Journal of Algebra 473 (2017) 324–351

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Graded cellularity and the monotonicity conjecture

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A R T I C L E   I N F O

Article history:
Received 18 April 2015
Available online 10 November 2016
Communicated by Gus I. Lehrer

Keywords:
Kazhdan–Lusztig polynomials
Graded cellular algebras
Double leaves basis
Soergel bimodules

A B S T R A C T

The graded cellularity of Libedinsky double leaves, which form a basis for the endomorphism ring of the Bott–Samelson–Soergel bimodules, allows us to view the Kazhdan–Lusztig polynomials as graded decomposition numbers. Using this interpretation, we can provide a proof of the monotonicity conjecture for any Coxeter system.

1. Introduction

In their seminal paper [12], Kazhdan and Lusztig defined, for each Coxeter system (W, S), a family of polynomials with integer coefficients indexed by pairs of elements of W. These polynomials are now known as the Kazhdan–Lusztig (KL) polynomials. We will denote them by $P_{x,w}(q) \in \mathbb{Z}[q]$, for all $x, w \in W$. Applications of the KL-polynomials have been found in the representation theory of semisimple algebraic groups, the topology of Schubert varieties, the theory of Verma modules, the Bernstein–Gelfand–Gelfand (BGG) category $\mathcal{O}$, etc. (see, e.g., [2] and references therein).

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[1] This research was supported by a FONDECYT Postdoctoral grant no. 3140612 and by PAI-CONICYT Concurso nacional de inserción en la academia 2015, 79150016.

http://dx.doi.org/10.1016/j.jalgebra.2016.10.040
0021-8693/

Aside from the importance of the KL-polynomials in the above-mentioned subjects, there are purely combinatorial reasons to study these polynomials. Perhaps the major reason is the longstanding Kazhdan–Lusztig positivity conjecture [12], which states that $P_{x,w}(q) \in \mathbb{N}[q]$, for all Coxeter groups W and all $x, w \in W$. In 2014, Elias and Williamson [6] gave a proof of this conjecture by proving a stronger result known as Soergel's conjecture.

For each Coxeter group W, Soergel constructed a category of graded R-bimodules (where R is a polynomial ring with coefficients in $\mathbb{R}$) known as the category of Soergel bimodules, which we will denote by $\mathbb{S}\mathrm{Bim}$. He proved that (up to degree shift) W parameterizes the set of all indecomposable objects in $\mathbb{S}\mathrm{Bim}$. For $w \in W$, let us denote by B_w the corresponding indecomposable object. Soergel proved in [18] that $\mathbb{S}\mathrm{Bim}$ is a categorification of the Hecke algebra $\mathcal{H}$ of W. This means that there is an algebra isomorphism

$$\epsilon : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}], \tag{1.1}$$

where $[\mathbb{S}\mathrm{Bim}]$ denotes the split Grothendieck group of $\mathbb{S}\mathrm{Bim}$. Soergel proposed the following conjecture, which came to be known as Soergel's conjecture:

$$\epsilon(\underline{H}_w) = [B_w], \tag{1.2}$$

where $\{\underline{H}_w\}_{w\in W}$ is the Kazhdan–Lusztig basis of $\mathcal{H}$. Assuming this conjecture, Soergel showed that the Kazhdan–Lusztig polynomials of W arise as graded ranks of Hom spaces between indecomposable Soergel bimodules and standard bimodules. It follows that these coefficients are non-negative, i.e., he proved that (1.2) implies the positivity conjecture.

More than mere positivity is conceivable for coefficients of KL-polynomials. In effect, a monotonicity property is known for these coefficients, when W is a finite or affine Weyl group. Namely, for these groups it is true that if $u, v, w \in W$ and $u \leq v \leq w$, then

$$P_{u,w}(q) - P_{v,w}(q) \in \mathbb{N}[q], \tag{1.3}$$

where $\leq$ denotes the usual Bruhat order on W. In other words, if we fix the second index of a KL-polynomial, and if the first one decreases in Bruhat order, all coefficients in the polynomial weakly increase in value. This result was originally proved by Irving [10, Corollary 4] using the interpretation of KL-polynomials as multiplicities of simple objects in the socle filtration of a Verma module, when W is a finite Weyl group, and by Braden and MacPherson [3, Corollary 3.7] using the interpretation of KL-polynomials as Poincaré polynomials of the local intersection cohomology of Schubert varieties, when W is a finite or affine Weyl group.

It is natural to conjecture that (1.3) holds for arbitrary Coxeter groups. In the literature, the latter conjecture is referred to as the *Monotonicity Conjecture*[2] for KL-

[2] To the best of the author's knowledge nobody conjectured the Monotonicity Conjecture.

polynomials. Elias and Williamson's work not only solved Soergel's conjecture, but it provides a "geometric" setting for arbitrary Coxeter groups. In this context, in order to prove (1.3), it then seems reasonable to try to adapt the geometric proof of Braden and MacPherson to the language of Soergel bimodules. However, a proof for the Monotonicity Conjecture has not been formally documented anywhere, to the best of the author's knowledge. In this paper, we adopt an alternative approach. Actually, we provide a proof of the Monotonicity Conjecture for arbitrary Coxeter groups completely contained in the language of Soergel bimodules. Therefore, the arguments used in our proof are different from the algebraic arguments used by Irving, and different from the geometric arguments used by Braden and MacPherson.

Let us briefly explain our approach to the Monotonicity Conjecture. For each reduced expression $\underline{w}$ of an element $w \in W$, one can explicitly define a Soergel bimodule $BS(\underline{w})$, called the Bott–Samelson bimodule. The endomorphisms ring of a Bott–Samelson bimodule, $\mathrm{End}(BS(\underline{w}))$, has a natural structure of free right R-algebra. Libedinsky constructed in [14] an R-basis for these spaces that he called *light leaves basis*. He generalized his construction in [15] to obtain another basis that he called the *double leaves basis*. The latter is more useful than the light leaves basis for our purposes because of its symmetry properties. In particular, Elias and Williamson noticed in [7] that the double leaves basis is a cellular basis for $\mathrm{End}(BS(\underline{w}))$, in the sense of Graham and Lehrer [8].

Let R^+ be the ideal of R generated by homogeneous elements of nonzero degree. We have $\mathbb{R} \cong R/R^+$. Therefore, we can reduce $\mathrm{End}(BS(\underline{w}))$ modulo R^+ to obtain an $\mathbb{R}$-algebra. The resulting algebra is equipped with a natural $\mathbb{Z}$-grading. The double leaves basis behaves satisfactorily with respect to reduction modulo R^+ and cellularity. Concretely, the image of the double leaves basis is a graded cellular basis of $\mathrm{End}(BS(\underline{w}))\otimes_R\mathbb{R}$ in the sense of Hu and Mathas [9]. The existence of a graded cellular basis allows us to define graded cell modules and graded simple modules, as well as graded decomposition numbers. We then prove using Soergel's conjecture that the KL-polynomials (suitably normalized) can be interpreted as graded decomposition numbers. Finally, we construct certain injective homomorphisms between cell modules that allow us to embed a cell module into another cell module. This embedding provides a family of inequalities for the respective graded decomposition numbers. This implies the Monotonicity Conjecture according to the aforementioned interpretation of the KL-polynomials as graded decomposition numbers.

The layout of this article is as follows. In Section 2, we recall a few useful results of the theory of graded cellular algebras. In Section 3, we define Hecke algebras and the category of Soergel bimodules, and conclude this section by recalling Libedinsky's construction of the double leaves basis. We establish the graded cellularity of double leaves basis in Section 4. Using graded cellularity, we can view the KL-polynomials as graded decomposition numbers. Finally, in Section 5, we show how to embed a cell module into another cell module. We then use this embedding to conclude our proof of the Monotonicity Conjecture.

2. Graded cellular algebras

In this section, we briefly recall the theory of graded cellular algebras. Graded cellular algebras were defined by Hu and Mathas in [9], following and extending the ideas of Graham and Lehrer [8]. A clear exposition of this theory (in the ungraded setting) can be found in [16].

Let $\mathbb{K}$ be a field. A graded $\mathbb{K}$-vector space M is a $\mathbb{K}$-vector space that has a direct sum decomposition $M = \bigoplus_{k\in\mathbb{Z}} M_k$. If M is a graded $\mathbb{K}$-vector space and $k \in \mathbb{Z}$, we denote by $M\langle k\rangle$ the graded $\mathbb{K}$-vector space obtained from M by shifting the grading on M, i.e., $M\langle k\rangle_i = M_{i-k}$, for all $i \in \mathbb{Z}$. Given a Laurent polynomial $f = \sum_{i\in\mathbb{Z}} a_i v^i \in \mathbb{N}[v, v^{-1}]$ and a graded vector space M, we set

$$fM = \bigoplus_{i\in\mathbb{Z}} M\langle i\rangle^{\oplus a_i}.$$

A graded $\mathbb{K}$-algebra A is a $\mathbb{K}$-algebra with a direct sum decomposition $A = \bigoplus_{k\in\mathbb{Z}} A_i$ as a $\mathbb{K}$-vector space such that $A_iA_j \subset A_{i+j}$, for all $i, j \in \mathbb{Z}$. A graded right A-module M is a graded $\mathbb{K}$-vector space that is an A-module in the usual (ungraded) sense, such that $A_iM_j \subset M_{i+j}$, for all $i, j \in \mathbb{Z}$. Given a graded finite-dimensional $\mathbb{K}$-vector space, $M = \bigoplus_{k\in\mathbb{Z}}, M_k$, we define its graded dimension $\dim_v M \in \mathbb{N}[v, v^{-1}]$ as follows

$$\dim_v M = \sum_{k\in\mathbb{Z}} \dim_{\mathbb{K}}(M_k)v^k. \tag{2.1}$$

We now define the concept of graded cellular algebra. This definition is provided in [9, Definition 2.1].

Definition 2.1. Let A be a graded finite-dimensional $\mathbb{K}$-algebra. A graded cell datum is an ordered quadruple $(\Lambda, T, C, \deg)$, where $(\Lambda, \geq)$ is a poset, $T(\lambda)$ is a finite set for $\lambda \in \Lambda$, and C and deg are two functions defined as follows:

$$C : \coprod_{\lambda\in\Lambda} T(\lambda) \times T(\lambda) \to A,\ (\mathfrak{s}, \mathfrak{t}) \to c^\lambda_{\mathfrak{s}\mathfrak{t}}; \qquad \deg : \coprod_{\lambda\in\Lambda} T(\lambda) \to \mathbb{Z}$$

such that C is injective and:

(a) $\mathcal{C} = \{c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda),\ \lambda \in \Lambda\}$ is a basis of A.
(b) The $\mathbb{K}$-linear map $* : A \to A$ determined by $(c^\lambda_{\mathfrak{s}\mathfrak{t}})^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$ is an anti-automorphism of A.
(c) For all $a \in A$, $\lambda \in \Lambda$, and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, there exist scalars $r_{\mathfrak{t}\mathfrak{v}}(a) \in \mathbb{K}$ that do not depend on $\mathfrak{s}$, such that

$$c^\lambda_{\mathfrak{s}\mathfrak{t}}a \equiv \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a)c^\lambda_{\mathfrak{s}\mathfrak{v}} \mod A^{>\lambda} \tag{2.2}$$

where $A^{>\lambda}$ is the vector subspace of A spanned by $\{c^\mu_{\mathfrak{a}\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in T(\mu), \mu > \lambda\}$.
(d) Each $c^\lambda_{\mathfrak{s}\mathfrak{t}}$ is a homogeneous element of degree $\deg(\mathfrak{s}) + \deg(\mathfrak{t})$.

A graded cellular algebra is a graded algebra with a graded cell datum. The set $\mathcal{C}$ is a graded cellular basis of A.

Remark 2.2. Ignoring the grading on A, the degree function, and axiom (d) in the above definition, one recovers the original definition of cellular algebras by Graham and Lehrer [8]. In this case, we say that A is a cellular algebra with a cellular basis and a cell datum.

Let A be a graded cellular algebra with graded cellular basis $\mathcal{C}$, as in the above definition. For each $\lambda \in \Lambda$, we define the *graded cell module*, $\Delta(\lambda)$, as the graded right A-module

$$\Delta(\lambda) = \bigoplus_{k\in\mathbb{Z}} \Delta(\lambda)_k$$

where $\Delta(\lambda)_k$ is a $\mathbb{K}$-vector space with basis $\{c_{\mathfrak{t}}^{\lambda} \mid \mathfrak{t} \in T(\lambda) \text{ and } \deg(\mathfrak{t}) = k\}$, and the symbols $c_{\mathfrak{t}}^{\lambda}$ are formal variables. The A-action on $\Delta(\lambda)$ is determined by the scalars that appear in (2.2), i.e.,

$$c_{\mathfrak{t}}^{\lambda} a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) c_{\mathfrak{v}}^{\lambda} \tag{2.3}$$

Suppose that $\lambda \in \Lambda$. Then, it follows from Definition 2.1 that there is a bilinear form $\langle\ ,\ \rangle$ on $\Delta(\lambda)$ which is determined by

$$c_{\mathfrak{a}\mathfrak{s}}^{\lambda} c_{\mathfrak{t}\mathfrak{b}}^{\lambda} \equiv \langle c_{\mathfrak{s}}^{\lambda}, c_{\mathfrak{t}}^{\lambda} \rangle c_{\mathfrak{a}\mathfrak{b}}^{\lambda} \mod A^{>\lambda} \tag{2.4}$$

For each $\lambda \in \Lambda$, $\langle\ ,\ \rangle$ satisfies $\langle x, y\rangle = \langle y, x\rangle$, and $\langle xa, y\rangle = \langle x, ya^*\rangle$, for all $x, y \in \Delta(\lambda)$ and $a \in A$. Accordingly, the *radical*

$$\operatorname{rad}(\Delta(\lambda)) := \{x \in \Delta(\lambda) \mid \langle x, y\rangle = 0, \text{ for all } y \in \Delta(\lambda)\} \tag{2.5}$$

of $\Delta(\lambda)$ is a graded A-submodule of $\Delta(\lambda)$ (see [9, Lemma 2.7]). Therefore, the quotient $D(\lambda) := \Delta(\lambda)/\operatorname{rad}(\Delta(\lambda))$ is a graded right A-module. Furthermore, if $D(\lambda) \neq 0$, then $D(\lambda)$ is a simple graded right A-module. Define

$$\Lambda_0 = \{\lambda \in \Lambda \mid D(\lambda) \neq 0\}.$$

The following theorem gives a classification of the simple graded A-modules for a graded cellular algebra A. This result is due to Hu and Mathas [9, Theorem 2.10], and is a graded version of [8, Theorem 3.4].

Theorem 2.3. *Let A be a graded cellular algebra, with a cell datum as in Definition 2.1. Then, the set $\{D(\lambda)\langle k\rangle \mid \lambda \in \Lambda_0 \text{ and } k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple right A-modules.*

Let Δ and D be graded right A-modules. If D is simple, we denote by $[\Delta : D\langle k\rangle]$ the multiplicity of the graded simple module $D\langle k\rangle$ as a graded composition factor in a graded composition series of Δ, for all $k \in \mathbb{Z}$. We then define the graded decomposition number, $d(\Delta, D)$, as the Laurent polynomial:

$$d(\Delta, D) = \sum_{k\in\mathbb{Z}} [\Delta : D\langle k\rangle] v^k. \tag{2.6}$$

In particular, if $\Delta = \Delta(\lambda)$ and $D = D(\mu)$, for some $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, we denote

$$d(\Delta, D) = d(\lambda, \mu). \tag{2.7}$$

Furthermore, we have

$$d(\mu, \mu) = 1 \text{ and } d(\lambda, \mu) \neq 0 \text{ only if } \lambda \geq \mu. \tag{2.8}$$

We end this section by relating the graded representation theory of the algebras A and eAe, where A is a graded (not necessarily cellular) algebra and $e \in A$ is an homogeneous idempotent. We remark that for each right A-module V, the subspace Ve of V has a natural structure of a right eAe-module.

Theorem 2.4. *Let A be a graded algebra. Let $e \in A$ be an homogeneous idempotent (and, therefore, of degree zero). We then have:*

(a) *If V is a simple graded right A-module and $Ve \neq 0$, Ve is a simple graded right eAe-module. Furthermore, all the simple right eAe-modules can be obtained in this manner.*
(b) *Let V and D be graded right A-modules. If D is simple and $De \neq 0$, then*

$$d(V, D) = d(Ve, De) \tag{2.9}$$

where the left (resp. right) side of (2.9) corresponds to the graded decomposition number for A-modules (resp. eAe-modules).

Proof. This is well-known (see, for example, [5, Appendix A1] or [17, Theorem 2.4]). □

3. Libedinsky double leaves

In this section, we introduce, for an arbitrary Coxeter system (W, S), its corresponding Hecke algebra, and its corresponding category of Soergel bimodules. We end this section by introducing the double leaves basis. This is a basis for morphism spaces between Bott–Samelson bimodules. Double leaves are the combinatorial tool that we use to prove the monotonicity conjecture in Section 5. This basis admits a convenient diagrammatic description. For the sake of brevity, we have omitted the diagrammatic approach in this paper. However, the diagrams allowed several calculations that helped us understand the problem. We refer the reader interested in the diagrammatic approach to [7, Part 3].

3.1. Hecke algebras and KL-polynomials

Let (W, S) be a *Coxeter system*. That is, W is a group with generators $s \in S$ and relations

$$(st)^{m_{st}} = e \qquad\qquad \text{for all } s, t \in S \tag{3.1}$$

where $e \in W$ is the identity, $m_{st} \in \{1, 2, \ldots, \infty\}$ satisfies: $m_{st} = 1$ if and only if $s = t$, and $m_{st} = m_{ts}$ for all $s, t \in S$. When $m_{st} = \infty$, relation (3.1) is omitted. We use the *underlined* letter $\underline{w} = (s_1, \ldots, s_k)$, $s_i \in S$ to denote a finite sequence of elements in S. We will call *expressions* to these sequences. If we consider an expression $\underline{w} = (s_1, \ldots, s_k)$, the corresponding Roman letter, w, will denote its product in W, i.e., $w = s_1 \ldots s_k$. We make this distinction between w and $\underline{w}$ because a few concepts defined in this paper and used throughout rely heavily on the considered expression for w rather than on w itself. We will often write $\underline{w} = s_1 \ldots s_k$, where the underlined letter reminds us that the entire sequence, and not merely w, is important. The group W is equipped with a length function $l : W \to \mathbb{N}$ and an order, called the *Bruhat order*, which is denoted by $\geq$ (see, e.g., [1, Chapter 1]). The *length* of an expression $\underline{w} = s_1 \ldots s_k$ is k. We say that an expression is reduced if $l(\underline{w}) = l(w)$.

Definition 3.1. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of the Laurent polynomials in v. The Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ is the $\mathcal{A}$-algebra that is associative and unital with generators $\{H_s | s \in S\}$ and relations

$$H_s^2 = (v^{-1} - v)H_s + 1 \tag{3.2}$$

$$\underbrace{H_s H_t H_s \ldots}_{m_{st} \text{ terms}} = \underbrace{H_t H_s H_t \ldots}_{m_{st} \text{ terms}}. \tag{3.3}$$

If $\underline{w} = s_1 \ldots s_k$ is a reduced expression for $w \in W$, we define $H_w := H_{s_1} \ldots H_{s_k}$. It is well-known that H_w does not depend on the choice of the reduced expression $\underline{w}$. The set $\{H_w | w \in W\}$ is a basis for $\mathcal{H}$ as an $\mathcal{A}$-module. There is a unique ring involution $^{-} : \mathcal{H} \to \mathcal{H}$ determined by $\overline{v} = v^{-1}$ and $\overline{H_w} = H_{w^{-1}}^{-1}$, for all $w \in W$.

Theorem 3.2. [12, Theorem 1.1] *There exists a unique basis $\{\underline{H}_w \mid w \in W\}$ for $\mathcal{H}$ as a $\mathcal{A}$-module such that $\underline{H}_w$ is invariant under $^{-}$ and*

$$\underline{H}_w = \sum_{x \leq w} h_{x,w} H_x \tag{3.4}$$

with $h_{x,w} \in v\mathbb{Z}[v]$ if $x \neq w$ and $h_{w,w} = 1$.

The set $\{\underline{H}_w \mid w \in W\}$ is called the *Kazhdan–Lusztig basis* of $\mathcal{H}$ and the polynomials $h_{x,w}$ are called the *Kazhdan–Lusztig polynomials.*

Remark 3.3. The reader should note that in this paper, we follow the normalization given by Soergel in [18] rather than the original normalization by Kazhdan and Lusztig in [12]. Therefore, we have $q = v^{-2}$, and the original Kazhdan–Lusztig polynomials $P_{x,w}(q) \in \mathbb{Z}[q]$ can be recovered from our Kazhdan–Lusztig polynomials $h_{x,w}(v) \in \mathbb{Z}[v]$ by the formula

$$h_{x,w}(v) = v^{l(w)-l(x)} P_{x,w}(v^{-2}). \tag{3.5}$$

3.2. The category of Soergel bimodules

Let us fix once and for all a *reflection-faithful* representation V of W over $\mathbb{R}$. In [19], Soergel constructed such a representation for arbitrary Coxeter groups. Let R be the $\mathbb{R}$-algebra of regular functions on V. We can grade this algebra by setting $R = \bigoplus_{i\in\mathbb{Z}} R_i$, with $R_2 = V^*$. Let R^+ be the ideal of R generated for all elements of positive degree. Of course, $R/R^+ \cong \mathbb{R}$. We will often consider $\mathbb{R}$ as an R-module via this isomorphism. There is a natural action of W on R induced by the action of W on V. For $s \in S$, let R^s be the subring of R fixed by s. Then, we define the graded (R,R)-bimodule

$$B_s = R \otimes_{R^s} R(1), \tag{3.6}$$

where for a graded (R,R)-bimodule B and every $k \in \mathbb{Z}$, we denote by $B(k)$ the graded (R,R)-bimodule defined by the formula

$$B(k)_i = B_{k+i} \tag{3.7}$$

For the expression $\underline{w} = s_1 \dots s_{s_k}$, we denote by $BS(\underline{w})$ the (R,R)-bimodule defined by

$$BS(\underline{w}) = B_{s_1} \otimes_R B_{s_2} \otimes_R \dots \otimes_R B_{s_k} \tag{3.8}$$

Bimodules of the type $BS(\underline{w})$ will be called *Bott–Samelson bimodules.* We introduce the convention that $BS(\emptyset) = R$. From now on, we denote the tensor product of (R,R)-bimodules, $\otimes_R$, simply juxtaposition. Thus, $BS(\underline{w})$ becomes $B_{s_1} B_{s_2} \dots B_{s_k}$. We then have the following isomorphism of (R,R)-bimodules

$$B_{s_1} B_{s_2} \dots B_{s_k} \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \otimes \dots \otimes_{R^{s_k}} R(k) \tag{3.9}$$

Therefore, we can write an element of this module as a sum of terms given by $k+1$ polynomials in R, one in each slot separated by the tensors. Let $x_s \in V^*$ be an equation of the hyperplane fixed by $s \in S$. Then, for all $s \in S$, we define the Demazure operator, $\partial_s : R(2) \to R^s$, as a morphism of graded R^s-modules given by

$$\partial_s(f) = \frac{f - s \cdot f}{2x_s} \tag{3.10}$$

It is not difficult to prove that $\partial_s(f)$ and $P_s(f) = f - x_s\partial_s(f)$ are s-invariant. Since $f = P_s(f) + x_s\partial_s(f)$, R is free as a graded right R^s-module with basis $\{1, x_s\}$, i.e., we have a decomposition $R \simeq R^s \oplus x_s R^s$. Using this decomposition, we can prove that $BS(s)$ is a free right R-module with basis $\{x_s \otimes 1, 1 \otimes 1\}$. Let $\underline{w} = s_1 \ldots s_k$ be an expression. Going once more through the above decomposition of R, we can see that $BS(\underline{w})$ is a free right R-module of rank 2^k, with basis

$$\{x_{s_1}^{e_1} \otimes x_{s_2}^{e_2} \otimes \ldots \otimes x_{s_k}^{e_k} \otimes 1 \mid e_i = 0, 1\}. \tag{3.11}$$

We now define the category of Soergel bimodules. This category categorifies the Hecke algebra, as will be made precise in Theorem 3.5.

Definition 3.4. The category of *Soergel bimodules*, $\mathbb{S}$Bim, is the category of $\mathbb{Z}$-graded (R, R)-bimodules whose objects are grading shifts and direct sums of direct summands of Bott–Samelson bimodules. The morphisms are all degree-preserving bimodule homomorphisms. For $B, B' \in \mathbb{S}$Bim, we denote by $\mathrm{Hom}(B, B')$ the corresponding set of morphisms. Moreover, we write

$$\mathrm{Hom}^{\mathbb{Z}}(B, B') = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(B(k), B') \tag{3.12}$$

to denote the bimodule homomorphisms between B and B' of all degrees. In particular, we write $\mathrm{End}^{\mathbb{Z}}(B) := \mathrm{Hom}^{\mathbb{Z}}(B, B)$. An element $f \in \mathrm{Hom}(B(k), B') \cong \mathrm{Hom}(B, B(-k))$ is called a homogeneous morphism of degree k and we write $\deg(f) = k$. Let $f, g \in \mathrm{End}^{\mathbb{Z}}(B)$ be homogeneous elements of degree k and l, respectively. By the above isomorphism, we can consider g as an element of $\mathrm{Hom}(B(l+k), B(k))$. Therefore, we can define $fg := f \circ g \in \mathrm{Hom}(B(l+k), B)$. Extending by linearity the above definition, we have equipped to $\mathrm{End}^{\mathbb{Z}}(B)$ with structure of $\mathbb{Z}$-graded R-algebra.

Let $[\mathbb{S}\mathrm{Bim}]$ be the split Grothendieck group of the category $\mathbb{S}$Bim. That is, $[\mathbb{S}\mathrm{Bim}]$ is the abelian group generated by the symbols $[B]$ for all objects $B \in \mathbb{S}$Bim, subject to the relation $[B] = [B'] + [B'']$ whenever we have $B \cong B' \oplus B''$ in $\mathbb{S}$Bim. The following theorem is known as Soergel's categorification theorem and relates $\mathbb{S}$Bim to $\mathcal{H}$.

Theorem 3.5. *For each $w \in W$, there exists a unique (up to isomorphism) indecomposable bimodule B_w that occurs as a direct summand of $BS(\underline{w})$ for any reduced expression $\underline{w}$ of w, and B_w does not appear in any $BS(\underline{x})$ for a word $\underline{x}$ shorter than $\underline{w}$. Furthermore, there is a unique $\mathcal{A}$-algebra isomorphism*

$$\epsilon : \mathcal{H} \longrightarrow [\mathbb{S}Bim] \tag{3.13}$$

such that $\epsilon(v) = R(1)$ and $\epsilon(\underline{H}_s) = [B_s]$ for all $s \in S$.

In order to explain the inverse of ϵ, known as Soergel's character map, we need to introduce standard bimodules. Given $x \in W$ we define the *standard bimodule* R_x as the (R, R)-bimodule, such that $R_x \cong R$ as a left R-module and the right action on R_x is the right multiplication on R deformed by the action of x on R, i.e.,

$$r \cdot r' := rx(r') \text{ for } r \in R_x \text{ and } r' \in R. \tag{3.14}$$

Theorem 3.6. *The categorification $\epsilon : \mathcal{H} \to [\mathbb{S}Bim]$ admits an inverse, $\eta : [\mathbb{S}Bim] \to \mathcal{H}$, given by the formula*

$$\eta([B]) = \sum_{x \in W} \dim_v(\mathrm{Hom}^{\mathbb{Z}}(B, R_x) \otimes_R \mathbb{R}) H_x. \tag{3.15}$$

We end this subsection by introducing Soergel's conjecture. For historical reasons, we call this a *conjecture* even though it was proven in 2014 [6].

Conjecture 3.7. *Let W be a Coxeter group. For all $w \in W$, we have*

$$\epsilon(\underline{H}_w) = B_w. \tag{3.16}$$

Remark 3.8. The Kazhdan–Lusztig positivity conjecture immediately follows from Soergel's conjecture by applying Soergel's character map to (3.16).

3.3. Double leaves basis

Let $\underline{w}$ and $\underline{v}$ be two (not necessarily reduced) expressions. In this section, we recall the construction of the double leaves basis (DLB), a basis of the space $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$, defined by Libedinsky in [15]. The DLB is, in some sense, an improvement over the light leaves basis, another basis for $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$ defined in [14]. In the remainder of this paper, we will work with the DLB rather than the light leaves basis because as we will see in Section 4, DLB is a (graded) cellular basis whereas the light leaves basis is not. We use the cellularity of the DLB to establish the monotonicity conjecture for Kazhdan–Lusztig polynomials.

To introduce the DLB, we begin by defining three morphisms between Bott–Samelson bimodules. The first one is the multiplication morphism, m_s, which is a degree 1 morphism determined by the formula:

$$\begin{aligned} m_s : BS(s) = R \otimes_{R^s} R(1) &\to R \\ p \otimes q &\to pq \end{aligned} \tag{3.17}$$

The second morphism is the unique (up to multiplication by a nonzero scalar) -1 degree morphism, j_s, determined by the formula

$$\begin{aligned} j_s : BS(ss) = R \otimes_{R^s} R \otimes_{R^s} R(2) &\to BS(s) \\ 1 \otimes p \otimes 1 &\to \partial_s(p) \otimes 1 \end{aligned} \tag{3.18}$$

For $s, r \in S$, consider the bimodule

$$X_{sr} := BS(srs \dots)$$

with the product having m_{sr} terms. We then define f_{sr} as the unique degree zero morphism from X_{sr} to X_{rs} sending $1 \otimes 1 \otimes \dots \otimes 1$ to $1 \otimes 1 \otimes \dots \otimes 1$.

We denote by $\mathbb{I}$ the identity on the endomorphism ring of a Bott–Samelson bimodule. Each time we use the symbol $\mathbb{I}$, the relevant Bott–Samelson bimodule will be clear from the context. For each expression $\underline{w} = s_1 \dots s_n \in S^n$, we inductively define a perfect binary directed tree, denoted by $\mathbb{T}_{\underline{w}}$, with nodes colored by Bott–Samelson bimodules and edges colored by morphisms from parent nodes to child nodes. At depth 1, the tree looks as in Fig. 1.

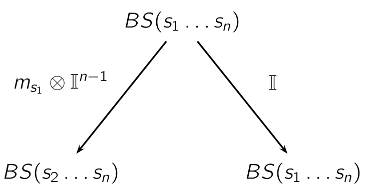

Fig. 1. Level one of $\mathbb{T}_{\underline{w}}$.

Now, let $1 < k \leq n$ and assume that we have constructed the tree to level $k-1$. Let $\underline{u} = t_1 \dots t_i \in S^i$ be a node N of depth $k-1$ colored by the bimodule $BS(t_1 \dots t_i)BS(s_k \dots s_n)$. We then have two possibilities:

a) $l(t_1 \dots t_i s_k) > l(t_1 \dots t_i)$. In this case the child nodes and edges of N are constructed as shown in Fig. 2.

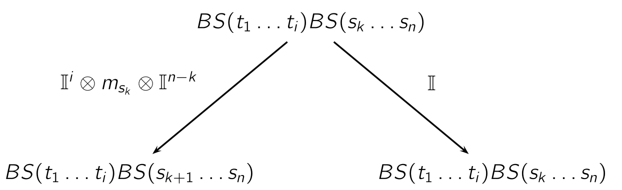

Fig. 2. Level k of $\mathbb{T}_{\underline{w}}$.

b) $l(t_1 \dots t_i s_k) < l(t_1 \dots t_i)$. In this case, it is a well-known fact for Coxeter groups that there exists a sequence of braid moves that converts $\underline{u} = t_1 \dots t_i$ into $\underline{u'} = t'_1 \dots t'_{i-1} s_k$. Of course, there are several ways to do this. However, we can fix a particular sequence of braid moves and construct a morphism $BS(\underline{u}) \to BS(\underline{u'})$ by replacing each braid move in the sequence by its respective morphism of type f_{sr}.

We denote this morphism by $F(\underline{u}, \underline{u'}, s_k)$. The child nodes of N are then colored by the two Bott–Samelson bimodules located at the bottom of Fig. 3, and the child edges are colored by morphisms obtained by composing the dashed arrows in Fig. 3.

$$\begin{tikzcd}
& BS(t_1 \ldots t_i)BS(s_k \ldots s_n) \arrow[d, dashed, "F(\underline{u},\underline{u'},s_k)\otimes \mathbb{I}^{n-k+1}"] \arrow[dddl, bend right] \arrow[dddr, bend left] & \\
& BS(t'_1 \ldots t'_{i-1}s_k)BS(s_k \ldots s_n) \arrow[d, dashed, "\mathbb{I}^{i-1}\otimes j_{s_k}\otimes \mathbb{I}^{n-k}"] & \\
& BS(t'_1 \ldots t'_{i-1}s_k \ldots s_n) \arrow[ddl, dashed, "\mathbb{I}^{i-1}\otimes m_{s_k}\otimes \mathbb{I}^{n-k}"'] \arrow[ddr, dashed, "\mathbb{I}"] & \\
& & \\
BS(t'_1 \ldots t'_{i-1})BS(s_{k+1} \ldots s_n) & & BS(t'_1 \ldots t'_{i-1})BS(s_k \ldots s_n)
\end{tikzcd}$$

Fig. 3. Level k of $\mathbb{T}_{\underline{w}}$.

By construction, each leaf of the tree $\mathbb{T}_{\underline{w}}$ is colored by a Bott–Samelson bimodule $BS(\underline{u})$, where the expression $\underline{u}$ is reduced. Note that it is possible for two leaves to be colored by Bott–Samelson bimodules $BS(\underline{u})$ and $BS(\underline{u'})$, where $\underline{u}$ and $\underline{u'}$ are two reduced expressions for the same element in $u \in W$. To avoid this ambiguity, we realize the following choices:

1. Fix, once and for all, a reduced expression $\underline{u}$ for all $u \in W$.
2. For all $u \in W$ and all reduced expression $\underline{u'}$ of u, choose a sequence of braid moves that converts $\underline{u'}$ into $\underline{u}$, where $\underline{u}$ is the fixed reduced expression selected in the previous step.
3. Finally, replace each braid move in the sequence selected in the previous step by its corresponding morphism of type f_{sr} to obtain a morphism from $BS(\underline{u'})$ to $BS(\underline{u})$, denoted by $F(\underline{u'}, \underline{u})$.

We now complete the construction of $\mathbb{T}_{\underline{w}}$ by composing each of the lower Bott–Samelson bimodules with its corresponding morphism $F(\underline{u'}, \underline{u})$. This procedure avoids the ambiguity in coloring the leaves. That is, if two leaves are colored by $BS(\underline{u})$ and $BS(\underline{u'})$, where $\underline{u}$ and $\underline{u'}$ are two reduced expressions for the same element $u \in W$, $\underline{u}$ and $\underline{u'}$ are the same expressions.

By composing the corresponding arrows, we can consider each leaf in $\mathbb{T}_{\underline{w}}$ colored by $BS(\underline{u})$ as a morphism from $BS(\underline{w})$ to $BS(\underline{u})$, where $\underline{u}$ is the reduced expression for $u \in W$ fixed above. Let $\mathbb{L}_{\underline{w}}(u)$ be the set of all leaves colored by $\underline{u}$. As mentioned before, we will consider the set $\mathbb{L}_{\underline{w}}(u)$ as a subset of $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{u}))$. Note that every leaf is a homogeneous morphism, since it was constructed as a composition of homogeneous morphisms. In fact, the degree of each leaf can be computed as the sum of $+1$ for each occurrence of a morphism of type m_s minus the sum of -1 for each occurrence of a morphism of type j_s.

Remark 3.9. The set $\mathbb{L}_{\underline{w}}(u)$ is not uniquely determined because it relies heavily on the choices realized along the way. Thus, when we refer to it, one must understand that we are implicitly assuming that we have fixed a particular choice for each of the non-canonical steps in the construction of $\mathbb{T}_{\underline{w}}$. For example, if $\underline{w}$ is a reduced expression for $w \in W$, there is exactly one leaf in $\mathbb{L}_{\underline{w}}(w)$. This leaf can be chosen as any morphism of the type $F(\underline{w}, \underline{w'})$, where $\underline{w'}$ is any reduced expression of w. However, we choose the identity in this case for the sake of simplicity. We will henceforth use this choice throughout the paper without reference to it.

In order to introduce the DLB, we need to define an adjoint leaf for each leaf. To do this, we must first define an adjoint morphism for each of m_s, j_s, and f_{sr}. The adjoint morphism of m_s is

$$\begin{aligned} \epsilon_s : R &\rightarrow BS(s) \\ 1 &\rightarrow x_s \otimes 1 + 1 \otimes x_s \end{aligned} \tag{3.19}$$

For j_s, the corresponding adjoint morphism is

$$\begin{aligned} p_s : BS(s) &\rightarrow BS(ss) \\ 1 \otimes 1 &\rightarrow 1 \otimes 1 \otimes 1 \end{aligned} \tag{3.20}$$

Finally, for f_{sr}, the adjoint morphism is f_{rs}. Each adjoint morphism is homogeneous and has the same degree as its corresponding morphism. For each leaf $l : BS(\underline{w}) \rightarrow BS(\underline{u})$ in $\mathbb{T}_{\underline{w}}$, we can thus define an adjoint leaf $l^a : BS(\underline{u}) \rightarrow BS(\underline{w})$ as the morphism obtained by replacing each morphism of type m_s, j_s and f_{sr} by its corresponding adjoint. We thus obtain an inverted tree, $\mathbb{T}^a_{\underline{w}}$, with the same nodes as $\mathbb{T}_{\underline{w}}$ but with the arrows pointing in the opposite direction.

For $f \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{u}))$ and $g \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{x}), BS(\underline{y}))$ we define

$$f \cdot g = \begin{cases} f \circ g, & \text{if } \underline{w} = \underline{y}; \\ \emptyset, & \text{if } \underline{w} \neq \underline{y}. \end{cases} \tag{3.21}$$

For an expression $\underline{w}$, we denote by $\mathbb{L}_{\underline{w}}$ (resp. $\mathbb{L}^a_{\underline{w}}$) the set of all leaves in $\mathbb{T}_{\underline{w}}$ (resp. $\mathbb{T}^a_{\underline{w}}$). We are now in a position to define the main object of interest in this paper.

Theorem 3.10. *[15, Theorem 3.2] For all expressions $\underline{w}$ and $\underline{v}$, the set $\mathbb{L}^a_{\underline{v}} \cdot \mathbb{L}_{\underline{w}}$ is a basis as right R-module of the space* $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$. *We call this set the* double leaves basis.

In order to prove the linear independence of $\mathbb{L}^a_{\underline{v}} \cdot \mathbb{L}_{\underline{w}}$, Libedinsky [15] introduced an order on the set $\mathbb{L}^a_{\underline{v}} \cdot \mathbb{L}_{\underline{w}}$ and applied a classical triangularity argument. This order is defined by indexing each leaf by two sequences of zeros and ones, which we denote by $\boldsymbol{i} = (i_1, \dots, i_n)$ and $\boldsymbol{j} = (j_1, \dots, j_n)$. Let us recall this assignment, since it will be important for our purposes.

Let $\underline{w} = s_1 \dots s_n$ be an expression of length n. Recall that l was constructed inductively in n steps. For all $1 \leq k \leq n$, we set $i_k = 1$ if a morphism of type m_s appears in the k-th step of the construction of l; otherwise, we set $i_k = 0$. In a similar manner, we set $j_k = 1$ if a morphism of type j_s appears in the k-th step of the construction of l, and otherwise set $j_k = 0$. Note that each leaf l is completely determined by these two sequences and the expression $\underline{w}$. Thus, we denote $l = f^{\boldsymbol{j}}_{\boldsymbol{i}}$. For $s \in S$, let us denote $x^0_s = 1$ and $x^1_s = x_s$. If $\boldsymbol{j} = (j_1, \dots, j_n)$ is a sequence of zeros and ones, we set

$$x^{\boldsymbol{j}} = x^{j_1}_{s_1} \otimes x^{j_2}_{s_2} \otimes \dots \otimes x^{j_n}_{s_n} \otimes 1 \in BS(\underline{s}) \tag{3.22}$$

In particular, if $\boldsymbol{j} = (0, \dots, 0)$, we denote $x^{\boldsymbol{j}}$ by $1^{\otimes}$. The indexing of each leaf by pairs of binary sequences is compatible with the lexicographic order in the sense of the following lemma [15, Section 5.5].

Lemma 3.11. *Denote by $\geq$ the lexicographic order on $\{0,1\}^n$. Then,*

$$f^{\boldsymbol{j}}_{\boldsymbol{i}}(x^{\boldsymbol{j}'}) = \begin{cases} 1^{\otimes}, & \text{if } \boldsymbol{j} = \boldsymbol{j}'; \\ 0, & \text{if } \boldsymbol{j} > \boldsymbol{j}'. \end{cases}$$

We end this section by introducing a basis for $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x)$, for all reduced expressions $\underline{w}$ of $w \in W$ and $x \in W$. We recall that R_x denotes the standard bimodule defined immediately following Theorem 3.6. We first need to introduce a new morphism. For all $s \in S$, consider the (R,R)-bimodule morphism $\beta_s : BS(s) \to R_s$ determined by $\beta_s(p \otimes q) = ps(q)$, for all $p, q \in R$. Let $\underline{x} = s_1 \dots s_k$ be a reduced expression for $x \in W$. Define $\beta_{\underline{x}} := \beta_{s_1} \otimes \dots \otimes \beta_{s_k} : BS(\underline{w}) \to R_{s_1} \otimes_R \dots \otimes_R R_{s_k}$. Since $R_y R_z \cong R_{yz}$, for all $y, z \in W$, $\beta_{\underline{x}}$ can be considered as a morphism from $BS(\underline{x})$ to R_x. If the underlying reduced expression $\underline{x}$ is clear from the context, then we often denote $\beta_{\underline{x}}$ simply as β_x. Let us define the set

$$\mathbb{L}^{\beta}_{\underline{w}}(x) = \{\beta_x \circ l \mid l \in \mathbb{L}_{\underline{w}}(x)\} \subset \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x). \tag{3.23}$$

Following Libedinsky, we will call $\mathbb{L}^{\beta}_{\underline{w}}(x)$ the standard leaves basis. This name is justified by the following lemma (see[15, Proposition 6.1]).

Lemma 3.12. *Let $\underline{w}$ be an expression and let $x \in W$. Then, $\mathbb{L}^{\beta}_{\underline{w}}(x)$ is an R-basis of $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x)$ as a right R-module.*

Corollary 3.13. *Let $\underline{w}$ be a reduced expression for w and let $x \in W$. Then,*

$$\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \neq 0$$

if and only if $x \leq w$.

Proof. The result is a direct consequence of Lemma 3.12 once we note that $\mathbb{L}_{\underline{w}}(x) \neq \emptyset$ if and only if $x \leq w$. □

4. KL-polynomials as graded decomposition numbers

In this section, we interpret KL-polynomials as graded decomposition numbers. For the rest of this section, we fix a reduced expression $\underline{w}$ for an element $w \in W$. As we saw in the previous section, $\mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ is $\mathbb{Z}$-graded R-algebra of finite rank, with a basis, the double leaves basis. Although our definition of graded cellular algebras applies only when the ground ring is a field, the original definition provided by Hu and Mathas in [9] does not require this assumption. However, in order to apply Theorem 2.3, the ground ring must be a field. On the other hand, if we recall that R is a polynomial ring with coefficients in $\mathbb{R}$, then it is clear that $\mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ is an infinite dimensional $\mathbb{Z}$-graded $\mathbb{R}$-algebra. Koenig and Xi [13] extend the framework of cellular algebras to algebras over a field not necessarily finite dimensional. Furthermore, they provide an analogous of Theorem 2.3 in this setting. However, the classification of simple modules in this case is rather complicated.

We remark that $\mathbb{R} \cong R/R^+$. This allows us to consider $\mathbb{R}$ as a left R-module. Given $r \in R$ we denote by $\hat{r} \in R/R^+ \cong \mathbb{R}$ its reduction modulo R^+. By the previous paragraph, it should be natural to consider $A_{\underline{w}} := \mathrm{End}^{\mathbb{Z}}(BS(\underline{w})) \otimes_R \mathbb{R}$ rather than $\mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$. By definition of $A_{\underline{w}}$ and the fact that $\mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ is a free $\mathbb{Z}$-graded R-algebra of finite rank, it is clear that $A_{\underline{w}}$ is a finite dimensional $\mathbb{Z}$-graded $\mathbb{R}$-algebra, where for a homogeneous element $a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ we have $\deg(a \otimes 1) := \deg(a)$. Furthermore, we have

Theorem 4.1. *The set $\mathcal{C} = \{l_1^a \circ l_2 \otimes_R 1 \mid l_1, l_2 \in \mathbb{L}_{\underline{w}}(x); x \in \Lambda(\underline{w})\}$ is a graded cellular basis for $A_{\underline{w}}$.*

Proof. Let us specify a graded cell datum for $A_{\underline{w}}$, as in Definition 2.1. Take

$$\Lambda = \Lambda(\underline{w}) := \{x \in W \mid w \geq x\}$$

partially ordered by reversing the usual Bruhat order. Accordingly, w and e (where e denotes the identity of W) are the minimal and the maximal element in $\Lambda(\underline{w})$, respectively. For each $x \in \Lambda(\underline{w})$, define $T(x) := \mathbb{L}_{\underline{w}}(x)$, i.e., $T(x)$ is the set of all leaves in $\mathbb{T}_{\underline{w}}$ with final target x. We also define $c^x_{l_1 l_2} =: l_1^a \circ l_2 \otimes 1$, for all $l_1, l_2 \in T(x)$ and $x \in \Lambda$. On the other hand, there is a natural degree function

$$\deg : \coprod_{x\in\Lambda(\underline{w})} T(x) \to \mathbb{Z} \tag{4.1}$$

given by the degree of the leaves.

Forgetting the grading for a moment, the cellularity of the basis $\mathcal{C}$ of $A_{\underline{w}}$ is clear from the cellularity of the double leaves basis of $\mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$, which was noticed by Elias and Williamson in [7, Proposition 6.22]. For the graded part, we only need to check that

$$\deg(l_1^a \circ l_2 \otimes 1) = \deg(l_1) + \deg(l_2), \tag{4.2}$$

for all $x \in \Lambda(\underline{w})$ and $l_1, l_2 \in T(x)$. This follows by the fact that composition of morphisms is additive with respect to degree, and to the fact that adjunctions preserve degrees, that is, $\deg(l^a) = \deg(l)$, for all $l \in \mathbb{T}_{\underline{w}}$. $\square$

Given the details of the graded cellular structure of $A_{\underline{w}}$ we have automatically defined the corresponding graded cell $A_{\underline{w}}$-modules and graded simple $A_{\underline{w}}$-modules, as well as the graded decomposition numbers for $A_{\underline{w}}$. However, by the abstract definition of cell modules and their bilinear forms given in Section 2, it is not clear how one ought to work with them. Fortunately, in this case, we can be a little more specific. Let us provide two definitions.

Definition 4.2. Let $\underline{w}$ and $\underline{v}$ be expressions of some elements in W. Given $u \in W$ we say that a double leaf $l_1^a \circ l_2 \in \mathbb{L}_{\underline{v}}^a \cdot \mathbb{L}_{\underline{w}}$ *factors through* u if $l_1 \in \mathbb{L}_{\underline{v}}(u)$ and $l_2 \in \mathbb{L}_{\underline{w}}(u)$.

Definition 4.3. Let $\underline{w}$ and $\underline{v}$ be expressions of some elements in W. For $u \in W$, we define the set $\mathbb{DL}_{<u}(\underline{w}, \underline{v}) \subset \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$ as the span of the double leaves in $\mathbb{L}_{\underline{v}}^a \cdot \mathbb{L}_{\underline{w}}$ that factor through $x < u$.

Let us denote by $\Delta_{\underline{w}}(x)$, $D_{\underline{w}}(x)$, and $d_{\underline{w}}(x, y)$ the graded cell module, the graded simple module, and the graded decomposition number of $A_{\underline{w}}$, for $x, y \in \Lambda(\underline{w})$, respectively, corresponding to the cellular structure determined by the double leaves basis. We now explain the action of $A_{\underline{w}}$ on a cell module. Let $x \in \Lambda(\underline{w})$. By definition, the graded cell module $\Delta_{\underline{w}}(x)$ is the $\mathbb{R}$-vector space spanned by $\mathbb{L}_{\underline{w}}(x)$.

Remark 4.4. If we want to be completely consistent with the notation introduced in Section 2, the cell module must be the $\mathbb{R}$-vector space with basis

$$\{c_l^x \mid l \in \mathbb{L}_{\underline{w}}(x)\}.$$

However, to avoid a subindex catastrophe, we prefer the previous notation.

Let $a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ and $l \in \mathbb{L}_{\underline{w}}(x)$. To determine $l(a \otimes 1) \in \Delta_{\underline{w}}(x)$, we first calculate the expansion of $l \circ a$ in terms of the double leaves basis for $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{x}))$. It is not difficult to see that

$$l \circ a \equiv \sum_{g \in \mathbb{L}_{\underline{w}}(x)} g r_g \mod \mathbb{DL}_{<x}(\underline{w}, \underline{x}), \tag{4.3}$$

for some scalars $r_g \in R$. Then, the action of $A_{\underline{w}}$ on $\Delta_{\underline{w}}(x)$ is given by

$$l(a \otimes 1) = \sum_{g \in \mathbb{L}_{\underline{w}}(x)} g \widehat{r_g} \in \Delta_{\underline{w}}(x). \tag{4.4}$$

In a similar manner, we can describe the bilinear form on $\Delta_{\underline{w}}(x)$ induced by the cellular structure. Let $l_1, l_2 \in \mathbb{L}_{\underline{w}}(x)$. Now, $l_1 \circ l_2^a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{x}))$. Thus, we can expand it in terms of the double leaves basis for $\mathrm{End}^{\mathbb{Z}}(BS(\underline{x}))$. Again, it is not hard to see that

$$l_1^a \circ l_2 \equiv \mathbb{I}_x r_{l_1,l_2} \mod \mathbb{DL}_{<x}(\underline{x}, \underline{x}), \tag{4.5}$$

for some $r_{l_1,l_2} \in R$, and where $\mathbb{I}_{\underline{x}}$ denotes the identity map of $BS(\underline{x})$. Then, the value of the bilinear form $\langle\ ,\ \rangle$ on $\Delta_{\underline{w}}(x)$ at two leaves l_1 and l_2 is $\langle l_1, l_2 \rangle = \widehat{r_{l_1,l_2}} \in \mathbb{R}$. We note that $\deg(\mathbb{I}_x) = 0$ implies that

$$\deg(l_1) + \deg(l_2) = \deg(r_{l_1,l_2}). \tag{4.6}$$

Thus, $\langle l_1, l_2 \rangle = 0$ unless $\deg(l_1) + \deg(l_2) = 0$.

It is a straightforward exercise to confirm that the descriptions of cell modules and bilinear form provided here coincide with those in Section 2. Furthermore, the explicit description given for the bilinear form makes clear that the graded dimension of $D_{\underline{w}}(x)$ coincides with the graded rank of the *local intersection form* $I_{\underline{w},x}$ of $\underline{w}$ at x (see [11, Definition 3.3]). For the sake of brevity, we do not recall here the definition of local intersection form. We only need this concept in order to establish the next lemma.

Let us denote by $\Lambda_0(\underline{w})$ the set that parameterizes the entire set (up to degree shift) of simple modules of $A_{\underline{w}}$, i.e.,

$$\Lambda_0(\underline{w}) = \{x \in \Lambda(\underline{w}) \mid D_{\underline{w}}(x) \neq 0\}. \tag{4.7}$$

Lemma 4.5. *Let $\underline{w}$ be a reduced expression of $w \in W$. Then,*

$$BS(\underline{w}) \cong \bigoplus_{x \in \Lambda_0(\underline{w})} \dim_v D_{\underline{w}}(x) B_x, \tag{4.8}$$

as graded (R, R)-bimodules.

Proof. By [20, Lemma 4.1] the multiplicity of B_x in $BS(\underline{w})$ is given by the graded rank of $I_{\underline{w},x}$. Then, the result follows since the graded rank of $I_{\underline{w},x}$ coincides with $\dim_v D_{\underline{w}}(x)$, for all $x \leq w$. $\square$

Lemma 4.6. *Let $\underline{w}$ be a reduced expression for $w \in W$. Then, there is an isomorphism of graded right $A_{\underline{w}}$-modules*

$$\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R} \cong \Delta_{\underline{w}}(x), \tag{4.9}$$

for all $x \leq w$.

Proof. Note first that $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R}$ has a natural structure of a right $A_{\underline{w}}$-module by composition of morphisms. Concretely, if $g \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x)$ and $a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$, the action of $A_{\underline{w}}$ on $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R}$ is given by

$$(g \otimes 1)(a \otimes 1) = (g \circ a) \otimes 1 \tag{4.10}$$

Furthermore, by Lemma 3.12, $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R}$ is an $\mathbb{R}$-vector space with basis

$$\{(\beta_x \circ l) \otimes 1 \mid l \in \mathbb{L}_{\underline{w}}(x)\},$$

where $\beta_x : BS(\underline{x}) \to R_x$ is the bimodule morphism defined following Lemma 3.11. Since $\Delta_{\underline{w}}(x)$ is defined as the $\mathbb{R}$-vector space with basis $\mathbb{L}_{\underline{w}}(x)$, there is a canonical $\mathbb{R}$-linear isomorphism determined by

$$\begin{array}{rccc} f: & \Delta_{\underline{w}}(x) & \to & \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R} \\ & l & \to & (\beta_x \circ l) \otimes 1 \end{array} \tag{4.11}$$

for all $l \in \mathbb{L}_{\underline{w}}(x)$. Then, by the $\mathbb{R}$-linearity of f, to finish the proof we need to show that

$$f(l(a \otimes 1)) = f(l)(a \otimes 1), \tag{4.12}$$

for all $a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ and $l \in \mathbb{L}_{\underline{w}}(x)$. We prove that both sides of (4.12) are equal to $(\beta_x \circ l \circ a) \otimes 1$. First, note that

$$f(l)(a \otimes 1) = ((\beta_x \circ l) \otimes 1)(a \otimes 1) = (\beta_x \circ l \circ a) \otimes 1,$$

proving that the right side of (4.12) is equal to $(\beta_x \circ l \circ a) \otimes 1$. To prove the other equality, we need the following

Claim 4.7. *If $g \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{x}))$ belongs to $\mathbb{DL}_{<x}(\underline{w}, \underline{x})$ then $\beta_x \circ g = 0$.*

Proof. It is enough to show that

$$\beta_x \circ (l_2^a \circ l_1) = 0, \tag{4.13}$$

for all double leaves $(l_2^a \circ l_1) \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{x}))$ that factor through $u < x$. Let us suppose that there exists a double leaf $(l_2^a \circ l_1) \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{x}))$ that factors through $u < x$ such that $\beta_x \circ (l_2^a \circ l_1) \neq 0$. In particular, we have $\beta_x \circ l_2^a \neq 0$. Note that $\beta_x \circ l_2^a$ belongs to $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{u}), R_x)$, for some reduced expression $\underline{u}$ of u. Therefore, $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{u}), R_x) \neq 0$. This contradicts Corollary 3.13 since $u < x$, proving (4.13) and Claim 4.7. □

Let us return to the proof of the lemma. To conclude the proof, we need to show that $f(l(a \otimes 1)) = (\beta_x \circ l \circ a) \otimes 1$. Write

$$l \circ a \equiv \sum_{g \in \mathbb{L}_{\underline{w}}(x)} g r_g \mod \mathbb{DL}_{<x}(\underline{w}, \underline{x}), \tag{4.14}$$

for some scalars $r_g \in R$. Composing with β_x on the left in (4.14) and using Claim 4.7, we obtain

$$\beta_x \circ l \circ a = \sum_{g \in \mathbb{L}_{\underline{w}}(x)} (\beta_x \circ g) r_g. \tag{4.15}$$

Thus, by reducing modulo R^+ we obtain

$$(\beta_x \circ l \circ a) \otimes 1 = \sum_{g \in \mathbb{L}_{\underline{w}}(x)} (\beta_x \circ g)\widehat{r_g} \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R}. \tag{4.16}$$

On the other hand, by (4.14) we know that

$$l(a \otimes 1) = \sum_{g \in \mathbb{L}_{\underline{w}}(x)} g\widehat{r_g} \in \Delta_{\underline{w}}(x). \tag{4.17}$$

Thus,

$$f(l(a \otimes 1)) = \sum_{g \in \mathbb{L}_{\underline{w}}(x)} (\beta_x \circ g)\widehat{r_g} \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R}. \tag{4.18}$$

Combining (4.16) with (4.18), we conclude that $f(l(a \otimes 1)) = (\beta_x \circ l \circ a) \otimes 1$. This completes the proof of the lemma. □

We are now in a position to interpret the Kazhdan–Lusztig polynomials as graded decomposition numbers.

Theorem 4.8. *Let $\underline{w}$ be a reduced expression of some element $w \in W$. Then,*

$$d_{\underline{w}}(x, y) = h_{x,y}, \tag{4.19}$$

for all $y \in \Lambda_0(\underline{w})$ and $x \leq w$.

Proof. By Lemma 4.5, we have the following isomorphism

$$BS(\underline{w}) \cong \bigoplus_{y \in \Lambda_0(\underline{w})} \dim_v D_{\underline{w}}(y) B_y \tag{4.20}$$

of (R, R)-bimodules. Fix $y \in \Lambda_0(\underline{w})$. Then, we can choose a projector (idempotent) $e_y \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$ whose image is isomorphic to $B_y(k)$, for some $k \in \mathbb{Z}$. We denote by $\hat{e}_y \in A_{\underline{w}}$ its reduction modulo R^+. Consider the idempotent truncation subalgebra

$$\hat{e}_y A_{\underline{w}} \hat{e}_y \cong \mathrm{End}^{\mathbb{Z}}(B_y(k)) \otimes_R \mathbb{R} \cong \mathrm{End}^{\mathbb{Z}}(B_y) \otimes_R \mathbb{R}. \tag{4.21}$$

We have the following two facts about $\hat{e}_y A_{\underline{w}} \hat{e}_y$.

Claim 4.9. *$\hat{e}_y A_{\underline{w}} \hat{e}_y$ has no non-trivial idempotents.*

Proof. First, we note that $\hat{e}_y$ is the identity of $\hat{e}_y A_{\underline{w}} \hat{e}_y$. Suppose that there exists a non-trivial idempotent $e \in \hat{e}_y A_{\underline{w}} \hat{e}_y$. Then, by (4.21), there exists a non-trivial idempotent $E \in \mathrm{End}^{\mathbb{Z}}(B_y) \otimes_R \mathbb{R}$. As is explicated in the proof of [11, Lemma 4.1], we can obtain an idempotent $\widetilde{E} \in \mathrm{End}^{\mathbb{Z}}(B_y)$ which is a lift of E. Since B_y is indecomposable, $\widetilde{E}$ must be trivial. This proves our claim. $\square$

Claim 4.10. *Up to isomorphism and degree shift, $D_{\underline{w}}(y)\hat{e}_y$ is the unique simple $\hat{e}_y A_{\underline{w}} \hat{e}_y$-module. Furthermore, $D_{\underline{w}}(y)\hat{e}_y \cong \Delta_{\underline{w}}(y)\hat{e}_y$.*

Proof. By the previous Claim 4.9 it is clear that $\hat{e}_y A_{\underline{w}} \hat{e}_y$ has a unique simple module. Furthermore, by combining Theorem 2.3 and Theorem 2.4(a) we know that this module is of the form $D_{\underline{w}}(z)\hat{e}_y$, for some $z \in \Lambda_0(\underline{w})$. On the other hand, by definition of e_y, we have the following isomorphism of right R-modules

$$\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) e_y \cong \mathrm{Hom}^{\mathbb{Z}}(B_y(k), R_x), \tag{4.22}$$

for all $x \in W$. Then, Lemma 4.6 implies

$$\Delta_{\underline{w}}(x)\hat{e}_y \cong (\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_x) \otimes_R \mathbb{R})\hat{e}_y \cong \mathrm{Hom}^{\mathbb{Z}}(B_y(k), R_x) \otimes_R \mathbb{R} \tag{4.23}$$

as graded $\mathbb{R}$-vector spaces. Since Soergel's conjecture is known to be true, by taking the graded dimension on both sides of (4.23) we obtain

$$\begin{aligned} \dim_v \Delta_{\underline{w}}(x)\hat{e}_y &= \dim_v \operatorname{Hom}^{\mathbb{Z}}(B_y(k), R_x) \otimes_R \mathbb{R} \\ &= v^k \dim_v \operatorname{Hom}^{\mathbb{Z}}(B_y, R_x) \otimes_R \mathbb{R} \\ &= v^k h_{x,y}, \end{aligned} \tag{4.24}$$

for all $x \in W$. In particular, by putting $x = y$ in (4.24), we have $\dim_v \Delta_{\underline{w}}(y)\hat{e}_y = v^k$. Hence, $\Delta_{\underline{w}}(y)\hat{e}_y \neq 0$. Finally, the triangularity of the graded decomposition numbers (2.8) and Corollary 3.13 allow us to conclude that the unique composition factor of $\Delta_{\underline{w}}(y)\hat{e}_y$ must be $D_{\underline{w}}(y)\hat{e}_y$. As a matter of fact, this also proves that $\Delta_{\underline{w}}(y)\hat{e}_y \cong D_{\underline{w}}(y)\hat{e}_y$. □

With Claim 4.10 at hand it is easy to finish the proof of Theorem 4.8. Actually, by using Theorem 2.4(b), we obtain

$$\begin{aligned} v^k h_{x,y} &= \dim_v \Delta_{\underline{w}}(x)\hat{e}_y \\ &= \dim_v D_{\underline{w}}(y)\hat{e}_y \cdot d(\Delta_{\underline{w}}(x)\hat{e}_y, D_{\underline{w}}(y)\hat{e}_y) \\ &= v^k d(\Delta_{\underline{w}}(x), D_{\underline{w}}(y)) \\ &= v^k d_{\underline{w}}(x,y), \end{aligned}$$

for all $x \leq w$. □

Note that the left hand side of (4.19) depends on the expression $\underline{w}$ whereas the right hand side does not. Thus, Theorem 4.8 seems to claim that $d_{\underline{w}}(x,y)$ does not depend on the choice of the reduced expression $\underline{w}$ of w. The above is not completely correct. Actually, $d_{\underline{w}}(x,y)$ is only defined for $y \in \Lambda_0(\underline{w})$ and $\Lambda_0(\underline{w})$ depends heavily on $\underline{w}$. Then, the existence of $d_{\underline{w}}(x,y)$ depends on $\underline{w}$. What Theorem 4.8 really claims is that once we know that $d_{\underline{w}}(x,y)$ exists (that is, it is defined), its value does not depend on $\underline{w}$ and coincides with $h_{x,y}$.

The previous paragraph may suggest that there are Kazhdan–Lusztig polynomials that can not be interpreted as graded decomposition numbers. Fortunately, this does not hold. Let $x, w \in W$. Assume that $x \leq w$. Then, the graded decomposition number $d_{\underline{w}}(x,w)$ always exists since $w \in \Lambda_0(\underline{w})$, for all reduced expression $\underline{w}$ of w. Thus, $d_{\underline{w}}(x,w) = h_{x,w}$. On the other hand, if $x \nleq w$ then $d_{\underline{w}}(x,w)$ does not make sense. However, in this case we know that $h_{x,w} = 0$. Therefore, this case is irrelevant for our purposes.

5. Monotonicity

In this section, we prove the Monotonicity Conjecture for the coefficients of the Kazhdan–Lusztig polynomials. More precisely, we prove:

Conjecture 5.1. *Let W be any Coxeter group. If $u, v, w \in W$ and $u \leq v \leq w$ then*

$$P_{u,w}(q) - P_{v,w}(q) \in \mathbb{N}[q]. \tag{5.1}$$

In terms of the polynomials $h_{x,w}(v) \in \mathbb{Z}[v]$, the above conjecture is equivalent (via Remark 3.3) to

$$h_{u,w}(v) - v^{l(v)-l(u)} h_{v,w}(v) \in \mathbb{N}[v] \tag{5.2}$$

We prove (5.2) in this section. To do this, we are first interested in a particular leaf.

Lemma 5.2. *Let W be a Coxeter group. Let $u, v \in W$ with $u \leq v$ and let $\underline{v}$ be a reduced expression for v. Then, there is a unique leaf in $\mathbb{L}_{\underline{v}}(u)$ of degree $l(v) - l(u)$.*

Proof. This is a direct consequence of the definition of the leaves and [15, Lemma 5.1] or [4, Proposition 2.3]. □

Let $u, v, w \in W$ with $u \leq v \leq w$. For the rest of the paper, we fix reduced expressions $\underline{u}$, $\underline{v}$, and $\underline{w}$ for u, v, and w, respectively. We denote by G_v^u the leaf in the above theorem, and refer to it as the *largest leaf* from $BS(\underline{v})$ to $BS(\underline{u})$. It follows directly from the construction of the leaves that for all $f \in \mathbb{L}_{\underline{v}}(u)$,

$$\begin{aligned} \deg(f) &= n_m(f) - n_j(f) \\ l(v) - l(u) &= n_m(f) + n_j(f), \end{aligned} \tag{5.3}$$

where $n_m(f)$ (resp. $n_j(f)$) denotes the number of times that morphisms of type m_s (resp. j_s) appear in the construction of leaf f. In particular, if we set $f = G_v^u$ in (5.3) and subtract the resulting equations, we obtain

$$n_j(G_v^u) = 0, \tag{5.4}$$

since $\deg(G_v^u) = l(v) - l(u)$. That is, morphisms of type j_s do not appear in the construction of the largest leaf. Consequently, G_v^u is the leaf in $\mathbb{L}_{\underline{v}}(u)$ of largest degree, which justifies its name. We define a map, $\Phi_{\underline{w}}^{u,v}$, from $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_v) \otimes_R \mathbb{R}$ to $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_u) \otimes_R \mathbb{R}$, determined in the standard basis by

$$\begin{array}{rccc} \Phi_{\underline{w}}^{u,v}: & \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_v) \otimes_R \mathbb{R} & \to & \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_u) \otimes_R \mathbb{R} \\ & (\beta_v \circ l) \otimes 1 & \to & (\beta_u \circ G_v^u \circ l) \otimes 1 \end{array} \tag{5.5}$$

for all $l \in \mathbb{L}_{\underline{w}}(v)$. The map $\Phi_{\underline{w}}^{u,v}$ will be the key to prove the monotonicity conjecture at the end of this section. In order to know the properties of $\Phi_{\underline{w}}^{u,v}$, we need some notation and a technical lemma.

Definition 5.3. Let $\underline{w}$ be an expression of some element in W. For $b \in BS(\underline{w})$, we define $\mathrm{coef}_{1^\otimes}(b)$ as the coefficient of $1^\otimes$ in the expansion of b in terms of the basis of $BS(\underline{w})$ described in (3.11).

Lemma 5.4. *Let $u, v, w \in W$ with $u \leq v \leq w$. If $h \in \mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$ belongs to $\mathbb{DL}_{<v}(\underline{w}, \underline{v})$, then,*

$$G_v^u \circ h \equiv \sum_{g \in \mathbb{L}_{\underline{w}}(u)} g r_g \mod \mathbb{DL}_{<u}(\underline{w}, \underline{u}), \tag{5.6}$$

for some scalars $r_g \in R^+$.

Proof. Let $l_2^a \circ l_1$ be a double leaf morphism in $\mathrm{Hom}^{\mathbb{Z}}(BS(\underline{w}), BS(\underline{v}))$ that factors through $z < v$. Write

$$G_v^u \circ (l_2^a \circ l_1) = \sum_{g \in \mathbb{L}_{\underline{w}}(u)} g r_g + f, \tag{5.7}$$

for some scalars $r_g \in R$ and some morphism $f \in \mathbb{DL}_{<u}(\underline{w}, \underline{u})$. To finish the proof, we need to show that $r_g \in R^+$, for all $g \in \mathbb{L}_{\underline{w}}(u)$.

We recall the indexing (given in Section 3) of the leaves by two sequences of zeros and ones, and define

$$\mathcal{J}_{\underline{w}}^u = \{\boldsymbol{j} \in \{0,1\}^{l(\underline{w})} \mid \text{there is a leaf } g \in \mathbb{L}_{\underline{w}}(u) \text{ such that } g = f_{\boldsymbol{i}}^{\boldsymbol{j}}\}. \tag{5.8}$$

By [15, Lemma 5.1], we know that each $\boldsymbol{j} \in \mathcal{J}_{\underline{w}}^u$ determines a unique leaf in $\mathbb{L}_{\underline{w}}(u)$. Index $\mathcal{J}_{\underline{w}}^u = \{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_m\}$ so that $\boldsymbol{j}_k < \boldsymbol{j}_{k+1}$ ($<$ here denotes the lexicographical order), for all $1 \leq k < m$. We denote by $g^{\boldsymbol{j}_k}$ the leaf determined by $\boldsymbol{j}_k$. With this notation at hand, we can rewrite (5.7) as

$$G_v^u \circ (l_2^a \circ l_1) = \sum_{k=1}^{m} g^{\boldsymbol{j}_k} r_k + f, \tag{5.9}$$

where $r_k := r_{g^{\boldsymbol{j}_k}} \in R$. We need the following

Claim 5.5. *For all $b \in BS(\underline{w})$ we have*

$$coef_{1\otimes}((G_v^u \circ l_2^a \circ l_1)(b)), \text{ and } coef_{1\otimes}(f(b)) \in R^+. \tag{5.10}$$

Proof. Let $b \in BS(\underline{w})$. Since $l_2^a \circ l_1$ factors through $z < v$, a morphism of type ϵ_s occurs in l_2^a. Then, by looking at the definition of the morphism ϵ_s, it is clear that

$$\mathrm{coef}_{1\otimes}((l_2^a \circ l_1)(b)) \in R^+. \tag{5.11}$$

On the other hand, since only morphisms of type m_s occur in G_v^u, (5.11) implies

$$\mathrm{coef}_{1\otimes}((G_v^u \circ l_2^a \circ l_1)(b)) \in R^+.$$

The same argument proves that $\mathrm{coef}_{1\otimes}(f(b)) \in R^+$. □

Let us return to the proof of the lemma. We proceed by induction. If we evaluate (5.9) at $x^{\boldsymbol{j}_1}$, then by Lemma 3.11 and Claim 5.5, we find that

$$r_1 = \mathrm{coef}_{1\otimes}((G_v^u \circ l_2^a \circ l_1)(x^{\boldsymbol{j}_1})) - \mathrm{coef}_{1\otimes}(f(x^{\boldsymbol{j}_1})) \in R^+, \tag{5.12}$$

which provides the basis of our induction. Now, let $1 < n \leq m$ and assume that we have already proven that $r_k \in R^+$, for all $1 \leq k < n$. Evaluating (5.9) at $x^{\boldsymbol{j}_n}$, and again by using Lemma 3.11, we obtain

$$r_n = \mathrm{coef}_{1\otimes}((G_v^u \circ l_2^a \circ l_1)(x^{\boldsymbol{j}_n})) - \sum_{k=1}^{n-1} \mathrm{coef}_{1\otimes}(g^{\boldsymbol{j}_k}(x^{\boldsymbol{j}_n}))r_k - \mathrm{coef}_{1\otimes}(f(x^{\boldsymbol{j}_n})). \tag{5.13}$$

By Claim 5.5 and our inductive hypothesis, we know that the right hand side of (5.13) belongs to R^+. Therefore, $r_n \in R^+$. This completes the induction and the proof of the lemma. □

Proposition 5.6. *Let $u, v, w \in W$ with $u \leq v \leq w$. Then, $\Phi_{\underline{w}}^{u,v}$ is an homogeneous $A_{\underline{w}}$-module homomorphism of degree $l(v) - l(u)$.*

Proof. The claim that $\Phi_{\underline{w}}^{u,v}$ is homogeneous with degree $l(v) - l(u)$ is a direct consequence of the definitions as well as the fact that $\deg(G_v^u) = l(v) - l(u)$. Now, in order to prove that $\Phi_{\underline{w}}^{u,v}$ is an $A_{\underline{w}}$-module homomorphism, it is enough to show that

$$\Phi_{\underline{w}}^{u,v}(((\beta_v \circ l) \otimes 1)(a \otimes 1)) = \Phi_{\underline{w}}^{u,v}((\beta_v \circ l) \otimes 1)(a \otimes 1) \tag{5.14}$$

for all $l \in \mathbb{L}_{\underline{w}}(v)$ and $a \in \mathrm{End}^{\mathbb{Z}}(BS(\underline{w}))$. We prove that both sides of (5.14) are equal to $(\beta_u \circ G_v^u \circ l \circ a) \otimes 1$. The desired equality for the right hand side of (5.14) is easy because by the definition of $\Phi_{\underline{w}}^{u,v}$, we have

$$\begin{aligned} \Phi_{\underline{w}}^{u,v}((\beta_v \circ l) \otimes 1)(a \otimes 1) &= ((\beta_u \circ G_v^u \circ l) \otimes 1)(a \otimes 1) \\ &= (\beta_u \circ G_v^u \circ l \circ a) \otimes 1 \end{aligned} \tag{5.15}$$

To obtain the equality for the left hand side of (5.14), we first write

$$l \circ a \equiv \sum_{f \in \mathbb{L}_{\underline{w}}(v)} f r_f \mod \mathbb{DL}_{<v}(\underline{w}, \underline{v}), \tag{5.16}$$

for some scalars $r_f \in R$. By Claim 4.7, we know that

$$((\beta_v \circ l) \otimes 1)(a \otimes 1) = \sum_{f \in \mathbb{L}_{\underline{w}}(v)} \beta_v \circ f r_f \otimes 1 \tag{5.17}$$

Thus, by applying $\Phi_{\underline{w}}^{u,v}$ to (5.17), we have

$$\Phi_{\underline{w}}^{u,v}(((\beta_v \circ l) \otimes 1)(a \otimes 1)) = \sum_{f \in \mathbb{L}(v)} (\beta_u \circ G_u^v \circ f r_f) \otimes 1. \tag{5.18}$$

On the other hand, by composing (5.16) with G_u^v on the left and using Lemma 5.4, we obtain

$$G_v^u \circ l \circ a \equiv \sum_{f \in \mathbb{L}_{\underline{w}}(v)} G_v^u \circ f r_f + \sum_{g \in \mathbb{L}_{\underline{w}}(u)} g \rho_g \mod \mathbb{DL}_{<u}(\underline{w}, \underline{u}), \tag{5.19}$$

for some scalars $\rho_g \in R^+$. Now, by composing with β_u to the left in (5.19) and using Claim 4.7, we have

$$\beta_u \circ G_v^u \circ l \circ a = \sum_{f \in \mathbb{L}_{\underline{w}}(v)} \beta_u \circ G_v^u \circ f r_f + \sum_{g \in \mathbb{L}_{\underline{w}}(u)} \beta_u \circ g \rho_g. \tag{5.20}$$

Following this, by reducing modulo R^+ and using the fact that $\rho_g \in R^+$ for all $g \in \mathbb{L}_{\underline{w}}(u)$, we obtain

$$\beta_u \circ G_v^u \circ l \circ a \otimes 1 = \sum_{f \in \mathbb{L}_{\underline{w}}(v)} \beta_u \circ G_v^u \circ f r_f \otimes 1. \tag{5.21}$$

Finally, combining (5.18) with (5.21), we obtain

$$\Phi_{\underline{w}}^{u,v}(((\beta_v \circ l) \otimes 1)(a \otimes 1)) = \beta_u \circ G_v^u \circ l \circ a \otimes 1.$$

This completes the proof of the proposition. $\square$

Proposition 5.7. *Let $u, v, w \in W$ with $u \leq v \leq w$. The map $\Phi_{\underline{w}}^{u,v}$ is injective.*

Proof. As in the proof of Lemma 5.4 we define subsets of $\{0,1\}^{l(\underline{w})}$,

$$\mathcal{J}_{\underline{w}}^u := \{\boldsymbol{j}_1 < \boldsymbol{j}_2 < \ldots < \boldsymbol{j}_m\} \text{ and } \mathcal{J}_{\underline{w}}^v := \{\boldsymbol{\iota}_1 < \boldsymbol{\iota}_2 < \ldots < \boldsymbol{\iota}_\mu\},$$

where $\boldsymbol{j}_k \in \mathcal{J}_{\underline{w}}^u$ (resp. $\boldsymbol{\iota}_\kappa \in \mathcal{J}_{\underline{w}}^v$) if there exists a leaf $g \in \mathbb{L}_{\underline{w}}(u)$ (resp. $l \in \mathbb{L}_{\underline{w}}(v)$) such that $g = f_{\boldsymbol{i}}^{\boldsymbol{j}_k}$ (resp. $l = f_{\boldsymbol{i}}^{\boldsymbol{\iota}_\kappa}$). We denote by $g^{\boldsymbol{j}_k}$ (resp. $l^{\boldsymbol{\iota}_\kappa}$) the leaf corresponding to $\boldsymbol{j}_k$ (resp. ι_κ). Given $\boldsymbol{\iota}_\kappa \in \mathcal{J}_{\underline{w}}^v$ we can write

$$G_v^u \circ l^{\boldsymbol{\iota}_\kappa} \equiv \sum_{\boldsymbol{j}_k \in \mathcal{J}_{\underline{w}}^u} g^{\boldsymbol{j}_k} r_{\boldsymbol{j}_k, \boldsymbol{\iota}_\kappa} \mod \mathbb{DL}_{<u}(\underline{w}, \underline{u}), \tag{5.22}$$

for some $r_{\boldsymbol{j}_k, \boldsymbol{\iota}_\kappa} \in R$. By using the same recursive argument as the one utilized in the proof of Lemma 5.4 we can conclude

$$r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa} \in R^+ \text{ (if } \boldsymbol{j}_k < \boldsymbol{\iota}_\kappa), \quad \boldsymbol{\iota}_\kappa \in \mathcal{J}^u_{\underline{w}} \quad \text{and} \quad r_{\boldsymbol{\iota}_\kappa,\boldsymbol{\iota}_\kappa} = 1. \tag{5.23}$$

We now suppose that

$$\Phi^{u,v}_{\underline{w}}\left(\sum_{\boldsymbol{\iota}_\kappa \in \mathcal{J}^v_{\underline{w}}} [(\beta_v \circ l^{\boldsymbol{\iota}_\kappa}) \otimes 1]\, \rho_{\boldsymbol{\iota}_\kappa} \right) = 0, \tag{5.24}$$

for some $\rho_{\boldsymbol{\iota}_\kappa} \in \mathbb{R}$. To prove the proposition we need to show that $\rho_{\boldsymbol{\iota}_\kappa} = 0$, for all $\boldsymbol{\iota}_\kappa \in \mathcal{J}^v_{\underline{w}}$. By definition of the map $\Phi^{u,v}_{\underline{w}}$ and (5.24) we have

$$\sum_{\boldsymbol{\iota}_\kappa \in \mathcal{J}^v_{\underline{w}}} [(\beta_u \circ G^u_v \circ l^{\boldsymbol{\iota}_\kappa}) \otimes 1]\, \rho_{\boldsymbol{\iota}_\kappa} = 0. \tag{5.25}$$

On the other hand, by combining Claim 4.7, (5.22) and (5.23) we obtain

$$(\beta_u \circ G^u_v \circ l^{\boldsymbol{\iota}_\kappa}) \otimes 1 = \sum_{\substack{\boldsymbol{j}_k \in \mathcal{J}^u_{\underline{w}} \\ \boldsymbol{j}_k \geq \boldsymbol{\iota}_\kappa}} \left[(\beta_u \circ g^{\boldsymbol{j}_k}) \otimes 1\right] \widehat{r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa}}, \tag{5.26}$$

where $\widehat{r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa}} \in \mathbb{R}$ denotes the reduction modulo R^+ of $r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa}$. By replacing (5.26) into (5.25) and by reordering the sum, we get

$$\sum_{\boldsymbol{j}_k \in \mathcal{J}^u_{\underline{w}}} \left[(\beta_u \circ g^{\boldsymbol{j}_k}) \otimes 1\right] \left(\sum_{\substack{\boldsymbol{\iota}_\kappa \in \mathcal{J}^v_{\underline{w}} \\ \boldsymbol{\iota}_\kappa \leq \boldsymbol{j}_k}} \widehat{r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa}} \rho_{\boldsymbol{\iota}_\kappa} \right) = 0. \tag{5.27}$$

Since, $\{(\beta_u \circ g^{\boldsymbol{j}_k}) \otimes 1 | \boldsymbol{j}_k \in \mathcal{J}^u_{\underline{w}}\}$ is a basis for $\operatorname{Hom}^{\mathbb{Z}}(BS(\underline{w}), R_u) \otimes_R \mathbb{R}$, we conclude that

$$\sum_{\substack{\boldsymbol{\iota}_\kappa \in \mathcal{J}^v_{\underline{w}} \\ \boldsymbol{\iota}_\kappa \leq \boldsymbol{j}_k}} \widehat{r_{\boldsymbol{j}_k,\boldsymbol{\iota}_\kappa}} \rho_{\boldsymbol{\iota}_\kappa} = 0, \tag{5.28}$$

for all $\boldsymbol{j}_k \in \mathcal{J}^u_{\underline{w}}$. We recall from (5.23) that $\mathcal{J}^v_{\underline{w}} \subset \mathcal{J}^u_{\underline{w}}$. Then, by considering the equations in (5.28) associated to elements in $\mathcal{J}^v_{\underline{w}}$ we obtain an homogeneous system of $\mu = |\mathcal{J}^v_{\underline{w}}|$ equations with μ unknowns $\rho_{\boldsymbol{\iota}_1}, \ldots, \rho_{\boldsymbol{\iota}_\mu}$. The matrix associated to this system (when suitably ordered) is unitriangular. Therefore, it has a unique solution $\rho_{\boldsymbol{\iota}_1} = \ldots = \rho_{\boldsymbol{\iota}_\mu} = 0$. Hence, the map $\Phi^{u,v}_{\underline{w}}$ is injective. □

Theorem 5.8. *Let $u, v, w \in W$ with $u \leq v \leq w$. Then,*

$$h_{u,w}(v) - v^{l(v)-l(u)} h_{v,w}(v) \in \mathbb{N}[v]. \tag{5.29}$$

That is, the Monotonicity Conjecture holds for arbitrary Coxeter groups.

Proof. A direct consequence of Lemma 4.6, Proposition 5.6 and Proposition 5.7 is that $\Delta_{\underline{w}}(v)\langle l(v)-l(u)\rangle$ can be seen as a graded right $A_{\underline{w}}$-submodule of $\Delta_{\underline{w}}(u)$. Therefore,

$$d_{\underline{w}}(u,y) - v^{l(v)-l(u)} d_{\underline{w}}(v,y) \in \mathbb{N}[v], \tag{5.30}$$

for all $y \in \Lambda_0(\underline{w})$. Then, Theorem 4.8 implies

$$h_{u,y}(v) - v^{l(v)-l(u)} h_{v,y}(v) \in \mathbb{N}[v], \tag{5.31}$$

for all $y \in \Lambda_0(\underline{w})$. Since we know that $w \in \Lambda_0(\underline{w})$, (5.29) is obtained from (5.31) by putting $y = w$. $\square$

Acknowledgments

We would like to thank Nicolas Libedinsky for many stimulating conversations on topics concerning this paper. We also thank Steen Ryom-Hansen for comments that greatly improved the manuscript. Finally, we thank the anonymous referee for his careful reading of our manuscript and his many insightful comments and suggestions.

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