary property (predicate) $\mathcal{P}$ on a semilattice $\mathcal{L}$ *(multi)monotone* if the property $\mathcal{P}(N_1, \dots, N_t)$, where $N_i \in \mathcal{L}$, implies $\mathcal{P}(N'_1, \dots, N'_t)$ for any $N'_i \leq N_i$. We call the property $\mathcal{P}$ *multilinear* if, for any i , $\mathcal{P}(N_1, \dots, N_{i-1}, N'_i, N_{i+1}, \dots, N_t)$ and $\mathcal{P}(N_1, \dots, N_{i-1}, N''_i, N_{i+1}, \dots, N_t)$ imply $\mathcal{P}(N_1, \dots, N_{i-1}, \sup(N'_i, N''_i), N_{i+1}, \dots, N_t)$.

We need also dual notions. A predicate $\mathcal{P}$ on a semilattice $\mathcal{L}$ is called *comonotone* if $\mathcal{P}(N_1, \dots, N_t)$, where $N_i \in \mathcal{L}$, implies $\mathcal{P}(N'_1, \dots, N'_t)$ for any $N'_i \geq N_i$. We call a predicate $\mathcal{P}$ *comultilinear* if, for any i , $\mathcal{P}(N_1, \dots, N_{i-1}, N'_i, N_{i+1}, \dots, N_t)$ and $\mathcal{P}(N_1, \dots, N_{i-1}, N''_i, N_{i+1}, \dots, N_t)$ imply $\mathcal{P}(N_1, \dots, N_{i-1}, \inf(N'_i, N''_i), N_{i+1}, \dots, N_t)$ for some lower bound $\inf(N'_i, N''_i)$ (we shall use the word *colinear* if $t = 1$).

We say that an endomorphism semigroup $\Phi \subseteq \text{End } \mathcal{L}$ of a semilattice $\mathcal{L}$ *preserves a property $\mathcal{P}$* (or the property $\mathcal{P}$ is Φ -invariant) if $\mathcal{P}(N_1, \dots, N_t)$ implies $\mathcal{P}(\varphi(N_1), \dots, \varphi(N_t))$ for any $N_i \in \mathcal{L}$ and $\varphi \in \Phi$. For example, the property $\mathcal{R}(X, Y, Z) = (X = \sup(Y, Z))$ is $(\text{End } \mathcal{L})$ -invariant by the definition of *endomorphisms of a semilat-*

tice. An element N of a semilattice is called Φ -invariant if $\varphi(N) \leq N$ for all $\varphi \in \Phi$ (this, in particular, means that $\varphi(N) = N$ for all $\varphi \in \Phi$ if the semigroup Φ is a subgroup of $\text{Aut } \mathcal{L}$).

The following assertion is a natural generalisation of the lemma from [3] (which is about the lattice of normal subgroups), see also Lemma 1 from [13] (which is about the lattice of normal subgroups in a multi-operator group).

Lemma 1. *Suppose that $\mathcal{M}$ is a directed semilattice with a largest element $\sup \mathcal{M}$, $\mathcal{P}$ is a monotone multilinear t -ary predicate on $\mathcal{M}$, m is a positive integer, and $\mathcal{N} \subseteq \mathcal{M}$ is a finite subset of $\mathcal{M}$ such that*

$$\mathcal{P}(\underbrace{N, N, \dots, N}_{m \text{ times}}, \sup \mathcal{M}, \sup \mathcal{M}, \dots, \sup \mathcal{M}) \quad \text{is true for all } N \in \mathcal{N}.$$

Then

$$\mathcal{P}(\underbrace{\widehat{N}, \widehat{N}, \dots, \widehat{N}}_{m-1 \text{ times}}, \widehat{G}, \widehat{G}, \dots, \widehat{G}), \quad \text{where } \widehat{N} = \inf \mathcal{N} \text{ and } \widehat{G} = \sup \mathcal{N}, \text{ is true too.}$$

Proof. Since $\mathcal{P}$ is monotone, $\widehat{G} \leq \sup \mathcal{M}$, and $\widehat{N} \leq N$ for all $N \in \mathcal{N}$, we have

$$\mathcal{P}(\underbrace{\widehat{N}, \widehat{N}, \dots, \widehat{N}}_{m-1 \text{ times}}, N, \widehat{G}, \dots, \widehat{G}) \quad \text{is true for all } N \in \mathcal{N}.$$

Now the multilinearity (to be more precise, the linearity with respect to the m th argument) implies the assertion of Lemma 1. $\square$

Let $\mathcal{L}$ be a Noetherian directed semilattice and let $\Phi \subseteq \text{End } \mathcal{L}$ be a semigroup of endomorphisms of $\mathcal{L}$. A function $\text{codim} : \mathcal{L} \rightarrow \mathbb{R}$ is called a (generalised) Φ -codimension if it has the following properties:

- 1) $\text{codim } N_1 \leq \text{codim } N_2$ if $N_1 \geq N_2$;
- 2) $\text{codim } \varphi(N) \leq \text{codim } N$ for any $N \in \mathcal{L}$ and $\varphi \in \Phi$;
- 3) $\text{codim } \inf(N_1, N_2) \leq \text{codim } N_1 + \text{codim } N_2$ for any $N_1, N_2 \in \mathcal{L}$ and some lower bound $\inf(N_1, N_2)$;
- 4) in any family $\mathcal{N} \subseteq \mathcal{L}$, there exists $r \leq \max_{N \in \mathcal{N}} \text{codim } N + 1$ elements $N_1, \dots, N_r$ such that $\sup \mathcal{N} = \sup(N_1, \dots, N_r)$.

This definition of codimension is a natural generalisation of the corresponding notion from [13] (which is about the lattice of normal subgroups of a multi-operator group). When we are talking about a codimension on a semilattice we suppose that the symbol $\inf$ always denotes a lower bound satisfying condition 3).

Main theorem. Suppose that $\mathcal{L}$ is a Noetherian directed semilattice, $\Phi \subseteq \text{End } \mathcal{L}$ is a semi-group of its endomorphisms, and $\mathcal{P}$ is a multi-monotone multilinear t -ary Φ -invariant predicate on $\mathcal{L}$. Then, if there exists an element $N \in \mathcal{L}$ with the property $\mathcal{P}(N, \dots, N)$, then there exists an element $H \in \mathcal{L}$ such that

- 1) H has the same property: $\mathcal{P}(H, \dots, H)$;
- 2) H is Φ -invariant;
- 3) if $\varphi(N) \leq J$ for any $\varphi \in \Phi$ and some $J \in \mathcal{L}$, then $H \leq J$;
- 4) if $\mathcal{L}$ is a lattice (i.e. each finite set has a greatest lower bound) and Φ consists of lattice endomorphisms (i.e. the mappings commuting with the taking the greatest lower bounds of finite sets), then H is contained in the sublattice generated by the set $\{\varphi(N); \varphi \in \Phi\}$;
- 5) if $\text{codim} : \mathcal{L} \rightarrow \mathbb{R}$ is a generalised Φ -codimension, then $\text{codim } H \leq f^{t-1}(\text{codim } N)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$.

Proof. Since $\mathcal{L}$ is Noetherian, it contains an element $G_1 = \sup_{\varphi \in \Phi} \varphi(N)$ and

$$G_1 = \sup(\varphi'_0(N), \dots, \varphi'_{p_1}(N)) \quad \text{for some endomorphisms } \varphi'_i \in \Phi.$$

Note that G_1 is Φ -invariant: $\varphi(G_1) = \sup(\varphi\varphi'_0(N), \dots, \varphi\varphi'_{p_1}(N)) \leq \sup_{\varphi \in \Phi} \varphi(N) = G_1$. Besides, $G_1 \leq J$ (if J is such as in assertion 3) of the theorem) and, for any codimension function on $\mathcal{L}$, we have

$$p_1 \leq l_0 \stackrel{\text{def}}{=} \text{codim } N \quad \text{by property 4) of codimension.}$$

Put $N_1 = \inf(\varphi'_0(N), \dots, \varphi'_{p_1}(N))$. By properties 2) and 3) of codim , we have

$$l_1 \stackrel{\text{def}}{=} \text{codim } N_1 \leq (p_1 + 1)\text{codim } N = (p_1 + 1)l_0 \leq (l_0 + 1)l_0 = f(l_0).$$

According to [Lemma 1](#) (applied to the semilattice $\mathcal{M} = \mathcal{L}$) we have the property

$$\mathcal{P}(N_1, \dots, N_1, G_1).$$

Similarly, we can choose elements

$$\begin{aligned} G_2 &= \sup_{\varphi \in \Phi} \varphi(N_1) = \sup(\varphi''_0(N_1), \dots, \varphi''_{p_2}(N_1)) \quad \text{and} \\ N_2 &= \inf(\varphi''_0(N_1), \dots, \varphi''_{p_2}(N_1)), \quad \text{where } \varphi''_i \in \Phi. \end{aligned}$$

Clearly,

$$G_2 \leq G_1 \leq J \quad (\text{because } N_1 \leq \varphi'_0(N) \text{ and, hence,})$$

$$G_2 = \sup_{\varphi \in \Phi} \varphi(N_1) \leq \sup_{\varphi \in \Phi} \varphi \varphi'_0(N) \leq \sup_{\varphi \in \Phi} \varphi(N) = G_1 \quad \text{and} \\ p_2 \leq \text{codim } N_1 = l_1 \leq f(l_0) \quad (\text{by properties 2) and 4) of codim}).$$

The element G_2 is obviously Φ -invariant (for the same reasons as G_1) and we have the estimates

$$\text{codim } G_2 \leq \text{codim } \varphi''_0 N_1 \leq \text{codim } N_1 = l_1 \leq f(l_0) \quad \text{and} \\ l_2 \stackrel{\text{def}}{=} \text{codim } N_2 \leq (p_2 + 1) \text{codim } N_1 = (p_2 + 1) l_1 \leq f(l_1) \leq f(f(l_0)).$$

Again by [Lemma 1](#) (applied to the semilattice $\mathcal{M} = \{X \in \mathcal{L} \mid X \leq G_1\}$), we obtain

$$\mathcal{P}(N_2, \dots, N_2, G_2, G_2).$$

Continuing in the same manner, at the t th step, we obtain a Φ -invariant element

$$G_t = \sup_{\varphi \in \Phi} \varphi(N_{t-1}) = \sup(\varphi_0^{(t)}(N_{t-1}), \dots, \varphi_{p_t}^{(t)}(N_{t-1})) \\ \text{and an element } N_t = \inf(\varphi_0^{(t)}(N_{t-1}), \dots, \varphi_{p_t}^{(t)}(N_{t-1})),$$

where $\varphi_i^{(t)} \in \Phi$. We have the required property $\mathcal{P}(G_t, \dots, G_t)$ and the inequalities

$$G_t \leq J, \quad \text{codim } G_t \leq \text{codim } N_{t-1} = l_{t-1} \leq f(l_{t-2}) \leq f(f(l_{t-3})) \leq \dots \leq f^{t-1}(l_0).$$

Thus, the element $H = G_t$ is as required and the theorem is proven (Assertion 4 is obviously satisfied by the construction). $\square$

The following lemma makes it possible to construct new multilinear properties from known ones.

Composition lemma. *Suppose that, on a lattice, there is a multilinear monotone predicate $Q(M_1, \dots, M_k)$ and a k -tuple of predicates $\mathcal{R} = \{\mathcal{R}_i(X_1, \dots, X_i)\}$ which are multilinear and monotone with respect to the first row (i.e. for any given second row) and colinear and comonotone with respect to the second row (i.e. for any given first row). Then the predicate*

$$\mathcal{Q} \circ \mathcal{R}(N_1, \dots, N_{kl}) \\ = \left(\exists M_1, \dots, M_k \quad Q(M_1, \dots, M_k) \text{ and, for } i \in \{1, \dots, k\}, \mathcal{R}_i \left(\begin{matrix} N_{(i-1)l+1}, \dots, N_{li} \\ M_i \end{matrix} \right) \right)$$

called the composition of the predicates Q and $\mathcal{R}$ is multilinear and monotone.

Proof. Let us verify that the composition is monotone, e.g., with respect to the first argument. If $N'_1 \leq N_1$ and the property $\mathcal{Q} \circ \mathcal{R}(N_1, \dots, N_{kl})$ holds (for some $M_1, \dots, M_k$ from the definition of composition), then $\mathcal{Q} \circ \mathcal{R}(N'_1, \dots, N_{kl})$ holds also (with the very same M_i) because $\mathcal{R}_1$ is monotone with respect to the first row.

Let us verify the multilinearity of $\mathcal{Q} \circ \mathcal{R}$, e.g., with respect to the first argument. Suppose that the properties $\mathcal{Q} \circ \mathcal{R}(N'_1, N_2, \dots, N_{kl})$ and $\mathcal{Q} \circ \mathcal{R}(N''_1, N_2, \dots, N_{kl})$ hold, i.e. we have

$$\begin{aligned} &\mathcal{R}_1 \left(\begin{array}{c} N'_1, N_2, \dots, N_l \\ M'_1 \end{array} \right), \quad \mathcal{R}_2 \left(\begin{array}{c} N_{l+1}, \dots, N_{2l} \\ M'_2 \end{array} \right), \dots, \quad \mathcal{Q}(M'_1, \dots, M'_k), \\ &\mathcal{R}_1 \left(\begin{array}{c} N''_1, N_2, \dots, N_l \\ M''_1 \end{array} \right), \quad \mathcal{R}_2 \left(\begin{array}{c} N_{l+1}, \dots, N_{2l} \\ M''_2 \end{array} \right), \dots, \quad \text{and} \quad \mathcal{Q}(M''_1, \dots, M''_k) \end{aligned}$$

for some $M'_1, \dots, M'_k, M''_1, \dots, M''_k$. We claim that then we have the properties

$$\begin{aligned} &\mathcal{R}_1 \left(\begin{array}{c} \sup(N'_1, N''_1), N_2, \dots, N_l \\ \sup(M'_1, M''_1) \end{array} \right), \quad \mathcal{R}_2 \left(\begin{array}{c} N_{l+1}, \dots, N_{2l} \\ \inf(M'_2, M''_2) \end{array} \right), \dots, \quad \text{and} \\ &\mathcal{Q}(\sup(M'_1, M''_1), \inf(M'_2, M''_2), \dots, \inf(M'_k, M''_k)) \end{aligned}$$

(i.e. the property $\mathcal{Q} \circ \mathcal{R}(\sup(N'_1, N''_1), N_2, \dots, N_k)$ holds with $\sup(M'_1, M''_1), \inf(M'_2, M''_2), \dots, \inf(M'_k, M''_k)$ in the roles of $M_1, \dots, M_k$, respectively). Indeed,

- the first property ($\mathcal{R}_1(\dots)$) holds because $\mathcal{R}_1$ is linear with respect to the first element of the first row and comonotone with respect to the second row;
- the second property ($\mathcal{R}_2(\dots)$) holds by virtue of the colinearity of $\mathcal{R}_2$ with respect to the second row;
- ...
- the last property ($\mathcal{Q}(\dots)$) holds because $\mathcal{Q}$ is linear with respect to the first argument and monotone with respect to the other arguments.

The lemma is proven. $\square$

The rest of the paper deals with applications of the Main theorem to groups, algebras, graphs, and other structures. In all applications, the semigroup Φ is taken to be a natural subgroup of $\text{Aut } \mathcal{L}$.

2. The lattice of large normal subgroups

Recall that an abstract class of groups $\mathcal{K}$ is called *radical* (or *Fitting*) if it is closed with respect to normal subgroups and finite products of normal subgroups, i.e.

- 1) any normal subgroup of a group from $\mathcal{K}$ lies in $\mathcal{K}$;
- 2) a group decomposable into a product of two normal subgroups lying in $\mathcal{K}$ belongs to $\mathcal{K}$.

A *coradical class* (or a *formation*) is an abstract class of groups $\mathcal{K}$ closed with respect to homomorphic images and subdirect products, i.e. such that:

- 1') any quotient group of a group from $\mathcal{K}$ lies in $\mathcal{K}$;
- 2') any subdirect product of two groups lying in $\mathcal{K}$ also lies in $\mathcal{K}$.

More details about radical and coradical classes can be found, e.g., in book [5].

The following classes of groups are *radical formations*, i.e. they are both radical and coradical:

- finite groups;
- finite p -groups;
- locally finite groups (radicality follows from a theorem of O.Yu. Schmidt: *an extension of a locally finite group by a locally finite is locally finite itself*, see [2]);
- periodic groups;
- Noetherian groups;
- Artinian groups;
- nilpotent groups (radicality follows from the Fitting theorem, see [2]);
- locally nilpotent groups (radicality follows from Plotkin's theorem, see [2]);
- solvable groups;
- almost solvable groups;
- locally polycyclic groups (radicality follows from Theorem 18.1.2 of [2]);
- groups satisfying nontrivial identities;
- groups without nonabelian free subgroups;
- amenable (discrete) groups;
- ...

Other examples of coradical classes are all varieties of groups, the class of all binary finite groups, groups with the maximality (or minimality) conditions for normal subgroups, etc.

Large-subgroup theorem. *Let N be a normal subgroup of a group G such that the quotient group G/N satisfies the maximality condition for normal subgroups. Then G contains characteristic subgroups $H_1, H_2, \dots$ such that*

- 1) *the quotient groups G/H_t lie in the coradical class (formation) $\mathcal{F}$ generated by G/N ; moreover, the subgroups H_t belong to the lattice (of subgroups of G) generated by the images of N under all automorphisms of G ;*
- 2) *for any multilinear commutator word w of degree $k \leq t$, the group $w(H_t, \dots, H_t)$ is contained in the radical class $\mathcal{R}_w$ generated by the group $w(N, \dots, N)$;*

- 3) if codim is a generalised codimension on the lattice of subgroups such that the corresponding quotients lie in $\mathcal{F}$, then $\text{codim } H_t \leq f^{t-1}(\text{codim } N)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$.

Proof. Let $K_1, \dots, K_t$ be normal subgroups of G and let $w(x_1, \dots, x_t)$ be an outer commutator. Then

- a) the subgroup $w(K_1, \dots, K_t) \stackrel{\text{def}}{=} \langle w(h_1, \dots, h_t) : h_i \in K_i \rangle$ is normal in G ;
- b) $w(K_1, \dots, K_t) = [u(K_1, \dots, K_r), v(K_{r+1}, \dots, K_t)]$ if $w(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)]$;
- c) $w(K_1, \dots, K_{i-1}, \prod_{N \in \mathcal{N}} N, K_{i+1}, \dots, K_t) = \prod_{N \in \mathcal{N}} w(K_1, \dots, K_{i-1}, N, K_{i+1}, \dots, K_t)$ for any family $\mathcal{N}$ of normal subgroups of G .

These facts are well known and easy to prove by induction.

Let us apply the Main theorem to the lattice $\mathcal{L}$ (of normal subgroups of G) generated by the images of N under all automorphisms of G . (This lattice is Noetherian and even the entire formation $\mathcal{F}$ consists of groups which are Noetherian with respect to normal subgroups.) Put $\Phi = \text{Aut } G$, and let $\mathcal{P}(N_1, \dots, N_t)$ be the following property:

for each multilinear commutator word w of degree at most t , the group $w(N_1, \dots, N_k)$ belongs to $\mathcal{R}_w$.

The predicate $\mathcal{P}$ is monotone because radical classes are closed with respect to normal subgroups; the multilinearity follows from the closedness of radical classes with respect to products of normal subgroups and property c) of outer commutators. The theorem is proven. $\square$

The Large-subgroup theorem generalises the Khukhro–Makarenko theorem in three directions:

- the finiteness of the quotient groups G/N and G/H is replaced by their belonging to any given formation with maximality condition for normal subgroups and the index (to be more precise, the logarithm of index) is replaced by an abstract codimension; this generalisation is not new; it was obtained in [13] for the first time;
- the identity (i.e. the triviality of a verbal subgroup) is replaced by the belonging of this verbal subgroup to any radical class; e.g., our theorem shows that *a group containing a finite-index subgroup whose 100th derived subgroup is periodic contains a characteristic finite-index subgroup with the same property*;
- finally, instead of one multilinear word w , we consider all multilinear words at once; this gives a substantial gain in estimation if, e.g., we want to construct a characteristic finite-index subgroup satisfying *all* multilinear identities of degree at most 100 satisfied by a given finite-index subgroup (of course, such a characteristic subgroup

can be constructed by iterated applications of the Khukhro–Makarenko theorem but this results in a very bad estimate of index).

The following simple facts make it possible to apply the Composition lemma, generalise the Khukhro–Makarenko theorem in yet another direction, and answer a question of Makarenko and Shumyatsky.

Quotient-group lemma. *The following two properties $\mathcal{A}(N, M)$ and $\mathcal{B}(N, M)$ of pairs of normal subgroups of a group G can be written in the form $\mathcal{R} \left(\begin{smallmatrix} N_1, \dots, N_t \\ M \end{smallmatrix} \right)$, where the predicate $\mathcal{R}$ on the lattice of normal subgroups is monotone and multilinear with respect to the first row and comonotone and colinear with respect to the second row:*

$$\begin{aligned} \mathcal{A}(N, M) &= (N/(N \cap M) \text{ satisfies a (given) outer commutator identity } w = 1), \\ \mathcal{B}(N, M) &= (N/(N \cap M) \text{ belongs to a (given) radical formations } \mathcal{F}). \end{aligned}$$

Proof. For the property $\mathcal{A}$, the predicate

$$\mathcal{R} \left(\begin{smallmatrix} N_1, \dots, N_t \\ M \end{smallmatrix} \right) = (w(N_1, \dots, N_t) \subseteq M)$$

obviously satisfies all conditions. As for $\mathcal{B}$, this property itself can play the role of $\mathcal{R}$:

$$\mathcal{R} \left(\begin{smallmatrix} N_1 \\ M \end{smallmatrix} \right) = \mathcal{B}(N_1, M).$$

This predicate is linear and monotone with respect to the first row because of the radicality of $\mathcal{F}$; colinearity and comonotonicity with respect to the second row follow from the coradicality of $\mathcal{F}$. $\square$

Extension lemma. *Suppose that $\mathcal{Q}(M_1, \dots, M_l)$ is a monotone multilinear predicate on the lattice of normal subgroups of a group G , w is an outer commutator word of degree d , and $\mathcal{F}$ is a radical formation. Then the following two properties $\mathcal{C}(N)$ and $\mathcal{D}(N)$ of normal subgroups of G can be written in the form $\mathcal{P}(N_1, \dots, N_t)$, where the predicate $\mathcal{P}(N_1, \dots, N_t)$ on the lattice of normal subgroups is monotone and multilinear, $t = ld$ for the property $\mathcal{C}$, and $t = l$ for the property $\mathcal{D}$:*

$$\begin{aligned} \mathcal{C}(N) &= (\exists M \triangleleft G \ M \subseteq N, \ \mathcal{Q}(M, \dots, M), \text{ and } N/M \text{ satisfies the identity } w = 1), \\ \mathcal{D}(N) &= (\exists M \triangleleft G \ M \subseteq N, \ \mathcal{Q}(M, \dots, M), \text{ and } N/M \in \mathcal{F}). \end{aligned}$$

Proof. Clearly, the property $\mathcal{C}(N)$ can be rewritten in the form

$$\begin{aligned} \mathcal{C}(N) &= (\exists M \triangleleft G \ \mathcal{A}(N, M) \text{ and } \mathcal{Q}(M, \dots, M)), \\ &\text{where } \mathcal{A} \text{ is from the Quotient-group lemma.} \end{aligned}$$

Note that the last formula is equivalent to

$$\mathcal{C}(N) = (\exists M_1, \dots, M_l \triangleleft G \mathcal{A}(N, M_1), \dots, \mathcal{A}(N, M_l), \text{ and } \mathcal{Q}(M_1, \dots, M_l)).$$

To verify this (in the non-obvious direction), it suffice to put $M = \bigcap M_i$ and recall that the variety of groups satisfying the identity $w = 1$ is closed with respect to subdirect products and the property $\mathcal{Q}$ is monotone. By the quotient group lemma, we can rewrite $\mathcal{C}$ in the form

$$\mathcal{C}(N) = \left(\exists M_1, \dots, M_l \triangleleft G \mathcal{R} \left(\begin{smallmatrix} N, \dots, N \\ M_1 \end{smallmatrix} \right), \dots, \mathcal{R} \left(\begin{smallmatrix} N, \dots, N \\ M_l \end{smallmatrix} \right), \text{ and } \mathcal{Q}(M_1, \dots, M_l) \right),$$

where the predicate $\mathcal{R}$ on the lattice of normal subgroups is monotone and multilinear with respect to the first row, and comonotone and colinear with respect to the second row.

Application of the Composition lemma completes the proof. Similar arguments prove the assertion about $\mathcal{D}$. $\square$

The following theorem was proven in [14] for the case, where the group G is locally finite and each class $\mathcal{K}_i$ is the class of all locally nilpotent groups.

Series theorem. *Suppose that a group G contains a finite-index subgroup N having normal (in G) series*

$$\{1\} = A_0 \subseteq \dots \subseteq A_n = N$$

such that each quotient A_i/A_{i-1} either satisfies a multilinear commutator identity $w_i = 1$ of weight t_i or lies in a radical class $\mathcal{K}_i$; moreover, all these classes, except possibly $\mathcal{K}_1$, are also coradical. Then G contains a characteristic subgroup H with the same property (i.e. with a series of the same length with quotients satisfying the same identities or lying in the same classes) and $\log_2 |G : H| \leq f^{t-1}(\log_2 |G : N|)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$ and $t = \prod t_i$.

Proof. The Extension lemma and an obvious induction show that the presence of such a normal series in a normal subgroup N can be written in the form $\mathcal{P}(N, \dots, N)$, where $\mathcal{P}$ is a multilinear monotone t -variable predicate on the lattice of normal subgroups of G . It remains to apply the Main theorem. $\square$

Note that, in the case where the group G is locally finite and each class $\mathcal{K}_i$ is the class of all locally nilpotent groups, in [14] a stronger proposition was proven: the group G has a characteristic finite-index subgroup having a *characteristic* series with the required property. In the general case, such a strengthening is impossible as the following example due to Yves Cornulier shows.

Example. (See [8].) There exists a group with a normal abelian countable-index subgroup but without characteristic abelian countable-index subgroups.

Take a countable-dimensional vector space V with a basis $\{e_q\}$, where $q \in \mathbb{Q}$, over a finite field K and consider the group G of “unitriangular” operators, i.e. such operators g in V that $ge_q - e_q \in \langle \{e_r : r < q\} \rangle$. Now, consider the subgroup $H \subset G$ consisting of matrices A with the following property: for any real r , there is only finitely many nonzero entries a_{pq} such that $p \neq q$ and either $p > r$ or $q < r$. This group H has an abelian normal countable-index subgroup consisting of matrices A whose all nonzero nondiagonal entries a_{pq} are such that either $p < 0$ or $q > 0$. However, H has no nontrivial characteristic abelian subgroups. Indeed, if $1 \neq h \in N \triangleleft H$, then the commutator c of h and any transvection lies in N and has only finitely many nonzero nondiagonal entries; hence, c belongs to a finite-dimensional unitriangular group $\mathbf{UT}_n(K)$ which is nilpotent and, therefore, any its nontrivial normal subgroup nontrivially intersects the centre (which consists of transvections). Thus, any nontrivial normal subgroup of H contains a transvection. It remains to note that the automorphism group of H acts transitively on the transvections, i.e. any nontrivial characteristic subgroup of H must contain all transvections and, hence, cannot be abelian.

3. The lattice of small subgroups. A dual of the Khukhro–Makarenko theorem

Consider a universal positive closed first-order formula in the group language, e.g.,

$$(\forall x)(\forall y) \quad ((x^3 = y^3 \wedge (xy)^4 = (yx)^4) \vee (xy)^{2014} = 1 \vee [x, y]^5 = 1).$$

Such a formula defines a class of groups consisting of groups where the formula holds. For instance, the formula $(\forall x) (x^2 = 1 \vee x^3 = 1)$ holds in the symmetric group of order six, but does not hold in the abelian group of order six.

The following theorem can be considered as dual to the Khukhro–Makarenko theorem.

Finite-subgroup theorem. *If a group G has a finite normal subgroup such that, in the quotient group, a given universal positive closed first-order formula holds, then G has a characteristic finite subgroup with the same property.*

We shall prove a more general assertion similar to the Large-subgroup theorem. Consider the following property $\mathcal{D}(N)$ of a normal subgroup N of a group G :

$$(\forall x)(\forall y) \dots \left(\mathcal{S}(x, y, \dots) \implies \bigvee_{i=1}^t (G/N, xN, yN, \dots) \in \mathcal{F}_i \right), \quad (*)$$

where $\mathcal{S}$ is some $(\text{Aut } G)$ -invariant property of a tuple of elements of G and $\mathcal{F}_i$ are some formations of groups with marked elements.

A *formation of groups with marked elements* is a class of tuples $(H, h_1, h_2, \dots)$, where H is a group and $h_i \in H$, closed with respect to homomorphic images and subdirect products (which are defined naturally). For example, the class of amenable groups with two marked elements whose commutator lie in the centre is a formation.

Properties (of normal subgroups) of the form $(*)$ are called *t-disjunctive*.

Small-subgroup theorem. *If all groups from the radical class generated by a normal subgroup N of a group G satisfy the minimality conditions for normal subgroups, then G contains characteristic subgroups $H_1, H_2, \dots$ such that*

- 1) H_t lie in the radical class $\mathcal{R}$, generated by the group N ; moreover, H_t are contained in the lattice (of subgroups of G) generated by all automorphic images of N ;
- 2) the quotient group G/H_t satisfies all t -disjunctive properties satisfied by G/N ;
- 3) if, in addition, codim is a generalised codimension (which may be called a dimension in this case), on the lattice dual to the lattice of subgroups lying in $\mathcal{R}$, then $\text{codim } H_t \leq f^{t-1}(\text{codim } N)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$.

Proof. Each t -disjunctive property $\mathcal{D}(N)$ can be rewritten in the form $\mathcal{D}(N) = \mathcal{P}_{\mathcal{D}}(N, \dots, N)$, where $\mathcal{P}_{\mathcal{D}}(N_1, \dots, N_t)$ is the following property of a tuple of normal subgroups:

$$(\forall x)(\forall y) \dots \left(\mathcal{S}(x, y, \dots) \implies \bigvee_{i=1}^t (G/N_i, xN_i, yN_i, \dots) \in \mathcal{F}_i \right).$$

Now, it suffice to apply the Main theorem to the lattice $\mathcal{L}$ (of subgroups of G) generated by all images of N under automorphisms of G ; but the order on $\mathcal{L}$ is opposite to the natural one: $A \leq B$ if $A \supseteq B$. This lattice is Noetherian, because it consists of normal subgroups that are Artinian with respect to normal subgroups. The semigroup Φ is the automorphism group of G in this case and the predicate $\mathcal{P}$ is the conjunction of all predicates $\mathcal{P}_{\mathcal{D}}$, where $\mathcal{D}$ runs over all t -disjunctive properties satisfied by N .

The predicate $\mathcal{P}$ is monotone because formations are closed with respect to quotient groups, the multilinearity follows from the closedness of formations with respect to subdirect products. Let us verify the linearity, e.g., with respect to the first argument. We have to prove that the property $\mathcal{P}_{\mathcal{D}}(N'_1 \cap N''_1, N_2, \dots)$ holds whenever the properties $\mathcal{P}_{\mathcal{D}}(N'_1, N_2, \dots)$ and $\mathcal{P}_{\mathcal{D}}(N''_1, N_2, \dots)$ hold. Thus, we known that for any set of elements $g, h, \dots \in G$ with property $\mathcal{S}$, either

- $(G/N_i, gN_i, hN_i, \dots) \in \mathcal{F}_i$ for some $i \geq 2$
- or the formation $\mathcal{F}_1$ contains two groups with marked elements:
 $(G/N'_1, gN'_1, hN'_1, \dots)$ and $(G/N''_1, gN''_1, hN''_1, \dots)$
 and, hence, it contains their subdirect product $(G/(N'_1 \cap N''_1), g(N'_1 \cap N''_1), h(N'_1 \cap N''_1), \dots)$.

This means that the property $\mathcal{P}_{\mathcal{D}}(N'_1 \cap N''_1, N_2, \dots)$ holds and the theorem is proven. $\square$

The Finite-subgroup theorem is obtained from the Small-subgroup theorem by taking all formations $\mathcal{F}_i$ to be the class of all groups whose marked elements satisfy a system of equations (depending on i). The following is a special case of the Finite-subgroup theorem.

Spectrum theorem. *For any finite normal subgroup N of a group G of bounded exponent, there exists a characteristic finite subgroup H such that the spectrum (i.e. the set of orders of all elements) of G/H is contained in the spectrum of G/N .*

Proof. It suffices to apply the Finite-subgroup theorem to the formula $\forall x \bigvee_{i=1}^t (x^{n_i} = 1)$, where $\{n_1, \dots, n_t\}$ is the spectrum of G/N . $\square$

The order of the characteristic subgroup H can be explicitly estimated via the order of N and the cardinality of the spectrum of G/N (because logarithm of the order of a subgroup is a natural example of codimension on the lattice of finite normal subgroups). The word “finite” (the both occurrences) in the Spectrum theorem can be replaced by “Artinian” or, e.g., “Chernikov”, etc. The word “spectrum” (the both occurrences) can be replaced, e.g., by “spectrum of the commutator subgroup”; to show this we may add to the property $\mathcal{S}(x)$ the condition that x lies in the commutator subgroup.

In conclusion, we give an example showing that the Finite-subgroup theorem cannot be extended to arbitrary (non-universal) positive first-order formulae.

Example. The group $G = \langle a \rangle_2 \times B$, where B is an abelian divisible group with infinitely many elements of order two (e.g., $B = (\mathbb{Z}_{2^\infty})^\infty$), has an obvious finite normal subgroup $N = \langle a \rangle_2$ such that, in the quotient group, all elements are squares (i.e., the formula $\forall x \exists y \ x = y^2$ holds). But there is no characteristic finite subgroup with the same property. Indeed, such a characteristic subgroup H cannot lie in B , obviously. Take an element $(a, b) \in H$ and consider its images under automorphisms that fix elements of B and map a into (a, x) , where x is an element of order two from B . These images (a, bx) form an infinite subset of H .

4. The lattices of ideals and subspaces

The term *algebra* in this section means not necessarily associative algebra over a field. A *characteristic* subspace of an algebra is a subspace invariant with respect to all automorphisms of this algebra.

Large-subspace theorem. *Let N be a subspace of an algebra G such that either*

- N is of finite codimension,
- N is a left ideal and the G -module G/N is Noetherian,
- or N is a two-sided ideal and the quotient algebra from G/N satisfies the maximality condition for two-sided ideals.

Then G contains a characteristic subspaces $H_1, H_2, \dots$ such that

- 1) H_t belong to the lattice (of subspaces of G) generated by the images of N under all automorphisms of G ; in particular, the subspaces H_t are ideals (one-sided or two-sided) if N is an ideal, the quotient algebras (quotient modules) G/H_t lie in the formation $\mathcal{F}$, generated by the algebra (module) G/N ;
- 2) for any multilinear element $w(x_1, \dots, x_n)$ of the free (nonassociative) algebra of rank $n \leq t$, the set $w(H_t, \dots, H_t)$ is contained in the linear hull of a finite number of images of the set $w(N, \dots, N)$ under automorphisms of G ;
- 3) If, in addition, codim is either the usual codimension (of a subspace of G) or a generalised codimension on the lattice of ideals such that the corresponding quotient algebras (modules) lie in $\mathcal{F}$ then $\text{codim } H_t \leq f^{t-1}(\text{codim } N)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$.

The proof almost literally repeats the proof of the Large-subgroup theorem; only obvious replacements should be made (“group” should be replaced with “algebra” and so on). $\square$

Similarly, we can prove an analogue of the small-subgroups theorem but we restrict ourselves to an analogue of the Finite-subgroup theorem.

Finite-dimensional-ideal theorem. *If an algebra G has a finite-dimensional two-sided ideal such that the quotient algebra satisfies a given universal positive closed first-order formula (in the language of algebras over the given field), then G has a characteristic finite-dimensional two-sided ideal with the same property.*

5. The lattice of finite subgraphs

The word *graph* in this section can be understood in any reasonable sense: all propositions are valid for directed and undirected graphs; multiple edges and loops may be allowed or prohibited; the vertices and/or edges may be coloured. In the Forbidden-subgraph theorem and the Local-embeddability theorem, the word “graph” may be even understood as “hypergraph”. All these variations do not affect the proofs; of course, automorphisms of a graph should be understood in the corresponding sense.

Forbidden-subgraph theorem. *Let $\{\Gamma_1, \dots, \Gamma_l\}$ be a finite set of finite graphs called forbidden and considered up to isomorphism, and let G be some graph. If G contains a finite set $\overline{N}$ of edges such that $G \setminus \overline{N}$ does not contain forbidden subgraphs, then G contains a finite set of edges $\overline{H}$ which is invariant with respect to all automorphisms of G and has the same property: $G \setminus \overline{H}$ does not contain forbidden subgraphs. Moreover, $|\overline{H}| \leq f^{t-1}(|\overline{N}|)$, where $f^k(x)$ is the k th iteration of the function $f(x) = x(x+1)$, and t is the maximal (in i) number of edges of Γ_i . In addition, if $\overline{H} \neq \emptyset$, then $\overline{H} \cap \overline{N} \neq \emptyset$.*

Proof. It suffices to apply the Main theorem to the lattice consisting of cofinite subsets of the edge set of G . Clearly, this lattice is Noetherian and the function

$$\text{codim}(X) \stackrel{\text{def}}{=} \text{the number of edges of the graph } G \setminus X$$

satisfies all conditions from the definition of codimension. The semigroup Φ is taken to be the automorphism group of G and $\mathcal{P}(N_1, \dots, N_t)$ is the following property:

*G contains no forbidden subgraph whose first edge lies in N_1 ,
second edge lies in $N_2, \dots$*

We assume that the edges of each forbidden graph are enumerated somehow. Clearly, the property $\mathcal{P}$ is monotone:

$$\mathcal{P}(N_1, \dots, N_t) \implies \mathcal{P}(N'_1, \dots, N'_t), \quad \text{if } N'_i \subseteq N_i.$$

The multilinearity is also obvious:

$$(\mathcal{P}(N'_1, N_2, \dots, N_t) \wedge \mathcal{P}(N''_1, N_2, \dots, N_t)) \implies \mathcal{P}(N'_1 \cup N''_1, N_2, \dots, N_t).$$

It remains to apply the Main theorem and note that the property $\mathcal{P}(N, \dots, N)$ means precisely the absence of forbidden subgraphs in N .

Now, $\bar{H} = G \setminus H$ and $\bar{N} = G \setminus N$ intersect, because, according to the Main theorem, H is contained in the sublattice generated by all images of N under automorphisms of G . In particular, $H \supseteq \bigcap_{\varphi \in \text{Aut } G} \varphi(N)$, i.e. $\bar{H} \subseteq \bigcup_{\varphi \in \text{Aut } G} \varphi(\bar{N})$. Therefore, if $\bar{H} \neq \emptyset$, then $\bar{H} \cap \varphi(\bar{N}) \neq \emptyset$ for some $\varphi \in \text{Aut } G$. By virtue of the invariance of $\bar{H}$, this means that $\bar{H} \cap \bar{N} \neq \emptyset$ as required. The theorem is proven. $\square$

This theorem can be significantly strengthened if we do not care about the estimate. We say that a graph X *locally embeds* into a graph Y if any finite subgraph of X is isomorphic to some subgraph of Y .

Local-embeddability theorem. *For any graph G and any finite set $\bar{N}$ of its edges, there exists a finite set of edges $\bar{H}$ invariant with respect to all automorphisms of G and such that the graph $H = G \setminus \bar{H}$ locally embeds into the graph $N = G \setminus \bar{N}$.*

Proof. Let $\Gamma_1, \Gamma_2, \dots$ be all finite graphs not embeddable into N . We have to show that

*all subgraphs of G isomorphic to Γ_i can be destroyed by removing a finite
(Aut G)-invariant set of edges $\bar{H}$ provided we know that these subgraphs can be
destroyed by removing some finite set of edges $\bar{M}$.*

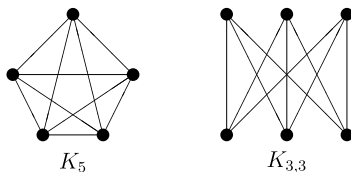


Fig. 1. Smallest nonplanar graphs.

To prove this assertion we use the induction on $|\overline{M}|$. Let us start with $\overline{M} = \overline{N}$. By the Forbidden-subgraph theorem, for each positive integer n , there exists a finite set $\overline{H}_n$ of edges such that

- 1) $\overline{H}_n$ is invariant with respect to all automorphisms of G ;
- 2) the graphs $\Gamma_1, \dots, \Gamma_n$ are not embeddable into $H_n = G \setminus \overline{H}_n$;
- 3) $\overline{H}_n \cap \overline{M} \neq \emptyset$ if $\overline{H}_n \neq \emptyset$.

If all sets $\overline{H}_n$ are empty, then we have nothing to prove. If there is a nonempty set $\overline{H}_k$, then we consider the graph $G' = H_k = G \setminus \overline{H}_k$. This graph contains a finite set $\overline{M}' = \overline{M} \setminus (\overline{M} \cap \overline{H}_k)$ of edges such that none of Γ_i embeds into $G' \setminus \overline{M}'$.

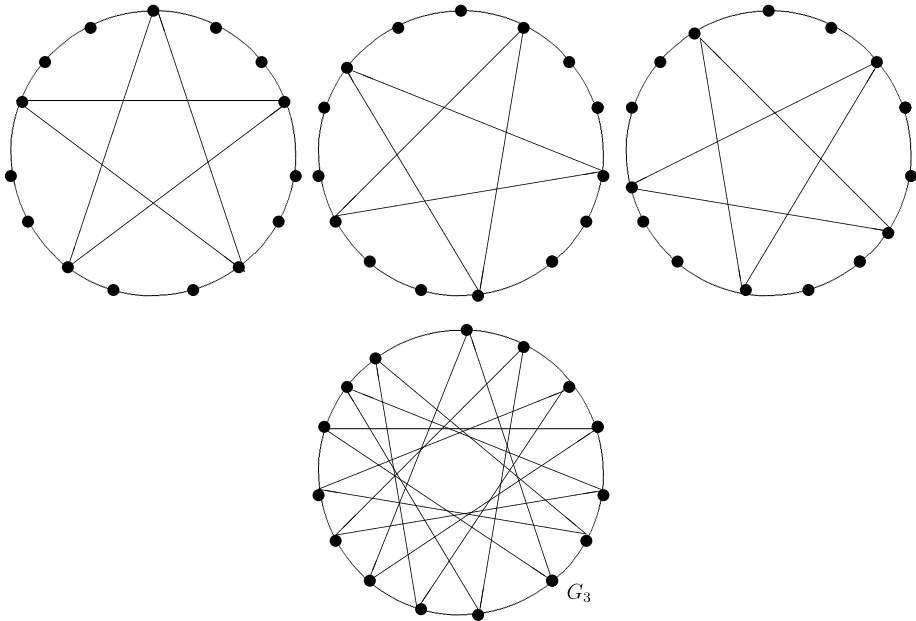
Moreover $|M'| < |M|$ by property 3) of $\overline{H}_k$. Therefore, by the induction hypothesis, G' contains a finite invariant set $\overline{H}'$ of edges such that none of Γ_i embeds into $G' \setminus \overline{H}' = G \setminus (\overline{H}_k \cup \overline{H}')$; this is what we want, because $\overline{H}_k \cup \overline{H}'$ is an invariant set. Indeed, $\overline{H}_k$ is invariant with respect to all automorphisms of G by definition; $\overline{H}'$ is invariant with respect to $\text{Aut } G'$, and, hence, with respect to $\text{Aut } G$, because G' is an $(\text{Aut } G)$ -invariant subgraph of G . This completes the proof. $\square$

Example. Consider the undirected graph G homoeomorphic to the straight line and the graph N obtained from G by removing one edge. Clearly, it is impossible to remove a finite automorphism-invariant set of edges from G in such a way that obtained graph H is embeddable in N (because the automorphism group of G acts transitively on edges). This example shows that we cannot replace local embeddability with embeddability in the theorem.

Planarity theorem. *If a graph can be made planar by removing a finite number of edges, then it can be made planar by removing a finite set of edges which is invariant with respect to all automorphisms of the graph.*

Proof. By the Kuratowski–Erdős–Wagner theorem [15] a graph G is planar if and only if

- the number of its edges is at most continuum;
- the number of its vertices of degree larger than two is countable (or finite);
- it does not contain subgraphs homoeomorphic to the complete graph on five vertices K_5 or the complete bipartite graph with three vertices in each fraction $K_{3,3}$ (Fig. 1); i.e. G does not contain subgraphs isomorphic to graphs obtained from K_5 or $K_{3,3}$ by subdivisions of edges.

Fig. 2. Construction of the graph G_3 .

The first two properties are not affected by removing or adding a finite number of edges; the third property is inherited by locally embeddable graphs. Therefore, the assertion follows immediately from the Local-embeddability theorem. $\square$

The following proposition shows that, in the Planarity theorem, no estimate of the cardinality of the invariant removed set of edges is possible.

Proposition. *For each positive integer n , there exists a finite graph G_n which becomes planar after removing five edges but cannot be made planar by removing an invariant set consisting of less than n edges.*

Proof. Let us take the graph K_5 and subdivide each edge of a length-five cycle onto n parts. Now, let us glue together n copies of the obtained graph along the cycle of length $5n$ with rotations (Fig. 2).

The obtained graph G_n becomes planar after removing five edges — each n th edge on the $5n$ -cycle (Fig. 3). However, removing small invariant set of edges cannot make this graph planar, because, among automorphisms of G_n , there is the rotation through one edge along the cycle of length $5n$ and, therefore, the orbit of each edge has at least n elements. Thus, any invariant set of edges containing less than n elements must be empty. It remains to note, that the graph G_n itself is not planar, because it contains a subgraph homoeomorphic to K_5 .

In conclusion, we note that, at least for countable graphs, there is an analogue of the Planarity theorem in which planarity is replaced with embeddability into any fixed

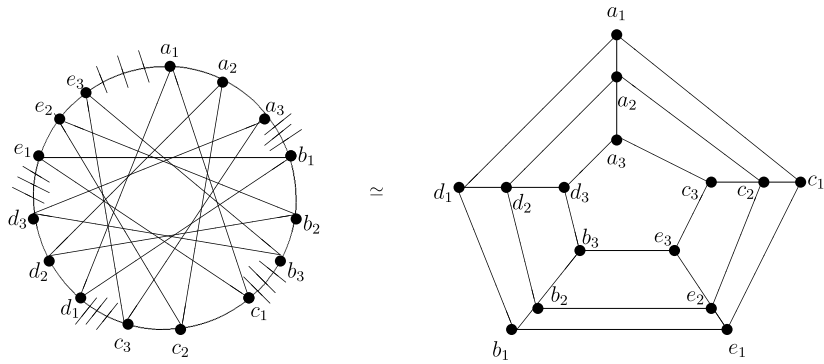


Fig. 3. G_3 without 5 edges is planar.

surface. To show this, it suffices to recall a theorem of Erdős which says that a countable graph embeds into a surface S if and only if each its finite subgraph embeds into S . $\square$

6. Elementary mathematics

Problem 1. In the three-dimensional Euclidean space, there is a set X . It is known that we can remove a finite set of points from X in such a way that no 2014 of the remaining points lie on the same sphere. Show that this finite set can be chosen invariant under all symmetries (= isometries) of X .

Solution. It suffices to apply the Main theorem to the lattice of cofinite subsets of X (this lattice is obviously Noetherian) and take Φ to be the symmetry group of X and $\mathcal{P}$ to be the following 2014-linear Φ -invariant monotone predicate:

$$\mathcal{P}(N_1, \dots, N_{2014}) = (\text{no points } x_1 \in N_1, \dots, x_{2014} \in N_{2014} \text{ lie on the same sphere}).$$

On the lattice, there is also a natural codimension: $\text{codim}(X \setminus K) \stackrel{\text{def}}{=} |K|$ that makes it possible to estimate the cardinality of the symmetric removed set via the cardinality of the initial (nonsymmetric) finite set.

Problem 2. There were chosen 10^{100} excellent candidates for a mission to Mars. The only problem is that they do not have enough respect for each other. The organisers noted that expelling some ten persons would result in an *efficient team* in the sense that, among any five of the remaining candidates, there is at least one respected by the majority (of this five). Show that a nonempty efficient team can be build *fairly*, i.e. in such a way that the expelled set is invariant under all permutations of candidates preserving the relation “respects”. (Certainly, the binary relation “respects” may be non-transitive, non-symmetric, and even non-reflexive.)

Solution. It suffices to apply the Main theorem to the lattice of all subsets of the set of candidates X putting Φ to be the group of permutations of X preserving the relation “respects” and $\mathcal{P}$ to be the following pentilinear Φ -invariant monotone predicate:

$$\mathcal{P}(N_1, \dots, N_5) = (\text{any candidates } x_1 \in N_1, \dots, x_5 \in N_5 \text{ form an efficient five}).$$

There is a natural codimension: $\text{codim}(X \setminus K) \stackrel{\text{def}}{=} |K|$, that makes it possible to estimate the number of fairly expelled candidates:

$$\begin{aligned} \text{codim } H &\leq f^{t-1}(\text{codim } N) = f^{5-1}(10) < \frac{11}{10} \cdot \left(\frac{11}{10} \cdot \left(\frac{11}{10} \cdot \left(\frac{11}{10} \cdot 10^2 \right)^2 \right)^2 \right)^2 \\ &= \left(\frac{11}{10} \right)^{15} \cdot 10^{16} = 11^{15} \cdot 10 \ll 10^{100}, \end{aligned}$$

i.e. the remaining efficient team is nonempty.

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