Journal of Algebra 568 (2021) 181–192

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Kazhdan–Lusztig polynomials and subexpressions

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A R T I C L E I N F O

*Article history:*
Received 22 May 2020
Available online 20 October 2020
Communicated by Nicolás Andruskiewitsch

*Keywords:*
Kazhdan-Lusztig polynomials
Hecke algebras
Soergel bimodules
Light leaves

A B S T R A C T

We refine an idea of Deodhar, whose goal is a counting formula for Kazhdan–Lusztig polynomials. This is a consequence of a simple observation that one can use the solution of Soergel's conjecture to make ambiguities involved in defining certain morphisms between Soergel bimodules in characteristic zero (double leaves) disappear.


## 1. Introduction

Let $(W, S)$ be a Coxeter system. To any pair of elements $(x, y)$ of $W$, Kazhdan and Lusztig [5] associated a polynomial

$$h_{x,y} \in \mathbb{Z}[v].$$

These polynomials are ubiquitous in representation theory; they appear in character formulas for simple representations of complex semi-simple Lie algebras, real Lie groups, quantum groups, finite reductive groups . . . On the other hand, they are still far from being well understood. For example, in several applications the coefficient of $v$ (the so-

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https://doi.org/10.1016/j.jalgebra.2020.10.009
0021-8693/

called $\mu$-coefficient) plays a crucial role, however even describing when it is non-zero appears extremely subtle.

In their original paper Kazhdan and Lusztig conjectured that the polynomials $h_{x,y}$ have non-negative coefficients. This conjecture was proved in [6] if the underlying Coxeter group is a Weyl or affine Weyl group. The proof proceeds by interpreting $h_{x,y}$ as the Poincaré polynomial of the local intersection cohomology of a Schubert variety.

Kazhdan and Lusztig's positivity conjecture was proved in general in [3]. The proof is via a study of Soergel bimodules associated to the underlying Coxeter system. Using Soergel bimodules one can produce a space $D_{x,y}$ which behaves as though it were the local intersection cohomology of a Schubert variety. The Kazhdan–Lusztig polynomial $h_{x,y}$ gives the graded dimension of $D_{x,y}$. This implies immediately that $h_{x,y}$ has non-negative coefficients. The theory also goes quite some way towards explaining what Kazhdan–Lusztig polynomials "are" for arbitrary Coxeter groups.

The aim of this paper is to explain a strategy to use Soergel bimodues to further our combinatorial understanding of Kazhdan–Lusztig polynomials. Our goal (not achieved in this paper) is a "counting formula" for Kazhdan–Lusztig polynomials. Ideally we would like to produce a canonical basis for the space $D_{x,y}$. That is, we would like to find a set $X_{x,y}$ and a degree statistic $d : X_{x,y} \to \mathbb{Z}_{\geqslant 0}$ such that if we use $X_{x,y}$ and $d$ to build a positively graded vector space, we have a canonical isomorphism:

$$\bigoplus_{e \in X_{x,y}} \mathbb{R}e \xrightarrow{\sim} D_{x,y}.$$

Taking graded dimensions we would deduce a counting formula:

$$h_{x,y} = \sum_{e \in X_{x,y}} v^{d(e)}.$$

We expect the sets $X_{x,y}$ to reflect in a subtle way the combinatorics of Kazhdan–Lusztig polynomials. If shown to exist, they would open the door to a deeper combinatorial study of Kazhan-Lusztig polynomials.

A proposal for such a counting formula was made by Deodhar in [1]. He considers the set $\widetilde{X}_{x,\underline{y}}$ of all subexpressions for $x$ of a fixed reduced expression $\underline{y}$ of $y$ (see Section 2.1 for more details on our notation). On this set he defines a statistic ("Deodhar's defect")

$$\mathrm{df} : \widetilde{X}_{x,\underline{y}} \to \mathbb{Z}.$$

Assuming that Kazhdan–Lusztig polynomials have non-negative coefficients (now known unconditionally), Deodhar proves the existence of a subset $X^D_{x,\underline{y}} \subset \widetilde{X}_{x,\underline{y}}$ such that

$$h_{x,y} = \sum_{e \in X^D_{x,\underline{y}}} v^{d(e)}. \tag{1.1}$$

Although initially appealing, Deodhar's proposal suffers from serious drawbacks, the principal one being that the set $X^D_{x,\underline{y}}$ is not canonical.

There are two sources of non-canonicity. The first is that $\widetilde{X}_{x,\underline{y}}$ depends on a reduced expression of $\underline{y}$. We do not regard this dependence as particularly worrisome. Indeed, there are many objects in Lie theory which depend on a choice of reduced expression, and (if canonical up to this point) relating them for different reduced expressions is potentially a fascinating question. The second source of non-canonicity is more concerning: even for a fixed reduced expression $\underline{y}$ there are in general many possible choices of subsets $X^D_{x,\underline{y}} \subset \widetilde{X}_{x,\underline{y}}$ satisfying (1.1). In Deodhar's framework there is no way to make a distinguished choice.

Let $x, \underline{y}$ be as above. Using Soergel bimodules one can produce a space $D_{x,\underline{y}}$ containing $D_{x,y}$ as a canonical direct summand. In other words, we have a canonical map $\pi : D_{x,\underline{y}} \twoheadrightarrow D_{x,y}$. The following is the main result of this paper.

**Theorem 1.1.** *There is a* canonical *isomorphism of graded vector spaces*

$$\mathrm{CLL} : \bigoplus_{e \in \widetilde{X}_{x,\underline{y}}} \mathbb{R}e \xrightarrow{\sim} D_{x,\underline{y}},$$

*where the left hand side is graded by Deodhar's defect, i.e. the generator $e \in \widetilde{X}_{x,\underline{y}}$ has degree* $\mathrm{df}(e)$*. (CLL stands for "Canonical light leaves".)*

This theorem leads to a natural refinement of Deodhar's proposal:

**Problem 1.2.** *Find a subset $X^L_{x,\underline{y}} \subset \widetilde{X}_{x,\underline{y}}$ such that the composition of the inclusion, canonical light leaves and the canonical surjection*

$$\bigoplus_{e \in X^L_{x,\underline{y}}} \mathbb{R}e \hookrightarrow \bigoplus_{e \in \widetilde{X}_{x,\underline{y}}} \mathbb{R}e \xrightarrow{\mathrm{CLL}} D_{x,\underline{y}} \twoheadrightarrow D_{x,y}$$

*is an isomorphism of graded vector spaces.*

If the choice of the subset $X^L_{x,\underline{y}}$ could be made canonically we would regard it as a solution to the counting problem above. Moreover, the map CLL has the potential to explain why a canonical choice is difficult in general, by recasting the problem as one of linear algebra.

The easiest situation is when the subset of non-zero elements in

$$\{\pi \circ \mathrm{CLL}(e) \,|\, e \in \widetilde{X}_{x,\underline{y}}\},$$

already constitutes a basis of $D_{x,y}$. Here we have no choice: we must define $X^L_{x,\underline{y}}$ to be those $e$ in $\widetilde{X}_{x,\underline{y}}$ whose image is non-zero under $\pi \circ \mathrm{CLL}$. This situation does occur "in nature". Namely it is the case for dihedral groups, universal Coxeter groups, and whenever $h_{x,y} = v^{\ell(y)-\ell(x)}$ ("rationally smooth case"). It is interesting to note that in these cases there already exist closed and combinatorial formulas for Kazhdan–Lusztig polynomials. We feel our result gives a satisfying explanation as to "why" there exist relatively straightforward formulas in these cases.

**Remark 1.3.** The basic observation in this paper is that certain morphisms ("light leaves") may be made canonical in the presence of Soergel's conjecture. The non-canonicity of light leaves has been a basic difficulty in the theory since their introduction in 2007. The basic canonicity observation was made during a visit of GW to NL at the Universidad de Chile in 2015, and has been shared with the community since. Subsequently, this idea has been pushed much further: in [11] Patimo studies the case of Grassmannians in detail; and in [10] the first author and Patimo study the case of affine type $A_2$. In both settings the authors find that the "canonical light leaves"[1] associated to different reduced expressions yield many different bases for intersection cohomology, and the question of relating them in interesting ways remains open. In particular, the easy case considered in the previous paragraph is certainly not indicative of the general setting, and the "potentially fascinating question" raised a few paragraphs ago is very much alive. We wrote this paper in order to record the basic observation in the hope that we and others may take it up in the future.

*Acknowledgements.* The first author was supported by FONDECYT No 1160152.

## 2. Background

In the following, we recall some standard background in Kazhdan–Lusztig theory and Soergel bimodules. References include [5,13,12,14,4,8]. There is also a book [2] on the way.

### 2.1. Coxeter group combinatorics

Let $(W, S)$ be a Coxeter group with length function $\ell$ and Bruhat order $\leqslant$. An *expression* $\underline{x} = (s_1, s_2, \dots, s_m)$ is a word in the alphabet $S$ (i.e. $s_i \in S$ for all $i$). Its *length* is $\ell(\underline{x}) = m$.

If $\underline{x} = (s_1, s_2, \dots, s_m)$ is an expression, we let $x := s_1 s_2 \dots s_m$ denote the product in $W$. Given an expression $\underline{x} = (s_1, s_2, \dots, s_m)$, a *subexpression* of $\underline{x}$ is a word $\underline{e} = e_1 e_2 \dots e_m$ of length $m$ in the alphabet $\{0, 1\}$. We will write $\underline{e} \subset \underline{x}$ to indicate that $\underline{e}$ is a subexpression of $\underline{x}$. We set

$$\underline{x}^{\underline{e}} := s_1^{e_1} s_2^{e_2} \dots s_m^{e_m} \in W$$

[1] In the setting of the Grassmannian considered in [11] these are singular variants (in the sense of singular Soergel bimodules) of the maps considered in the present work.

and say that $\underline{e} \subset \underline{x}$ *expresses* $\underline{x}^{\underline{e}}$.

For $1 \leqslant i \leqslant m$, we define $w_i := s_1^{e_1} s_2^{e_2} \dots s_i^{e_i}$. We also define $d_i \in \{U, D\}$ (where $U$ stands for *Up* and $D$ for *Down*) in the following way:

$$d_i := \begin{cases} U & \text{if } w_{i-1}s_i > w_{i-1}, \\ D & \text{if } w_{i-1}s_i < w_{i-1}. \end{cases}$$

We write the decorated sequence $(d_1e_1, \dots, d_me_m)$. *Deodhar's defect* df is defined by

$$\mathrm{df}(e) := |\{\, i \mid d_ie_i = U0\}| - |\{\, i \mid d_ie_i = D0\}|$$

## 2.2. Hecke algebras

For the basic definitions of Hecke algebras and Kazhdan–Lusztig polynomials we follow [13]. Let $(W, S)$ be a Coxeter system. Recall that the *Hecke algebra* $\mathcal{H}$ of $(W, S)$ is the algebra with free $\mathbb{Z}[v, v^{-1}]$-basis given by symbols $\{h_x\}_{x \in W}$ and multiplication given by

$$h_x h_s := \begin{cases} h_{xs} & \text{if } xs > x, \\ (v^{-1} - v)h_x + h_{xs} & \text{if } xs < x. \end{cases}$$

We can define a $\mathbb{Z}$-module morphism $\overline{(-)} : \mathcal{H} \to \mathcal{H}$ by the formula $\overline{v} = v^{-1}$ and $\overline{h_x} = (h_{x^{-1}})^{-1}$. It is a ring morphism, and we call it the *duality* in the Hecke algebra. The *Kazhdan–Lusztig basis* of $\mathcal{H}$ is denoted by $\{b_x\}_{x \in W}$. It is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$ and it is characterised by the two conditions

$$\overline{b_x} = b_x \qquad \text{and} \qquad b_x \in h_x + \sum_{y \in W} v\mathbb{Z}[v]h_y$$

for all $x \in W$. If we write $b_x = h_x + \sum_{y \in W} h_{y,x}h_y$ then the *Kazhdan–Lusztig polynomials* (as defined in [5]) $p_{y,x}$ are defined by the formula $p_{y,x} = v^{l(x)-l(y)}h_{y,x}$, and $C'_x = b_x$ (their $q^{-1/2}$ is our $v$).

Let us define the $\mathbb{Z}[v, v^{-1}]$-bilinear form

$$(-, -) : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v, v^{-1}],$$

given by $(h_x, h_y) := \delta_{x,y}$. A useful property of this pairing is that $(b_x, b_y) \in v\mathbb{Z}[v]$ if $x \neq y$ and $(b_x, b_x) \in 1 + v\mathbb{Z}[v]$.

## 2.3. Soergel bimodules

We fix a realisation $\mathfrak{h}$ of our Coxeter system $(W, S)$ over the real numbers $\mathbb{R}$. That is, $\mathfrak{h}$ is a real vector space and we have fixed roots $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ and coroots $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$

such that the familiar formulas from Lie theory define a representation of $W$ of $\mathfrak{h}$ and $\mathfrak{h}^*$.

Throughout, we assume that this is a realisation for which Soergel's conjecture holds (i.e. Conjecture 1.13 in [14]. This conjecture is also recalled in Section 2.4). For example, as it is proved in [3], we could take $\mathfrak{h}$ to be the realisation from [14,3]. We could also take $\mathfrak{h}$ to be the *geometric representation* [7] so that $\mathfrak{h} = \bigoplus \mathbb{R}\alpha_s^\vee$ and for $t \in S$, the element $\alpha_t \in \mathfrak{h}^*$ is defined by $\langle \alpha_t, \alpha_s^\vee \rangle = -2\cos(\pi/m_{st})$, where $m_{st}$ denotes the order (possibly $\infty$) of $st \in W$.

Having fixed $\mathfrak{h}$ we define $R = S(\mathfrak{h}^*) = \mathcal{O}(\mathfrak{h})$ to be the symmetric algebra on $\mathfrak{h}^*$ (alias the polynomials functions on $\mathfrak{h}$), graded so that $\mathfrak{h}^*$ has degree 2. We denote by $\mathrm{Bim}_R$ the category of $\mathbb{Z}$-graded $R$-bimodules which are finitely generated both as left and right $R$-modules. Given an object $M = \bigoplus M^i \in \mathrm{Bim}_R$ we denote by $M(k)$ the shifted bimodule, with $M(k)^i := M^{k+i}$. Given objects $M, N \in \mathrm{Bim}_R$ we denote their tensor product by juxtaposition: $MN := M \otimes_R N$. This operation gives $\mathrm{Bim}_R$ the structure of a monoidal category. The Krull-Schmidt theorem holds in $\mathrm{Bim}_R$. For any $s \in S$ we denote by $R^s \subset R$ the $s$-invariants in $R$. We consider the bimodule

$$B_s := R \otimes_{R^s} R(1).$$

Given an expression $\underline{w} = (s_1, \ldots, s_m)$ we consider the *Bott-Samelson bimodule*

$$B_{\underline{w}} := B_{s_1} B_{s_2} \ldots B_{s_m}.$$

The category $\mathcal{B}$ of *Soergel bimodules* is defined to be the strictly full (i.e. closed under isomorphism), additive (i.e. $M, N \in \mathcal{B} \Rightarrow M \oplus N \in \mathcal{B}$), monoidal (i.e. $M, N \in \mathcal{B} \Rightarrow MN \in \mathcal{B}$) category of $\mathrm{Bim}_R$ which contains $B_s$ for all $s \in S$ and is closed under shifts $(m)$ and direct summands.

**Notation 2.1.** For Soergel bimodules $M$ and $N$, we denote by $\mathrm{Hom}^i(M, N)$ the degree $i$ morphisms in $\mathrm{Hom}(M, N)$, where the latter is the set of all $R$-bimodule morphisms.

### *2.4. Soergel's theorems and Soergel's conjecture*

Soergel proved the following facts (usually known as *Soergel's categorification theorem*). For all $w \in W$ there exists a unique (up to isomorphism) bimodule $B_w$ which occurs as a direct summand of $B_{\underline{w}}$ for any reduced expression $\underline{w}$ of $w$, and is not a summand of (some shift of) $B_{\underline{y}}$ for any shorter sequence $\underline{y}$. The set $\{B_w \mid w \in W\}$ constitutes a complete set of non-isomorphic indecomposable Soergel bimodules, up to isomorphism and grading shift. There is a unique isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras between the split Grothendieck group of $\mathcal{B}$ and the Hecke algebra

$$\mathrm{ch} : [\mathcal{B}] \to \mathcal{H},$$

satisfying $\mathrm{ch}([B_s]) = b_s$ and $\mathrm{ch}([R(1)]) = v$.

Soergel gave a formula to calculate the graded dimensions of the Hom spaces in $\mathcal{B}$ in the Hecke algebra. We need some notation to explain it. Given a finite dimensional graded $\mathbb{R}$-vector space $V = \oplus V^i$, we define

$$\mathrm{gdim}(V) = \sum \dim(V^i)v^i \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}].$$

Given a finitely-generated and free graded right $R$-module $M$, we define

$$\mathrm{grk}(M) := \mathrm{gdim}(M \otimes_R \mathbb{R}).$$

The following is *Soergel's hom formula.* Let $M, N \in \mathcal{B}$, then $\mathrm{Hom}(M, N)$ is finitely-generated and free as a right $R$-module, and

$$\mathrm{grkHom}(M, N) = (\mathrm{ch}(M), \overline{\mathrm{ch}(N)}).$$

*Soergel's conjecture* is the following statement:

$$\mathrm{ch}([B_x]) = b_x \quad \text{for all } x \in W.$$

We remark that when Soergel's conjecture is satisfied (the case considered in this paper), by Soergel's hom formula and by the useful property at the end of Section 2.2, we obtain a complete description of the degree zero morphisms between indecomposable objects:

$$\mathrm{Hom}^0(B_x, B_y) \cong \delta_{x,y}\mathbb{R}. \tag{2.1}$$

### *2.5. Double leaves*

An important result in the theory of Soergel bimodules is a theorem of the first author giving a "double leaves" basis of morphisms between Soergel bimodules. Let $\underline{w} = (s_1, \ldots, s_m)$ denote an expression. For any subexpression $\underline{e}$ of $\underline{w}$ the first author associates a morphism

$$\mathrm{LL}_{\underline{w},\underline{e}} : B_{\underline{w}} \to B_{\underline{x}}(\mathrm{df}(\underline{e})).$$

Here $\underline{x}$ is a fixed but arbitrary reduced expression of $x = \underline{w}^{\underline{e}}$. The definition of $\mathrm{LL}_{\underline{w},\underline{e}}$ is inductive, and will not be given here, as we will not need it. However it is important to note that the definition of $\mathrm{LL}_{\underline{w},\underline{e}}$ depends on choices (fixed reduced expressions for elements and fixed sequences of braid relations between reduced expressions) which seem difficult to make canonical.

However, once one has fixed such choices one can produce a basis of homomorphisms between any two Bott-Samelson bimodules. Indeed, a theorem of the first author [9, Thm. 3.2] (see also [4, Thm 6.11]) asserts that the set

$$\bigsqcup_{x\in W}\{\mathrm{LL}^*_{\underline{w},\underline{e}}\circ \mathrm{LL}_{\underline{z},\underline{f}} \mid \underline{e}\subset \underline{w}, \underline{f}\subset \underline{z} \text{ such that } \underline{w}^{\underline{e}} = \underline{z}^{\underline{f}} = x\}$$

is a free $R$-basis for $\mathrm{Hom}(B_{\underline{z}}, B_{\underline{w}})$. Here the superscript $^*$ stands for flipping the diagram upside-down, when the map is seen diagrammatically. Let us now give a non-diagrammatic definition. Any light leaf $l$ is constructed using three kind of maps. In the language of [9, §3.1.1] they are $m_s, j_s$ and $f_{sr}$ for $s, r \in S$. These maps admit adjoint maps $m_s^*, j_s^*$ and $f_{sr}^*$. To obtain the map $l^*$, one starts with $l$ and then inverse the arrows by replacing each appearance of $m_s$ (resp. $j_s$ and $f_{sr}$) by $m_s^*$ (resp. $j_s^*$ and $f_{sr}^*$).

*2.6. The sets $D_{x,\underline{y}}$ and $D_{x,y}$*

Let $M, N \in \mathcal{B}$. For $x \in W$ we denote by

$$\mathrm{Hom}_{<x}(M,N) \subset \mathrm{Hom}(M,N)$$

the vector space generated by all morphisms $f : M \to N$ that factor through $B_y(n)$ for some $y < x$ and $n \in \mathbb{Z}$. Let

$$\mathrm{Hom}_{\not< x}(M,N) := \mathrm{Hom}(M,N)/\,\mathrm{Hom}_{<x}(M,N).$$

We denote by $\mathcal{B}_{\not< x}$ the category whose objects coincide with those of $\mathcal{B}$ and for any $M, N \in \mathcal{B}_{\not< x}$ we have $\mathrm{Hom}_{\mathcal{B}_{\not< x}}(M,N) := \mathrm{Hom}_{\not< x}(M,N)$.

Consider the sets

$$\widehat{D}_{x,\underline{y}} := \mathrm{Hom}_{\not< x}(B_{\underline{y}}, B_x),$$
$$D_{x,\underline{y}} := \mathrm{Hom}_{\not< x}(B_{\underline{y}}, B_x) \otimes_R \mathbb{R} \text{ and}$$
$$D_{x,y} := \mathrm{Hom}_{\not< x}(B_y, B_x) \otimes_R \mathbb{R}.$$

The set $D_{x,y}$ is a canonical direct summand of $D_{x,\underline{y}}$. This is because, when Soergel's conjecture is satisfied, there is one element in $\mathrm{End}(B_{\underline{y}})$ projecting to $B_y$ called the *favourite projector* (see [9, §4.1]). Let us give the construction of this projector. Let us assume (by induction) that projection and inclusion maps have been constructed

$$B_{\underline{y}} \overset{p_{\underline{y}}}{\twoheadrightarrow} B_y \overset{i_{\underline{y}}}{\hookrightarrow} B_{\underline{y}}$$

for some reduced expression $\underline{y}$ of $y$. Suppose $y < ys$, then

$$b_y b_s = b_{ys} + \sum_{x<ys} m_x b_x, \quad \text{with } m_x \in \mathbb{Z}_{\geqslant 0}.$$

By Soergel's conjecture this implies

$$B_y B_s = B_{ys} \oplus \bigoplus_{x<ys} B_x^{\oplus m_x}.$$

By (2.1), there is only one projector in this space projecting to $B_{ys}$, which we write as

$$B_y B_s \overset{p_{y,s}}{\twoheadrightarrow} B_{ys} \overset{i_{y,s}}{\hookrightarrow} B_y B_s.$$

We now define the inclusion and projection maps of our favourite projector to be the compositions

$$B_{\underline{y}} B_s \overset{p_{\underline{y}}\,\mathrm{id}_{B_s}}{\twoheadrightarrow} B_y B_s \overset{p_{y,s}}{\twoheadrightarrow} B_{ys} \overset{i_{y,s}}{\hookrightarrow} B_y B_s \overset{i_{\underline{y}}\,\mathrm{id}_{B_s}}{\hookrightarrow} B_{\underline{y}} B_s.$$

## 3. Canonical light leaves

This section contains the new observations of this paper. We explain that certain canonical elements and maps allow one to define canonical light leaves, from which our main theorem (Theorem 1.1) follows easily.

**Remark 3.1.** In this paper we use "canonical" to mean "not depending on any choices". We do not use it in the stronger sense that is typical in Lie theory (i.e. to refer to the Kazhdan–Lusztig basis of the Hecke algebra, or the canonical basis of quantum groups).

### *3.1. Some canonical elements*

What do we really mean when we write $B_x$? In the general setting of Soergel bimodules, we mean a representative of an equivalence class of isomorphic bimodules, where each isomorphism is not canonical. In our setting (where Soergel's conjecture is available), we mean a representative of an equivalence class of isomorphic bimodules, where each isomorphism is canonical up to an invertible scalar (in our case the real numbers without zero $\mathbb{R}_{\neq 0}$). We now explain a somewhat *ad hoc* way to fix this scalar, so that $B_x$ is defined up to unique isomorphism.

Consider an expression $\underline{x}$, and the corresponding Bott-Samelson bimodule $B_{\underline{x}}$. It contains a canonical element

$$c_{\mathrm{bot}}^{\underline{x}} := 1 \otimes 1 \otimes \cdots \otimes 1 \in B_{\underline{x}}.$$

(Note that $B_{\underline{x}}$ is zero below degree $-\ell(\underline{x})$ and is spanned by $c_{\mathrm{bot}}$ in degree $-\ell(\underline{x})$; bot stands for "bottom".) We denote by $c_{\mathrm{bot}}^{x} \in B_x$ the image of $c_{\mathrm{bot}}^{\underline{x}}$ under the favourite projector, where $\underline{x}$ is a reduced expression for $x$.

From now on we will always understand $B_x$ to mean $B_x$ together with the element $c_{\mathrm{bot}} \in B_x$. Given two representatives $(B_x, c_{\mathrm{bot}}^x)$ and $(\tilde{B}_x, \tilde{c}_{\mathrm{bot}}^x)$, there is a unique isomorphism $B_x \to \tilde{B}_x$ which sends $c_{\mathrm{bot}}^x$ to $\tilde{c}_{\mathrm{bot}}^x$.

**Remark 3.2.** Consider the following commutative diagram

$$\begin{tikzcd} B_{\underline{x}} \arrow[r, "\varphi"] \arrow[d, "p_{\underline{x}}"'] & B_{\underline{x}'} \arrow[d, "p_{\underline{x}'}"] \\ B_x \arrow[r, "\sim", "\zeta"'] & B_x \end{tikzcd}$$

where: $\varphi$ is a braid move (see [4, §4.2], where they are called *rex moves*); $p_{\underline{x}}$ (resp. $p_{\underline{x}'}$) are the projections in the favourite projector associated to $\underline{x}$ and $\underline{x}'$; and $\zeta$ is the induced isomorphism. One may check that $\zeta(c^x_{\text{bot}}) = c^x_{\text{bot}}$. (We will not need this fact below.) This gives another sense to which $c_{\text{bot}}$ is canonical.

*3.2. Some canonical maps*

In this section we introduce the canonical maps which will be our building blocks for the definition of canonical light leaves, in the next section.

**Lemma 3.3.** *Let $x \in W$ and $s \in S$ and suppose that $x < xs$. The spaces*

$$\text{Hom}^0(B_x B_s, B_{xs}), \quad \text{Hom}^{-1}(B_{xs}B_s, B_{xs}) \quad \textit{and} \quad \text{Hom}^1(B_{xs}, B_x)$$

*are all one-dimensional.*

**Proof.** We consider the spaces one at a time. As in last section, we have

$$B_x B_s = B_{xs} \oplus \bigoplus_{y<xs} B_y^{\oplus m_y}$$

and (2.1) allows us to conclude that $\text{Hom}^0(B_x B_s, B_{xs})$ is one dimensional.

We now consider the second space. By Soergel's hom formula and Soergel's conjecture, the dimension of

$$\text{Hom}^{-1}(B_{xs}B_s, B_{xs})$$

is the coefficient of $v^{-1}$ in the Laurent polynomial $(b_{xs}b_s, b_{xs})$. But

$$b_{xs}b_s = (v + v^{-1})b_{xs}.$$

As $(b_{xs}, b_{xs}) \in 1 + v\mathbb{Z}[v]$, we conclude that $\text{Hom}^{-1}(B_{xs}B_s, B_{xs}) \cong \mathbb{R}$.

For the last case, we need to calculate the coefficient of $v$ in $(b_{xs}, b_x)$, i.e. in

$$(h_{xs} + vh_x + \sum_{\substack{y<xs \\ y\neq x}} P_y h_y \ , \ h_x + \sum_{z<x} Q_z h_z)$$

where $P_y, Q_z \in v\mathbb{Z}[v]$. By definition of the pairing, it is clear that the coefficient of $v$ is 1. □

Let $x \in W$ and $s \in S$ be as in the lemma above (i.e. $x < xs$). Both $B_xB_s$ and $B_{xs}$ are one-dimensional in degree $-\ell(x) - 1$, where they are spanned by $c^x_{\mathrm{bot}}c^s_{\mathrm{bot}}$ and $c^{xs}_{\mathrm{bot}}$ respectively. (We write $c^x_{\mathrm{bot}}c^s_{\mathrm{bot}}$ instead of $c^x_{\mathrm{bot}} \otimes c^s_{\mathrm{bot}}$.) Hence there exists a unique map

$$\alpha_{x,s} : B_xB_s \to B_{xs} \tag{3.1}$$

which maps $c^x_{\mathrm{bot}}c^s_{\mathrm{bot}}$ to $c^{xs}_{\mathrm{bot}}$. Similar considerations show that there exists a unique map

$$\beta_{x,s} : B_{xs}B_s \to B_{xs}(1) \tag{3.2}$$

resp.

$$\gamma_{x,s} : B_{xs} \to B_x(1) \tag{3.3}$$

mapping $c^{xs}_{\mathrm{bot}}c^s_{\mathrm{bot}}$ to $c^{xs}_{\mathrm{bot}}$ (resp. $c^{xs}_{\mathrm{bot}}$ to $c^x_{\mathrm{bot}}$).

*3.3. The construction*

We will use the maps $\alpha_{x,s}, \beta_{x,s}$ and $\gamma_{x,s}$ constructed above. We will also use the *multiplication map*

$$m_s : B_s \to R(1) : f \otimes g \mapsto fg.$$

**Remark 3.4.** The reader may easily check that in fact $m_s = \gamma_{\mathrm{id},s}$.

Consider the following data:

(1) an expression (not necessarily reduced) $\underline{y} = (s_1, \dots, s_n)$;
(2) elements $x \in W$, $s \in S$; and
(3) $f : B_{\underline{y}} \to B_x$.

To this data, we will associate two new maps:

$$f0 : B_{\underline{y}}B_s \to B_x \quad \text{and} \quad f1 : B_{\underline{y}}B_s \to B_{xs}.$$

These maps are constructed as follows: If $x < xs$, define

$$f0 := f \otimes m_s \quad \text{and} \quad f1 := \alpha_{x,s} \circ (f \otimes \mathrm{id}).$$

If $xs < x$, define

$$f0 := \beta_{xs,s} \circ (f \otimes \mathrm{id}) \quad \text{and} \quad f1 := \gamma_{xs,s} \circ \beta_{xs,s} \circ (f \otimes \mathrm{id}).$$

Given an expression $\underline{w}$ and a subexpression $\underline{e}$ define the *canonical light leaf*

$$\mathrm{CLL}_{\underline{w},\underline{e}} := \mathrm{id}\underline{e},$$

where id means $\mathrm{id} \in \mathrm{End}(R)$ and for example $\mathrm{id}(0, 1, 0)$ means $(((\mathrm{id}0)1)0)$.

**Example 3.5.** If $\underline{x} = (s_1, \ldots, s_m)$ is reduced, and $\underline{e} = (1, 1, \ldots, 1)$ then $\mathrm{CLL}_{\underline{w},\underline{e}}$ agrees with the projection in the favourite projector. If $\underline{e} = (0, 0, \ldots, 0)$ then $\mathrm{CLL}_{\underline{w},\underline{e}} = m_{s_1} \otimes \cdots \otimes m_{s_m}$.

The proof of the following theorem is essentially the same as in [9, Thm. 3.2] and [4, Thm 6.11].

**Theorem 3.6.** *The set*

$$\bigsqcup_{x \in W} \{\mathrm{CLL}^*_{\underline{w},\underline{e}} \circ \mathrm{CLL}_{\underline{z},\underline{f}} \mid \underline{e} \subset \underline{w}, \underline{f} \subset \underline{z} \text{ such that } \underline{w}^{\underline{e}} = \underline{z}^{\underline{f}} = x\}$$

*is a free R-basis for* $\mathrm{Hom}(B_{\underline{z}}, B_{\underline{w}})$.

Now we can explain why this theorem proves Theorem 1.1. By Theorem 3.6, the graded set $\{\mathrm{CLL}_{\underline{y},\underline{e}} \mid \text{with } \underline{e} \text{ expressing } x\}$ is naturally an $R$-basis of $\widehat{D}_{x,\underline{y}}$, thus it gives an $\mathbb{R}$-basis of $D_{x,\underline{y}}$. So, in summary, the canonical map CLL in Theorem 1.1 is the $\mathbb{R}$-linear map defined on the generators $\underline{e} \in \widetilde{X}_{x,\underline{y}}$ by $\underline{e} \mapsto \mathrm{CLL}_{\underline{y},\underline{e}} \otimes_R \mathrm{id}_{\mathbb{R}}$.

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