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PI-algebras with slow codimension growth

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Abstract

Let $c_n(A)$, $n = 1, 2, \dots$, be the sequence of codimensions of an algebra A over a field F of characteristic zero. We classify the algebras A (up to PI-equivalence) in case this sequence is bounded by a linear function. We also show that this property is closely related to the following: if $l_n(A)$, $n = 1, 2, \dots$, denotes the sequence of colengths of A, counting the number of S_n-irreducibles appearing in the nth cocharacter of A, then $\lim_{n\to\infty} l_n(A)$ exists and is bounded by 2.

Keywords: Polynomial identity; T-ideal; Codimensions

1. Introduction

Given an algebra A over a field F one can associate to A a numerical sequence $c_n(A)$, $n = 1, 2, \dots$, called the sequence of codimensions of A, giving a measure of the polynomial identities satisfied by A. In general $c_n(A)$ is bounded from above by $n!$, but in case A is a PI-algebra, i.e., satisfies a non-trivial polynomial identity, a celebrated theorem of Regev asserts that $c_n(A)$ is exponentially bounded [17]. When the field F is of characteristic zero, it turns out that the sequence of codimensions of any PI-algebra is either polynomially bounded or grows exponentially (see [11]). For general PI-algebras the exponential rate of growth was computed in [5] and [6] and it turns out to be a non-negative

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integer. In case the codimensions are polynomially bounded, Kemer in [12] gave the following characterization. Let G be the infinite dimensional Grassmann algebra and let UT_2 be the algebra of 2×2 upper triangular matrices. Then $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $G, UT_2 \notin \mathrm{var}(A)$, where $\mathrm{var}(A)$ denotes the variety of algebras generated by A.

The aim of this paper is to try to refine Kemer's result through the knowledge of the polynomial rate of growth of $c_n(A)$. In general, given an integer $t \geqslant 0$ can one find a (finite) number of algebras $M_1, \ldots, M_d$ depending on t, such that $\limsup_{n\to\infty} \log_n c_n(A) \leqslant t$ if and only if $M_1, \ldots, M_d \notin \mathrm{var}(A)$? This question seems out of reach in this generality but we shall give a complete answer for some values of t. We first need to formulate an apparently different problem about representations of the symmetric group S_n.

An equivalent formulation of Kemer's result can be given as follows. Let $F\langle X\rangle$ be the free algebra on a countable set $X = \{x_1, x_2, \ldots\}$ and let $\mathrm{Id}(A)$ be the T-ideal of polynomial identities of the algebra A. The permutation action of S_n on the space V_n of multilinear polynomials in the first n variables induces a structure of S_n-module on $\frac{V_n}{V_n \cap \mathrm{Id}(A)}$ and let $\chi_n(A)$ be its character. By complete reducibility we can write $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ where χ_λ is the irreducible S_n-character associated to the partition λ of n and $m_\lambda \geqslant 0$ is the corresponding multiplicity. Then $l_n(A) = \sum_{\lambda\vdash n} m_\lambda$ is the nth colength of A. Now Kemer's result can be stated as follows [15]: $c_n(A)$ is polynomially bounded if and only if the sequence of colengths is bounded by a constant, i.e., $l_n(A) \leqslant k$, for some $k \geqslant 0$ and for all $n \geqslant 1$ (see [3]).

In this setting one can ask if it is possible to give a finer classification depending on the value of the constant k. In particular, given $k \geqslant 0$, can one find a finite number of algebras $M_1, \ldots, M_d$, depending on k, such that $\limsup_{n\to\infty} l_n(A) \leqslant k$ if and only if $M_1, \ldots, M_d \notin \mathrm{var}(A)$? In this paper we are able to answer this question in the positive in case $k \leqslant 2$. We shall also show that this is strictly related to the codimensions of A being linearly bounded. As a consequence we are able to classify up to PI-equivalence the algebras A such that $l_n(A) \leqslant 2$ or $c_n(A) \leqslant kn$. It turns out that for n large enough the only sequences of codimensions allowed are $c_n(A) = 0, 1, n$ and $c_n(A) = 2n - 1$.

2. Generalities

Throughout this paper, we shall denote by F a field of characteristic zero and by A an associative algebra over F. We refer the reader to [4] and [18] for the basic definitions and properties of PI-algebras.

Let $F\langle X\rangle$ be the free associative algebra on the countable set $X = \{x_1, x_2, \ldots\}$ and let $\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ be the set of polynomial identities of A. Clearly $\mathrm{Id}(A)$ is a T-ideal of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$. It is well known that in characteristic zero $\mathrm{Id}(A)$ is completely determined by its multilinear polynomials and we denote by

$$V_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

the space of multilinear polynomials in the indeterminates $x_1, \dots, x_n$. The symmetric group S_n acts on the left on V_n: if $\sigma \in S_n$ and $f(x_1, \dots, x_n) \in V_n$, then

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Since the subspace $V_n \cap \mathrm{Id}(A)$ is invariant under this action,

$$V_n(A) = \frac{V_n}{V_n \cap \mathrm{Id}(A)}$$

has a structure of S_n-module. The S_n-character of $V_n(A)$, denoted $\chi_n(A)$, is called the nth cocharacter of A and $c_n(A) = \dim_F V_n(A)$ is the nth codimension of A.

It is well known that in characteristic zero there is a one-to-one correspondence between irreducible S_n-characters and partitions $\lambda \vdash n$. If χ_λ denotes the irreducible S_n-character corresponding to λ then, since char $F = 0$, by complete reducibility we can write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1}$$

where $m_\lambda \geqslant 0$ is the multiplicity of χ_λ in the given decomposition. Also

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the nth colength of A.

The following remark lists some basic properties of the sequence of cocharacters, codimensions and colengths.

Remark 1. Let A and B be F-algebras and let $A \oplus B$ be their direct sum. If $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, $\chi_n(B) = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda$, and $\chi_n(A \oplus B) = \sum_{\lambda \vdash n} m''_\lambda \chi_\lambda$ are the nth cocharacters of A, B and $A \oplus B$, respectively, then $m''_\lambda \leqslant m_\lambda + m'_\lambda$,

$$l_n(A \oplus B) \leqslant l_n(A) + l_n(B) \quad \text{and} \quad c_n(A \oplus B) \leqslant c_n(A) + c_n(B).$$

In case B is a nilpotent algebra and $B^k = 0$, then for all $n \geqslant k$ we have $m''_\lambda = m_\lambda$. Hence $l_n(A \oplus B) = l_n(A)$ and $c_n(A \oplus B) = c_n(A)$ in this case.

Proof. Consider the map of S_n-modules

$$\alpha : V_n \to \frac{V_n}{V_n \cap \mathrm{Id}(A)} \oplus \frac{V_n}{V_n \cap \mathrm{Id}(B)}$$

such that $\alpha(f) = (f + (V_n \cap \mathrm{Id}(A)), f + (V_n \cap \mathrm{Id}(B)))$. Since $\ker(\alpha) = V_n \cap \mathrm{Id}(A) \cap \mathrm{Id}(B) = V_n \cap \mathrm{Id}(A \oplus B)$, we have an embedding of S_n-modules

$$\frac{V_n}{V_n \cap \mathrm{Id}(A \oplus B)} \hookrightarrow \frac{V_n}{V_n \cap \mathrm{Id}(A)} \oplus \frac{V_n}{V_n \cap \mathrm{Id}(B)}.$$

Thus $m''_\lambda \leqslant m_\lambda + m'_\lambda$ and $l_n(A \oplus B) \leqslant l_n(A) + l_n(B)$, $c_n(A \oplus B) \leqslant c_n(A) + c_n(B)$.

If $B^k = 0$ then, for all $n \geqslant k$, $\frac{V_n}{V_n \cap \mathrm{Id}(B)}$ is the zero module. It follows that $\frac{V_n}{V_n \cap \mathrm{Id}(A \oplus B)} = \frac{V_n}{V_n \cap \mathrm{Id}(A)}$ for all $n \geqslant k$. $\square$

Given an algebra A, let $\mathrm{var}(A)$ denote the variety of algebras generated by A. Another fact that we shall use throughout the paper is that if A and B are F-algebras and $B \in \mathrm{var}(A)$ then

$$l_n(B) \leqslant l_n(A) \quad \text{and} \quad c_n(B) \leqslant c_n(A).$$

This is clear since in this case, $\mathrm{Id}(A) \subseteq \mathrm{Id}(B)$ and so, $V_n(B)$ can be embedded in $V_n(A)$.

In the next sections we prefer to work with the representation theory of the general linear group which is well related to that of the symmetric group. To this end we need to introduce the space of homogeneous polynomials in a given set of variables. Let $F_m\langle X\rangle = F\langle x_1, \dots, x_m\rangle$ denote the free associative algebra in m variables and let $U = \mathrm{span}_F\{x_1, \dots, x_m\}$. The group $GL(U) \cong GL_m$ acts naturally on the left on the space U and we can extend this action diagonally to get an action on $F_m\langle X\rangle$.

The space $F_m\langle X\rangle \cap \mathrm{Id}(A)$ is invariant under this action, hence

$$F_m(A) = \frac{F_m\langle X\rangle}{F_m\langle X\rangle \cap \mathrm{Id}(A)}$$

inherits a structure of left GL_m-module. Let F_m^n be the space of homogeneous polynomials of degree n in the variables $x_1, \dots, x_m$. Then

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap \mathrm{Id}(A)}$$

is a GL_m-submodule of $F_m(A)$ and we denote its character by $\psi_n(A)$. Write

$$\psi_n(A) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

where ψ_λ is the irreducible GL_m-character associated to the partition λ and $\bar{m}_\lambda$ is the corresponding multiplicity. It was proved in [1] and [2] that if the nth cocharacter of A has the decomposition given in (1) then $m_\lambda = \bar{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most m.

It is also well known (see for instance [4, Theorem 12.4.12]) that any irreducible submodule of $F_m^n(A)$ corresponding to λ is generated by a non-zero polynomial f_λ, called highest weight vector, of the form

$$f_\lambda = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1, \dots, x_{h_i(\lambda)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \tag{2}$$

where $\alpha_\sigma \in F$, the right action of S_n on $F_m^n(A)$ is defined by place permutation, $h_i(\lambda)$ is the height of the ith column of the diagram of λ and

$$St_r(x_1, \dots, x_r) = \sum_{\tau \in S_r} (\operatorname{sgn} \tau) x_{\tau(1)} \cdots x_{\tau(r)}$$

is the standard polynomial of degree r. Recall that f_λ is unique up to a multiplicative constant.

For a Young tableau T_λ, denote by f_{T_λ} the highest weight vector obtained from (2) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \dots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\sigma(h_1(\lambda)+1), \dots, \sigma(h_1(\lambda)+h_2(\lambda))$ the second column of T_λ, etc.

We also have the following (see for instance [4, Proposition 12.4.14]).

Remark 2. If

$$\psi_n(A) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

is the GL_m-character of $F_m^n(A)$, then $\bar{m}_\lambda \neq 0$ if and only if there exists a tableau T_λ such that the corresponding highest weight vector f_{T_λ} is not a polynomial identity for A. Moreover $\bar{m}_\lambda$ is equal to the maximal number of linearly independent highest weight vectors f_{T_λ} in $F_m^n(A)$.

3. Computing the identities of some PI-algebras

The purpose of this section is to compute the cocharacters, the codimensions, the T-ideals, etc. of some PI-algebras that will play a basic role in the next section.

Given polynomials $f_1, \dots, f_n \in F\langle X\rangle$ let us denote by $\langle f_1, \dots, f_n\rangle_{\mathrm{T}}$ the T-ideal generated by $f_1, \dots, f_n$. We shall also denote by y, z, t, w the variables of X. Also, given an algebra A let $J(A)$ denote its Jacobson radical.

In order to shorten the proof of next lemma and of Lemma 6 we use a result of Guterman and Regev [9] even though a direct proof can be easily found.

Lemma 3. *Let* $M_1 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ *and* $M_2 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. *Then for all* $n > 1$,

1. $\chi_n(M_1) = \chi_n(M_2) = \chi_{(n)} + \chi_{(n-1,1)}$.
2. $l_n(M_1) = l_n(M_2) = 2$.
3. $\{x_{i_1} \cdots x_{i_n},\ i_2 < \cdots < i_n\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_1)}$ *and* $\{x_{i_1} \cdots x_{i_n},\ i_1 < \cdots < i_{n-1}\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_2)}$.
4. $c_n(M_1) = c_n(M_2) = n$.
5. $\mathrm{Id}(M_1) = \langle z[x,y]\rangle_{\mathrm{T}}$ *and* $\mathrm{Id}(M_2) = \langle [x,y]z\rangle_{\mathrm{T}}$.

Proof. The conclusion clearly holds for $n = 2$, hence we may assume that $n \geqslant 3$. By [9, Theorem 1], $\mathrm{Id}(M_1) = \langle z[x,y]\rangle_T$, $\mathrm{Id}(M_2) = \langle [x,y]z\rangle_T$ and $c_n(M_1) = c_n(M_2) = n$. Hence the elements

$$x_{i_1}\cdots x_{i_n}, \quad i_2 < \cdots < i_n$$

span $V_n \pmod{V_n \cap \mathrm{Id}(M_1)}$. Since their number equals $n = c_n(M_1) = \dim V_n/(V_n \cap \mathrm{Id}(M_1))$ we have that $\{x_{i_1}\cdots x_{i_n} \mid i_2 < \cdots < i_n\}$ is a basis of $V_n \pmod{V_n \cap \mathrm{Id}(M_1)}$.

We now determine the decomposition of the nth cocharacter of M_1. Let e_{ij} be the usual matrix units. Since $c_n(M_1) = n$ is polynomially bounded and $J(M_1) = Fe_{12}$ satisfies $J(M_1)^2 = 0$, by [7, Theorem 3] we have that

$$\chi_n(M_1) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 < 2}} m_\lambda \chi_\lambda = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)}.$$

If $\lambda = (n)$, the corresponding highest weight vector $f_\lambda = x^n$ is not an identity of M_1 since $f_\lambda(e_{22}) = e_{22} \neq 0$. Then, being x^n the only highest weight vector corresponding to $\lambda = (n)$, it follows that $m_{(n)} = 1$.

Since $c_n(M_1) = m_{(n)} \deg \chi_{(n)} + m_{(n-1)} \deg \chi_{(n-1)}$ and by the hook formula [10], $\deg \chi_{(n)} = 1$, $\deg \chi_{(n-1,1)} = n - 1$, it follows that $n = 1 + m_{(n-1)}(n-1)$ and, so, $m_{(n-1)} = 1$. Therefore

$$\chi_n(M_1) = \chi_{(n)} + \chi_{(n-1,1)}$$

and

$$l_n(M_1) = \sum_{\lambda \vdash n} m_\lambda = 2.$$

A similar proof gives the desired results about M_2. $\square$

In what follows we use the left normed notation for Lie commutators. Hence we write $[\ldots[[x_1,x_2],x_3],\ldots,x_n] = [x_1,x_2,\ldots,x_n]$.

Lemma 4. *Let* $M_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a,b,c,d \in F \right\}$. *Then for all* $n > 3$,

1. $\chi_n(M_3) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)}$.
2. $l_n(M_3) = 3$.
3. $\{x_1\cdots x_n, x_{i_1}\cdots x_{i_{n-2}}[x_i,x_j],\ i_1 < \cdots < i_{n-2},\ i > j\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_3)}$.
4. $c_n(M_3) = \frac{n(n-1)+2}{2}$.
5. $\mathrm{Id}(M_3) = \langle [x,y,z], [x,y][z,w]\rangle_T$.

Proof. Let $Q = \langle [x,y,z], [x,y][z,w]\rangle_T$. Since $[M_3,M_3] \subseteq \mathrm{span}\{e_{13}\}$ it is clear that $[x,y,z]$ and $[x,y][z,w]$ are identities of M_3 and $Q \subseteq \mathrm{Id}(M_3)$. Before proving the opposite inclusion, we find a generating set of V_n modulo $V_n \cap Q$.

It is well known (see for instance [4, Theorem 5.2.1]) that any multilinear polynomial of degree n can be written, modulo $\langle[x, y][z, w]\rangle_{\mathrm{T}}$, as a linear combination of polynomials of the type

$$x_{i_1} \cdots x_{i_m} [x_k, x_{j_1}, \ldots, x_{j_{n-m-1}}]$$

where $i_1 < \cdots < i_m$, $j_1 < \cdots < j_{n-m-1}$, $k > j_1$, $m \neq n-1$. Thus, because of the identity $[x, y, z]$, the elements

$$x_1 \cdots x_n,\ x_{i_1} \cdots x_{i_{n-2}} [x_i, x_j], \quad i_1 < \cdots < i_{n-2},\ i > j \tag{3}$$

span V_n modulo $V_n \cap Q$. We next prove that these elements are linearly independent modulo $\mathrm{Id}(M_3)$.

Suppose that

$$f = \sum_{\substack{i,j=1 \\ i>j}}^{n} \alpha_{ij} x_{i_1} \cdots x_{i_{n-2}} [x_i, x_j] + \beta x_1 \cdots x_n \equiv 0 \quad \bigl(\mathrm{mod}\ V_n \cap \mathrm{Id}(M_3)\bigr).$$

By making the evaluation $x_k = e_{11} + e_{22} + e_{33}$, for all $k = 1, \ldots, n$, we get $\beta = 0$. Also, for fixed i and j, the evaluation $x_i = e_{12}$, $x_j = e_{23}$ and $x_k = e_{11} + e_{22} + e_{33}$ for $k \notin \{i, j\}$ gives $\alpha_{ij} = 0$. Thus the elements in (3) are linearly independent modulo $V_n \cap \mathrm{Id}(M_3)$. Since $V_n \cap Q \subseteq V_n \cap \mathrm{Id}(M_3)$, this proves that $\mathrm{Id}(M_3) = Q$ and the elements in (3) are a basis of V_n modulo $V_n \cap \mathrm{Id}(M_3)$. By counting we obtain

$$c_n(M_3) = \dim \frac{V_n}{V_n \cap \mathrm{Id}(M_3)} = \frac{n(n-1)+2}{2}.$$

Since $\deg \chi_{(n)} + \deg \chi_{(n-1,1)} + \deg \chi_{(n-2,1,1)} = 1 + (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)+2}{2}$, if we find for each $\lambda \in \{(n), (n-1, 1), (n-2, 1, 1)\}$ a highest weight vector which is not an identity of M_3, we may conclude that $\chi_n(M_3) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)}$ and the nth cocharacter has the wished decomposition.

Clearly x^n, the highest weight vector corresponding to $\lambda = (n)$, is not an identity of M_3.

Let $f_{(n-1,1)} = [x_1, x_2]x_1^{n-2}$ be a highest weight vector corresponding to $\lambda = (n-1, 1)$. Taking $a_1 = (e_{11} + e_{22} + e_{33}) + e_{12}$ and $a_2 = e_{23}$, we get that

$$f_{(n-1,1)}(a_1, a_2) = e_{13} \neq 0.$$

Thus $f_{(n-1,1)}$ is not an identity of M_3.

Finally $f_\lambda = St_3(x_1, x_2, x_3)x_1^{n-3}$ is a highest weight vector corresponding to $\lambda = (n-2, 1, 1)$, where St_3 is the standard polynomial of degree 3. By choosing $a_1 = e_{11} + e_{22} + e_{33}$, $a_2 = e_{12}$ and $a_3 = e_{23}$, we obtain

$$f_\lambda(a_1, a_2, a_3) = e_{13} \neq 0.$$

Thus, since these polynomials are not identities of M_3, it follows that

$$\chi_n(M_3) = \chi_{(n)} + \chi_{(n-1,1)} + \chi_{(n-2,1,1)}$$

and

$$l_n(M_3) = 3.$$

The proof of the lemma is complete. □

Lemma 5. *Let $B = M_1 \oplus M_2$. Then for all $n \geqslant 3$,*

1. $\chi_n(B) = \chi_{(n)} + 2\chi_{(n-1,1)}$.
2. $l_n(B) = 3$.
3. $\{[x_1, x_i]x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}},\ x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}}[x_1, x_i],\ x_1 \cdots x_n,\ j_1 < \cdots < j_{n-2}\}$ *is a basis of V_n $(\mathrm{mod}\ V_n \cap \mathrm{Id}(B))$, where the symbol $\hat{x}_i$ means that the variable x_i is omitted.*
4. $c_n(B) = 2n - 1$.
5. $\mathrm{Id}(B) = \langle St_3(x, y, z), z[x, y]w, [x, y][z, w]\rangle_T$.

Proof. Let $Q = \langle St_3(x, y, z), z[x, y]w, [x, y][z, w]\rangle_T$. It is easy to check that $Q \subseteq \mathrm{Id}(B)$. Next we claim that the set of polynomials

$$\bigl\{[x_1, x_i]x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}},\ x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}}[x_1, x_i],\ x_1 \cdots x_n,\ j_1 < \cdots < j_{n-2}\bigr\} \tag{4}$$

span V_n modulo $V_n \cap Q$.

Since for $n = 3$ this is clear, we assume that $n > 3$. Now, as in the proof of the previous lemma, any multilinear polynomial of degree n can be written, modulo $\langle [x, y][z, w]\rangle_T$, as a linear combination of polynomials of the type

$$x_{i_1} \cdots x_{i_m}[x_k, x_{j_1}, \ldots, x_{j_{n-m-1}}]$$

where $i_1 < \cdots < i_m$, $j_1 < \cdots j_{n-m-1}$, $k > j_1$, $m \neq n - 1$.

Since

$$\begin{aligned}[x_k, x_{j_1}, \ldots, x_{j_{n-m-1}}] \equiv{}& [x_k, x_{j_1}]x_{j_2} \cdots x_{j_{n-m-1}} \\ &\pm x_{j_{n-m-1}} \cdots x_{j_2}[x_k, x_{j_1}] \quad \bigl(\mathrm{mod}\ \langle z[x, y]w\rangle_T\bigr)\end{aligned}$$

and

$$x_{i_1} \cdots x_{i_m}[x_k, x_{j_1}]x_{j_2} \cdots x_{j_{n-m-1}} \equiv 0 \quad \bigl(\mathrm{mod}\ \langle z[x, y]w\rangle_T\bigr),$$

we have that for $m \geqslant 1$,

$$x_{i_1} \cdots x_{i_m}[x_k, x_{j_1}, \ldots, x_{j_{n-m-1}}] \equiv x_{k_1} \cdots x_{k_{n-2}}[x_k, x_{j_1}] \quad (\mathrm{mod}\ Q).$$

Also, since $[x, y][z, w] \in Q$, we may assume that $k_1 < \cdots < k_{n-2}$. Now, since $n \geqslant 4$, if $j_1 \neq 1$ we have that

$$x_{k_1} \cdots x_{k_{n-2}}[x_k, x_{j_1}] \equiv w x_1 x_k x_{j_1} - w x_1 x_{j_1} x_k \equiv w x_k x_1 x_{j_1} - w x_{j_1} x_1 x_k \\ \equiv w x_k [x_1, x_{j_1}] - w x_{j_1}[x_1, x_k] \pmod{Q}$$

where $w = x_{k_2} \cdots x_{k_{n-2}}$. Thus

$$x_{k_1} \cdots x_{k_{n-2}}[x_k, x_{j_1}] \equiv x_{h_1} \cdots x_{h_{n-2}}[x_1, x_{j_1}] - x_{l_1} \cdots x_{l_{n-2}}[x_1, x_k] \pmod{Q}$$

where $h_1 < \cdots < h_{n-2}$ and $l_1 < \cdots < l_{n-2}$. Similarly

$$[x_k, x_{j_1}] x_{k_1} \cdots x_{k_{n-2}} \equiv [x_1, x_{j_1}] x_{h_1} \cdots x_{h_{n-2}} - [x_1, x_k] x_{l_1} \cdots x_{l_{n-2}} \pmod{Q}.$$

It follows that the polynomials

$$[x_1, x_i] x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}},\ x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}}[x_1, x_i], \quad x_1 \cdots x_n,\ j_1 < \cdots < j_{n-2}$$

generate $V_n \pmod{V_n \cap Q}$ for $n \geqslant 3$.

We next show that the elements in (4) are linearly independent modulo $\mathrm{Id}(B)$.

Let $f \in \mathrm{Id}(B)$ be a linear combination of the elements in (4):

$$f = \sum_{\substack{i=1 \\ j_1 < \cdots < j_{n-2}}}^{n} \alpha_i [x_1, x_i] x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}} \\ + \sum_{\substack{j=1 \\ j_1 < \cdots < j_{n-2}}}^{n} \beta_j x_{j_1} \cdots \hat{x}_i \cdots x_{j_{n-2}}[x_1, x_j] + \gamma x_1 \cdots x_n.$$

By making the evaluation $x_i = (e_{22}, 0)$, for all $i = 1, \ldots, n$, we get $\gamma = 0$. Also for a fixed k, the evaluation $x_k = (e_{12}, 0)$ and $x_i = (e_{22}, 0)$ for $i \neq k$, gives $\alpha_k = 0$. Similarly $x_k = (0, e_{12})$ and $x_i = (0, e_{11})$ for $i \neq k$, gives $\beta_k = 0$. Therefore the elements in (4) are linearly independent modulo $V_n \cap \mathrm{Id}(B)$. Since

$$V_n \cap \mathrm{Id}(B) \supseteq V_n \cap Q$$

it follows that $\mathrm{Id}(B) = Q$ and the elements in (4) are a basis of V_n modulo $V_n \cap \mathrm{Id}(B)$ for $n \geqslant 3$. Thus

$$c_n(B) = \dim \frac{V_n}{V_n \cap \mathrm{Id}(B)} = 2n - 1, \quad \text{for } n \geqslant 3.$$

We now determine the decomposition of the nth cocharacter of this algebra. For an algebra A write $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda$. Then, since $B = M_1 \oplus M_2$, by Remark 1, $m_\lambda(B) \leqslant m_\lambda(M_1) + m_\lambda(M_2)$ for all $\lambda \vdash n$. Hence by Lemma 3, it follows that

$$\chi_n(B) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)}$$

and $m_{(n)}, m_{(n-1,1)} \leqslant 2$.

Since $\deg \chi_{(n)} = 1$ and x^n is not an identity of B, clearly $m_{(n)} = 1$. Moreover, since $c_n(B) = 2n - 1$ and $\deg \chi_{(n-1,1)} = n - 1$, it follows that $m_{(n-1,1)} = 2$. Thus

$$\chi_n(B) = \chi_{(n)} + 2\chi_{(n-1,1)}$$

and

$$l_n(B) = \sum_{\lambda \vdash n} m_\lambda = 3. \qquad \square$$

Lemma 6. *Let* $M_4 = \begin{pmatrix} F & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, $M_5 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & F \end{pmatrix}$ *and* $M_6 = \begin{pmatrix} 0 & F & F \\ 0 & F & F \\ 0 & 0 & 0 \end{pmatrix}$. *Then for all* $n > 3$,

1. $\chi_n(M_4) = \chi_n(M_5) = \chi_n(M_6) = \chi_{(n)} + 2\chi_{(n-1,1)} + \chi_{(n-2,2)} + \chi_{(n-2,1,1)}$.
2. $l_n(M_4) = l_n(M_5) = l_n(M_6) = 5$.
3. $\{x_{i_1} \cdots x_{i_{n-2}} x_i x_j,\ i_1 < \cdots < i_{n-2}\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_4)}$, $\{x_i x_j x_{i_1} \cdots x_{i_{n-2}},\ i_1 < \cdots < i_{n-2}\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_5)}$ *and* $\{x_i x_{i_1} \cdots x_{i_{n-2}} x_j,\ i_1 < \cdots < i_{n-2}\}$ *is a basis of* $V_n \pmod{V_n \cap \mathrm{Id}(M_6)}$.
4. $c_n(M_4) = c_n(M_5) = c_n(M_6) = n(n-1)$.
5. $\mathrm{Id}(M_4) = \langle [x,y]zw \rangle_{\mathrm{T}}$, $\mathrm{Id}(M_5) = \langle zw[x,y] \rangle_{\mathrm{T}}$ *and* $\mathrm{Id}(M_6) = \langle z[x,y]w \rangle_{\mathrm{T}}$.

Proof. By [9] we have that $\mathrm{Id}(M_4) = \langle [x,y]zw \rangle_{\mathrm{T}}$, $\mathrm{Id}(M_5) = \langle zw[x,y] \rangle_{\mathrm{T}}$, $\mathrm{Id}(M_6) = \langle z[x,y]w \rangle_{\mathrm{T}}$ and $c_n(M_4) = c_n(M_5) = c_n(M_6) = n(n-1)$. As in the proof of the previous lemmas, it is easy to see that the elements

$$x_{i_1} \cdots x_{i_{n-2}} x_i x_j, \quad i_1 < \cdots < i_{n-2}$$

form a basis of $V_n \pmod{V_n \cap \mathrm{Id}(M_4)}$. A similar remark holds for M_5 and M_6.

We next determine the decomposition of the nth cocharacter of M_4. A similar proof will give the decomposition of the nth cocharacter of M_5 and M_6. Since $J(M_4) = Fe_{12} + Fe_{13} + Fe_{23}$, and $J(M_4)^3 = 0$, by [7, Theorem 3] we have that

$$\chi_n(M_4) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)} + m_{(n-2,2)}\chi_{(n-2,2)} + m_{(n-2,1,1)}\chi_{(n-2,1,1)}.$$

Since M_4 is not nilpotent, clearly $m_{(n)} = 1$.

Let $\lambda = (n-1, 1)$ and denote by $T_\lambda^{(i)}$ the standard tableau containing the integer $i = 2, \ldots, n$ in the only box of the second row. Then

$$f_{T_\lambda^{(i)}} = x_1 x_1^{i-2} x_2 x_1^{n-i} - x_2 x_1^{i-2} x_1 x_1^{n-i}$$

is the highest weight vector corresponding to $T_\lambda^{(i)}$. By making the evaluation $x_1 = e_{11} + e_{23}$ and $x_2 = e_{12}$, we get that

$$f_{T_\lambda^{(n)}} = e_{12} \quad \text{and} \quad f_{T_\lambda^{(n-1)}} = e_{13}.$$

This says that $f_{T_\lambda^{(n)}}$ and $f_{T_\lambda^{(n-1)}}$ are not identities of M_4. Moreover these polynomials are linearly independent (mod $\mathrm{Id}(M_4)$). In fact if

$$f = \alpha_1 f_{T_\lambda^{(n)}} + \alpha_2 f_{T_\lambda^{(n-1)}} \equiv 0 \quad \big(\mathrm{mod}\ \mathrm{Id}(M_4)\big),$$

make the evaluation $x_1 = e_{11}$ and $x_2 = e_{12}$ to get $\alpha_1 = 0$. Similarly, choose $x_1 = e_{11} + e_{23}$ and $x_2 = e_{12}$ to obtain $\alpha_2 = 0$. Thus $m_{(n-1,1)} \geqslant 2$. Since $c_n(M_4) = n(n-1)$ and

$$\begin{aligned} &\deg \chi_{(n)} + 2\deg \chi_{(n-1,1)} + \deg \chi_{(n-2,2)} + \deg \chi_{(n-2,1,1)} \\ &\quad = 1 + 2(n-1) + \frac{n(n-3)}{2} + \frac{(n-2)(n-1)}{2} = n(n-1), \end{aligned}$$

in order to prove the given decomposition of the nth cocharacter, it is enough to find for each $\lambda \in \{(n-2,2), (n-2,1,1)\}$ a highest weight vector which is not an identity of M_4.

For $\lambda = (n-2,2)$, consider the following standard tableau

1	2	$\cdots$
$n-1$	n	

and the corresponding highest weight vector

$$f_{T_\lambda} = \sum_{\sigma,\tau \in S_2} (\mathrm{sgn}\, \sigma\tau) x_{\sigma(1)} x_{\tau(1)} x_1^{n-4} x_{\sigma(2)} x_{\tau(2)}.$$

Evaluating $x_1 = e_{11}$ and $x_2 = e_{12} + e_{23}$, we get $f_{T_\lambda} = e_{13} \neq 0$ and f_{T_λ} is not an identity of M_4.

Finally consider $f_{T_\lambda} = \sum_{\sigma \in S_3} (\mathrm{sgn}\ \sigma) x_{\sigma(1)} x_1^{n-3} x_{\sigma(2)} x_{\sigma(3)}$, the highest weight vector corresponding to the standard tableau

1	$\cdots$
$n-1$	
n	

.

Since

$$f_{T_\lambda}(e_{11}, e_{12}, e_{23}) = e_{13} \neq 0$$

we have that f_{T_λ} is not an identity of M_4. Hence $\chi_n(M_4) = \chi_{(n)} + 2\chi_{(n-1,1)} + \chi_{(n-2,2)} + \chi_{(n-2,1,1)}$ and $l_n(M_4) = 5$. $\square$

4. When codimensions are bounded by a constant

A main tool in this and the next section is the following result on the decomposition of the Jacobson radical of a finite dimensional algebra.

Lemma 7 [8, Lemma 2]. *Let A be a finite dimensional algebra over F and suppose that $A = B + J$, where B is a semisimple subalgebra and $J = J(A)$ is its Jacobson radical. Then J can be decomposed into the direct sum of B-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where, for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B.

An obvious remark is that if $A = B + J$ is an algebra satisfying the hypothesis of the previous lemma and 1_B is the unit element of B, we have that

$$1_B a 1_B = a, \quad 1_B b = b \quad \text{and} \quad c 1_B = c$$

for all $a \in J_{11}, b \in J_{10}, c \in J_{01}$.

In the sequel we shall also use the following notation.

Definition 8. Let A and B be algebras. We say that A is PI-equivalent to B and we write $A \sim_{\mathrm{PI}} B$, if $\mathrm{Id}(A) = \mathrm{Id}(B)$.

In the following lemmas we shall assume that A is a finite dimensional algebra of the type $A = F + J$ where $J = J(A)$ is the Jacobson radical of A. Also we shall tacitly assume that J has the decomposition given in Lemma 7. We start with the following.

Lemma 9. *Let $A = F + J$. If $[J_{11}, J_{11}] \neq 0$, then $M_3 \in \mathrm{var}(A)$.*

Proof. Notice that $A' = F + J_{11}$ is a subalgebra of A. We shall prove that $M_3 \in \mathrm{var}(A') \subseteq \mathrm{var}(A)$. Hence without lost of generality we may assume that $A = F + J$ and $J = J_{11}$.

Since by hypothesis J is not commutative, $J^2 \neq 0$ and let k be the least integer such that $J^k = 0$ and $J^{k-1} \neq 0$. We proceed by induction on k.

If $k = 3$ we shall prove that A itself is PI-equivalent to M_3. In fact, in this case, since 1_F commutes with J, $[A, A] \subseteq J^2$; hence $[A, A, A] = 0$ and $[A, A][A, A] = 0$. By Lemma 4 this says that $\mathrm{Id}(M_3) \subseteq \mathrm{Id}(A)$.

Conversely, let $f \in \mathrm{Id}(A)$ be a multilinear polynomial of degree n. By Lemma 4, one can write f as

$$f = \alpha x_1 \cdots x_n + \sum_{\substack{i>j \\ i_1<\cdots<i_{n-2}}} \alpha_{ij} x_{i_1} \cdots x_{i_{n-2}} [x_i, x_j] + g$$

where $g \in \mathrm{Id}(M_3)$. Choosing $x_i = 1_F$ for all $i = 1, \dots, n$ we get $\alpha = 0$. Since $[J, J] \neq 0$, there exist two elements $a, b \in J$ such that $[a, b] \neq 0$. Let $i_0, j_0 \in \{1, \dots, n\}$ be such that $\alpha_{i_0 j_0} \neq 0$. Take $c_1, \dots, c_n \in A$, with $c_{i_0} = a$, $c_{j_0} = b$ and $c_k = 1_F$ for all $k \notin \{i_0, j_0\}$. It is easy to check that

$$f(c_1, \dots, c_n) = \alpha_{i_0 j_0}[a, b] \neq 0,$$

a contradiction. Hence $f = g \in \mathrm{Id}(M_3)$ and $\mathrm{Id}(A) = \mathrm{Id}(M_3)$. This settles the case $k = 3$.

Let now $k > 3$ and suppose first that $[J^{k-2}, J] \neq 0$. Let $a \in J^{k-2}$ and $b \in J$ be such that $[a, b] \neq 0$ and let B be the subalgebra of A generated by $1_F, a, b$. Then B is the linear span of the elements $\{1_F, a, b, ab, ba, b^2, \dots, b^{k-1}\}$, and an easy computation shows that $[B, B] \subseteq J^{k-1}$; hence $[B, B, B] = 0$ and $[B, B][B, B] = 0$. By Lemma 4 this says that $\mathrm{Id}(M_3) \subseteq \mathrm{Id}(B)$. The other inclusion is proved as above. Thus $B \sim_{\mathrm{PI}} M_3$, $M_3 \in \mathrm{var}(A)$ and we are done in this case too.

Therefore we may assume that $[J^{k-2}, J] = 0$. Suppose first that $[J, J] \subseteq J^{k-2}$. Then, since $J^k = 0$ we have that

$$[J, J, J] = 0, \qquad [J, J][J, J] = 0 \tag{5}$$

and since $A = F + J$, $J = J_{11}$ we also have that $[x, y, z]$, $[x, y][z, w]$ are identities of A. Then by Lemma 4, $\mathrm{Id}(M_3) \subseteq \mathrm{Id}(A)$. Notice that, by hypothesis, there exist $a, b \in J$ such that $[a, b] \neq 0$. Let

$$f = \alpha x_1 \cdots x_n + \sum_{\substack{i>j \\ i_1 < \cdots < i_{n-2}}} \alpha_{i>j} x_{i_1} \cdots x_{i_{n-2}} [x_i, x_j] + g$$

be an identity of A with $g \in \mathrm{Id}(M_3)$. The same procedure as above proves that $f = g \in \mathrm{Id}(M_3)$, and $\mathrm{Id}(A) = \mathrm{Id}(M_3)$ and we are done.

In order to finish the proof of the lemma we have only to study the case $[J, J] \nsubseteq J^{k-2}$. This says that if $\bar{A} = A/J^{k-2}$, $\bar{A}$ is not commutative. Then, from Lemma 7, $J(\bar{A}) = J_{11}(\bar{A}) = J/J^{k-2}$ is still non-commutative and $J(\bar{A})^{k-2} = 0$ with $k > 4$. By the inductive hypothesis $M_3 \in \mathrm{var}(\bar{A})$ and since $\bar{A} \in \mathrm{var}(A)$, we have that $M_3 \in \mathrm{var}(\bar{A}) \subseteq \mathrm{var}(A)$. The proof of the lemma is now complete. □

Lemma 10. *Let $A = F + J$ with $J_{01} \neq 0$ (respectively $J_{10} \neq 0$). Then $B = F + J_{01}$ or $B = F + J_{01} + J_{11}$, in case J_{11} is commutative, is a subalgebra PI-equivalent to M_1 (respectively $B = F + J_{10}$ or $B = F + J_{10} + J_{11}$ is PI-equivalent to M_2).*

Proof. It is clear that $B = F + J_{01}$ or $B = F + J_{01} + J_{11}$ is a subalgebra of A. In any case by the hypotheses we have that $[B, B] \subseteq J_{01}$ and so, $B[B, B] = 0$. This says, by Lemma 3, that $\mathrm{Id}(M_1) \subseteq \mathrm{Id}(B)$. Let $f \in \mathrm{Id}(B)$ be a multilinear polynomial of degree n. By Lemma 3, f can be written as

$$f = \sum_{\substack{i_1=1 \\ i_2<\cdots<i_n}}^{n} \alpha_{i_1} x_{i_1} \cdots x_{i_n} + g$$

where $g \in \mathrm{Id}(M_1)$. Suppose that there exists $t \in \{1, \ldots, n\}$ such that $\alpha_t \neq 0$. Choose $c_i = 1_F$, for $i \neq t$ and $c_t = b \in J_{01}$, $b \neq 0$. Since g is an identity of B, we have

$$f(c_1, \ldots, c_n) = \alpha_t b 1_F = \alpha_t b \neq 0,$$

a contradiction. Hence $f = g \in \mathrm{Id}(M_1)$ and $\mathrm{Id}(B) = \mathrm{Id}(M_1)$ follows. Thus B is a subalgebra of A which is PI-equivalent to M_1. Similarly it is proved that $B = F + J_{10}$ and $B = F + J_{10} + J_{11}$, in case J_{11} is commutative, are PI-equivalent to M_2. $\square$

Let G denotes the Grassmann algebra of countable rank over F. Hence G is generated by the set $\{e_1, e_2, \ldots\}$ over F satisfying the relations $e_i e_j = -e_j e_i$, $i, j = 1, 2, \ldots$. Let also UT_2 denote the algebra of 2×2 upper triangular matrices over F. We have

Remark 11. $M_3 \in \mathrm{var}(UT_2) \cap \mathrm{var}(G)$.

Proof. By [14], $\mathrm{Id}(UT_2) = \langle [x, y][z, w] \rangle_{\mathrm{T}}$ and by [13], $\mathrm{Id}(G) = \langle [x, y, z] \rangle_{\mathrm{T}}$. Hence, since $[x, y][z, w], [x, y, z] \in \mathrm{Id}(M_3)$ it follows that $\mathrm{Id}(UT_2) \cup \mathrm{Id}(G) \subseteq \mathrm{Id}(M_3)$ and, so, $M_3 \in \mathrm{var}(UT_2) \cap \mathrm{var}(G)$. $\square$

We next state a result that will be used throughout the paper.

Theorem 12 [7, Theorem 2]. *Let A be an F-algebra whose codimensions are polynomially bounded. Then there exists an algebra B such that $A \sim_{\mathrm{PI}} B$ and $B = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional algebras such that $\dim B_i / J(B_i) \leqslant 1$ with $J(B_i)$ the Jacobson radical of B_i, $1 \leqslant i \leqslant m$.*

We are now in a position to prove the main result of this section.

Theorem 13. *Let A be an F-algebra. Then $M_1, M_2, M_3 \notin \mathrm{var}(A)$ if and only if $\lim_{n\to\infty} l_n(A)$ exists and is bounded by 1.*

Proof. Suppose first that $\lim_{n\to\infty} l_n(A) \leqslant 1$. Then if $M_i \in \mathrm{var}(A)$ for some $i \in \{1, 2, 3\}$, it would follow that $l_n(M_i) \leqslant l_n(A)$. But by Lemmas 3 and 4 we have that for all $n > 3$, $l_n(M_i) > 1$ and this contradicts the assumption that $\lim_{n\to\infty} l_n(A) \leqslant 1$.

Conversely, suppose that $M_1, M_2, M_3 \notin \mathrm{var}(A)$. By Remark 11, UT_2, $G \notin \mathrm{var}(A)$, hence by [12] the codimensions of A are polynomially bounded, i.e., $c_n(A) \leqslant dn^t$, for some constants d, t. Since the codimensions and the cocharacters of an algebra do not change upon extension of the base field (see for instance [5]) we assume, as we may, that F is algebraically closed. Also by Theorem 12 we may assume that

$$A = A_1 \oplus \cdots \oplus A_m$$

where $A_1, \ldots, A_m$ are finite dimensional algebras such that, $\dim A_i/J(A_i) \leqslant 1$, $1 \leqslant i \leqslant m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent algebra.

If $A_i = J(A_i)$ is nilpotent for all i, then A is a nilpotent algebra and for n large enough $l_n(A) = 0$. Hence we are done in this case.

Therefore we may assume that there exists $i \in \{1, \ldots, m\}$ such that $A_i = F + J(A_i)$ and let $J(A_i) = J_{00} + J_{11} + J_{10} + J_{01}$ where $J_{00}, J_{11}, J_{10}, J_{01}$ are the bimodules defined in Lemma 7. If J_{11} is non-commutative, by Lemma 9, $M_3 \in \mathrm{var}(A_i) \subseteq \mathrm{var}(A)$, a contradiction. Therefore J_{11} must be commutative. Since $M_1, M_2 \notin \mathrm{var}(A)$, by Lemma 10 we have that $J_{10} = J_{01} = 0$. Thus $A_i = (F + J_{11}) \oplus J_{00}$ is a direct sum of algebras and $F + J_{11}$ is a commutative subalgebra of A_i.

We have proved that if for some i, A_i is not nilpotent, then A_i is the direct sum of a commutative algebra and a nilpotent algebra. Recalling that $A = A_1 \oplus \cdots \oplus A_m$ and putting together all pieces, it turns out that we can write

$$A = C \oplus N$$

where C is a commutative non-nilpotent algebra and N is a nilpotent algebra. Since for all n, $l_n(C) = 1$ and for n large enough $l_n(N) = 0$, by Remark 1 we obtain $l_n(A) = l_n(C) = 1$ for n large enough, and the conclusion of the theorem follows. □

Notice that the proof of the previous theorem actually shows that $M_1, M_2, M_3 \notin \mathrm{var}(A)$ if and only if there exists $n_0 > 3$ such that $l_{n_0}(A) \leqslant 1$. Also in this case A is PI-equivalent to either a nilpotent algebra or to the direct sum of a nilpotent algebra and a commutative algebra. Thus $l_n(A)$ has constant value (equal to 0 or 1) for n large. Reading these conclusions in terms of codimensions we get the following corollary.

Corollary 14. *For an F-algebra A, the following conditions are equivalent.*

1. $M_1, M_2, M_3 \notin \mathrm{var}(A)$.
2. *There exists $n_0 > 3$ such that $l_{n_0}(A) \leqslant 1$.*
3. $\lim_{n\to\infty} l_n(A)$ *exists and is bounded by* 1.
4. *Either $A \sim_{\mathrm{PI}} N$ or $A \sim_{\mathrm{PI}} C \oplus N$ where N is a nilpotent algebra and C is a commutative algebra.*
5. *$c_n(A) \leqslant k$ for some constant $k \geqslant 0$, for all $n \geqslant 1$.*

We remark that as a consequence, if the codimensions of an algebra A are bounded, then they are eventually bounded by 1. This result and essentially property 4 of the above corollary were proved by Olsson and Regev in [16].

5. Characterizing varieties of colength $\leqslant 2$

In this section we shall deal with the case $l_n(A) \leqslant 2$. We start with the following.

Lemma 15. *Let $A = F + J$ be an F-algebra.*

1. *If $J_{10}J_{00} \neq 0$, then $M_4 \in \mathrm{var}(A)$.*
2. *If $J_{00}J_{01} \neq 0$, then $M_5 \in \mathrm{var}(A)$.*

Proof. Suppose that $J_{10}J_{00} \neq 0$ and let $k \geqslant 1$ be the largest integer such that $J_{10}J_{00}^k \neq 0$. Then there exist $a \in J_{10}$ and $b \in J_{00}^k$ such that $ab \neq 0$. Let B be the subalgebra of A generated by $1_F, a, b$. Since $ab^2 = ba = a^2 = a1_F = 1_F b = b1_F = 0$, it is easily seen that $[x, y]zw \in \mathrm{Id}(B)$. Hence by Lemma 6, $\mathrm{Id}(M_4) \subseteq \mathrm{Id}(B)$. Let now $f \in \mathrm{Id}(B)$ be a multilinear polynomial of degree n. By Lemma 6 we can write

$$f \equiv \sum_{\substack{i,j=1 \\ i_1<\cdots<i_{n-2}}}^{n} \alpha_{ij} x_{i_1} \cdots x_{i_{n-2}} x_i x_j \quad \bigl(\mathrm{mod}\ \mathrm{Id}(M_4)\bigr)$$

and suppose that there exist $i_0, j_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0 j_0} \neq 0$. Take $c_1, \ldots, c_n \in B$, such that $c_{i_0} = a$, $c_{j_0} = b$ and $c_k = 1_F$, for all $k \notin \{i_0, j_0\}$. It is easy to check that

$$f(c_1, \ldots, c_n) = \alpha_{i_0 j_0} ab \neq 0,$$

a contradiction. Thus $f \in \mathrm{Id}(M_4)$ and $\mathrm{Id}(B) = \mathrm{Id}(M_4)$ implies $M_4 \in \mathrm{var}(A)$.

Property 2 is proved similarly. □

Lemma 16. *Let $A = F + J$ be such that $J_{10} \neq 0$, $J_{01} \neq 0$ and $J_{10}J_{01} = J_{01}J_{10} = 0$. If J_{11} is commutative, $B = F + J_{10} + J_{01} + J_{11}$ is a subalgebra PI-equivalent to $M_1 \oplus M_2$.*

Proof. Under the hypotheses of the lemma, B is a subalgebra of A. Since $[B, B] \subseteq J_{01} + J_{10}$, it is immediate that $z[x, y]w$, $[x, y][z, w]$ are identities of B. Also it can be checked that St_3 vanishes in B. By Lemma 5 this implies that $\mathrm{Id}(M_1 \oplus M_2) \subseteq \mathrm{Id}(B)$. Now, $\mathrm{Id}(B) \subseteq \mathrm{Id}(F + J_{10} + J_{11}) \cap \mathrm{Id}(F + J_{01} + J_{11})$. Since by Lemma 10, $\mathrm{Id}(F + J_{10} + J_{11}) = \mathrm{Id}(M_2)$ and $\mathrm{Id}(F + J_{01} + J_{11}) = \mathrm{Id}(M_1)$ we obtain $\mathrm{Id}(B) \subseteq \mathrm{Id}(M_1) \cap \mathrm{Id}(M_2) = \mathrm{Id}(M_1 \oplus M_2)$. Thus $\mathrm{Id}(B) = \mathrm{Id}(M_1 \oplus M_2)$ and we are done. □

Lemma 17. *Let $A = A_1 \oplus A_2$, where $A_1 = F + J(A_1)$ with $J(A_1)_{10} \neq 0$ and $A_2 = F + J(A_2)$ with $J(A_2)_{01} \neq 0$. Then $M_1 \oplus M_2 \in \mathrm{var}(A)$.*

Proof. It is clear that $B = (F + J(A_1)_{10}) \oplus (F + J(A_2)_{01})$ is a subalgebra of A. Hence

$$\mathrm{Id}(B) = \mathrm{Id}\bigl(F + J(A_1)_{10}\bigr) \cap \mathrm{Id}\bigl(F + J(A_2)_{01}\bigr)$$

and by Lemma 10, $\mathrm{Id}(B) = \mathrm{Id}(M_1) \cap \mathrm{Id}(M_2) = \mathrm{Id}(M_1 \oplus M_2)$ Thus $M_1 \oplus M_2 \in \mathrm{var}(A)$. □

Theorem 18. *Let A be an F-algebra. Then $M_1 \oplus M_2, M_3, M_4, M_5 \notin \mathrm{var}(A)$ if and only if $\lim_{n\to\infty} l_n(A)$ exists and is bounded by* 2.

Proof. Suppose first that there exists an integer $n_0 > 3$ such that $l_{n_0}(A) \leqslant 2$. Then, since for $n \geqslant 4$, $M_1 \oplus M_2, M_3, M_4, M_5$ have colength sequence bounded from below by 3, it is clear that $M_1 \oplus M_2, M_i \notin \mathrm{var}(A)$, for every $i = 3, 4, 5$.

Conversely, suppose that $M_1 \oplus M_2, M_i \notin \mathrm{var}(A)$, for $i = 3, 4, 5$. Since, by Remark 11, $UT_2, G \notin \mathrm{var}(A)$, as in the proof of Theorem 13, we may assume that F is algebraically closed and that $A = A_1 \oplus \cdots \oplus A_m$ with A_i either a nilpotent algebra or A_i isomorphic to $F + J(A_i)$, $i = 1, \ldots, m$.

By eventually reordering the algebras A_i, we may write

$$A = A_1 \oplus \cdots \oplus A_k \oplus A_{k+1} \oplus \cdots \oplus A_m,$$

where $A_i = F + J(A_i)$ for $i = 1, \ldots, k$, and $A_{k+1}, \ldots, A_m$ are nilpotent algebras. Since $A_{k+1} \oplus \cdots \oplus A_m$ is still a nilpotent algebra, by Remark 1

$$l_n(A) = l_n(A_1 \oplus \cdots \oplus A_k)$$

for n large enough. Hence, without lost of generality we may assume that $A = A_1 \oplus \cdots \oplus A_m$ and for each $i = 1, \ldots, m$, $A_i = F + J(A_i)$.

Since $M_1 \oplus M_2 \notin \mathrm{var}(A)$, by Lemma 17, A can be only of one of the following types:

(1) for every $i = 1, \ldots, m$, $A_i = F + J(A_i)$ with $J(A_i)_{01} = 0$;
(2) for every $i = 1, \ldots, m$, $A_i = F + J(A_i)$ with $J(A_i)_{10} = 0$;
(3) there exists i such that $A_i = F + J(A_i)$ with $J(A_i)_{10} \neq 0$, $J(A_i)_{01} \neq 0$ and for every $j \neq i$, $A_j = F + J(A_j)$ with $J(A_j)_{10} = J(A_j)_{01} = 0$.

Notice that since $M_3 \notin \mathrm{var}(A_i) \subseteq \mathrm{var}(A)$, by Lemma 9, we have that $J(A_i)_{11}$ is commutative, for all $i = 1, \ldots, m$.

We start by considering the first case, i.e., $J(A_i)_{01} = 0$, for all i. Write $A_i = F + J$ with $J = J(A_i)$. If $J_{10}J_{00} \neq 0$ then, by Lemma 15 we would get $M_4 \in \mathrm{var}(A_i) \subseteq \mathrm{var}(A)$, a contradiction. Thus $J_{10}J_{00} = 0$ and this says that

$$A_i = (F + J_{10} + J_{11}) \oplus J_{00},$$

a direct sum of algebras. But then by Lemma 10 we have that either

$$A_i \sim_{\mathrm{PI}} M_2 \oplus J_{00} \quad \text{or} \quad A_i \sim_{\mathrm{PI}} C \oplus J_{00},$$

for some commutative algebra C, according as $J_{10} \neq 0$ or $J_{10} = 0$, respectively.

Summing up over all algebras A_i, recalling that $C \in \mathrm{var}(M_2)$, it turns out that either

$$A \sim_{\mathrm{PI}} M_2 \oplus \cdots \oplus M_2 \oplus C \oplus N \sim_{\mathrm{PI}} M_2 \oplus C \oplus N \sim_{\mathrm{PI}} M_2 \oplus N$$

or

$$A \sim_{\mathrm{PI}} M_2 \oplus \cdots \oplus M_2 \oplus N \sim_{\mathrm{PI}} M_2 \oplus N$$

or

$$A \sim_{\mathrm{PI}} C \oplus N$$

where N is a nilpotent algebra and C is a commutative non-nilpotent algebra. Thus for n large, the sequence of colengths of A is constant and $l_n(A)$ equals either 1 or 2 as wished.

It is clear that in case the second possibility occurs, the same proof with the due changes shows that either

$$A \sim_{\mathrm{PI}} M_1 \oplus N$$

or

$$A \sim_{\mathrm{PI}} C \oplus N$$

and $l_n(A)$ takes constant value equal to 1 or 2, for n large.

Next we show that the third possibility cannot occur. Since $M_4, M_5 \notin \mathrm{var}(A_i)$, by Lemma 15, $J_{10}J_{00} = J_{00}J_{01} = 0$. This implies that $J_{10}J_{01} + J_{01}J_{10}$ is a two-sided ideal of A_i. Let $\bar{A}_i$ be the algebra

$$\bar{A}_i = A_i/(J_{10}J_{01} + J_{01}J_{10}).$$

It is easy to check that $\bar{A}_i = F + J(\bar{A}_i)$ and the Jacobson radical $J(\bar{A}_i)$ satisfies

$$J(\bar{A}_i)_{01} J(\bar{A}_i)_{10} = 0 \quad \text{and} \quad J(\bar{A}_i)_{10} J(\bar{A}_i)_{01} = 0.$$

Obviously J_{10}, J_{01} are not contained in $J_{10}J_{01} + J_{01}J_{10}$, therefore $J(\bar{A}_i)_{10}, J(\bar{A}_i)_{01} \neq 0$. Hence by Lemma 16, $M_1 \oplus M_2 \in \mathrm{var}(\bar{A}_i) \subseteq \mathrm{var}(A_i) \subseteq \mathrm{var}(A)$, a contradiction. This completes the proof of the theorem. $\square$

From the proof of Theorem 18 it follows that $M_1 \oplus M_2, M_3, M_4, M_5 \notin \mathrm{var}(A)$ if and only if there exists $n_0 > 3$ such that $l_{n_0}(A) \leqslant 2$. Also in this case A is PI-equivalent to either one of the algebras of Corollary 14 or to $M_1 \oplus N$ or to $M_2 \oplus N$ with N a nilpotent algebra. Thus $l_n(A)$ has constant value (equal to 0 or 1 or 2) for n large. Reading also these conclusions in terms of codimensions we have.

Corollary 19. *For an F-algebra A, the following conditions are equivalent.*

1. $M_1 \oplus M_2, M_3, M_4, M_5 \notin \mathrm{var}(A)$.
2. *There exists $n_0 > 3$ such that $l_{n_0}(A) \leqslant 2$.*
3. $\lim_{n\to\infty} l_n(A)$ *exists and is bounded by* 2.

4. *A is PI-equivalent to one of the algebras N, $C \oplus N$, $M_1 \oplus N$ or $M_2 \oplus N$, where N is a nilpotent algebra and C is a commutative algebra.*
5. *$c_n(A) \leqslant n$ for n large enough.*

In conclusion we have the following classification where N denotes a nilpotent algebra and C is a commutative non-nilpotent algebra: for any algebra A and n large enough,

1. $l_n(A) = 0$ if and only if $A \sim_{\mathrm{PI}} N$.
2. $l_n(A) = 1$ if and only if $A \sim_{\mathrm{PI}} C \oplus N$.
3. $l_n(A) = 2$ if and only if either $A \sim_{\mathrm{PI}} M_1 \oplus N$ or $A \sim_{\mathrm{PI}} M_2 \oplus N$.

6. Algebras with linear codimension growth

In this section we introduce a new algebra denoted M_7 defined as follows:

$$M_7 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in F \right\}.$$

We claim that the sequence of codimensions of M_7 is bounded from below by n^2. In fact, in order to show this, it is enough to find an S_n-character appearing with non-zero multiplicity in $\chi_n(M_7)$, whose degree is $\geqslant n^2$. Consider the partition $\lambda = (n-2, 1, 1)$ and let f_{T_λ} be the highest weight vector corresponding to the standard Young tableau containing the integers $1, 2, 3$ into the first column. Choosing $a_1 = e_{11} + e_{33}$, $a_2 = e_{12}$ and $a_3 = e_{23}$ we get that $f_{T_\lambda} = 2e_{13} \neq 0$. This says that $\chi_{(n-2,1,1)}$ appears with non-zero multiplicity in $\chi_n(M_7)$. Therefore, since by the hook formula, $\deg \chi_{(n-2,1,1)} = (n-1)(n-2)/2$, $c_n(M_7)$ grows asymptotically at least as $n^2/2$.

Lemma 20. *Let $A = F + J$ be an F-algebra. If $J_{10}J_{00} = J_{00}J_{01} = 0$ and $J_{01}J_{10} \neq 0$, then $M_6 \in \mathrm{var}(A)$.*

Proof. From the hypotheses it follows that $J_{10}J_{01}$ is a two-sided ideal of A. Then the quotient algebra $\bar{A} = \frac{A}{J_{10}J_{01}}$ still satisfies the hypotheses and moreover

$$J(\bar{A})_{01} J(\bar{A})_{10} \neq 0 \quad \text{and} \quad J(\bar{A})_{10} J(\bar{A})_{01} = 0.$$

We shall prove that $M_6 \in \mathrm{var}(\bar{A})$, hence we may assume that A itself satisfies $J_{01}J_{10} \neq 0$ and $J_{10}J_{01} = 0$.

Let $a \in J_{01}, b \in J_{10}$ be such that $ab \neq 0$ and let B be the algebra generated by $1_F, a$ and b. Since $ba = aba = 0$ and $1_F a = b 1_F = 0, a 1_F = a$, $1_F b = b$, then $B = \mathrm{span}\{1_F, a, b, ab\}$ and $[B, B] \subseteq \mathrm{span}\{a, b, ab\}$. Thus $B[B, B] \subseteq \mathrm{span}\{b, ab\}$ and $B[B, B]B = 0$. By Lemma 6, $\mathrm{Id}(M_6) \subseteq \mathrm{Id}(B)$.

Conversely, let $f \in \mathrm{Id}(B)$ be a multilinear polynomial of degree n. By Lemma 6, one can write

$$f \equiv \sum_{\substack{i,j=1 \\ i_1<\cdots<i_{n-2}}}^{n} \alpha_{ij} x_i x_{i_1} \cdots x_{i_{n-2}} x_j \pmod{\mathrm{Id}(M_6)}.$$

Suppose that there exist $i_0, j_0 \in \{1, \dots, n\}$ such that $\alpha_{i_0 j_0} \neq 0$. Take $c_1, \dots, c_n \in B$, with $c_{i_0} = a$, $c_{j_0} = b$ and $c_k = 1_F$ for all $k \notin \{i_0, j_0\}$ it is easy to check that

$$f(c_1, \dots, c_n) = \alpha_{i_0 j_0} ab \neq 0,$$

a contradiction. Therefore $f \in \mathrm{Id}(M_6)$ and $\mathrm{Id}(B) = \mathrm{Id}(M_6)$. Hence $M_6 \in \mathrm{var}(A)$. □

Lemma 21. *If $A = F + J$ with $J_{10}J_{01} \neq 0$ and $J_{01}J_{10} = 0$. Then $M_7 \in \mathrm{var}(A)$.*

Proof. Let $a \in J_{10}, b \in J_{01}$ be such that $ab \neq 0$. We claim that the subalgebra B generated by $1_F, a$ and b over F is isomorphic to M_7. This is easily seen since $J_{01}^2 = J_{10}^2 = J_{01}J_{10} = 0$ implies that $a^2 = b^2 = ba = 0$. Hence $B = \mathrm{span}\{1_F, a, b, ab\}$ is isomorphic to M_7 through the map φ such that

$$\varphi(1_F) = e_{11} + e_{33}, \qquad \varphi(a) = e_{12}, \qquad \varphi(b) = e_{23}, \qquad \varphi(ab) = e_{13}. \qquad \square$$

Theorem 22. *Let A be an F-algebra. Then the following conditions are equivalent.*

1. *$c_n(A) \leqslant kn$ for all $n \geqslant 1$, for some constant k.*
2. *$M_3, M_4, M_5, M_6, M_7 \notin \mathrm{var}(A)$.*
3. *A is PI-equivalent to either N or $C \oplus N$ or $M_1 \oplus N$ or $M_2 \oplus N$ or $M_1 \oplus M_2 \oplus N$ where N is a nilpotent algebra and C is a commutative algebra.*

Proof. First suppose that the sequence of codimensions of A is linearly bounded. By Lemmas 4 and 6 and by the claim proved at the beginning of the section, $M_3, M_4, M_5, M_6, M_7 \notin \mathrm{var}(A)$. Suppose now that property 2 holds. Then, since $M_3 \notin \mathrm{var}(A)$, by Remark 11, $UT_2, G \notin \mathrm{var}(A)$ and, as in the proof of the previous theorems, we may assume that F is algebraically closed and

$$A = A_1 \oplus \cdots \oplus A_m$$

where for every $i = 1, \dots, m$, either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent algebra. Suppose that for some i, A_i is not a nilpotent algebra. Since $M_3 \notin \mathrm{var}(A)$, by Lemma 9, J_{11} is commutative. Also, since $M_4, M_5, M_6, M_7 \notin \mathrm{var}(A)$, by Lemmas 15, 20 and 21 we have that

$$J_{10}J_{00} = J_{00}J_{01} = J_{01}J_{10} = J_{10}J_{01} = 0.$$

Under these conditions J_{00} is a two-sided nilpotent ideal of A_i and $A_i = F + J_{01} + J_{10} + J_{11} \oplus J_{00}$. By Lemmas 10 and 16 we obtain that $A_i \sim_{\mathrm{PI}} B$ where B is either $M_1 \oplus N$, or

$M_2 \oplus N$ or $M_1 \oplus M_2 \oplus N$ or $C \oplus N$. The four cases appear according as $J_{01} \neq 0$, $J_{10} = 0$ or $J_{01} = 0$, $J_{10} \neq 0$ or $J_{01} \neq 0$, $J_{10} \neq 0$ or $J_{01} = J_{10} = 0$, $J_{11} \neq 0$ respectively. Since $A = A_1 \oplus \cdots \oplus A_m$ we obtain that property 3 holds.

It is clear that since each of the algebras N, $C \oplus N$, $M_1 \oplus N$, $M_2 \oplus N$, $M_1 \oplus M_2 \oplus N$ has codimensions linearly bounded then property 3 implies property 1 and we are done. $\square$

It is worth noticing that the previous theorem allows us to classify all possible linearly bounded codimension sequences.

Corollary 23. *Let A be an F-algebra such that $c_n(A) \leqslant kn$ for all $n \geqslant 0$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n(A) = 0$ or $c_n(A) = 1$ or $c_n(A) = n$ or $c_n(A) = 2n - 1$.*

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