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Coherent functors and asymptotic stability

Adson Banda [a,1], Leif Melkersson [b,*]

[a] Department of Mathematics and Statistics, University of Zambia, Lusaka, Zambia
[b] Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

A R T I C L E I N F O

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A B S T R A C T

Asymptotic properties of high powers of an ideal related to a coherent functor F are investigated. It is shown that when N is an artinian module the sets of attached prime ideals $\mathrm{Att}_A F(0 :_N \mathfrak{a}^n)$ are the same for n large enough. Also it is shown that for an artinian module N if the modules $F(0 :_N \mathfrak{a}^n)$ have finite length and for a finitely generated module M if the modules $F(M/\mathfrak{a}^n M)$ have finite length, their lengths are given by polynomials in n, for large n. When A is local it is shown that, the Betti numbers $\beta_i(F(M/\mathfrak{a}^n M))$ and the Bass numbers $\mu^i(F(M/\mathfrak{a}^n M))$ are given by polynomials in n for large n.

1. Introduction

In this paper, A will always be assumed to be a noetherian commutative ring. We are concerned with a subcategory of the abelian category of A-linear functors on the category of A-modules, namely the category of coherent functors. An A-linear functor F is called coherent (or finitely presented) if there is an exact sequence

* Corresponding author.
E-mail addresses: adson.banda@unza.zm (A. Banda), leif.melkersson@liu.se (L. Melkersson).
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$$\mathrm{h}_Y \to \mathrm{h}_X \to F \to 0$$

Here X and Y are finite A-modules. When W is an A-module we denote by h_W the covariant functor $\mathrm{Hom}_A(W, -)$.

Coherent functors in the general setting of abelian categories were introduced by Auslander [1]. In our setting, coherent functors on the category of modules over a noetherian commutative ring, an extensive treatment was made by Hartshorne [4].

By Yoneda's theorem [12, Theorem 1.17] every morphism $\mathrm{h}_Y \to \mathrm{h}_X$ is induced by a unique A-linear map $f : X \to Y$. If we let $Z = \mathrm{Coker}\, f$, then

$$0 \to \mathrm{h}_Z \to \mathrm{h}_Y \to \mathrm{h}_X \to F \to 0$$

is exact.

Any coherent functor F has the following useful properties:

(1) F preserves direct limits of directed systems of A-modules.
(2) F maps finite A-modules to finite A-modules, artinian A-modules to artinian A-modules, and modules of finite length to modules of finite length.
(3) More generally for any thick subcategory $\mathcal{S}$ of the category of A-modules $F(X)$ is in $\mathcal{S}$ when X is in $\mathcal{S}$.

A thick subcategory of an abelian category is a full subcategory closed under kernels, cokernels and extensions. A Serre subcategory of an abelian category is a full subcategory closed under subobjects, quotient objects and extensions. Each Serre subcategory is also a thick subcategory. In the category of A-modules the subcategory consisting of the finite A-modules and the subcategory consisting of the artinian A-modules are both Serre subcategories. In the abelian category of A-linear endofunctors on the category of A-modules the subcategory consisting of the coherent functors is thick. (See [4] for proofs.)

2. Associated and attached prime ideals

In this section, we study the asymptotic stability of sequences of prime ideals related to coherent functors. The topic of asymptotic stability is a classic one in commutative algebra. The theory of asymptotic prime divisors was initiated by Ratliff [11]. Brodmann [2] showed that when M is a finite module over a noetherian commutative ring A and $\mathfrak{a}$ is an ideal of A, then the sequences of sets of associated prime ideals $\mathrm{Ass}_A(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M)$ and $\mathrm{Ass}_A(M/\mathfrak{a}^n M)$, $n = 1, 2, \ldots$ are ultimately constant.

Sharp [14] proved a similar result in the dual case of an artinian A-module N. He showed that, the sequences of sets of attached prime ideals

$$\mathrm{Att}_A((0 :_N \mathfrak{a}^n)/(0 :_N \mathfrak{a}^{n-1})) \text{ and } \mathrm{Att}_A(0 :_N \mathfrak{a}^n),\ n = 1, 2 \ldots$$

are both ultimately constant. These results were generalized by Schenzel and the present second author [10]. There was shown the asymptotic stability for every i, of the sequences

$$\{\mathrm{Ass}_A(\mathrm{Tor}_i^A(\mathfrak{a}^n/\mathfrak{a}^{n+1},M))\}_{n=1}^{\infty}$$

and

$$\{\mathrm{Ass}_A(\mathrm{Tor}_i^A(A/\mathfrak{a}^n,M))\}_{n=1}^{\infty})$$

for M a finite A-module,

$$\{\mathrm{Att}_A(\mathrm{Ext}_A^i(A/\mathfrak{a}^n,N))\}_{n=1}^{\infty}$$

and

$$\{\mathrm{Att}_A(\mathrm{Ext}_A^i(\mathfrak{a}^n/\mathfrak{a}^{n-1},N))\}_{n=1}^{\infty}$$

for N an artinian A-module.

In [13], Se proved that if $\mathfrak{a}$ is an ideal of A and F is a coherent functor then for any finite A-module M the sets $\mathrm{Ass}_A(F(M/\mathfrak{a}^nM))$ are independent of n when n is large enough. We will give a simple proof and also prove the dual result involving an artinian module and attached prime ideals.

Let R be a standard graded A-algebra, that is, R is generated over A by finitely many elements of degree one. In the category of graded modules over R, the finite graded modules form a Serre subcategory, hence a thick subcategory.

Proposition 2.1. *Let R be a standard graded A-algebra and let $\mathcal{S}$ be a thick subcategory of the category of graded R-modules. Then for each coherent functor F on A, $F(U)$ is in $\mathcal{S}$ for every U in $\mathcal{S}$.*

Proof. Let W be a finite A-module. Then there are $m,n \in \mathbb{N}$ and an exact sequence $A^m \to A^n \to W \to 0$. Applying $\mathrm{Hom}_A(-,U)$ to this exact sequence we get the exact sequence

$$0 \to \mathrm{Hom}_A(W,U) \to \mathrm{Hom}_A(A^n,U) \to \mathrm{Hom}_A(A^m,U).$$

Since $\mathrm{Hom}_A(A^n,U) \cong U^n$ and $\mathrm{Hom}_A(A^m,U) \cong U^m$ are both in $\mathcal{S}$ and $\mathcal{S}$ is closed under kernels, we have that $\mathrm{h}_W(U) = \mathrm{Hom}_A(W,U)$ is in $\mathcal{S}$. Let $\mathrm{h}_Y \to \mathrm{h}_X \to F \to 0$ be a presentation of F and consider the exact sequence $\mathrm{h}_Y(U) \to \mathrm{h}_X(U) \to F(U) \to 0$. Since as just has been shown $\mathrm{h}_Y(U)$ and $\mathrm{h}_X(U)$ are in $\mathcal{S}$, also $F(U)$ is in $\mathcal{S}$. $\square$

Theorem 2.2. *Let R be a standard graded A-algebra and let $\mathcal{S}$ be a Serre subcategory of the category of graded R-modules. Let U and V be graded R-modules and $f\colon U \to V$ a* *graded homomorphism. If* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *are in* $\mathcal{S}$*, then for every coherent functor* F *on the category of* A*-modules,* $\operatorname{Ker} F(f)$ *and* $\operatorname{Coker} F(f)$ *are also in* $\mathcal{S}$*.*

Proof. Let W be a finite A-module. We will first show that $\operatorname{Ker} \mathrm{h}_W(f)$ and $\operatorname{Coker} \mathrm{h}_W(f)$ are in $\mathcal{S}$. The first assertion follows from 2.1 and the isomorphism $\operatorname{Ker} \mathrm{h}_W(f) \cong \mathrm{h}_W(\operatorname{Ker} f)$. The second assertion follows from 2.1 and [9, Lemma 3.1], applied to the functors $\operatorname{Hom}_A(W, -)$ and $\operatorname{Ext}^1_A(W, -)$.

Next consider an exact sequence $0 \to F_2 \to F_1 \to F_0 \to 0$ of coherent functors. We claim that if the kernels and cokernels of $F_2(f)$, $F_1(f)$ belong to $\mathcal{S}$, then so do $\operatorname{Ker} F_0(f)$ and $\operatorname{Coker} F_0(f)$.

In order to prove the claim we apply the snake lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_2(U) \arrow[r] \arrow[d, "F_2(f)"'] & F_1(U) \arrow[r] \arrow[d, "F_1(f)"'] & F_0(U) \arrow[r] \arrow[d, "F_0(f)"'] & 0 \\
0 \arrow[r] & F_2(V) \arrow[r] & F_1(V) \arrow[r] & F_0(V) \arrow[r] & 0
\end{tikzcd}$$

We get an exact sequence

$$\begin{aligned} 0 \to \operatorname{Ker} F_2(f) \to \operatorname{Ker} F_1(f) \to \operatorname{Ker} F_0(f) \to \\ \to \operatorname{Coker} F_2(f) \to \operatorname{Coker} F_1(f) \to \operatorname{Coker} F_0(f) \to 0 \end{aligned}$$

Using this exact sequence we immediately get the proof of our claim.

Now let F be any coherent functor, so there is an exact sequence

$$0 \to \mathrm{h}_Z \to \mathrm{h}_Y \to \mathrm{h}_X \to F \to 0$$

where X, Y and Z are finite A-modules.

We split this exact sequence into two exact sequences

$$0 \to \mathrm{h}_Z \to \mathrm{h}_Y \to G \to 0$$

and

$$0 \to G \to \mathrm{h}_X \to F \to 0$$

Applying the result in the claim, it follows first that $\operatorname{Ker} G(f)$ and $\operatorname{Coker} G(f)$ are in $\mathcal{S}$ and then that $\operatorname{Ker} F(f)$ and $\operatorname{Coker} F(f)$ are in $\mathcal{S}$. □

Corollary 2.3. *Let* U *and* V *be graded* R*-modules and let* $f \colon U \to V$ *be a graded* R*-homomorphism. If both* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *are finite (or artinian), then so are* $\operatorname{Ker} F(f)$ *and* $\operatorname{Coker} F(f)$*.*

Proof. The subcategories of the category of graded R-modules consisting of the finite graded R-modules and that of the artinian graded R-modules are Serre subcategories of the category of graded R-modules. □

Before we prove the main results of this section, we first prove a technical result.

Lemma 2.4. *Let $\{A(n)\}_{n=1}^{\infty}$ and $\{B(n)\}_{n=1}^{\infty}$ be sequences of finite sets that satisfy*

$$B(n) \subset A(n+1) \subset A(n) \cup B(n)$$

for all n. Assume that the sequence $\{B(n)\}_{n=1}^{\infty}$ is ultimately constant. Then the sequence $\{A(n)\}_{n=1}^{\infty}$ is ultimately constant.

Proof. Since $\{B(n)\}_{n=1}^{\infty}$ stabilizes, there exists m such that $B(n) = B(m)$ for all $n \geq m$. Let X be in $A(n+1)$ where $n \geq m+1$. If $X \notin A(n)$, then $X \in B(n)$. Since by assumption $n-1 \geq m$, we have $X \in B(n-1) \subset A(n)$ and that is a contradiction. Hence, $X \in A(n)$ for $n \geq m+1$. Since $A(n+1) \subset A(n)$ for all $n \geq m+1$ and these are finite sets, there is an l such that $A(l) = A(l+1) = \cdots$. □

The following theorem was proved by Se [13]. We will give a different proof using the machinery we have built up.

Theorem 2.5. *Let M be a finite A-module, $\mathfrak{a}$ an ideal in A and F a coherent functor. Then the sequences $\{\operatorname{Ass}_A F(\mathfrak{a}^n M/\mathfrak{a}^{n+1} M)\}_{n=1}^{\infty}$ and $\{\operatorname{Ass}_A F(M/\mathfrak{a}^n M)\}_{n=1}^{\infty}$ are ultimately constant.*

Proof. Let $R = A[\mathfrak{a}t]$ be the Rees algebra of $\mathfrak{a}$ and consider the R-modules U, V and W which in degree n, $n = 0, 1, 2, \ldots$ are defined by $U_n = \mathfrak{a}^n M/\mathfrak{a}^{n+1} M$, $V_n = M/\mathfrak{a}^{n+1} M$ and $W_n = M/\mathfrak{a}^n M$ respectively. Since U is a finite graded module, so is $F(U)$ by 2.1 and the first assertion follows.

We have an exact sequence

$$0 \to U \to V \xrightarrow{f} W \to 0$$

In degree n for $n = 0, 1, 2, \ldots$, we get exact sequences of A-modules

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^n M \to 0.$$

Since U is a finite R-module, we can apply Corollary 2.3 and conclude that $K = \operatorname{Ker} F(f)$ is a finite R-module.

Now it follows from [8, Theorem 3.1] that the sequence $\{\operatorname{Ass}_A K_n\}_{n=1}^{\infty}$ is ultimately constant.

Consider for each $n \geq 0$ the exact sequence

$$0 \to K_n \to F(M/\mathfrak{a}^{n+1}M) \to F(M/\mathfrak{a}^n M).$$

We get the inclusions

$$\operatorname{Ass}_A K_n \subset \operatorname{Ass}_A F(M/\mathfrak{a}^{n+1}M) \subset \operatorname{Ass}_A F(M/\mathfrak{a}^n M) \cup \operatorname{Ass}_A K_n$$

Lemma 2.4 implies that the sequence $\{\operatorname{Ass}_A F(M/\mathfrak{a}^n M)\}_{n=1}^{\infty}$ is ultimately constant. □

Corollary 2.6.

(1) *The closed sets* $\operatorname{Supp}_A F(M/a^n M)$ *are independent of* n *for large* n.
(2) *The ideals* $\sqrt{\operatorname{Ann}_A F(M/\mathfrak{a}^n M)}$ *are independent of* n *for large* n.
(3) *There is an* m *such that either* $F(M/\mathfrak{a}^n M) = 0$ *for all* $n \geq m$ *or* $F(M/\mathfrak{a}^n M) \neq 0$ *for all* $n \geq m$.
(4) *There is an* m *such that either* $F(M/\mathfrak{a}^n M)$ *have finite length for all* $n \geq m$ *or none of the modules* $F(M/\mathfrak{a}^n M)$ *for* $n \geq m$ *has finite length.*

Proof. (1) The support of a module is determined by its (minimal) associated primes.
(2) Finite modules with the same support have the same annihilator up to radical.
(3) A module vanishes if and only if its set of associated prime ideals is empty.
(4) A finite module has finite length if and only if all of its associated prime ideals are maximal ideals in the ring. □

Theorem 2.7. *Let* A *be a noetherian ring,* $\mathfrak{a}$ *an ideal in* A *and* N *an artinian* A*-module. When* F *is a coherent functor, the sequences* $\{\operatorname{Att}_A(F(0 \underset{N}{:} \mathfrak{a}^n)/(0 \underset{N}{:} \mathfrak{a}^{n-1}))\}_{n=1}^{\infty}$ *and* $\{\operatorname{Att}_A(F(0 \underset{N}{:} \mathfrak{a}^n))\}_{n=1}^{\infty}$ *are both ultimately constant.*

Proof. Let $R = A[\mathfrak{a}t]$ be the Rees algebra of $\mathfrak{a}$ and consider the graded R-modules U, V and W, which are 0 in non-negative degrees and in degree $-n$, where $n > 0$ are defined by $U_{-n} = 0 \underset{N}{:} \mathfrak{a}^{n-1}$, $V_{-n} = 0 \underset{N}{:} \mathfrak{a}^n$ and $W_{-n} = 0 \underset{N}{:} \mathfrak{a}^n/0 \underset{N}{:} \mathfrak{a}^{n-1}$. The graded R-module W is artinian, by the proof of [8, Theorem 1.2], and the first assertion follows. There is an exact sequence $0 \to U \xrightarrow{f} V \to W \to 0$, which in degree $-n$, where $n > 0$ is

$$0 \to 0 :_N \mathfrak{a}^{n-1} \to 0 :_N \mathfrak{a}^n \to (0 :_N \mathfrak{a}^n)/(0 :_N \mathfrak{a}^{n-1}) \to 0$$

Since W is an artinian R-module, we get from Lemma 2.3 that $C = \operatorname{Coker} F(f)$ is an artinian R-module. From the exact sequence

$$F(0 \underset{N}{:} \mathfrak{a}^{n-1}) \to F(0 \underset{N}{:} \mathfrak{a}^n) \to C_{-n} \to 0$$

we obtain the inclusions of sets of attached prime ideals

$$\operatorname{Att}_A C_{-n} \subset \operatorname{Att}_A F(0 \underset{N}{:} \mathfrak{a}^n) \subset \operatorname{Att}_A F(0 \underset{N}{:} \mathfrak{a}^{n-1}) \cup \operatorname{Att}_A C_{-n}$$

Since $\operatorname{Att}_A C_{-n}$ are independent of n for large n, it follows from 2.4 that the sets $\operatorname{Att}_A F(0 \underset{N}{:} \mathfrak{a}^n)$ are independent of n for n sufficiently large. □

3. Coherent functors and Hilbert polynomials

Let A be a noetherian ring, N an artinian A-module and $\mathfrak{a}$ an ideal of A. If $0 \underset{N}{:} \mathfrak{a}$ has finite length, then $0 \underset{N}{:} \mathfrak{a}^n$ also has finite length for all n. The Hilbert function of N (with respect to $\mathfrak{a}$) is the function $H(n) = l_A(0 \underset{N}{:} \mathfrak{a}^n)$. It is a polynomial function in the sense that there is a polynomial P with rational coefficients such that $H(n) = P(n)$ for large n [5, Proposition 2].

Theorem 3.1. *Let A be a commutative noetherian ring, $\mathfrak{a}$ an ideal in A and let F be a coherent functor. If N is an artinian A-module such that $0 \underset{N}{:} \mathfrak{a}$ has finite length, then there is a polynomial P, such that $l_A(F(0 \underset{N}{:} \mathfrak{a}^n)) = P(n)$ for large n.*

Proof. Consider first the case $F = \mathrm{h}_W = \operatorname{Hom}_A(W, -)$, where W is a finite A-module Observe that we have the following congruences

$$\begin{aligned} \operatorname{Hom}_A(W, 0 \underset{N}{:} \mathfrak{a}^n) &\cong \operatorname{Hom}_A(W, \operatorname{Hom}_A(A/\mathfrak{a}^n, N)) \\ &\cong \operatorname{Hom}_A(W \otimes_A A/\mathfrak{a}^n, N) \\ &\cong \operatorname{Hom}_A(A/\mathfrak{a}^n, \operatorname{Hom}_A(W, N)) \\ &\cong 0 \underset{\mathrm{h}_W(N)}{:} \mathfrak{a}^n. \end{aligned}$$

Since $\mathrm{h}_W(N) = \operatorname{Hom}_A(W, N)$ is artinian, there is by [5, Proposition 2] a polynomial P such that $l_A(0 \underset{\mathrm{h}_W(N)}{:} \mathfrak{a}^n) = P(n)$ for all large n. Next turn to the case of an arbitrary coherent functor F. There are finite A-modules X, Y and Z such that there is an exact sequence $0 \to \mathrm{h}_Z \to \mathrm{h}_Y \to \mathrm{h}_X \to F \to 0$. Then for each n, we have an exact sequence of A-modules

$$0 \to \mathrm{h}_Z(0 \underset{N}{:} \mathfrak{a}^n) \to \mathrm{h}_Y(0 \underset{N}{:} \mathfrak{a}^n) \to \mathrm{h}_X(0 \underset{N}{:} \mathfrak{a}^n) \to F(0 \underset{N}{:} \mathfrak{a}^n) \to 0.$$

It follows that

$$l_A(F(0 \underset{N}{:} \mathfrak{a}^n)) = l_A(\mathrm{h}_X(0 \underset{N}{:} \mathfrak{a}^n)) - l_A(\mathrm{h}_Y(0 \underset{N}{:} \mathfrak{a}^n)) + l_A(\mathrm{h}_Z(0 \underset{N}{:} \mathfrak{a}^n))$$

for all n. Hence by the first case proved there is a polynomial P such that $l_A(F(0 \underset{N}{:} \mathfrak{a}^n)) = P(n)$ for large n. $\square$

We next prove a far reaching generalisation of 3.1, and a quite different technique must be used for its proof. It will be similar to the proof of 2.7.

Theorem 3.2. *Let A be a noetherian ring, N an artinian A-module and F a coherent functor. If $\mathfrak{a}$ is an ideal in A, such that $F(0 \underset{N}{:} \mathfrak{a}^n)$ have finite length for all n, then there is a polynomial P, such that $l_A(F(0 \underset{N}{:} \mathfrak{a}^n)) = P(n)$ for all large n.*

Proof. We use the same notation as in the proof of 2.7.

With $K = \operatorname{Ker} f$ and $C = \operatorname{Coker} f$ we get for each $n > 0$ the exact sequence

$$0 \to K_{-n} \to F(0 \underset{N}{:} \mathfrak{a}^{n-1}) \to F(0 \underset{N}{:} \mathfrak{a}^n) \to C_{-n} \to 0$$

Hence the A-modules K_{-n} and C_{-n} have finite length. Since the graded R-modules K and L are by 2.3 artinian, there are polynomials P_1 and P_2 such that $l_A(K_{-n}) = P_1(n)$ and $l_A(C_{-n}) = P_2(n)$ for large n. Put $H(n) = l_A(F(0 \underset{N}{:} \mathfrak{a}^n))$ and $\Delta H = H(n) - H(n-1)$. We get from the exact sequences above that $\Delta H(n) = P_2(n) - P_1(n)$ for large n. By [7, Chap. X, Lemma 6.4], there is a polynomial P, such that $H(n) = P(n)$ for large n. $\square$

In a similar way we prove the dual result.

Theorem 3.3. *Let M be a finite A-module and F a coherent functor, such that $F(M/\mathfrak{a}^n M)$ have finite length for all n. Then there is a polynomial P, such that $l_A(F(M/\mathfrak{a}^n M)) = P(n)$ for all large n.*

Proof. We use the same notation as in the proof of Theorem 2.5. With $K = \operatorname{Ker} F(f)$ and $C = \operatorname{Coker} F(f)$, we have in each degree n an exact sequence of A-modules

$$0 \to K_n \to F(M/\mathfrak{a}^{n+1} M) \to F(M/\mathfrak{a}^n M) \to C_n \to 0$$

It follows that for all n the A-modules K_n and C_n have finite length. By 2.3 the graded R-modules K and C are finitely generated, and therefore there are polynomials P_1 and P_2, such that $l_A(K_n) = P_1(n)$ and $l_A(C_n) = P_2(n)$ for all large n. With $H(n) = l(F(M/\mathfrak{a}^n M))$ and $\Delta H(n) = H(n+1) - H(n)$ we get by taking lengths in the above sequences, that $\Delta H(n) = P_1(n) - P_2(n)$ for all large n. It follows that there is a polynomial P, such that $H(n) = P(n)$ for large n. (See e.g. [7, Chap. X, Lemma 6.4].) $\square$

Theorem 3.4. *Let F and G be coherent functors. Then $G \circ F$ is coherent.*

Proof. Let X and Y be finite A-modules, such that there is an exact sequence

$$\mathrm{h}_Y \to \mathrm{h}_X \to G \to 0.$$

Composing with F on the right, we get the exact sequence

$$\mathrm{h}_Y \circ F \to \mathrm{h}_X \circ F \to G \circ F \to 0.$$

Since the cokernel of a morphism between coherent functors is again coherent, we just have to show that $\mathrm{h}_Z \circ F$ is coherent when Z is finite and F is coherent.

To this end let $A^m \to A^n \to Z \to 0$ be exact. Then we get the exact sequence

$$0 \to \mathrm{h}_Z \circ F \to \mathrm{h}_{A^n} \circ F \to \mathrm{h}_{A^m} \circ F$$

Here the two last terms are isomorphic to the sum of n resp. m copies of F and thus coherent. Since the kernel of a morphism between two coherent functors is again coherent, $\mathrm{h}_Z \circ F$ is coherent for any finite A-module Z. $\square$

Our first application of Theorem 3.4 is a generalisation of Brodmann's theorem [3] about the asymptotic stability of depth. This was also done by Se [13], but our method is different and we think simpler.

Theorem 3.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of the noetherian ring A. Let M be a finite A-module and F a coherent functor. Suppose that for some l, the modules $A/\mathfrak{b} \otimes_A F(M/\mathfrak{a}^n M) \neq 0$ for all $n \geq l$.*

Then there is an $m \geq l$, such that $\operatorname{depth}_{\mathfrak{b}} F(M/\mathfrak{a}^n M)$ *are the same for all $n \geq m$.*

Proof. The functors $\operatorname{Ext}^i_A(A/\mathfrak{b}, F(-))$ are by 3.4 coherent. By 2.6 there are integers m_i such that $\operatorname{Supp}_A \operatorname{Ext}^i_A(A/\mathfrak{b}, F(M/\mathfrak{a}^n M)) = Y_i$ for all $n \geq m_i$. Let s be the minimal number of generators of $\mathfrak{b}$. If $Y_i = \emptyset$ for $i = 0, \dots, s$, then $\operatorname{depth}_{\mathfrak{b}} F(M/\mathfrak{a}^n M) > s$ for large n. This is impossible since the depth with respect to an ideal is always bounded above by the minimal number of generators of the ideal. Hence there is $r \leq s$, such that $Y_i = \emptyset$ for $i < r$, but $Y_r \neq \emptyset$. Therefore $\operatorname{depth}_{\mathfrak{b}} F(M/\mathfrak{a}^n M) = r$ for all $n \geq m$, where $m = \max(l, m_0, \dots, m_r)$. $\square$

Corollary 3.6. $\operatorname{grade} F(M/\mathfrak{a}^n M)$ *is independent of n for large n.*

Proof. The grade of a nonzero finite module is by definition the depth of A with respect to the annihilator of the module. Moreover the depth with respect to an ideal is the same as the depth with respect to any other ideal with the same radical.

Now by 2.6 the radicals $\sqrt{\operatorname{Ann}_A F(M/\mathfrak{a}^n M)}$ are independent of n for all large n. $\square$

Next we study the asymptotic properties of Betti numbers and Bass numbers in our context of coherent functors. The next theorem generalises [6, Corollary 7].

Theorem 3.7. *Let A be a local noetherian ring with residue field k. Let $\mathfrak{a}$ be an ideal of A, M a finite A-module and F a coherent functor.*

Then for $i \geq 0$ there are polynomials P_i and Q_i, such that for all large n we have

$$\beta_i(F(M/\mathfrak{a}^n M)) = P_i(n)$$

and

$$\mu^i(F(M/\mathfrak{a}^n M)) = Q_i(n)$$

Proof. The functors $\mathrm{Tor}_i^A(k,-)$ and $\mathrm{Ext}_A^i(k,-)$ are coherent for all i. Thus when we compose with F on the right we get coherent functors by 3.4.

Applying Theorem 3.3 to these functors will finish the proof. $\square$

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