Journal of Algebra 490 (2017) 518–554

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Global Okounkov bodies for Bott–Samelson varieties ☆

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A R T I C L E I N F O

Article history:
Received 1 July 2016
Available online 5 August 2017
Communicated by Steven Dale Cutkosky

Keywords:
Bott–Samelson variety
Mori dream space
Linear series
Okounkov body

A B S T R A C T

We use the theory of Mori dream spaces to prove that the global Okounkov body of a Bott–Samelson variety with respect to a natural flag of subvarieties is rational polyhedral. As a corollary, Okounkov bodies of effective line bundles over Schubert varieties are shown to be rational polyhedral. In particular, it follows that the global Okounkov body of a flag variety G/B is rational polyhedral. As an application we show that the asymptotic behaviour of dimensions of weight spaces in section spaces of line bundles is given by the volume of polytopes.

Introduction

Okounkov bodies were first introduced by A. Okounkov in his famous paper [20] as a tool for studying multiplicities of group representations. The idea is that one should

☆ The first author was supported by DFG grant BA 1559/6-1. The second author was supported by the DFG Priority Programme 1388 "Representation Theory".

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http://dx.doi.org/10.1016/j.jalgebra.2017.07.018
0021-8693/

be able to approximate these multiplicities by counting the number of integral points in a certain convex body in $\mathbb{R}^n$. More precisely, the setting is the following. Let G be a complex reductive group which acts as automorphisms on an effective line bundle L over a projective variety X, and hence defines a representation on the space of sections $H^0(X, L^k)$ for each integral power, L^k, of L. For an n-dimensional variety, Okounkov constructs a convex compact set $\Delta \subseteq \mathbb{R}^n$ whose most important property can be interpreted as follows: for each irreducible finite-dimensional representation V_λ the multiplicity $m_{k\lambda,k} := \dim \operatorname{Hom}_G(V_{k\lambda}, H^0(X, L^k))$ of $V_{k\lambda}$ in $H^0(X, L^k)$ is "asymptotically given" by the volume of the convex body $\Delta_\lambda := \Delta \cap H_\lambda$. Here, λ—the so-called *highest weight*—is a parameter and $H_\lambda \subseteq \mathbb{R}^{n+1}$ is a certain affine subspace. Concretely,

$$\lim_{k\to\infty} \frac{m_{k\lambda,k}}{k^m} = \operatorname{vol}_m(\Delta_\lambda), \tag{1}$$

where m is the dimension of Δ_λ, and the volume on the right hand side denotes the m-dimensional Euclidean volume–normalized by a certain sublattice of $\mathbb{Z}^m$–of Δ_λ. In fact, this gives a Euclidean interpretation of a Duistermaat–Heckman-measure, cf. [20]. An approximation of the integral $\operatorname{vol}_m(\Delta_\lambda)$ by Riemann sums yields that the multiplicity $m_{k\lambda,k}$ is asymptotically given by the number of points of the set $\Delta_\lambda \cap \frac{1}{k}\mathbb{Z}^m$.

The construction of the body Δ is purely geometric and depends on a choice of a flag $Y_\bullet$, $Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_0 = X$ of irreducible subvarieties of X, and the "successive orders of vanishing" of unipotent invariant sections $s \in H^0(X, L^k)$ along this flag. It was later realized by Kaveh and Khovanskii ([9]), and independently by Lazarsfeld and Mustaţă, ([15]), that Okounkov's construction makes sense for more general subseries of the section ring $R(X, L)$ of a line bundle over a variety X, and that the asymptotics of dimensions of linear series can be expressed as volumes of convex bodies. Specifically, the analog of (1) for the complete linear series of a big line bundle L is given by the identity

$$\lim_{k\to\infty} \frac{h^0(X, L^k)}{n!k^n} = \frac{1}{n!}\operatorname{vol}_n(\Delta_{Y_\bullet}(L)),$$

where $\Delta_{X_\bullet}(L)$ denotes the Okounkov body of the line bundle L with respect to the flag $Y_\bullet$.

The above formula shows in particular that the volume of the Okounkov body is an invariant of the line bundle L, and thus does not depend on the choice of the flag $Y_\bullet$. However, the shape of $\Delta_{X_\bullet}(L)$ depends heavily on the flag, and it is a notoriously hard problem to explicitly describe these bodies, or even to show that they possess some nice properties, such as being polyhedral. A yet more difficult problem is to determine the global Okounkov body $\Delta_{Y_\bullet}(X)$ of a variety X (cf. [15]), which is a convex cone in $\mathbb{R}^n \times N^1(X)_\mathbb{R}$ such that for each big divisor D the fibre of the second projection over $[D]$ is exactly $\Delta_{Y_\bullet}(D)$.

Returning to the original motivation by Okounkov of studying multiplicities of representations, there is also another approach to describing multiplicities by counting lattice points in convex bodies, namely Littelmann's construction of string polytopes ([16]). The setting here is the following. Let G, again, be a complex reductive group, and let $H \subseteq G$ be a maximal torus in G. Then any irreducible finite-dimensional G-representation V_λ admits a basis of weight vectors with respect to H, and this basis is parametrized by the integral points in a rational polytope C^λ, the *string polytope* of V_λ. Moreover, the approximative lattice counting problem is even exact here. Since the irreducible representations V_λ can be realized as section spaces $H^0(X, L_\lambda)$, where $X = G/B$ for a Borel subgroup $B \subseteq G$, and L_λ is a line bundle over X, it would be interesting to recover Littelmann's string polytopes C^λ as Okounkov bodies, or at least to construct rational polyhedral Okounkov bodies which describe asymptotic multiplicities of weight spaces.

In the present paper we study both problems described above—namely the Okounkov bodies for complete linear series, and the asymptotics of weight multiplicities—for general Bott–Samelson varieties $Z = Z_w$ (given by a reduced expression w for an element $\overline{w}$ in the Weyl group of G), that is, Bott–Samelson varieties which desingularize some Schubert variety X_w in a flag variety G/B.

Our main result is the following

Theorem A. *Let $Z = Z_w$ be a Bott–Samelson variety defined by a reduced expression w of an element $\overline{w}$ in the Weyl group of G. Then there exists a natural flag $Y_\bullet$ on Z, the so called* vertical flag*, such that the corresponding global Okounkov-body $\Delta_{Y_\bullet}(Z)$ is rational polyhedral.*

Among other reasons, one is interested in rational polyhedrality of Okounkov bodies as a necessary condition of finite generation of the valuation semi-groups $\Gamma_{Y_\bullet}(D)$ for divisors D, which in turn would yield toric degenerations of X by [1]. It would be nice to know whether in our situation the valuation semi-groups are indeed finitely generated. However, this is unclear to the authors.

In order to facilitate cleanness of presentation we separate the proof into two steps, proving a result formulated in the more general context of Mori dream spaces.

Any Mori dream space X admits finitely many small $\mathbb{Q}$-factorial modifications such that any movable divisor D on X is the pullback of a nef divisor under one of these modifications. In particular, for any effective divisor E, all necessary flips in the E-MMP exist and terminate. We define an admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ to be *good*, if

(1) Y_i is a Mori dream space for each $0 \leq i \leq n$, and for $i = 1, \ldots, n$, Y_i defines a Cartier divisor of Y_i, and
(2) any small $\mathbb{Q}$-factorial modification $f : Y_i \dashrightarrow Y_i'$ restricts to a small $\mathbb{Q}$-factorial modification of Y_{i+1}.

We prove the following result.

Theorem B. *Let X be a Mori dream space and assume that there exists a good flag $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$. Then the global Okounkov body $\Delta_{Y_\bullet}(X)$ of X with respect to the flag $Y_\bullet$ is a rational polyhedral cone.*

Note that the existence of a good flag is in general very hard to check. In the case of a Mori dream surface S, condition (2) is vacuous, so S admits a good flag if and only if it contains a rational curve. As far as the authors know, apart from trivial examples like projective space, the following theorem provides the only known instance in higher dimensions. It is not unreasonable to expect that there are other Mori dream spaces also admitting a good flag and we hope to determine those in the future. We discuss the similar looking result from [18] in Remark 3.6.

Theorem C. *Let $Z = Z_w$ be a Bott–Samelson variety defined by a reduced expression w of an element $\overline{w}$ in the Weyl group of G.*

(i) The variety Z is log-Fano and hence a Mori dream space. (Theorem 2.2)

(ii) The vertical flag $Y_\bullet$ on Z is good. (Proposition 4.7)

As a consequence, all Okounkov bodies $\Delta_{Y_\bullet}(L)$ of line bundles L over Z are rational polyhedral. Using the desingularization $Z_w \longrightarrow X_w$ of the Schubert variety X_w, we also see that line bundles over Schubert varieties admit rational polyhedral Okounkov bodies.

On the representation-theoretic side, we obtain Okounkov bodies describing weight multiplicities. Indeed, the flag $Y_\bullet$ of subvarieties is B-invariant, which allows for the construction of affine subspaces H_μ mentioned before. We then get the following result on asymptotics of weight multiplicities in a section ring $R(Z, L)$.

Theorem D. *Let L be an effective line bundle over the Bott–Samelson variety Z. Let $H \subseteq B$ be a torus contained in a maximal torus of G lying in B, and let μ be an rational H-weight. Then there exists an affine subspace H_μ (in $\mathbb{R}^{n+1}$, where $n = \dim Z$) such that the asymptotics of the multiplicity function $m_{k\mu,k}$ defined above is given by*

$$\lim_{k\to\infty} \frac{m_{k\mu,k}}{k^m} = vol_m(\Delta_{Y_\bullet}(L) \cap H_\mu),$$

where $\Delta_{Y_\bullet}(L)$ is the rational polyhedral Okounkov body of L.

If we apply this to the situation when the torus H is a maximal torus, Z is of maximal dimension, and thus admits a birational morphism $f : Z \longrightarrow G/B$ to the flag variety of G, and $L = f^*(L_\lambda)$ is the pull-back of the line bundle L_λ over G/B, we obtain the following corollary, which can be seen as an analogue of Littelmann's result which describes weight multiplicities using string polytopes.

Corollary E. *Let $V_\lambda \cong H^0(G/B, L_\lambda)$ be the irreducible G-representation of highest weight λ. If μ is rational weight, let $m_{k\mu,k}$, for $k\mu$ integral, denote the multiplicity of the weight $k\mu$ in the G-module $V_{k\lambda}$. Then there exists an $m \in \mathbb{N}$ such that*

$$\lim_{k\to\infty} \frac{m_{k\mu,k}}{k^m} = vol_m(\Delta_{Y_\bullet}(f^*(L_\lambda)) \cap H_\mu).$$

We would finally like to mention the recent preprint by Postinghel–Urbinati ([21]), where the authors show that any Mori dream space X admits a birational model $h : \overline{X} \to X$ with an admissible flag $\overline{Y}_\bullet$ such that the corresponding global Okounkov body of X is rational polyhedral. It is not clear to us, however, whether our valuation can be seen as a special case of theirs.

The present paper is organized as follows: we begin by recalling basic facts about Okounkov bodies and Bott–Samelson varieties in sections 1 and 2, respectively. The general result on Mori dream spaces admitting good flags is proved in section 3. Finally, in section 4 the result is applied to Bott–Samelson varieties, furthermore we address there the representation-theoretic consequences.

We work throughout over the complex numbers $\mathbb{C}$ as our base field.

Acknowledgements. The authors would like to thank Thomas Bauer and Rob Lazarsfeld for helpful discussions and suggestions, as well as Jesper Funch Thomsen for having answered several questions about Bott–Samelson varieties. We would also like to thank Dave Anderson for interesting remarks on an earlier version of this paper. Finally, we are very grateful to the referee for useful comments and suggestions that helped improve the exposition.

1. Okounkov bodies

For the convenience of the reader not familiar with the construction of Okounkov bodies we give here a quick overview. For a thorough discussion, we refer the reader to [9] and [15].

To a graded linear series $W_\bullet$ on a normal projective variety X of dimension n we want to assign a convex subset of $\mathbb{R}^n$ carrying information on $W_\bullet$. In practice, more often than not, $W_\bullet$ will be the complete graded linear series $\bigoplus_k H^0(X, \mathcal{O}_X(kD))$ corresponding to an effective divisor D. However, at times it is convenient to work with an arbitrary linear series associated to some divisor D, i.e., a series $W_\bullet = \{W_k\}$ of subspaces $W_k \subseteq H^0(X, \mathcal{O}_X(kD))$ satisfying the condition $W_k \cdot W_l \subseteq W_{k+l}$. The construction will depend on the choice of a valuation-like function

$$\nu : \bigsqcup_{k\geq 0} W_k \setminus \{0\} \longrightarrow \mathbb{Z}^n.$$

Instead of recounting the conditions on ν, we describe a certain type of valuation which automatically satisfies these conditions. To this end, let

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

be a flag of irreducible subvarieties such that $\mathrm{codim}_X(Y_i) = i$ and such that Y_n is a smooth point of each Y_i. Then, for a section $s \in W_k \subseteq H^0(X, \mathcal{O}_X(kD))$, we set $\nu_1(s) := \mathrm{ord}_{Y_1}(s)$. If we choose a local equation for Y_1, we obtain a unique section $\widetilde{s_1} \in H^0(X, \mathcal{O}_X(kD - \nu_1(s)Y_1))$ not vanishing identically along Y_1, and thus determining a section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$. We then set $\nu_2(s) = \mathrm{ord}_{Y_2}(s_1)$, and proceed as before to obtain the valuation vector $\nu_{Y_\bullet}(s) = (\nu_1(s), \ldots, \nu_n(s))$.

One then defines the valuation semi-group of $W_\bullet$ with respect to $Y_\bullet$ as

$$\Gamma_{Y_\bullet}(W_\bullet) := \left\{ (\nu_{Y_\bullet}(s), m) \in \mathbb{Z}^{n+1} \mid 0 \neq s \in W_m \right\}.$$

Furthermore, we define the Okounkov body of $W_\bullet$ as

$$\Delta_{Y_\bullet}(W_\bullet) := \Sigma(\Gamma_{Y_\bullet}(W_\bullet)) \cap (\mathbb{R}^n \times \{1\})$$

where $\Sigma(\Gamma_{Y_\bullet}(W_\bullet))$ denotes the closed convex cone in $\mathbb{R}^{n+1}$ spanned by $\Gamma_{Y_\bullet}(W_\bullet)$.

If $W_\bullet$ is the complete graded linear series of a divisor D, we write $\Delta_{Y_\bullet}(D)$ for the Okounkov body of $W_\bullet$. In this case, by [15, Theorem 2.3], we have the important identity

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!}\mathrm{vol}_X(D),$$

showing in particular, that the volume of the body $\Delta_{Y_\bullet}(D)$ is independent of the choice of the flag $Y_\bullet$. Another important observation made in [15] is that even the shape of the Okounkov body $\Delta_{Y_\bullet}(D)$ for a divisor D only depends on the numerical equivalence class of D. It is therefore a natural question how the bodies $\Delta_{Y_\bullet}(D)$ change as $[D]$ varies in the Néron–Severi vector space $N^1(X)_{\mathbb{R}}$. An answer to this question is given in [15, Theorem 4.5] by proving the existence of the global Okounkov body: there exists a closed convex cone

$$\Delta_{Y_\bullet}(X) \subset \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

such that for each big divisor D the fibre of the second projection over $[D]$ is exactly $\Delta_{Y_\bullet}(D)$.

The concrete determination or even the description of geometric properties of Okounkov bodies associated to some graded linear series is extremely difficult in general. As is to be expected this will be even more true of the global Okounkov body of a given variety. In particular, it is an intriguing question under which conditions on X it is possible to pick a flag such that the corresponding global Okounkov body is rational polyhedral. Already in [15] this was shown to be possible for toric varieties. Based on this evidence it is conjectured to work also for any Mori dream space X. In [22], we introduce a possible technique to prove rational polyhedrality of global Okounkov bodies by constructing so-called Minkowski bases on X. Work in progress hints at the feasibility of this approach for any Mori dream space. In this paper however, we use a more direct strategy to prove the rational polyhedrality of global Okounkov bodies with respect to a natural choice of flag for a special class of Mori dream spaces, namely Bott–Samelson varieties, which we introduce in the following section.

Let us make a small remark on how the construction by Littelmann mentioned in the introduction compares to Okounkov bodies. Littelmann's string polytopes are constructed by purely algebraic and combinatorial means, notably using quantum enveloping algebras of Lie algebras, and the result thus only shows formal analogies with the outcome of Okounkov's approach. However, since—by the Borel–Weil theorem—every irreducible G-module V_λ can be realized as the space of sections $H^0(X, L_\lambda)$ of a line bundle L_λ over a flag variety $X := G/B$, where B is a Borel subgroup of G, Okounkov's approach makes sense for the study of asymptotics of weight spaces in the section ring $R(X, L_\lambda)$. For the approach to work, the flag $Y_\bullet$ should consist of H-invariant subvarieties. A natural candidate for such a flag would then be a flag of Schubert varieties, and indeed this approach was taken by Kaveh in [11]. For technical reasons, notably for having a flag of Cartier divisors, Kaveh passed to a Bott–Samelson resolution $Z \longrightarrow X$ of X, pulled back L_λ to Z, and replaced the flag $Y_\bullet$ by a flag $Z_\bullet$ of (translations of) Bott–Samelson subvarieties of Z. The main result in [11] is that Littelmann's string bases can be interpreted in terms of a $\mathbb{Z}^n$-valued valuation on the function field $\mathbb{C}(Z)$ of Z, depending on the flag $Z_\bullet$. This valuation, however, differs from those introduced by Okounkov: whereas orders of vanishing of a regular function f are described in local coordinates $x_1, \dots, x_n$ by the smallest monomial term of $f(x) = \sum_{a\in\mathbb{N}^n} c_a x^a$, with respect to some ordering of the variables $x_1, \dots, x_n$, Kaveh's valuation is locally defined by the highest monomial term. In geometric language, this valuation thus tells how often f can be differentiated in the various directions defined by the x_i in the given order. Moreover, Fujita studied used this same valuation to study string polytopes for more general line bundles on Bott–Samelson varieties ([6]), being interested the closely related *Demazure modules* for the group B.

It still remains an open problem to interpret Littelmann's string polytopes as Okounkov bodies, or indeed, more generally, to construct rational polyhedral Okounkov bodies for line bundles over flag varieties using some H-invariant flag $Y_\bullet$.

2. Bott–Samelson varieties

Let us recall the basics of Bott–Samelson varieties, following [14].

Let G be a connected and simply connected reductive complex linear group, let $B \subseteq G$ be a Borel subgroup, and let W be the Weyl group of G. If $s_i \in W$ is a simple reflection, let P_i denote the associated minimal parabolic subgroup containing B. Then the quotient space P_i/B is isomorphic to $\mathbb{P}^1$. For a sequence $w = (s_1, \dots, s_n)$ (where the s_i are not necessarily distinct), let $P_w := P_1 \times \dots \times P_n$ be the product of the corresponding parabolic subgroups, and consider the right action of B^n on P_w given by

$$(p_1, \dots, p_n)(b_1, \dots, b_n) := (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \dots, b_{n-1}^{-1} p_n b_n).$$

The Bott–Samelson variety Z_w is the quotient

$$Z_w := P_w/B^n.$$

An alternative description of this quotient can be given as follows. Suppose that X and Y are two varieties, such that X is equipped with a right action and Y with a left action of B. Consider the right action of B on the product given by

$$(x, y).b := (xb, b^{-1}y), \quad (x, y) \in X \times Y, \quad b \in B,$$

and let $X \times^B Y := (X \times Y)/B$ denote the quotient space. Then the map $X \times^B Y \longrightarrow X/B,\ [(x, y)] \mapsto xB$ exhibits $X \times^B Y$ as a fibre bundle over X/B and with fibre Y. Now, we can alternatively describe Z_w as

$$Z_w = (P_1 \times^B \cdots \times^B P_n)/B,$$

where B acts on the right on $P_1 \times^B \cdots \times^B P_n$ by

$$[(p_1, \ldots, p_n)].b := [(p_1, \ldots, p_{n-1}, p_n b)], \quad (p_1, \ldots, p_n) \in P_w, \quad b \in B.$$

As a consequence, using the fact that each quotient P_i/B is isomorphic to $\mathbb{P}^1$, Z_w is given as an iteration of $\mathbb{P}^1$-bundles. In particular, Z_w is a smooth variety. To describe this iterated fibre bundle structure in more detail, let, for $j \in \{1, \ldots, n\}$, $w[j]$ denote the truncated sequence $(s_1, \ldots, s_{n-j})$, and let $Z_{w[j]} := P_{w[j]}/B^{n-j}$ denote the associated Bott–Samelson variety. Then the projections $P_w \longrightarrow P_{w[j]}$ are B^n-equivariant, where B^n acts on $P_{w[j]}$ by the factor B^{n-j}, and thus induce a projections $\pi_{w[j]} : Z_w \longrightarrow Z_{w[j]}$, which can be factorized as a sequence of $\mathbb{P}^1$-fibrations

$$\pi_{w[j]} : Z_w \longrightarrow Z_{w[1]} \longrightarrow \cdots \longrightarrow Z_{w[j]}.$$

Let $\pi : Z_w \longrightarrow P_1/B$ denote the composition of all these projections, i.e., π is the projection morphism onto P_1/B defined by the description of Z_w as the bundle

$$Z_w := P_1 \times^B (P_2 \times^B \times \cdots \times^B P_n).$$

Now, each $\mathbb{P}^1$-bundle admits a natural section as follows. Let $w(j) := (s_1, \ldots, \hat{s_j}, \ldots, s_n)$, so that $P_{w(j)}$ embeds naturally as a subgroup of P_w. The embedding $\sigma^0_{w,j} : P_{w(j)} \longrightarrow P_w$ is B^{n-1}-equivariant, and thus induces an embedding

$$\sigma_{w,j} : Z_{w(j)} \longrightarrow Z_w$$

of $Z_{w(j)}$ as a divisor in Z_w such that the divisors $Z_{w(j)}$, $j = 1, \ldots, n$, intersect transversely in a point. In particular, $\sigma_{w,n} : Z_{w(n)} \cong Z_{w[1]} \longrightarrow Z_w$ defines a section of the $\mathbb{P}^1$-bundle $\pi_{w[1]} Z_w \longrightarrow Z_{w[1]}$, and identifies $Z_{w[1]}$ with a divisor which is transversal to the fibres of $\pi_{w[1]}$. Now, the Picard group $\mathrm{Pic}(Z_w)$ splits as the direct sum

$$\mathrm{Pic}(Z_w) \cong \mathrm{Pic}(Z_{w[1]}) \oplus \mathbb{Z},$$

where $\mathbb{Z}$ is identified with the subgroup generated by the line bundle $\mathcal{O}_{Z_w}(Z_{w(n)})$. Iterating the above splitting yields that the line bundles $\mathcal{O}_{Z_w}(Z_{w(j)})$, $j = 1, \dots, n$, define a basis for $\mathrm{Pic}(Z_w)$. Clearly, they are all effective. Conversely, if w is a reduced sequence, i.e., if the length of the product $\overline{w} := s_1 \cdots s_n$ equals n, a divisor

$$m_1 Z_{w(1)} + \cdots + m_n Z_{w(n)}, \quad m_1, \dots, m_n \in \mathbb{Z},$$

is effective if and only if $m_1, \dots, m_n \geq 0$ (cf. [14, Prop. 3.5]). The basis $\{Z_{w(1)}, \dots, Z_{w(n)}\}$ is called the *effective basis* for $\mathrm{Pic}(Z_w)$. Notice that, since $Z_{w(n)}$ defines a section of the bundle $\pi_{w[1]}$, the restricted divisors

$$Z_{w(1)} \cdot Z_{w(n)}, \dots, Z_{w(n-1)} \cdot Z_{w(n)} \tag{2}$$

form the effective basis for $Z_{w[1]} \cong Z_{w(n)}$.

2.1. The vertical flag

We also recall the so-called $\mathcal{O}(1)$-basis for $\mathrm{Pic}(Z_w)$ defined as follows. Each product $P_1 \times \cdots \times P_k$, $k = 1, \dots, n$, defines a morphism

$$\varphi_k : Z_{w[n-k]} \longrightarrow G/B, \quad [(p_1, \dots, p_k)] \mapsto p_1 p_2 \cdots p_k B.$$

Put $\mathcal{O}_{w[n-k]}(1) := \varphi_k^* L_{\omega_\alpha}$, where α is simple root corresponding to the simple reflection s_k, ω_α is its fundamental weight, viewed as a character of B, and $L_{\omega_\alpha} := G \times^B \mathbb{C}$ is the associated line bundle over the flag variety G/B. Let $\mathcal{O}_k(1) := \pi_{w[n-k]}^* \mathcal{O}_{w[n-k]}(1)$. The line bundles $\mathcal{O}_1(1), \dots, \mathcal{O}_n(1)$, being pullbacks of globally generated line bundles, are then globally generated and form a basis for $\mathrm{Pic}(Z_w)$. Moreover, a line bundle $\mathcal{O}_1(1)^{m_1} \otimes \cdots \otimes \mathcal{O}_n(1)^{m_n}$ is very ample (nef) if and only if $m_1, \dots, m_n > 0$ ($m_1, \dots, m_n \geq 0$) ([14, Thm. 3.1]). Notice here that the morphism φ_k above is induced by the B-equivariant multiplication map

$$P_1 \times \cdots \times P_k \longrightarrow G, \quad (p_1, \dots, p_k) \mapsto p_1 p_2 \cdots p_k,$$

where B acts on $P_1 \times \cdots \times P_k$ by the right multiplication on the factor P_k. We can therefore also view the line bundle $\mathcal{O}_{w[n-k]}(1)$ over $Z_{w[n-k]}$ as the product

$$\mathcal{O}_{w[n-k]}(1) = P_1 \times \cdots \times P_k \times^{B^k} \mathbb{C},$$

where B^k acts on the product $P_1 \times \cdots \times P_k$ by

$$(p_1, \dots, p_k).(b_1, \dots, b_k) := (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{k-1}^{-1} p_k b_k),$$

and on $\mathbb{C}$ by the character

$$(b_1, \dots, b_k) \mapsto \omega_\alpha(b_k).$$

Thus, the sections of the sheaf $\mathcal{O}_{w[n-k]}(1)$ over an open set $U \subseteq Z_{w[n-k]}$ correspond to the regular functions f on the B^k-invariant open subset

$$\widetilde{U}_k := \{(p_1, \dots, p_k) \in P_1 \times \cdots \times P_k \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^k-equivariance property

$$\begin{aligned}
&f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{k-1}^{-1} p_k b_k) = \omega_\alpha(b_k)^{-1} f(p_1, \dots, p_k),\\
&(p_1, \dots, p_k) \in \widetilde{U}_k, \quad (b_1, \dots, b_k) \in B^k.
\end{aligned}$$

It follows that sections of the sheaf $\mathcal{O}_k(1)$ over the open subset $\pi_{w[n-k]}^{-1}(U)$ then correspond to the regular functions f on the B^n-invariant open subset

$$\widetilde{U} := \{(p_1, \dots, p_n) \in P_1 \times \cdots \times P_n \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^n-equivariance property

$$\begin{aligned}
&f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n) = \omega_\alpha(b_k)^{-1} f(p_1, \dots, p_n),\\
&(p_1, \dots, p_n) \in \widetilde{U}, \quad (b_1, \dots, b_n) \in B^n.
\end{aligned}$$

In other words, $\mathcal{O}_k(1)$ is the line bundle

$$\mathcal{O}_k(1) = P_1 \times \cdots \times P_n \times^{B^n} \mathbb{C},$$

where B^n acts on $\mathbb{C}$ by the character

$$\xi_k : B^n \to \mathbb{C}^\times, \quad \xi_k(b_1, \dots, b_n) := \omega_\alpha(b_k).$$

Since the $\mathcal{O}_k(1)$, $k = 1, \dots, n$, form a basis for the Picard group, we see that each line bundle corresponds to a unique character $\xi_1^{m_1} \cdots \xi_n^{m_n}$, for an $(m_1, \dots, m_n) \in \mathbb{Z}^n$ in such a way that the sections of the sheaf $\mathcal{O}_1(1)^{m_1} \otimes \cdots \otimes \mathcal{O}_n(1)^{m_n}$ over an open subset $U \subseteq Z_w$ correspond to the regular functions f on the B^n-invariant open subset

$$\widetilde{U} := \{(p_1, \dots, p_n) \in P_1 \times \cdots \times P_n \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^n-equivariance property

$$\begin{aligned}
&f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n) = \xi_1(b_1)^{-m_1} \cdots \xi_n(b_n)^{-m_n} f(p_1, \dots, p_n),\\
&(p_1, \dots, p_n) \in \widetilde{U}, \quad (b_1, \dots, b_n) \in B^n. \qquad (3)
\end{aligned}$$

Thus, every line bundle L on Z_w is of the form

$$L = P_1 \times \cdots \times P_n \times^{B^n} \mathbb{C},$$

where B^n acts on $\mathbb{C}$ by the character $\xi_1^{m_1} \cdots \xi_n^{m_n}$, for a unique $(m_1, \ldots, m_n) \in \mathbb{Z}^n$. Let $\mathcal{O}(m_1, \ldots, m_n)$ denote the line bundle corresponding to $(m_1, \ldots, m_n)$. In particular, each line bundle $L = \mathcal{O}(m_1, \ldots, m_n)$ admits an action of P_1 as bundle automorphisms by

$$p.[((p_1, \ldots, p_n), z)] \mapsto [((pp_1, p_2, \ldots, p_n), z)],$$

which is clearly well-defined since the left multiplication of P_1 on the P_1-factor in $P_1 \times \cdots \times P_n$ commutes with the B^n-action on this product. The induced representation of P_1 on the space of global sections $H^0(Z_w, \mathcal{O}(m_1, \ldots, m_n))$ is given by

$$(p.f)(p_1, \ldots, p_n) := p.(f(p^{-1}p_1, p_2, \ldots, p_n)), \tag{4}$$

for $p \in P_1$ and $(p_1, \ldots, p_n) \in P_1 \times \cdots \times P_n$, where we have used the identification (3) of sections with equivariant regular functions on $P_1 \times \cdots \times P_n$. Clearly, the B^n-equivariance property is preserved by the left action of P_1, so that this indeed defines a representation of P_1 on $H^0(Z_w, \mathcal{O}(m_1, \ldots, m_n))$.

Consider now the fibration

$$\pi : Z_w \longrightarrow P_1/B \cong \mathbb{P}^1$$

given by mapping $[(p_1, \ldots, p_n)]$ to the class $[p_1] = p_1 \cdot B$. All of its fibres are isomorphic to the Bott–Samelson variety $Z_{[s_2, \ldots, s_n]}$. Note that B operates on the quotient P_1/B as the upper triangular matrices act on $\mathbb{P}^1$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} [z_0 : z_1] = [az_0 + bz_1 : cz_1],$$

and thus with exactly one fixed point $p_0 = [1 : 0]$. Denote by Y the fibre over p_0. Note also that for any $p \in P_1$ and $[(p_1, \ldots, p_n)] \in Z_w$ we have

$$\begin{aligned} \pi(p.[(p_1, \ldots, p_n)]) &= \pi([(pp_1, p_2, \ldots, p_n)]) \\ &= pp_1B = p.\pi([(p_1, \ldots, p_n)]), \end{aligned}$$

i.e., π is P_1-equivariant, and hence P_1 acts as automorphisms of the fibre bundle π.

We can reiterate this construction to obtain a natural flag of Bott–Samelson varieties

$$Z_w \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

where Y_i is given as the fibre of the corresponding bundle $Y_{i-1} \longrightarrow \mathbb{P}^1$ over the B-fixed point. Thus,

$$Y_i := \{[(p_1, \ldots, p_n)] \in Z_w \mid p_1 = \cdots = p_i = e\}, \quad i = 1, \ldots, n.$$

In particular, being Bott–Samelson varieties of lower dimensions, all the Y_i are smooth varieties. We call this flag the *vertical flag* on Z_w.

Remark 2.1. The i-th piece $Y_i = \{[(p_1, \ldots, p_n)] \mid p_1, \ldots, p_i \in B\}$ is mapped onto the Schubert variety $X_{s_{i+1} \cdots s_n} \subseteq G/B$ by the map $[(p_1, \ldots, p_n)] \mapsto p_1 \cdots p_n B$ (cf. [4, Section 2.1]), so that the flag $Y_\bullet$ defines a desingularization of the flag $X_{s_n} \subseteq X_{s_{n-1}s_n} \subseteq \cdots \subseteq X_{s_2 \cdots s_n} \subseteq G/B$ of the flag variety.

Other authors, such as Kaveh ([11]) and Kiritchenko ([12]), have considered flags of subvarieties on Bott–Samelson varieties that desingularize flags of translates of Schubert varieties by various Weyl group elements. In fact, Kiritchenko's flag, which is defined for Bott–Samelson varieties for the group GL_n, has the same image in the flag variety GL_n/B as our flag, up to translation of the individual pieces, meaning that the occurring Schubert varieties are the same. Harada and Yang ([7]), on the other hand, use a "horizontal" flag, defining the i-th piece as the set of points represented by tuples $(p_1, \ldots, p_n)$ having the last—instead of the first—i entries equal to the identity element. Although formally similar, this flag is of another nature than ours: it is not defined by intersections of globally generated divisors, but by the effective basis for the divisor class group.

Example 1. Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the point $\mathbb{C} \subsetneq \mathbb{C}^2$ defined as the subspace spanned by the first standard basis vector $(1, 0)$. The variety X is isomorphic to the incidence variety of all pairs (V_1, V_2) of linear subspaces of $\mathbb{C}^2$ where V_i is i-dimensional and $V_1 \subseteq V_2, \mathbb{C} \subseteq \mathbb{C}^2$. For such a pair, the subspace V_2 can be viewed as a point in $\mathbb{P}(\mathbb{C}^3/\mathbb{C})$. The projection p_2, $(V_1, V_2) \mapsto V_2$ then describes as a $\mathbb{P}^1$-bundle over $\mathbb{P}(\mathbb{C}^2)$, and p_2 also admits the section $V_2 \mapsto (\mathbb{C}, V_2)$. In fact, this bundle is the projective bundle of the tautological vector bundle over $\mathbb{P}(\mathbb{C}^3/\mathbb{C})$ whose fibre over a point V is precisely the space V viewed as a two-dimensional linear subspace of $\mathbb{C}^3$.

The blow-up morphism is given by $\pi(V_1, V_2) := V_1$, so that $\pi^{-1}(\mathbb{C}) = p_2(\mathbb{P}(\mathbb{C}^3/\mathbb{C}))$, i.e., the exceptional divisor E of π equals the image of the section p_2. The morphism π is also the birational morphism from the Bott–Samelson variety X to the partial flag variety $\mathbb{P}^2$.

The pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ is generated by the classes $[H - E]$ and $[E]$, where H is the divisor class of $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. The nef cone $\mathrm{Nef}(X)$ is generated by $[H - E]$ and $[H]$, and the other chamber in $\overline{\mathrm{Eff}}(X)$ is generated by $[H]$ and $[E]$.

The vertical flag $Y_\bullet$ is here given by a divisor C in $|H - E|$ with $C \cong \mathbb{P}^1$, and a point $x \in C$ such that $x \notin E$.

Example 2. Consider the three-dimensional incidence variety X consisting of all triples (V_1, V_2, V_2') of linear subspaces of $\mathbb{C}^3$ such that V_1 is one-dimensional, V_2, V_2' are two-dimensional, and the inclusions

$$V_1 \subset V_2, \quad V_1 \subset V_2', \quad \mathbb{C} \subset V_2$$

hold, where $\mathbb{C} \subset \mathbb{C}^3$ denotes the span of the first standard basis vector $(1, 0, 0)$. Moreover, let S denote the set of all tuples (V_1, V_2) of subspaces of the type above, satisfying the relations that do not include V_2'. It is well-known that S is the blow-up of $\mathbb{P}^2$ at the point $\mathbb{C} \subset \mathbb{C}^3$, and that the blow-up morphism is given by $q : S \longrightarrow \mathbb{P}^2$, $q(V_1, V_2) := V_1$.

Now, the map $\pi_X : X \longrightarrow S$, $\pi_S(V_1, V_2, V_2') := (V_1, V_2)$ clearly defines a $\mathbb{P}^1$-bundle. Moreover, the map $\pi_S : S \longrightarrow \mathbb{P}(\mathbb{C}^3/\mathbb{C})$, $\pi_S(V_1, V_2) := V_2$ defines S as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Together, these two maps describe X as an iterated $\mathbb{P}^1$-bundle. This yields a special case of a Bott–Samelson variety for the group SL_3 (cf. [17, Section 2]).

The surface S comes equipped with the tautological vector bundles $\mathcal{V}_1$ and $\mathcal{V}_\in$, where $\mathcal{V}_i$ has the fibre V_i over the point (V_1, V_2), $i = 1, 2$. Then $(\mathcal{V}_2/\mathbb{C})^* = \pi_S^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{V}_1^* = p^* \mathcal{O}_{\mathbb{P}^2}(1)$. Define D_1 to be the divisor class of $(\mathcal{V}_2/\mathbb{C})^*$ and D_2 to be divisor class of $\mathcal{V}_1^*$. Then the exceptional divisor, E_2, of q is given by the equation $V_1 = \mathbb{C}$, that is, as the zero set of the section

$$s_{E_2} \in H^0(S, \mathcal{V}_2/\mathbb{C} \otimes \mathcal{V}_1^*), \quad s_{E_2}(V_1, V_2)(v) := v + V_2, \quad v \in V_1.$$

This shows that $E_2 = D_2 - D_1$. Pulling back $E_1 := D_1$ and E_2 to X by π_X, yields the first two elements, also denoted by E_1 and E_2 of the Picard group of X.

The final effective generator is the surface E_3, obtained by embedding S into X by putting $V_2' = \mathbb{C}^2$. In order to describe the defining section of E_3, we define a third tautological vector bundle $\mathcal{V}_2'$ on X–denoting by $\mathcal{V}_1$ and $\mathcal{V}_2$ also the pullbacks to X of these vector bundles on S–by requiring the fibre of $\mathcal{V}_2'$ over (V_1, V_2, V_2') to be the vector space V_2'. The section defining section of E_3 is then given by

$$\begin{aligned} &s_{E_3} \in H^0(X, \mathbb{C}^3/\mathcal{V}_2 \otimes (\mathcal{V}_2'/\mathcal{V}_1)^*), \quad s_{E_3}(V_1, V_2, V_3)(v + V_1) := v + V_2, \\ &v + V_1 \in V_2'/V_1. \end{aligned}$$

Defining D_3 to be the divisor class of $\mathbb{C}^3/\mathcal{V}_2'$, using the identities

$$\mathcal{V}_2'/\mathcal{V}_1 \otimes \mathcal{V}_1 \cong \det \mathcal{V}_2'$$
$$\det \mathcal{V}_2' \otimes \mathbb{C}^3/\mathcal{V}_2' \cong \det \mathbb{C}^3,$$

one obtains $E_2 = D_1 - D_2 + D_3$.

The change of basis from (E_1, E_2, E_3) to (D_1, D_2, D_3) for the $\mathbb{R}$-Picard group of X is then given by

$$aE_1 + bE_2 + cE_3 = (a + c - b)D_1 + (b - c)D_2 + cD_3.$$

Finally, we consider the projection onto the appropriate flag variety. Consider therefore the flag variety $\mathcal{F}(\mathbb{C}^3)$ given by all pairs (W_1, W_2) of linear subspaces of $\mathbb{C}^3$, where W_i is i-dimensional for $i = 1, 2$ and $W_1 \subset W_2$. This variety is equipped with the obvious tautological vector bundles $\mathcal{W}_1$ and $\mathcal{W}_\in$, and the Picard group of $\mathcal{F}(\mathbb{C}^3)$ is generated by the globally generated line bundles $\mathcal{W}_1^*$ and $\mathbb{C}^3/\mathcal{W}_2$. The map

$$p : X \longrightarrow \mathcal{F}(\mathbb{C}^3), \quad (V_1, V_2, V_2') := (V_1, V_2'),$$

defines a birational morphism, and we have that $p^*\mathcal{W}_1^* = \mathcal{O}_X(D_2), p^*\mathbb{C}^3/\mathcal{W}_2 = \mathcal{O}_X(D_3)$. The Picard group of $\mathcal{F}(\mathbb{C}^3)$ thus embeds into $\mathrm{Pic}(X)$ as the subgroup generated by $\mathcal{O}_X(D_2)$ and $\mathcal{O}_X(D_3)$.

We now study the vertical flag on X. The divisor $Y = Y_1$ should be a fibre for the projection $\pi := \pi_S \circ \pi_X : X \longrightarrow \mathbb{P}(\mathbb{C}^3/\mathbb{C})$, and we choose it to be the fibre above $\mathbb{C}^2$, the span of the first two standard basis vectors, which we identify with a point in $\mathbb{P}(\mathbb{C}^3/\mathbb{C})$. Thus, $Y \in |E_1|$, and is given as the set of all (V_1, V_2') with $V_1 \subset \mathbb{C}$ and $V_1 \subset V_2'$. This is again the blow-up of $\mathbb{P}^2$ at a point, the blow-up projection being given by $(V_1, V_2') \mapsto V_2' \in Gr(2,3) \cong \mathbb{P}^2$, and the exceptional divisor is the curve E defined by $V_2' = \mathbb{C}^2$. The projection of Y onto $\mathbb{P}^1$ is the map $(V_1, V_2') \mapsto V_1$. Defining H to be the fibre above $\mathbb{C}$ for this projection, the pseudoeffective cone is spanned by the numerical equivalence classes of E and $H - E$.

The restriction of line bundles is given by the following three identities, which follow immediately from the defining equations for the respective divisors on X and Y:

$$\begin{aligned} \mathcal{O}_X(D) \mid_Y &= \mathcal{O}_Y, \\ \mathcal{O}_Y(E_2) \mid_Y &= \mathcal{O}_Y(H - E), \\ \mathcal{O}_X(E_3) \mid_Y &= \mathcal{O}_Y(E). \end{aligned}$$

The vertical flag $Y_\bullet$ on X is given by $x \subset C \subset Y$, where $C \in |H - E|$ is the fibre above $\mathbb{C}$ for the projection $(V_1, V_2') \mapsto V_1$, and $x \in C \setminus E$ is a generic point on C.

2.2. Bott–Samelson varieties as Mori dream spaces

Now assume that w is a reduced sequence. Recall that the product map $P_w \longrightarrow G, (p_1, \ldots, p_n) \mapsto p_1 \cdots p_n$, induces a morphism

$$p_w : Z_w \longrightarrow Y_{\overline{w}} := \overline{B\overline{w}B}$$

into the Schubert subvariety $Y_{\overline{w}}$ of the flag variety G/B, and that this morphism is in fact birational. Moreover, it is B-equivariant with respect to the left action of B on Z_w defined by

$$b[(p_1, \ldots, p_n)] := [(bp_1, p_2, \ldots, p_n)], \quad (p_1, \ldots, p_n) \in P_w, \quad b \in B.$$

In particular, if $\overline{w}$ is the longest element of the Weyl group, p_w defines a birational map $Z_w \longrightarrow G/B$.

The following theorem is the crucial result for our application of the theory from the following section.

Theorem 2.2. *Let G be a complex reductive group with Weyl group W, and let $Z = Z_w$ be a Bott–Samelson variety defined by a reduced sequence w of simple reflections. Then Z admits a divisor Δ such that (Z, Δ) is a log Fano pair, i.e., it is Kawamata log terminal and $-(K_X + \Delta)$ is ample. In particular, Z is a Mori dream space.*

Proof. Let $Y = G/B$, and let D_ρ be the divisor on $Y_{\overline{w}}$ which corresponds to the restriction to $Y_{\overline{w}}$ of the square root of the anticanonical bundle of Y. Then D_ρ is an ample divisor on $Y_{\overline{w}}$, so that $p_w^*(D_\rho)$ is a nef divisor on Z.

In order to facilitate the notation, let $\{D_1, ..., D_n\}$ be the basis of effective divisors for $\mathrm{Pic}(Z)$. Now choose integers $a_1, ..., a_n > 0$ so that $\sum_{i=1}^n a_i D_i$ is an ample divisor. Then, for every $N > 0$, the divisor $p_w^*(-D_\rho) - \sum_{i=1}^n a_i/N D_i$ is anti-ample. Now let $N \in \mathbb{N}$ be so big that $a_i/N < 1$ for every i, and put

$$\Delta := \sum_{i=1}^{n} (1 - a_i/N) D_i.$$

If K_Z is the canonical divisor of Z, we then have that

$$K_Z + \Delta = \pi^*(-D_\rho) - \sum_{i=1}^{n} D_i + \sum_{i=1}^{n} (1 - a_i/N) D_i = \pi^* L_{-\rho} - \sum_{i=1}^{n} (a_i/N) D_i$$

(cf. [14, Lemma 5.1]) is anti-ample. Since Z is nonsingular, and all subsets of the set of smooth divisors $\{D_1, \ldots, D_n\}$ intersect transversely and smoothly, the pair (Z, Δ) thus defines a log Fano pair. □

Remark 2.3. In the context of the above theorem, it is worth mentioning an analogous result by Anderson and Stapledon ([2]) on the log Fano property of Schubert varieties.

3. Good flags on Mori dream spaces

In this section we prove the main theorem of this paper. The main objective is to establish conditions on a flag on a Mori dream space, such that its global Okounkov body is rational polyhedral.

First let us recall that a Mori dream space X is a normal $\mathbb{Q}$-factorial variety such that $\mathrm{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$ and with a Cox ring $\mathrm{Cox}(X)$ which is a finitely generated $\mathbb{C}$-algebra. We make use of the theory of Mori dream spaces developed by Hu and Keel in [8] and we refer the reader to this beautiful paper for a detailed investigation of Mori dream spaces.

Note that for any effective divisor D on a Mori dream space X, the ring of sections $R(X,D) := \bigoplus_{k\geq 0} H^0(X,\mathcal{O}_X(kD))$ is finitely generated, so we obtain a natural rational map

$$f_D : X \dashrightarrow \mathrm{Proj}(R(X,D))$$

which is regular outside the stable base locus of D. One obtains an equivalence relation of effective divisors as follows: two effective divisors D and D' are *Mori-equivalent* if up to isomorphism they yield the same rational maps. Hu and Keel prove ([8, Proposition 1.11]) that there are only finitely many equivalence classes, indexed by contracting rational maps $f : X \dashrightarrow X'$ and that the closure Σ_f of a maximal dimensional equivalence class can be described as the closed convex cone spanned by f-exceptional rays together with the face $f^*(\mathrm{Nef}(X'))$ of the moving cone. These subcones Σ_f, which decompose the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$, are in the remainder of this paper following [8] referred to as *Mori-chambers*.

Recall that a birational map $f : X \dashrightarrow X'$ to a $\mathbb{Q}$-factorial variety X' is called a *small* $\mathbb{Q}$*-factorial modification* if it defines an isomorphism

$$f\,|_U : U \to V$$

between open subsets $U \subseteq X, V \subseteq X'$ with complements of codimension at least two. Then f^* induces an isomorphism of pseudo-effective cones $\overline{\mathrm{Eff}}(X) \cong \overline{\mathrm{Eff}}(X')$ as well as an isomorphism $\mathrm{Cox}(X) \cong \mathrm{Cox}(X')$ of Cox rings. We further recall that f is induced by GIT in the following manner.

Let $R = \mathrm{Cox}(X)$. The variety X can be written as the GIT quotient $X = \mathrm{Spec}(R)^{ss}(\chi)//G$, where G is the complex torus of rank equal to the Picard number of X, and $\mathrm{Spec}(R)^{ss}(\chi)$ denotes the set of semistable points in $\mathrm{Spec}(R)$ with respect to the character χ of G. We even have that $\mathrm{Spec}(R)^{ss}(\chi) = \mathrm{Spec}(R)^{s}(\chi)$; the set of χ-semistable points equals the set of χ-stable points, so that the quotient is even a geometric quotient. Also X' is a geometric quotient of the set of stable points with respect to a character χ': $X' = \mathrm{Spec}(R)^s(\chi')//G$. Let $\pi_\chi : \mathrm{Spec}(R)^s(\chi) \longrightarrow X$ and $\pi_{\chi'} : \mathrm{Spec}(R)^s(\chi') \longrightarrow X'$ denote the respective quotient morphisms. In both cases the sets of unstable points (with respect to χ and χ') are of codimension at least two in $\mathrm{Spec}(R)$, and the rational map f is induced on the level of quotients by the inclusion

$$\mathrm{Spec}(R)^s(\chi) \cap \mathrm{Spec}(R)^s(\chi') \subseteq \mathrm{Spec}(R)^s(\chi')$$

of the subset of common stable points into the set of χ'-stable points. In particular, the exceptional locus of f equals the complement of the domain of definition of f and is given as the image of the χ'-unstable and χ-stable points;

$$\mathrm{exc}(f) = \pi_\chi(\mathrm{Spec}(R)^s(\chi) \cap \mathrm{Spec}(R)^{us}(\chi')),$$

so that f induces an isomorphism

$$f\,|_U\colon U \xrightarrow{\cong} V, \quad U := \pi_\chi(\operatorname{Spec}(R)^s(\chi)\cap\operatorname{Spec}(R)^s(\chi')),$$
$$V := \pi_{\chi'}(\operatorname{Spec}(R)^s(\chi)\cap\operatorname{Spec}(R)^s(\chi')).$$

Now let $f : X \dashrightarrow X'$ be a small $\mathbb{Q}$-factorial modification as above, and assume that $Y \subseteq X$ is an irreducible hypersurface given as the zero set $Y = Z(s)$ of a section $s \in H^0(X, L)$ of some line bundle L. Let $L' := (f^{-1})^*L$ denote the corresponding line bundle on X', let $s' \in H^0(X', L')$ be the section of L' corresponding to s, and put $Y' := Z(s')$. The restriction of f to Y then defines a birational map

$$f_Y : Y \dashrightarrow Y',$$

yielding the commuting diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & X' \\ Y \arrow[u] \arrow[r, dashed, "f_Y"] & Y', \arrow[u] \end{tikzcd}$$

where the vertical arrows denote the respective inclusion morphisms.

We now turn to Okounkov bodies on a Mori dream space X equipped with an admissible flag $Y_\bullet$. Our strategy is to deduce properties of the global Okounkov body of X from those of Okounkov bodies of line bundles restricted to Y_1 and to argue inductively. We are thus particularly interested in a comparison of the Okounkov bodies of a graded linear series coming from restricting sections to Y_1 and of a restricted line bundle. More concretely, for a divisor D on X we consider the restriction map

$$R : \bigoplus_k H^0(X, \mathcal{O}_X(kD)) \longrightarrow \bigoplus_k H^0(Y, \mathcal{O}_Y(kD\cdot Y))$$

and hope for an identity

$$\Delta_{Y^1_\bullet}(\operatorname{im} R) = \Delta_{Y^1_\bullet}(D\cdot Y),$$

where $Y^1_\bullet$ denotes the flag $Y_2 \supseteq \ldots \supseteq Y_n$ on Y_1. Note that in case D is ample the above identity holds. This follows from the exact sequence

$$H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(Y, \mathcal{O}_Y(mD)) \longrightarrow H^1(X, \mathcal{O}_X(mD - Y))$$

together with the fact that for large m the last cohomology group is trivial by Serre's vanishing theorem. In order to get the desired identity for any movable divisor D, we will consider the corresponding small modification.

Proposition 3.1. *Let X be a Mori dream space, and let $Y_\bullet$ be an admissible flag of normal subvarieties, write $Y := Y_1$, and let $Y_\bullet^1$ denote the admissible flag*

$$Y_n \subseteq \cdots \subseteq Y_1 = Y$$

of subvarieties of Y.

Let $f : X \dashrightarrow X'$ be a SQM, and let $f_Y : Y \dashrightarrow Y'$ be the induced birational morphism. Assume that there exist open subsets $U \subset Y$ and $V \subset Y'$ with $codim_Y(Y \setminus U) \geq 2$ and $codim_{Y'}(Y' \setminus V) \geq 2$ such that $f_Y : U \longrightarrow V$ is an isomorphism.

If D' is an ample divisor on X' and $D := f^*(D')$ is the corresponding divisor on X, then the identity

$$\Delta_{Y_\bullet^1}(im\, R) = \Delta_{Y_\bullet^1}(D \cdot Y) \tag{5}$$

of Okounkov bodies holds.

Furthermore, if D' is merely nef, then, for any $a \in \Delta_{Y_\bullet^1}(D \cdot Y)$, the point $(0, a) \in \mathbb{R}^n$ is contained in the Okounkov body $\Delta_{Y_\bullet}(D)$, i.e., the inclusion

$$\{0\} \times \Delta_{Y_\bullet^1}(D \cdot Y) \subseteq \Delta_{Y_\bullet}(D) \tag{6}$$

holds.

Proof. Let us first assume that D' is ample. Since $S := \mathrm{im} R$ is a graded linear subseries of $\bigoplus_k H^0(Y, \mathcal{O}_Y(kD \cdot Y))$, the claimed identity of Okounkov bodies follows if we can show that both series have the same volume.

We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & X' \\ Y \arrow[u] \arrow[r, dashed, "f_Y"] & Y' \arrow[u] \end{tikzcd},$$

where the vertical arrows denote the respective inclusion morphisms By assumption, there are open subsets $U \subset Y$ and $V \subset Y'$ with $\mathrm{codim}_Y(Y \backslash U) \geq 2$ and $\mathrm{codim}_{Y'}(Y' \backslash V) \geq 2$ such that $f_Y : U \longrightarrow V$ is an isomorphism.

Denote the restricted divisors $D \cdot Y$ and $D' \cdot Y'$ by D_Y and $D'_{Y'}$, respectively.

Since D defines the map f, we have the identity

$$D = f^*(\mathcal{O}_{X'}(1)).$$

This restricts to U as the identity $f_Y^*(\mathcal{O}_{Y'}(1))|_U = D_Y|_U$. By assumption, $Y \setminus U$ has codimension at least 2 in Y, so that we get

$$f_Y^*(\mathcal{O}_{Y'}(1)) = D_Y.$$

Being ample, $\mathcal{O}_{Y'}(1)$ does not have a divisorial base component in $Y' \setminus V$, so we can represent it as a divisor $A = \overline{A \cap V}$. Since in case of small modifications pulling back is functorial, under f_Y^{-1} the divisor D_Y pulls back to A.

Since Y' might not be normal we consider the normalization

$$\pi : \widetilde{Y'} \longrightarrow Y'.$$

Note that π defines an isomorphism $\pi^{-1}(V) \longrightarrow V$ since V is contained in the normal locus of Y'. In particular, we have the identity

$$(f_Y^{-1} \circ \pi)^*(D_Y) = \pi^* f_Y^{-1*}(D_Y) = \pi^*(A).$$

Since $f_Y^{-1} \circ \pi$ is a contracting birational map between normal varieties, this implies the identity of volumes

$$\mathrm{vol}(D_Y) = \mathrm{vol}(\pi^*(A)).$$

Since π is a birational morphism, the right hand side is just $\mathrm{vol}(A)$. On the other hand,

$$\mathrm{vol}(A) = \mathrm{vol}(\mathcal{O}_{Y'}(1)) = \mathrm{vol}(S),$$

since $Y' = \mathrm{Proj}(S)$. This proves the claim in case D' is ample.

If D' is merely nef, write $[D']$ as a limit $[D'] = \lim_{i\to\infty}[D'_i]$ of numerical equivalence classes of ample divisors $D'_i, i \in \mathbb{N}$. Put $D_i := f^*D_i$. Then $[D] = \lim_{i\to\infty}[D_i]$. Now, let $a \in \Delta_{Y^1_\bullet}(D \cdot Y)$. Then $(a, [D \cdot Y]) \in \Delta_{Y^1_\bullet}(Y)$. Now choose points $a_i \in \Delta_{Y^1_\bullet}(Y \cdot D_i)$ so that $(a, [D \cdot Y]) = \lim_{i\to\infty}(a_i, [D_i \cdot Y])$. By the above, the identity (5) holds when D is replaced by $D_i, i \in \mathbb{N}$. Hence, by [15, Theorem 4.26],

$$((0, a_i), [D_i]) \in \Delta_{Y_\bullet}(X)$$

for each i, so that

$$((0, a), [D]) = \lim_{i\to\infty}((0, a_i), [D_i]) \in \Delta_{Y_\bullet}(X),$$

i.e., $(0, a) \in \Delta_{Y_\bullet}(D)$. This shows that the inclusion (6) holds for an arbitrary nef divisor D' on X'. $\square$

In order to apply the above proposition to obtain information on the structure of the global Okounkov body of a Mori dream space, we need the following construction formulated in a more general context. Here $Y_\bullet$ can be any admissible flag on a normal projective variety X.

If $s \in H^0(X, \mathcal{O}_X(m_1D_1 + \cdots + m_nD_n))$ is a section which does not vanish on Y_1, so that $\nu(s) = (\nu_1(s), \ldots, \nu_n(s))$ with $\nu_1(s) = 0$, then the restriction of s to Y_1 defines a section of the line bundle $\mathcal{O}_{Y_1}(D \cdot Y_1)$ over Y_1 with value $\nu^1(s) = (\nu_2(s), \ldots \nu_n(s))$ with respect to the truncated flag

$$Y_n \subseteq \cdots \subseteq Y_1 \tag{7}$$

on Y_1.

For a finite set $F_1, \ldots, F_r$ of movable divisors on X, let

$$\Gamma(F_1, \ldots, F_r) \subseteq \mathrm{Mov}(X)$$

be the semigroup generated by the divisors $F_1, \ldots, F_r$, and let

$$C(F_1, \ldots, F_r) \subseteq \mathrm{Mov}(X)$$

be the cone generated by $F_1, \ldots, F_r$. Define the semigroups

$$\begin{aligned} S(F_1, \ldots, F_r) := \{(\nu(s), [D]) \in \mathbb{N}_0^n \times \Gamma(F_1, \ldots, F_r) \mid s \in H^0(X, \mathcal{O}_X(D)), \\ D \in \Gamma(F_1, \ldots, F_r),\, \nu_1(s) = 0\} \end{aligned}$$

and

$$\begin{aligned} S_1(F_1, \ldots, F_r) := \{(\nu^1(s), [D \cdot Y_1]) \in \mathbb{N}_0^{n-1} \times N^1(Y_1)_{\mathbb{R}} \mid [D] \in \Gamma(F_1, \ldots, F_r), \\ s \in H^0(Y_1, \mathcal{O}_{Y_1}(D \cdot Y_1))\}, \end{aligned}$$

as well as the morphism of semigroups

$$q_0 : S \to S_1, \quad q_0(\nu(s), [D]) := (\nu^1(s), [D \cdot Y_1]),$$

which extends to the linear map

$$\begin{aligned} q: \qquad \mathbb{R}^n \oplus V(F_1, \ldots, F_r) &\longrightarrow \mathbb{R}^{n-1} \oplus N^1(Y_1)_{\mathbb{R}}, \\ ((x_1, \ldots, x_n), [D]) &\mapsto ((x_2, \ldots, x_n), [D \cdot Y_1]), \end{aligned} \tag{8}$$

where $V(F_1, \ldots, F_r) \subseteq N^1(X)_{\mathbb{R}}$ is the $\mathbb{R}$-vector space generated by the numerical equivalence classes $[F_1], \ldots, [F_r]$. Furthermore, we denote by $C(S(F_1, \ldots, F_r))$ and $C(S_1(F_1, \ldots, F_r))$ the closed convex cones in $\mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ and $\mathbb{R}^{n-1} \times N^1(Y_1)_{\mathbb{R}}$ spanned by the semigroups $S(F_1, \ldots, F_r)$ and $S_1(F_1, \ldots, F_r)$, respectively.

We now recall that for a Mori dream space X the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ is the union of finitely many Mori chambers, $\Sigma_1, \ldots, \Sigma_m$, where each Mori chamber Σ_j is the convex hull of finitely many integral divisors $D_1^j, \ldots, D_{\ell_j}^j$. More concretely, by [8, Proposition 1.1], the chambers are in correspondence to contracting birational maps $f_j : X \dashrightarrow X_j$ with image a Mori dream space, and are given as the convex cone spanned by $f_j^*(\mathrm{Nef}(X_j))$ together with the rays spanning the exceptional locus $\mathrm{exc}(f_j)$.

The corresponding decomposition of a divisor $D \in \Sigma_j$ is exactly its decomposition into its fixed and movable parts. We can thus reorder the divisors spanning each chamber in such a way that the first n_j of them are movable and the remaining ones are fixed. Let $\sigma_i^j \in H^0(X, \mathcal{O}_X(N_i^j))$ be the defining section of D_i^j, for $j = 1, \dots, m$, $i = n_j + 1, \dots, \ell_j$.

Corollary 3.2. *Let $Y_\bullet$ be an admissible flag on a Mori dream space X such that the conclusion of Proposition 3.1 holds for any SQM $f : X \dashrightarrow X'$ and any nef divisor $D' \subset X'$. Furthermore let $D_1, \dots, D_r$ those generators of a Mori chamber Σ which are movable. Then we have the identity*

$$C(S(D_1, \dots, D_r)) = q^{-1}(C(S_1(D_1, \dots, D_r))) \cap (\{0\} \times \mathbb{R}^{n-1}_{\geq 0} \times C(D_1, \dots, D_r)). \quad (9)$$

Proof. This follows from the condition together with the fact that there exists a SQM $\pi : X \dashrightarrow X'$ such that each divisor in the cone $C(D_1, \dots, D_r)$ is a pullback by π of a nef divisor on X' ([8, Proposition 1.11(3)]). □

We can now prove the following theorem which will—together with identity (9)—enable us to inductively infer information on the shape of global Okounkov bodies of certain Mori dream spaces.

Theorem 3.3. *Suppose in the above situation that each of the cones $C(S(D_1^j, \dots, D_{n_j}^j))$ is rational polyhedral with generators given by vectors $w_1^j, \dots, w_{r_j}^j$. Then the global Okounkov body $\Delta_{Y_\bullet}(X)$ is the cone generated by the vectors*

$$(\nu(s_{Y_1}, [Y_1])), (\nu(\sigma_i^j), [D_i^j]), w_h^j, \quad (10)$$

for

$$j = 1, \dots, m, \quad i = n_j + 1, \dots, \ell_j, \quad h = 1, \dots, r_j.$$

Proof. Let E be an effective integral divisor on X, and let $s \in H^0(X, \mathcal{O}_X(E))$ be a nonzero section of E. Let $\nu_1(s) = a$. Then, $\zeta := s/s_{Y_1}^a$, where $s_{Y_1} \in H^0(X, \mathcal{O}_X(Y_1))$ is the defining section of Y_1, is a section of $\mathcal{O}_X(E - aY_1)$ which vanishes to order 0 along Y_1. Now, let $\Sigma = \operatorname{conv}\{D_1, \dots, D_\ell\}$ be a Mori chamber such that $E - aY_1 \in \Sigma$, and with generators ordered so that $D_1, \dots, D_r$ are the movable generators. Let $E - aY_1 = P + N$ be the corresponding decomposition of $E - aY_1$ into its movable part P and fixed part N. Choose $M \in \mathbb{N}$ large enough such that all the divisors $MP = c_1D_1 + \cdots + c_\ell D_r$, $MN = c_{r+1}D_{r+1} + \cdots + c_\ell D_\ell$, where $c_1, \dots, c_\ell \in \mathbb{N}_0$, and all c_iD_i are integral divisors. Let $\sigma_i \in H^0(X, \mathcal{O}_X(D_i))$ be the defining section of N_i, $i = r + 1, \dots, \ell$. The section $\zeta^M \in H^0(X, \mathcal{O}_X(m(E - aY_1)))$ now decomposes uniquely as a product

$$\zeta^M = \eta\sigma,$$

where $\eta \in H^0(X, \mathcal{O}_X(c_1D_1 + \cdots + c_rD_r))$, and $\sigma = \sigma_{r+1}^{c_{r+1}} \cdots \sigma_\ell^{c_\ell}$. Since $\nu_1(\zeta) = 0$, we also have $\nu_1(\eta) = 0$. Now, by assumption we have integral generators $w_1, \ldots, w_k \in \mathbb{R}^n \times \overline{\mathrm{Eff}}(X)$ for the cone $C(S(D_1, \ldots, D_r))$, so that $(\nu(\eta), MP) = s_1w_1 + \cdots + s_kw_k$, for some $s_1, \ldots, s_k \geq 0$. Hence,

$$(\nu(s), E) = a(\nu(s_{Y_1}), Y_1) + \frac{c_{r+1}}{M}(v(\sigma_{r+1}), [D_{r+1}]) + \cdots + \frac{c_\ell}{M}(v(\sigma_\ell), [D_\ell]) + \frac{s_1}{M}w_1 + \cdots + \frac{s_k}{M}w_k.$$

It follows that $\Delta_{Y_\bullet}(X)$ lies in the closed convex cone generated by the vectors (10). Since all these vectors clearly belong to $\Delta_{Y_\bullet}(X)$, this finishes the proof. $\square$

We are now in the position to prove the main result of this paper. Let us first define what we mean by a good flag on a Mori dream space.

Definition 3.4. Let X be a Mori dream space. An admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ is *good*, if

(1) Y_i is a Mori dream space for each $0 \leq i \leq n$, and for $i = 1, \ldots, n$, Y_i is cut out by a global section $s_i \in H^0(Y_{i-1}, \mathcal{O}_{Y_{i-1}}(Y_i))$, and
(2) any small $\mathbb{Q}$-factorial modification $f : Y_i \dashrightarrow Y_i'$, restricts to a small $\mathbb{Q}$-factorial modification of Y_{i+1}.

Our main theorem now follows from the above results.

Theorem 3.5. *Assume that the Mori dream space X admits a good flag $Y_\bullet$. Then $\Delta_{Y_\bullet}(X)$ is rational polyhedral.*

Proof. We prove the theorem by induction over n. Every Mori dream curve is isomorphic to $\mathbb{P}^1$, which for any choice of flag (i.e., choice of a point) has rational polyhedral global Okounkov body, namely the cone in $\mathbb{R}^2$ spanned by the points $(0, 1)$ and $(1, 1)$.

For the inductive step assume that $\Delta_{Y_\bullet^1}(Y_1)$ is rational polyhedral. By Theorem 3.3, what we need to prove is that for any Mori chamber Σ in $\overline{\mathrm{Eff}}(X)$ the set of movable generators $D_1, \ldots, D_r$ of Σ yield a rational polyhedral cone $C(S(D_1, \ldots, D_r))$.

Since $Y_\bullet$ is a good flag, in particular the assumptions of Proposition 3.1 are satisfied for $Y_1 \subset X$, so we can apply Corollary 3.2. Since the linear map q (cf. (8)) is defined over $\mathbb{Z}$, equality (9) implies that $C(S(D_1, \ldots, D_r))$ is rational polyhedral if $C(S_1(D_1, \ldots, D_r))$ is. Now the rational polyhedrality of $C(S_1(D_1, \ldots, D_r))$ follows from the rational polyhedrality of $\Delta_{Y_\bullet^1}(Y_1)$ since

$$C(S_1(D_1, \ldots, D_r)) = pr_2^{-1}(\Gamma(D_1 \cdot Y_1, \ldots, D_r \cdot Y_1)) \cap \Delta_{Y_\bullet^1}(Y_1),$$

where $\Gamma(D_1 \cdot Y_1, \cdots, D_r \cdot Y_1) \subseteq \overline{\mathrm{Eff}}(Y_1)$ is the convex cone generated by the numerical equivalence classes of the divisors $D_1 \cdot Y_1, \ldots, D_r \cdot Y_1$ on Y_1. $\square$

Remark 3.6. Formally, the above theorem looks similar to Okawa's [18, Theorem 1.5]. Note however that the conditions for a good flag differ substantially. In particular, our condition (2) enables us to control the behaviour of Okounkov bodies under small $\mathbb{Q}$-factorial modifications, an issue that Okawa appears not to consider.

Remark 3.7. It should be noted that the above result does not hold for general admissible flags of subvarieties of a Mori dream space. Indeed, [13, Example 3.4] shows that $X := \mathbb{P}^2 \times \mathbb{P}^2$ can be equipped with an admissible flag $Y_\bullet$ such that the Okounkov body $\Delta_{Y_\bullet}(D)$, where D is a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,1)$, is not polyhedral. The flag $Y_\bullet$ here is of course not a flag of Mori dream spaces: if $Y_1 := \mathbb{P}^2 \times E$, where $E \subseteq \mathbb{P}^2$ is a general elliptic curve, were a Mori dream space, then its image E under the second projection would also be a Mori dream space by [19, Theorem 1.1]. However, $\mathbb{P}^1$ is the only Mori dream curve (cf. [5, p. 6]).

Finally, [13, Prop. 3.5] gives another example of a non-polyhedral Okounkov body of a divisor an a Mori dream space Z with respect to a family of admissible flags $Y_\bullet$. However, the description of the pseudo-effective cone $\overline{\mathrm{Eff}}(Y_1)$ shows that this cone is defined by quadratic equations and is thus not polyhedral; hence the divisor Y_1 on Z is not a Mori dream space.

4. Okounkov bodies on Bott–Samelson varieties

In this section we apply the general results from the previous section to Bott–Samelson varieties.

4.1. Bott–Samelson varieties as quotients

Being a Mori dream space, we know that a Bott–Samelson variety $X = Z_w$ is given as a geometric quotient of an open subset of the spectrum of its Cox ring. In this section we shall see that the structure of X as a fibre bundle over $\mathbb{P}^1$ is reflected in the Cox ring of X – a fact that will turn to be useful for the study of the images of X under small $\mathbb{Q}$-factorial modifications.

Let now X be an n-dimensional Bott–Samelson variety, and write X as $X = P \times^B Y$, where Y is a Bott–Samelson variety of dimension $n-1$ and $P/B = \mathbb{P}^1$. Let $V := \mathrm{Spec}(\mathrm{Cox}(Y))$ and let $U_0 \subseteq V$ be the open subset, invariant under the $(n-1)$-dimensional torus T^{n-1}, of stable points of a T^{n-1}-linearized line bundle such that $Y = U_0//T^{n-1}$. Since B acts on the Cox ring of Y by graded automorphisms, B acts on V, and the action commutes with that of T^{n-1}, so that U_0 is B-invariant. We can thus form the bundle $P \times^B V$ over X with fibre V, as well as the bundle $P \times^B U_0$ which embeds as an open subset of $P \times^B V$. Let $\pi_0 : P \times_B V \longrightarrow P/B$ denote the bundle projection. By the commutativity of the actions of B and T^{n-1} on V, the torus T^{n-1} then acts on $P \times^B V$ by

$$t.[(p,v)] := [(p,t.v)], \quad [(p,v)] \in P \times^B V, \quad t \in T^{n-1},$$

and X can be written as the geometric quotient

$$X = (P \times^B U_0)//T^{n-1}.$$

Let $q : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1 = P/B$ be the quotient morphism defined by the standard action of T^1 and consider the pullback $q^*(P \times^B V)$, which is a bundle over $\mathbb{A}^2 \setminus \{0\}$ with fibre V. Let $\tilde{\varphi} : q^*(P \times^B V) \longrightarrow \mathbb{A}^2 \setminus \{0\}$ denote the bundle projection. As a topological space,

$$q^*(P \times^B U_0) = \{(a,\xi) \in (\mathbb{A}^2 \setminus \{0\}) \times (P \times^B V) \mid q(a) = \pi_0(\xi)\},$$

and $\tilde{\varphi}(a,\xi) = a$.

The action of T^{n-1} on $P \times^B V$ lifts to an action on $q^*(P \times^B V)$ by fibre-preserving morphisms:

$$t'.(a,[(p,v)]) := (a,[(p,t'.v)]), \quad t' \in T^{n-1}, \quad (a,[(p,v)]) \in q^*(P \times^B V).$$

Combining this action with the standard action of T^1, we obtain an action of $T^n = T^1 \times T^{n-1}$ given by

$$(t,t').(a,[(p,v)]) := (t.a,[(p,t'.v)]), \quad (t,t') \in T^1 \times T^{n-1}, \quad (a,[(p,v)]) \in q^*(P \times^B V),$$

and X is the geometric quotient of the open subset $q^*(P \times^B U_0) \subseteq q^*(P \times^B V)$ by this action;

$$X = q^*(P \times^B U_0)//T^n.$$

Let $\pi : q^*(P \times^B U_0) \longrightarrow X$ denote the quotient morphism. In the above discussion, we have now proved the following lemma.

Lemma 4.1. *Let X be the Bott–Samelson variety $X := P \times^B Y$ with projection*

$$\varphi : P \times^B Y \longrightarrow P/B \cong \mathbb{P}^1$$

Then, there exist a commuting diagram

$$\begin{tikzcd} q^*(P \times^B U_0) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi"] & \mathbb{A}^2 \setminus \{0\} \arrow[d, "q"] \\ X \arrow[r, "\varphi"] & \mathbb{P}^1, \end{tikzcd}$$

where $\widetilde{\varphi}$ defines a locally trivial fibre bundle, and q is the quotient morphism with respect to the standard action of the one-dimensional torus on $\mathbb{A}^2 \setminus \{0\}$.

Although the quotient description above does not coincide with the standard one, giving X as a GIT quotient of Spec(Cox(X)), the following lemma shows that the variety $q^*(P \times^B V)$ shares an important property with Spec(Cox(X)).

Lemma 4.2. *The variety $q^*(P \times^B V)$ has torsion Picard group, and the $\mathbb{Q}$-vector space $Pic^{T^n}(q^*(P \times^B V)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the vector space $\widehat{T^n} \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-characters of T^n.*

Proof. First of all, for a T^{n-1}-character χ', the χ'-linearized trivial line bundle $L_{\chi'}$ over V carries a B-action, and $P \times^B L_{\chi'}$ is then a χ'-linearization of the trivial line bundle over $P \times^B V$ whose descent to each fibre of $P \times^B Y$ coincides with the descent of $L_{\chi'}$ to Y. The pullback $q^*(P \times^B L_{\chi'})$ is then a T^{n-1}-linearization of the trivial line bundle over $q^*(P \times^B V)$ by the character χ'. We have thus embedded the character group of T^{n-1} into the T^n-Picard group of $q^*(P \times^B V)$ in such a way that the descents to X of these line bundles yield an embedding of Pic(Y) into Pic(X). Moreover, the identity character of T^1 gives a T^1-linearization of the trivial line bundle on $q^*(P \times^B V)$, and this T^n-line bundle descends to a line bundle on X generating the subgroup Pic(P/B) of Pic(X). Thus we have embedded the character group of T^n into $\mathrm{Pic}^{T^n}(q^*(P \times^B V))$.

On the other hand, since the open subset $q^*(P \times^B U_0)$ has a complement of codimension at least two, we have $\mathrm{Pic}^{T^n}(q^*(P \times^B V)) = \mathrm{Pic}^{T^n}(q^*(P \times^B U_0))$. Since the quotient π is a geometric one, there exists an $N \in \mathbb{N}$ such that the N-th power of every T^n-line bundle descends to X, and hence the latter T^n-Picard group is a finite extension of the group Pic(X), which in turn is isomorphic to the character group of T^n. This proves the claim about the rational T^n-Picard group.

For the first claim, since T^n has trivial Picard group, every line bundle L on $q^*(P\times^B V)$ admits a T^n-linearization. Hence, by the above, L^N is isomorphic to the trivial line bundle with the a linearization given by a character of T^N. This shows that the Picard group of $q^*(P \times^B V)$ is torsion. □

Theorem 4.3. *The Cox ring of X is a quotient of Cox(P/B) ⊗ Cox(Y). More precisely, Cox(X) is generated by homogeneous elements of Cox(P/B) and homogeneous elements of Cox(Y), with respect to the gradings defined by the character groups of the tori T^1 and T^{n-1}.*

Proof. By the above lemma, and the fact that the open subset $q^*(P\times^B U_0)$ of $q^*(P\times^B V)$ has a complement of codimension at least two, Hartog's theorem implies that Cox(X) with its grading by the effective basis for Pic(X) is isomorphic as a graded ring to the ring of regular functions on $q^*(P \times^B V)$ with the grading defined by the characters of T^n.

Now, if $\chi \otimes \chi'$ is a character of $T^1 \times T^{n-1}$, a regular T^n-equivariant function, with respect to the character $\chi\otimes\chi'$, on the bundle $q^*(P\times^B V)$ over $\mathbb{A}^2 \setminus \{0\}$ is a T^1-equivariant function f on $\mathbb{A}^2 \setminus \{0\}$, with respect to the character χ, such that for each $a \in \mathbb{A}^2 \setminus \{0\}$,

$f(a)$ defines a regular T^{n-1}-equivariant function on the fibre $\tilde{\varphi}^{-1}(a)$ with respect to the character χ'. In other words, f corresponds to a section s of the vector bundle $q^*(P \times^B H^0(V, L_{\chi'})^{T^{n-1}})$. Moreover, $s(t.a) = \chi(t)s(a)$, for $t \in T^1$ and $a \in \mathbb{A}^2 \setminus \{0\}$.

In order to express $\mathrm{Cox}(X)$ as a quotient ring, we first consider a T^{n-1}-equivariant closed embedding $V \hookrightarrow \mathbb{A}^m = \mathrm{Spec}(\mathbb{C}[s_1, \dots, s_m])$, where the s_i form a set of homogeneous generators of $\mathrm{Cox}(Y)$ with respect to T^{n-1}-characters. By collecting the generators s_i into groups according to their T^{n-1}-characters, the underlying vector space, $\mathbb{C}^m$, of $\mathbb{A}^m$ decomposes further into a sum of subspaces which are B-invariant and on which T^{n-1} acts by a fixed character:

$$\mathbb{C}^m = E_1 \oplus \cdots \oplus E_r.$$

Due to the B-invariance of the E_j, we can for each j define the vector bundle $P \times^B E_j$, which then, by a theorem of Grothendieck, splits as a direct sum of line bundles:

$$P \times^B E_j \cong \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(m_\ell),$$

for some $m_1, \dots, m_\ell \in \mathbb{Z}$, so that

$$q^*(P \times^B E_j) \cong q^*\mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \cdots \oplus q^*\mathcal{O}_{\mathbb{P}^1}(m_\ell).$$

Since $\mathbb{A}^2 \setminus \{0\}$ has a trivial Picard group, the vector bundle $q^*(P \times^B E_j)$ is isomorphic to the trivial vector bundle of rank ℓ $(= \ell(j))$. Moreover, since T^{n-1} acts by a fixed character, χ_j, on the fibres of $P \times^B E_j$, both the above isomorphisms of vector bundles respect this action, so that the sections of $q^*(P \times^B E_j)$ correspond in the trivialized model to T^{n-1}-equivariant functions, with respect to the character χ_j, with values in $\mathbb{C}^\ell$.

It follows that the ring of regular functions on $q^*(P \times^B \mathbb{C}^m)$ is isomorphic as a T^n-graded ring to the ring $\mathbb{C}[x_1, x_2] \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_m]$, where T^1 acts naturally on the first tensor factor, and T^{n-1} acts on the second tensor factor by the grading described above.

It now suffices to show that the restriction of regular functions from $q^*(P \times^B \mathbb{C}^m)$ to $q^*(P \times^B V)$ defines a surjective and T^n-equivariant homomorphism

$$\mathbb{C}[x_1, x_2] \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_m] \longrightarrow \mathrm{Cox}(X),$$

where the T^n-equivariance is a consequence of the T^{n-1}-equivariance of the embedding $V \hookrightarrow \mathbb{A}^m$. For the surjectivity, we again return to the identification of a T^n-equivariant regular function f on $q^*(P \times^B V)$ with character $\chi_1 \times \chi'$ with a section s of the bundle $q^*(P \times^B H^0(V, L_{\chi'})^{T^{n-1}})$, and consider the (finite dimensional) subspace $W \subseteq \mathbb{C}[s_1, \dots, s_m]$ of all polynomials of homogeneous degree χ' with respect to the grading by characters of T^{n-1}. The restriction of such polynomials to V defines a surjection

$W \longrightarrow H^0(V, L_{\chi'})^{T^{n-1}}$, and if F denotes the kernel of this restriction map, we have a short exact sequence of vector spaces

$$0 \longrightarrow F \longrightarrow W \longrightarrow H^0(V, L_{\chi'})^{T^{n-1}} \longrightarrow 0,$$

which is equivariant with respect to both T^{n-1} and B. By forming the corresponding vector bundles over P/B and pulling back to $\mathbb{A}^2 \setminus \{0\}$, we get the short exact sequence

$$0 \longrightarrow q^*(P \times^B F) \longrightarrow q^*(P \times^B W) \longrightarrow q^*(P \times^B H^0(V, L_{\chi'})^{T^{n-1}}) \longrightarrow 0$$

of vector bundles over $\mathbb{A}^2 \setminus \{0\}$. By the same argument as earlier in the proof, these vector bundles are all trivial, and hence they can be identified with vector bundles over $\mathbb{A}^2$. Moreover, by Hartog's theorem, the restriction map of sections

$$H^0(\mathbb{A}^2 \setminus \{0\}, q^*(P \times^B W)) \longrightarrow H^0(\mathbb{A}^2 \setminus \{0\}, q^*(P \times^B H^0(V, L_{\chi'})^{T^{n-1}}))$$

can be identified with the restriction of sections

$$H^0(\mathbb{A}^2, q^*(P \times^B W)) \longrightarrow H^0(\mathbb{A}^2, q^*(P \times^B H^0(V, L_{\chi'})^{T^{n-1}}))$$

of the corresponding vector bundles over $\mathbb{A}^2$. The surjectivity of the latter restriction now follows from the fact that $H^1(\mathbb{A}^2, q^*(P \times^B F)) = 0$. This finishes the proof. $\square$

The fibre bundle $\varphi : X = P \times^B Y \longrightarrow P/B$ defines an embedding $\mathrm{Cox}(P/B) \hookrightarrow \mathrm{Cox}(X)$ of Cox rings, and the image of this embedding is precisely the subring of T^{n-1}-invariants. The induced morphism

$$p : \mathrm{Spec}(\mathrm{Cox}(X)) \longrightarrow \mathrm{Spec}(\mathrm{Cox}(P/B)) = \mathrm{Spec}(\mathrm{Cox}(X)^{T^{n-1}}) \tag{11}$$

then defines a quotient with respect to the T^{n-1}-action on $\mathrm{Spec}(\mathrm{Cox}(X))$. The morphism p is clearly T^1-equivariant. In particular, since T^1 fixes the point $0 \in \mathbb{A}^2 = \mathrm{Spec}(\mathrm{Cox}(P/B))$, T^1 acts on the fibre $p^{-1}(0)$.

Lemma 4.4. *The torus T^1 acts trivially on $p^{-1}(0)$.*

Proof. Let $s_1, s_2, \ldots, s_N$ be a set of homogeneous generators of $\mathrm{Cox}(X)$ according to Theorem 4.3, where s_1, s_2 are generators of $\mathrm{Cox}(P/B)$, and $s_3, \ldots, s_N$ are generators of $\mathrm{Cox}(Y)$. Let $\mathbb{A}^N$ be the affine space with coordinates corresponding to the generators $s_1, \ldots, s_N$, so that we have a closed T^n-equivariant embedding $\mathrm{Spec}(\mathrm{Cox}(X)) \hookrightarrow \mathbb{A}^N$. On $\mathbb{A}^N$, the torus T^n acts on each coordinate function s_i by the character associated to s_i. In particular, T^1 fixes $s_3, \ldots, s_N$, and T^{n-1} fixes s_1, s_2, and s_1, s_2 generate the subring of T^{n-1}-invariants in the polynomial ring $\mathbb{C}[s_1, \ldots, s_N]$. The projection of $\mathbb{A}^N$ onto the $\mathbb{A}^2$ defined by the first two coordinates defines the Hilbert quotient $\mathbb{A}^N \longrightarrow \mathbb{A}^N//T^{n-1} \cong \mathbb{A}^2$ of $\mathbb{A}^N$ by T^{n-1}. Hence,

$$p^{-1}(0) = \mathrm{Spec}(\mathrm{Cox}(X)) \cap (\{0\}^2 \times \mathbb{A}^{N-2}),$$

and the claim thus follows. $\square$

Let $W := \mathrm{Spec}(\mathrm{Cox}(X))$. Then, X is the GIT-quotient by T^n of the semistable locus $W^{ss}(\chi) \subseteq W$ with respect to a character χ of T^n. We also have the following analogon of Lemma 4.1.

Lemma 4.5. *Let X be the Bott–Samelson variety $X := P \times^B Y$ with projection*

$$\varphi : P \times^B Y \longrightarrow P/B \cong \mathbb{P}^1$$

Then, there exists a commuting diagram

$$\begin{tikzcd} W^{ss}(\chi) \arrow[r, "p"] \arrow[d, "\pi_\chi"] & \mathbb{A}^2 \setminus \{0\} \arrow[d, "q"] \\ X \arrow[r, "\varphi"] & \mathbb{P}^1. \end{tikzcd}$$

Proof. Since p is induced by the projection $\varphi : X \longrightarrow P/B$, we only need to show that $W^{ss}(\chi)$ does not intersect the zero fibre $p^{-1}(0)$. But this is clear from Lemma 4.4 since the semistable locus $W^{ss}(\chi)$ in fact equals the stable locus $W^s(\chi)$ so that T^n acts $W^{ss}(\chi)$ with finite stabilizers. $\square$

The following lemma is the main result of this section. It will be used in the next section to show that images of X under SQM's are also fibre bundles over $\mathbb{P}^1$ (cf. Proposition 4.8).

Lemma 4.6. *Let $X = P \times^B Y$, with projection*

$$\varphi : P \times^B Y \longrightarrow P/B \cong \mathbb{P}^1,$$

be a Bott–Samelson variety, and let $f : X \dashrightarrow X'$ be an SQM. If ξ is a section of the line bundle $\mathscr{L} := \varphi^*\mathcal{O}_{\mathbb{P}^1}(1)$ on X, and ξ' is the corresponding section of the corresponding line bundle $\mathscr{L}' := (f^{-1})^*\mathscr{L}$, then the restrictions of $\mathscr{L}$ and $\mathscr{L}'$ to $Z(\xi)$ and $Z(\xi')$, respectively, are trivial.

Proof. The zero locus of any section of $\mathscr{L}$ is the φ-fibre over some point in $\mathbb{P}^1$, which proves the claim for X.

Now, if the zero loci $Z(\xi_x)$ and $Z(\xi_y)$ of sections ξ_x and ξ_y of $\mathscr{L}$ are φ-fibres of points x and y in $\mathbb{P}^1$, then we have $\xi_x = \varphi^* s_x$ and $\xi_y = \varphi^* s_y$, where $s_x, s_y \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ are the defining sections of the divisors $\{x\}$ and $\{y\}$. Using the commuting diagram of Lemma 4.5, and the fact that $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ is the trivial line bundle, T^1-linearized by the identity character, we see that the pullbacks $f := \pi_\chi^*\xi_x$ and $g := \pi_{\chi'}^*\xi_y$ are regular T^n-eigenfunctions on $W^{ss}(\chi)$ with respect to the character defined by pulling back the identity character of T^1 by the projection $T^n = T^1 \times T^{n-1} \longrightarrow T^1$. Now, the zero loci $Z(f)$ and $Z(g)$ are disjoint since their images–the intersections with $p(W^{ss}(\chi))$ of $q^{-1}(x)$ and $q^{-1}(y)$, respectively–under p in $\mathbb{A}^2 \setminus \{0\}$ are disjoint. Since $W^{ss}(\chi)$ has a complement of codimension at least two in W, using the normality of W, we can extend f and g to functions on W by Hartog's theorem. Viewed as functions on W, the common zero set of f and g is precisely the zero fibre $p^{-1}(0) \subseteq W$.

The variety X' can be written as a quotient

$$X' = W^{ss}(\chi')//T^n = W^s(\chi')//T^n,$$

with quotient morphism $\pi_{\chi'} : W^s(\chi') \longrightarrow X'$, for a character χ'. We can now argue as in the proof of Lemma 4.5: since the points in $W^s(\chi')$ have finite stabilizers, and the stabilizer of any point in $p^{-1}(0)$ contains the one-dimensional subgroup $T^1 \times \{(1, \dots, 1)\}$ the zero sets of f and g have no common point in $W^s(\chi')$. The sections ξ'_x and ξ'_y of $\mathscr{L}'$ with $f = (\pi_{\chi'})^*\xi'_x$ and $g = (\pi_{\chi'})^*\xi'_y$ on $W^s(\chi')$ therefore have no common zero set. This proves the claim for X'. $\square$

4.2. Global Okounkov bodies

On a Bott–Samelson variety $X := Z_w$ defined by a reduced sequence w let $Y_\bullet$ be the natural vertical flag described in section 2.1. In order to apply Theorem 3.5 we will show that $Y_\bullet$ is in fact a good flag. The fact that each Y_i is a Mori dream space follows from Theorem 2.2, and the second condition is a consequence of the following proposition together with the fact that the exceptional locus of an SQM equals the stable base locus of the defining movable divisor.

Proposition 4.7. *Let D be a Cartier divisor on $X := Z_\omega$, and F a base component of the complete linear series $|D|$. Then F intersects $Y := Y_1$ properly, i.e.,*

$$\operatorname{codim}_Y(F|_Y) = \operatorname{codim}_X(F)$$

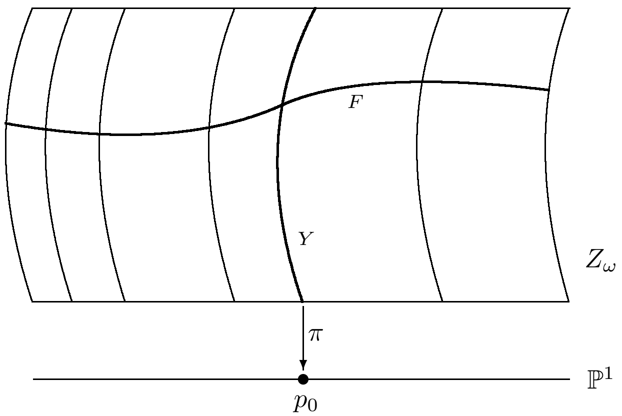
Proof. We first prove that the base locus of $|D|$ is invariant under the group P_1. Let therefore $s \in H^0(X, \mathcal{O}_X(D))$ be a section, and let $p \in P_1$. Then for any $x \in Z_\omega$

$$s(px) = pp^{-1}s(px) = p(p^{-1}s(px)).$$

By (4), the right hand side is exactly $p((p^{-1}s)(x))$ and $(p^{-1}s \in H^0X, \mathcal{O}_X(D))$ vanishes in x if x is in the base locus $B(D)$. Therefore,

$$s(px) = p0_{L_x} = 0_{L_{px}},$$

and the claim follows. Then, every irreducible component of $B(D)$ is also P_1-invariant. In particular, F is invariant under P_1.



Now, the P_1-action on $\mathbb{P}^1$ has only one orbit, so P_1 operates transitively on the fibres of π. Hence, the base component F is given by the union of orbits of elements in the restriction $F|_Y$. Therefore, the generic fibre dimension holds for all fibres of π, so that

$$\dim F = \dim F|_Y + 1,$$

which implies the statement. $\square$

In order to construct a good flag on a Bott–Samelson variety, we next prove that the above result generalizes to the birational images under a small $\mathbb{Q}$-factorial modification.

Proposition 4.8. *Let $X = P_1 \times^B \cdots \times^B P_n/B$ be a Bott–Samelson variety, and let $f : X \dashrightarrow X'$ be an SQM. Then there exists a P_1-equivariant locally trivial fibre bundle $q : X' \longrightarrow \mathbb{P}^1$.*

Consequently, for every P_1-invariant subvariety $W \subseteq X'$ of codimension at least two, and every point $y \in \mathbb{P}^1$, the subvariety $W \cap q^{-1}(y)$ of the fibre $q^{-1}(y)$ is of codimension at least two in $q^{-1}(y)$.

Proof. Consider the fibre bundle $\varphi : X = P_1 \times^B \cdots \times^B P_n/B \longrightarrow P_1/B \cong \mathbb{P}^1$, and let $\mathscr{L} := \varphi^* \mathscr{O}_{\mathbb{P}^1}(1)$. Putting $\mathscr{L}' := (f^{-1})^* \mathscr{L}$, we have the commuting diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\varphi"] & X' \arrow[d, dashed, "\varphi'"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1, \end{tikzcd}$$

where φ is the morphism defined by the globally generated line bundle $\mathscr{L}$, φ' is the rational map defined by the line bundle $\mathscr{L}'$, and the bottom horizontal morphism $\mathbb{P}(H^0(X,\mathscr{L})) \cong \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \cong \mathbb{P}(H^0(X',\mathscr{L}'))$ is the isomorphism induced by the isomorphism of section spaces $H^0(X,\mathscr{L}) \cong H^0(X',\mathscr{L}')$.

If $X' = \mathrm{Proj}(R(X,\mathcal{O}_X(D)))$, for the movable divisor D, the stable base locus $\mathbb{B}(D)$ of D intersects each fibre of φ along a subvariety of codimension at least two. Since φ' can be identified with φ on an open subset $V \subseteq X'$ isomorphic to $X \setminus \mathbb{B}(D)$, this means that $\varphi' : V \longrightarrow \mathbb{P}^1$ is surjective.

Since the fibres of the bundle $\varphi : X \longrightarrow \mathbb{P}^1$ are precisely the zero loci of the sections of $\mathscr{L}$, X is naturally isomorphic to the incidence variety

$$\{(x,[\xi]) \in X \times \mathbb{P}(H^0(X,\mathscr{L})) \mid \xi(x) = 0\}. \tag{12}$$

In order to describe X' as a fibre bundle over $\mathbb{P}^1$, we therefore first consider the analogous incidence variety

$$Z := \{(x,[\xi]) \in X' \times \mathbb{P}(H^0(X',\mathscr{L}')) \mid \xi(x) = 0\},$$

and let

$$p : Z \longrightarrow X', \quad q : Z \longrightarrow \mathbb{P}^1$$

be the morphism given as the restrictions to Z of the projections onto the respective factors.

The group P_1 operates naturally on $X' := \mathrm{Proj}(R(X,\mathcal{O}_X D))$ and on $\mathbb{P}(H^0(X',\mathscr{L}'))$ by the actions induced from those on $\mathcal{O}_X(D)$ and $\mathscr{L}$, and hence the restriction to Z of the product action of P_1 on $X' \times \mathbb{P}(H^0(X',\mathscr{L}'))$ makes the morphism $q : Z \longrightarrow \mathbb{P}^1$ P_1-equivariant. By generic smoothness, the generic fibre of q is smooth. On the other hand, the transitivity of the P_1-action on $\mathbb{P}^1$ implies that any two q-fibres are isomorphic via an element from P_1. It thus follows that Z has the structure of a smooth, and locally trivial, fibre bundle over $\mathbb{P}_1$. In fact, for any point $[\xi] \in \mathbb{P}^1$, q is trivial over the open set $\mathbb{P}^1 \setminus \{[\xi]\} \cong \mathbb{A}^1$. Indeed, if $y_0 \in \mathbb{A}^1$ is a fixed point, for every $y \in \mathbb{A}^1$ there is a canonical $g_y \in P_1$ such that $g_y(y_0) = y$ in such a way that the map $y \mapsto g_y$ defines a morphism $\mathbb{A}^1 \longrightarrow P_1$. (This morphism is essentially given by embedding $\mathbb{A}^1$ into SL_2 as the group of upper triangular matrices with ones on the diagonal.) A local trivialization over $\mathbb{A}^1$ is then given by the isomorphism

$$\mathbb{A}^1 \times q^{-1}(y_0) \longrightarrow q^{-1}(\mathbb{A}^1), \quad (y,z) \mapsto g_y(z) \in q^{-1}(y) \subseteq q^{-1}(\mathbb{A}^1).$$

Moreover, for each $[\xi] \in \mathbb{P}^1$, the fibre

$$q^{-1}([\xi]) = \{x \in X' \mid (x,[\xi]) \in Z\}$$

is a Weil divisor for the line bundle $q^*\mathcal{O}_{\mathbb{P}^1}(1)$, and we have the short exact sequence of Picard groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(q^{-1}(\mathbb{A}^1)) \longrightarrow 0, \tag{13}$$

where $k \in \mathbb{Z}$ is identified with the line bundle $q^*\mathcal{O}_{\mathbb{P}^1}(k)$. By the local triviality of q, we further have

$$\mathrm{Pic}(q^{-1}(\mathbb{A}^1)) \cong \mathrm{Pic}(\mathbb{A}^1 \times q^{-1}(y_0)) \cong \mathrm{Pic}(q^{-1}(y_0)). \tag{14}$$

In order to prove the claim by the same argument as for X, we now show that X' is in fact isomorphic to Z. First of all, we have an isomorphism

$$q^*\mathcal{O}_{\mathbb{P}^1}(k) \cong p^*\mathscr{L}' \tag{15}$$

of line bundles on X', for some $k \in \mathbb{N}$. Indeed, as shown above, a Weil divisor of $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ is given by any q-fibre. On the other hand, the zero set of $p^*\xi \in H^0(Z, p^*\mathscr{L}')$ is given by

$$Z(p^*\xi) = \{(x, [\eta]) \in X' \times \mathbb{P}(H^0(X', \mathscr{L}')) \mid \eta(x) = 0, \quad \xi(x) = 0\}.$$

Now, if we restrict $p^*\xi$ to the q-fibre $q^{-1}([\eta])$, for some $[\eta] \in \mathbb{P}^1 \setminus \{[\xi]\} \cong \mathbb{A}^1$, we obtain the divisor

$$Z(p^*\xi) \cap q^{-1}([\eta]) = \{x \in X' \mid (x, [\eta]) \in Z, \quad \xi(x) = 0\} \cong Z(\xi) \cap Z(\eta) \subseteq X'$$

of $q^{-1}([\eta]) \cong Z(\eta)$. Now, by Lemma 4.6, this divisor represents the trivial line bundle on the fibre $q^{-1}([\eta])$, i.e., the restriction of the line bundle $p^*\mathscr{L}'$ to the fibre $q^{-1}(\eta)$ is trivial. By the isomorphism (14), the restriction of $p^*\mathscr{L}'$ is then trivial on the open subset $q^{-1}(\mathbb{A}^1) \subseteq Z$ as well. Since this also holds for the line bundle $q^*\mathcal{O}_{\mathbb{P}^1}(1)$, the exact sequence (13) now yields that

$$p^*\mathscr{L}' = q^*\mathcal{O}_{\mathbb{P}^1}(k),$$

for some integer k. Since $p^*\mathscr{L}'$ is effective with a space of sections of dimension at least two, k is in fact positive.

The identity (15) now shows that $p^*\mathscr{L}'$ is nef, and hence $\mathscr{L}'$ is itself also nef, and therefore even semi-ample since X' is a Mori dream space. Since the section ring $R(X', \mathscr{L}')$ is generated in degree one, $\mathscr{L}'$ is even globally generated. It follows that φ' in fact defines a morphism $\varphi' : X' \longrightarrow \mathbb{P}^1$.

We now show that X' is naturally isomorphic to Z. For a point $y \in \mathbb{P}^1$, let $s_y \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ be the defining section of the Weil divisor $\{y\}$, and let $\xi_y \in H^0(X', (\varphi')^*\mathcal{O}_{\mathbb{P}^1}(1))$ be its pullback to X'. Now, let

$$j : X' \longrightarrow Z, \quad j(x) := (x, [\xi_{\varphi'(x)}]), \quad x \in X'.$$

Then, j defines an isomorphism $j : V \longrightarrow j(V)$, from which we see that the image of j is of full dimension in Z, so that j is indeed surjective. The injectivity is clear from the identity

$$p \circ j = id_{X'}.$$

Moreover, the identity

$$j \circ p = id_Z$$

holds on the open subset $j(V) \subseteq Z$ since every point in V corresponds to a point $x \in X$ which lies in a unique zero locus of a section of $\mathscr{L}$ (cf. (12)). By the separability of Z (over $\mathbb{C}$), the identity then holds on all of Z, and hence p is an inverse to j. Thus, $j : X' \longrightarrow Z$ defines an isomorphism. This finishes the proof. $\square$

Theorem 4.9. *Let Z_w be a Bott–Samelson variety defined by a reduced sequence w. Then the global Okounkov body $\Delta_{Y_\bullet}$ is rational polyhedral.*

Proof. By Theorem 2.2, the variety Z_w is a Mori dream space. The same is true for each Y_i. Moreover, by construction, Y_i defines a Cartier divisor on Y_{i-1} and is cut out by a global section s_i which is just the pullback of a section $t_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ vanishing in the B-fixed point p_0. Inductive application of Proposition 4.7 and Proposition 4.8 then shows that $Y_\bullet$ is a good flag, and the result follows from Theorem 3.5. $\square$

Example 3. As a basic example consider for any n the n-dimensional Bott–Samelson variety $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Its big cone and ample cone agree and are spanned by the fibres $D_1, \ldots, D_n$ under the various projections. In other words, there is one Mori chamber spanned by the divisors D_i. The vertical flag $Y_\bullet$ here is just the flag of complete intersections of the D_i. Thus the Okounkov bodies of the generators are

$$\Delta_{Y_\bullet}(D_i) = \{0\} \times \cdots \times \{0\} \times [0,1] \times \{0\} \times \cdots \times \{0\}$$

It follows from the main theorem that the global Okounkov body is spanned by the vectors $((0, \ldots, 0, 1, 0, \ldots, 0), [D_i])$, for $i = 1, \ldots, n$.

Example 4. Let X be the Bott–Samelson surface studied in Example 1.

We determine the global Okounkov body of X with respect to the vertical flag $Y_\bullet$ given by a divisor C in $|H - E|$ with $C \cong \mathbb{P}^1$, and a point $x \in C$ such that $x \notin E$ (cf. Example 1).

Since $H - E$ has trivial restriction to C, and $E\mid_C \in |\mathrm{div}\mathcal{O}_C(1)|$, the nef divisor H restricts to C as $H\mid_C = (H-E)\mid_C + E\mid_C = E\mid_C$.

Theorem 3 in [22] now shows that the global Okounkov body of X with respect to $Y_\bullet$ is given by

$$\Delta_{Y_\bullet}(X) = \text{cone}\{((0,0),[H]),((0,0),[H-E]),((1,0),[H-E]),((0,1),[H]), \\ ((0,0),[E])\}.$$

Moreover, since H is a big divisor on X, the Okounkov body $\Delta_{Y_\bullet}(H)$ equals the fibre over $[H]$ in $\Delta_{Y_\bullet}(H)$ with respect to the projection onto the first factor. We thus consider all linear combinations

$$\alpha((0,0),[H]) + \beta((0,0),[H-E]) + \gamma((1,0),[H-E]) + \delta((0,1),[H]) \\ + \epsilon((0,0),[E]),$$

with $\alpha,\dots,\epsilon \geq 0$ and

$$\alpha+\beta+\gamma+\delta = 1, \quad \beta+\gamma = \epsilon.$$

Thus, $\Delta_{Y_\bullet}(H)$ is given as the convex hull of $\{(0,0),(1,0),(0,1)\}$.

Example 5. Consider the three-dimensional incidence variety X from Example 2.

We now compute a subcone, $\Delta_{Y_\bullet}(E_1, E_2+E_3)$, of the global Okounkov body $\Delta_{Y_\bullet}(X)$, namely the part that lies above the cone $C(E_1, E_2+E_3)$ in $\overline{\text{Eff}}(X)$ spanned by the classes of the two divisors E_1 and E_2+E_3. Note that, for an effective divisor $kE_1 + \ell(E_1+E_2)$ in $C(E_1, E_2+E_3)$, the subtraction of a multiple (the order of vanishing of a section) of E_1 results in a divisor still lying in this cone. Moreover, $kE_1 + \ell(E_1+E_2) = kD_1 + \ell D_3$ is nef, and it restricts to the nef and big divisor H on Y. By Proposition 3.1, the volume of the restricted linear series $R(X, kD_1+\ell D_2)\mid_Y$ equals the volume of the full linear series $R(Y,H)$ on H. Moreover, if two divisors on X have the same restriction to Y, they differ by a multiple of E_1. By the expression for the Okounkov body of the divisor H on Y in Example 4 above, and Corollary 3.2 (applied to the subcone $C(E_1, E_2+E_3)$ of the nef cone), the cone $\Delta_{Y_\bullet}(E_1, E_2+E_3)$ is spanned by the vectors

$$((0,0,0),[E_2+E_3]),((0,1,0),[E_2+E_3]),((0,0,1),[E_2+E_3]),((0,0,0),[E_1]), \\ ((1,0,0),[E_1]).$$

We can now also show that the Okounkov bodies of effective line bundles over Schubert varieties, with respect to a natural valuation-like function, are rational polyhedral. Indeed, Schubert varieties have rational singularities, so that the projection morphism $p_w : Z_w \longrightarrow Y_{\overline{w}}$ satisfies the property $(p_w)_*\mathcal{O}_{Z_w} = \mathcal{O}_{Y_{\overline{w}}}$ (cf. [4, Section 2.2]). Hence, for any effective line bundle L on $Y_{\overline{w}}$, we have

$$H^0(Y_{\overline{w}}, L) \cong H^0(Z_w, p_w^* L). \tag{16}$$

Let now

$$\nu : \text{Cox}(Z_w)_h \setminus \{0\} \longrightarrow \mathbb{N}_0^n,$$

where $\mathrm{Cox}(Z_w)_h$ denotes the set of homogeneous elements in the Cox ring of Z_w with respect to the effective basis, be the valuation-like function defined by the flag $Y_\bullet$, and let

$$\nu_L : \bigsqcup_{k\geq 0} H^0(Y_{\overline{w}}, L^k) \setminus \{0\} \longrightarrow \mathbb{N}_0^n$$

be the valuation-like function naturally defined by the isomorphisms (16) (for all powers L^k) and restriction of ν. Then, the Okounkov body $\Delta_{\nu_L}(L)$ coincides with the slice $p_2^{-1}(p_w^* L) \cap \Delta_{Y_\bullet}(X)$ of $\Delta_{Y_\bullet}(X)$, and hence is rational polyhedral. Thus, we have proved the following corollary.

Corollary 4.10. *Let L be an effective line bundle over the Schubert variety $Y_{\overline{w}}$ of G/B. Then, the Okounkov body $\Delta_{Y_\bullet}(L)$ defined by the natural valuation-like function v_L defined by the flag $Y_\bullet$ in Z_w is a rational polytope.*

If $Y_{\overline{w}} = G/B$ is a flag variety, the Picard group $\mathrm{Pic}(G/B)$ has an effective basis, namely the line bundles $L_i = G \times^{\omega_i} \mathbb{C}$ defined by the fundamental weights ω_i, $i = 1, \ldots, r$, with respect to a choice of simple roots for the root system of G. Let $\Sigma \subseteq \overline{\mathrm{Eff}}(Z_w)$ be the closed convex cone generated by the divisors of the line bundles $p_w^* L_i$, $i = 1, \ldots, r$. By the isomorphisms (16) we now have

$$\Delta_{Y_\bullet}(G/B) \cong p_2^{-1}(\Sigma) \cap \Delta_{Y_\bullet}(Z_w).$$

Since the cone Σ is finitely generated, the cone on the right hand side is rational polyhedral, so that we have proved the following corollary.

Corollary 4.11. *The global Okounkov body $\Delta_{Y_\bullet}(G/B)$ of the flag variety G/B, with respect to the valuation defined by the flag $Y_\bullet$ of subvarieties of Z_w, is a rational polyhedral cone.*

4.3. Weight multiplicities

We now turn our attention to the action of a torus $H \subseteq B$, contained in a maximal torus of G lying in B, on the section ring $R(D) := \bigoplus_{k\geq 0} H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ of an effective divisor D on Z_w. Recall that each section space $H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ carries a representation of B given by the action of B as automorphisms of the line bundle $\mathcal{O}_{Z_w}(kD)$ (cf. [14]). Moreover, the flag $Y_\bullet$ consists of B-invariant subvarieties of Z_w, so that the valuation-like function

$$\nu_D : \bigsqcup_{k\geq 0} H^0(Z_w, \mathcal{O}_{Z_w}(kD)) \setminus \{0\} \longrightarrow \mathbb{N}_0^n$$

is B-invariant, i.e., the identity $\nu(b.s) = \nu(s)$ holds for any non-zero section $s \in H^0(Z_w, \mathcal{O}_{Z_w}(kD))$, and $b \in B$. Hence, there is a well-defined projection

$$q : \Delta_{Y_\bullet}(D) \longrightarrow \Pi_D$$

onto the weight polytope (cf. [3]) of the section ring $R(D)$ for the action of the torus H (cf. [20] [10]). If $\mathfrak{h} = \mathrm{Lie}(H)$ is the Lie algebra of H, and the $\mu \in \Pi_D \subseteq \mathfrak{h}$ is a rational point in the interior of the weight polytope, we then have that the asymptotics of the weight spaces $W_{k\mu} \subseteq H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ are given by

$$\lim_{k\to\infty} \frac{\mathrm{dim} W_{k\mu}}{k^{d-r}} = \mathrm{vol}_{d-r}(q^{-1}(\mu) \cap \Delta_{Y_\bullet}(D)),$$

where r is the dimension of the moment polytope Π_D, d is the dimension of the Okounkov body $\Delta_{Y_\bullet}(D)$ (and which equals the Iitaka dimension of the line bundle $\mathcal{O}_{Z_w}(D)$), and the right hand side denotes the $(d-r)$-dimensional Lebesgue measure of the slice $q^{-1}(\lambda) \cap \Delta_{Y_\bullet}(D)$ of the Okounkov body $\Delta_{Y_\bullet}(D)$. We thus get the following result, saying the asymptotics of weight spaces are given by polyhedral expressions.

Corollary 4.12. *For any effective divisor D on Z_w, and rational point $\mu \in \Pi_D$ in the interior of Π_D, the asymptotic multiplicity*

$$\lim_{k\to\infty} \frac{dim W_{k\mu}}{k^{d-r}}$$

is the volume of a rational polytope. As a consequence, the same holds for the weight spaces

$$W_{k\mu} \subseteq H^0(Y_{\overline{w}}, L^k)$$

for an effective line bundle L over a Schubert variety $Y_{\overline{w}}$.

Proof. We only need to prove the second claim about Schubert varieties. Here we notice that the projection morphism $p_w : Z_w \longrightarrow Y_{\overline{w}}$ is B-equivariant, and so in particular H-invariant. Hence, the isomorphisms (16) for the powers L^k are H-equivariant, so that the claim thus follows from the first part about Bott–Samelson varieties. □

References

[1] D. Anderson, Okounkov bodies and toric degenerations, Math. Ann. 356 (3) (2013) 1183–1202.
[2] D. Anderson, A. Stapledon, Schubert varieties are log Fano over the integers, Proc. Amer. Math. Soc. 142 (2) (2014) 409–411.
[3] M. Brion, Sur l'image de l'application moment, in: Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, Paris, 1986, in: Lecture Notes in Math., vol. 1296, Springer, 1987, pp. 177–192.
[4] M. Brion, Lectures on the geometry of flag varieties, Lecture notes.
[5] C. Casagrande, Mori dream spaces and Fano varieties, Lecture notes, 2012.
[6] N. Fujita, Newton–Okounkov bodies for Bott–Samelson varieties and string polytopes for generalized Demazure modules, preprint, arXiv:1503.08916, 2015.
[7] M. Harada, J. Yang, Newton–Okounkov bodies of Bott–Samelson varieties and Grossberg–Karshon twisted cubes, preprint, arXiv:1504.00982, 2015.

[8] Y. Hu, S. Keel, Mori dream spaces and GIT, Michigan Math. J. 48 (2000) 331–348.
[9] K. Kaveh, A.G. Khovanskii, Newton convex bodies, semigroups of integral points, graded algebras and intersection theory, Ann. of Math. (2) 176 (2) (2012) 925–978.
[10] K. Kaveh, A.G. Khovanskii, Convex bodies associated to actions of reductive groups, Mosc. Math. J. 12 (2012) 369–396, 461.
[11] K. Kaveh, Crystal bases and Newton–Okounkov bodies, Duke Math. J. 164 (13) (2015) 2461–2506.
[12] V. Kiritchenko, Newton–Okounkov polytopes of flag varieties, preprint, arXiv:1506.00362 [.org/abs].
[13] A. Küronya, V. Lozovanu, C. Maclean, Convex bodies appearing as Okounkov bodies of divisors, Adv. Math. 229 (2012) 2622–2639.
[14] N. Lauritzen, J.F. Thomsen, Line bundles on Bott–Samelson varieties, J. Algebraic Geom. 13 (2004) 461–473.
[15] R. Lazarsfeld, M. Mustaţă, Convex bodies associated to linear series, Ann. Sci. Éc. Norm. Supér. 42 (2009) 783–835.
[16] P. Littelmann, Cones, crystals, and patterns, Transform. Groups 3 (2) (1998) 145–179.
[17] P. Magyar, Schubert polynomials and Bott–Samelson varieties, Comment. Math. Helv. 73 (1998) 603–636.
[18] S. Okawa, On global Okounkov bodies of Mori dream spaces, preprint, 2010.
[19] S. Okawa, On images of Mori dream spaces, preprint, arXiv:1104.1326 [.org/abs], 2011.
[20] A. Okounkov, Brunn–Minkowski inequality for multiplicities, Invent. Math. 125 (1996) 405–411.
[21] E. Postinghel, S. Urbinati, Newton–Okounkov bodies and toric generations of Mori dream spaces by tropical compactifications, preprint, arXiv:1612.03861, 2016.
[22] D. Schmitz, H. Seppänen, On the polyhedrality of global Okounkov bodies, Adv. Geom. 16 (1) (2016) 83–91.