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On the representation dimension of tilted and laura algebras ☆

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Abstract

We prove that the representation dimension of a tilted, or of a strict laura algebra, is at most three.

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0. Introduction

The chief objective of the representation theory of Artin algebras is to characterise such an algebra by properties of its module category. For this purpose, homological dimensions

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are useful invariants. They are meant to measure how much an algebra or a module deviates from a situation considered to be "nice." Among these invariants is the representation dimension, introduced by Maurice Auslander in the early seventies, see [7]. It measures the least global dimension of all endomorphism rings of those finitely generated modules which are both generators and cogenerators of the module category. The interest in the representation dimension was recently revived by works of Xi [25,26] and also because of its relationship with the finitistic and the Nakayama conjectures: it was shown by Igusa and Todorov in [18] that if the representation dimension of an algebra is at most 3, then its finitistic dimension is finite. It was already proven by Auslander in [7] that an Artin algebra A is representation-finite if and only if its representation dimension rep.dim. A is at most two and, also, that if A is either hereditary or has radical square zero, then rep.dim. $A \leqslant 3$. Many important classes of algebras have been shown to have representation dimension at most 3, see, for instance, [14,15]. It was shown by Iyama that, for any Artin algebra A, rep.dim. $A < \infty$, see [19], and Rouquier has constructed examples of algebras with representation dimension larger than or equal to 4, see [23].

In this paper we consider two rather large classes of algebras which play an important role in representation theory, namely, the tilted algebras (see, for instance, [1,21]) and the laura algebras (see, for instance, [2,3,24]). We prove the following theorem, which generalises [14, (2.2), (2.3)].

Theorem. *Let A be a tilted, or a strict laura algebra. Then* rep.dim. $A \leqslant 3$.

As a direct consequence, the weak representation dimension of any laura algebra is at most three, and hence the finitistic dimension conjecture holds for laura algebras (which, we recall, may have infinite global dimension and even infinitely many isomorphism classes of indecomposables with infinite projective dimension). We conjecture that, if A is quasi-tilted, in the sense of [16], then rep.dim. $A \leqslant 3$. We do not prove here this conjecture, but we show that the representation dimension of a quasi-tilted algebra is at most 4.

The paper is organised as follows. After a short preliminary section we prove in Section 2 that the representation dimension of a tilted algebra is at most 3. Sections 3 and 4 are respectively devoted to the cases of quasi-tilted, and strict laura algebras.

1. Representation dimension of Artin algebras

1.1. Notation

Throughout this paper, all algebras are connected Artin algebras and all modules are finitely generated right modules. For an Artin algebra A, we denote by $\operatorname{mod} A$ the category of A-modules and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ containing exactly one representative of each isomorphism class of indecomposable A-modules. We denote by gl.dim. A the global dimension of A and by D the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$.

If $\mathcal{C}$ is a subcategory of $\operatorname{mod} A$, we sometimes write $X \in \mathcal{C}$ to express that X is an object of $\mathcal{C}$. We denote by $\operatorname{add} \mathcal{C}$ the full subcategory having as objects the direct sums of indecomposable summands of objects in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$

as $\operatorname{add} M$. We denote by $\operatorname{Gen} M$ (or $\operatorname{Cogen} M$) the full subcategory having as objects those modules X such that there is an epimorphism $M_0 \to X$ (or a monomorphism $X \to M_0$, respectively), with $M_0 \in \operatorname{add} M$. Finally, we denote the projective (or injective) dimension of a module X by $\operatorname{pd} X$ (or $\operatorname{id} X$, respectively).

Unexplained notions and facts needed on $\operatorname{mod} A$ can be found in [8,21].

1.2. Representation dimension

We refer the reader to [7] for the original definition. We shall rather use the following characterisation, used in [7].

Definition. Let A be a nonsemisimple Artin algebra. The representation dimension $\operatorname{rep.dim.} A$ of A is the infimum of the global dimensions of the algebras $\operatorname{End} M$, where M is a generator and a cogenerator of $\operatorname{mod} A$.

The following language is useful when dealing with representation dimension. Given an A-module M, a functor F from $(\operatorname{add} M)^{\mathrm{op}}$ to the category $\mathcal{A}b$ of abelian groups is called *finitely presented* (or *coherent*) if there exists a morphism $f: M_1 \to M_0$ in add M inducing an exact sequence of abelian groups

$$\operatorname{Hom}_A(M, M_1) \xrightarrow{\operatorname{Hom}_A(M,f)} \operatorname{Hom}_A(M, M_0) \to F(M) \to 0.$$

We denote by $\mathcal{F}_M$ the category of all finitely presented functors from $(\operatorname{add} M)^{\mathrm{op}}$ to $\mathcal{A}b$. Thus, a functor $F: (\operatorname{add} M)^{\mathrm{op}} \to \mathcal{A}b$ is finitely presented if and only if there exists a morphism $f: M_1 \to M_0$ inducing an exact sequence of functors

$$\operatorname{Hom}_A(-, M_1) \xrightarrow{\operatorname{Hom}_A(-,f)} \operatorname{Hom}_A(-, M_0) \to F \to 0$$

from $(\operatorname{add} M)^{\mathrm{op}}$ to $\mathcal{A}b$. It was shown in [7] that the categories $\mathcal{F}_M$ and $\operatorname{mod}(\operatorname{End} M)$ are equivalent. The next lemma is well known [7,14,15,26].

Lemma. *Let A be an Artin algebra, n be a positive integer, and M be a generator–cogenerator of* $\operatorname{mod} A$. *Then* $\operatorname{gl.dim.End} M \leqslant n+1$ *if and only if for each A-module X, there exists an exact sequence*

$$0 \to M_n \to \cdots \to M_1 \to X \to 0$$

with M_i in $\operatorname{add} M$ *for all i, such that the induced sequence of functors*

$$0 \to \operatorname{Hom}_A(-, M_n) \to \cdots \to \operatorname{Hom}_A(-, M_1) \to \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$. In particular, $\operatorname{rep.dim.} A \leqslant n+1$.

The above considerations may equivalently be expressed in the language of relative homological algebra, as developed by Auslander and Solberg in [10]: indeed, the lemma above says exactly that, for each module X, there exists an exact sequence

$$0 \to M_n \to \cdots \to M_1 \to X \to 0$$

(with M_i in $\operatorname{add} M$ for all i), which is $\operatorname{add} M$-exact.

1.3. The following lemma is also well known and follows from the fact that, for any (finitely generated) module M over an Artin algebra A, any A-module X admits an $\operatorname{add} M$-approximation. We include the proof because it is useful for our future considerations.

Lemma. *Let A be an Artin algebra and M be any A-module. Then, for any A-module X, the functor* $\operatorname{Hom}_A(-, X) : (\operatorname{add} M)^{\mathrm{op}} \to \mathcal{A}b$ *is finitely presented.*

Proof. Let $\{g_1, \ldots, g_d\}$ be a set of generators of the $\operatorname{End} M$-module $\operatorname{Hom}_A(M, X)$. The morphism $g_0 = [g_1, \ldots, g_d]$ from $M_0 = M^d$ to X has the property that the induced sequence

$$\operatorname{Hom}_A(-, M_0) \xrightarrow{\operatorname{Hom}_A(-, g_0)} \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$. Considering the kernel of g_0 yields similarly a module M_1 in $\operatorname{add} M$, and a morphism $g_1 : M_1 \to M_0$ such that the sequence

$$\operatorname{Hom}_A(-, M_1) \xrightarrow{\operatorname{Hom}_A(-, g_1)} \operatorname{Hom}_A(-, M_0) \xrightarrow{\operatorname{Hom}_A(-, g_0)} \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$. $\square$

We note that the displayed projective presentation of $\operatorname{Hom}_A(-, X)$ is usually not induced by an exact sequence

$$M_1 \to M_0 \to X \to 0.$$

This is however clearly the case when both X and $\operatorname{Ker}(g_0)$ are generated by M. In Section 2 we give conditions for this to be the case.

1.4. We have considered a projective presentation for the functor $\operatorname{Hom}_A(-, X)$. We now look at a projective cover.

Lemma. *Let A be an Artin algebra, and M be any A-module. If $X \in \operatorname{Gen} M$, then there exists an epimorphism $f_0 : M_0 \to X$, with $M_0 \in \operatorname{add} M$, and such that*

$$\operatorname{Hom}_A(-, M_0) \xrightarrow{\operatorname{Hom}_A(-, f_0)} \operatorname{Hom}_A(-, X) \to 0$$

is a projective cover in $\mathcal{F}_M$.

Proof. Since $X \in \operatorname{Gen} M$, there exists, by the discussion in 1.3, an epimorphism $f_1 : M_1 \to X$, with $M_1 \in \operatorname{add} M$ such that

$$\operatorname{Hom}_A(-, M_1) \to \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$. Since $\mathcal{F}_M$ is equivalent to $\operatorname{mod}(\operatorname{End} M)$, we have a projective cover

$$\operatorname{Hom}_A(-, M_0) \xrightarrow{\pi} \operatorname{Hom}_A(-, X) \to 0$$

in $\mathcal{F}_M$, with $M_0 \in \operatorname{add} M$. We now claim that there exists a morphism $f_0 : M_0 \to X$ such that $\pi = \operatorname{Hom}_A(-, f_0)$. The projectivity of $\operatorname{Hom}_A(-, M_0)$ in $\mathcal{F}_M$ yields a morphism $\sigma : \operatorname{Hom}_A(-, M_0) \to \operatorname{Hom}_A(-, M_1)$ such that $\pi = \operatorname{Hom}_A(-, f_1)\sigma$. Since $M_0, M_1 \in \operatorname{add} M$, Yoneda's lemma gives a morphism $h : M_0 \to M_1$ such that $\sigma = \operatorname{Hom}_A(-, h)$. Hence $\pi = \operatorname{Hom}_A(-, f_1)\operatorname{Hom}_A(-, h) = \operatorname{Hom}_A(-, f_1 h)$ and setting $f_0 = f_1 h$ establishes our claim.

There remains to show that f_0 is surjective. Since $M_1 \in \operatorname{add} M$, the morphism f_0 induces an exact sequence

$$\operatorname{Hom}_A(M_1, M_0) \xrightarrow{\operatorname{Hom}_A(M_1, f_0)} \operatorname{Hom}_A(M_1, X) \to 0$$

in $\mathcal{A}b$. Thus, we find $g : M_1 \to M_0$ such that $f_0 g = f_1$. Since f_1 is surjective, so is f_0. $\square$

1.5. We leave to the reader the straightforward proof of the following lemma.

Lemma. *Let A be an Artin algebra and $f_0 : P_0 \to X$ be a projective cover in* mod A. *If we have a commutative diagram*

$$\begin{tikzcd} P_0 \arrow[r, "f_0"] \arrow[d, "h"'] & X \\ P \arrow[ur, "f"'] & \end{tikzcd}$$

with P projective, then h is a section.

2. Tilting and tilted algebras

2.1. Let A be an Artin algebra. An A-module T is a *tilting module* if $\operatorname{pd} T_A \leqslant 1$, $\operatorname{Ext}_A^1(T, T) = 0$ and there exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$, with $T', T'' \in \operatorname{add} T$. It is well known that any tilting module T_A induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$, where $\mathcal{T}(T) = \operatorname{Gen} T = \{X_A \mid \operatorname{Ext}_A^1(T, X) = 0\}$ and $\mathcal{F}(T) = \{X_A \mid \operatorname{Hom}_A(T, X) = 0\}$. Thus, in particular, $DA \in \mathcal{T}(T)$.

The endomorphism algebra of a tilting module over a hereditary algebra is said to be *tilted*.

We introduce some further terminology. Let A be an Artin algebra, a *path* in $\operatorname{ind} A$ from X to Y is a sequence of nonzero morphisms

$$X = X_0 \to X_1 \to \cdots \to X_n = Y \tag{$*$}$$

with all the X_i indecomposable. A set Σ of indecomposable modules is *convex* if for any $X, Y \in \Sigma$ and any path $(*)$ from X to Y in $\operatorname{ind} A$, all the X_i lie in Σ. A tilting module T is *convex* provided the set $\Sigma_T = \operatorname{ind} A \cap \operatorname{add} T$ of all indecomposable summands of T is convex.

A class of pairwise nonisomorphic indecomposable A-modules is called a *complete slice* in $\operatorname{mod} A$ (see [21,22]) if it satisfies the following conditions:

(1) $U = \bigoplus_{M \in \Sigma} M$ is a sincere module (that is, $\operatorname{Hom}_A(P, U) \neq 0$ for every projective A-module P).
(2) Σ is convex.
(3) If $0 \to L \to M \to N \to 0$ is an almost split sequence, then at most one of L and N lies on Σ. Furthermore, if an indecomposable summand of M lies in Σ, then either L or N lie in Σ.

It is shown in [21] (see also [1, (5.2)]) that an algebra A is tilted if and only if $\operatorname{mod} A$ contains a complete slice. Furthermore, in this case $U = \bigoplus_{M \in \Sigma} M$ is a tilting module with $\operatorname{End} U$ hereditary.

Lemma. *An Artin algebra A is tilted if and only if there exists a convex tilting A-module T. In this case, $(\mathcal{T}(T), \mathcal{F}(T))$ is a split torsion pair and Σ_T is a complete slice.*

Proof. Assume that A is tilted, and let T be the direct sum of the indecomposables lying on a complete slice. Then T is convex and $\operatorname{End} T$ hereditary. Therefore, by [1, (3.6)], T is separating hence $(\mathcal{T}(T), \mathcal{F}(T))$ splits.

Since the necessity follows from the well-known properties of complete slices, we prove the sufficiency using Bakke's theorem (see [11] or [1, (5.3)]). Let T be a convex tilting A-module. We define a torsion pair $(\mathcal{T}, \mathcal{F})$ as follows: Let $\mathcal{T}$ be the full additive subcategory of $\operatorname{mod} A$ having as indecomposable objects the modules X such that there is a path $T' \to \cdots \to X$, with $T' \in \Sigma_T$, and let $\mathcal{F}$ be the full additive subcategory generated by the remaining indecomposables. Then $(\mathcal{T}, \mathcal{F})$ is a split torsion pair. It is shown in [11], [1, (5.3)] that, if U denotes the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-projectives in $\mathcal{T}$ (in the sense of Auslander and Smalø [9]), then U is a tilting module and $\mathcal{T} = \mathcal{T}(U)$, $\mathcal{F} = \mathcal{F}(U)$. Moreover, $\operatorname{End} U$ is hereditary (so that A is tilted).

In order to complete the proof, it suffices to show that $U = T$. For this purpose, we claim that T is Ext-projective in $\mathcal{T}$: indeed, assume there exist $T' \in \Sigma_T$, an indecomposable module X' in $\mathcal{T}$ and a nonsplit short exact sequence

$$0 \to X' \to E \to T' \to 0.$$

Then there exist an indecomposable summand E' of E and a path $X' \to E' \to T'$ in $\operatorname{ind} A$. On the other hand, $X' \in \mathcal{T}$, so there exist $T'' \in \Sigma_T$ and a path $T'' \to \cdots \to X'$ in $\operatorname{ind} A$. Considering the composed path $T'' \to \cdots \to X' \to E' \to T'$ and applying convexity yields $X' \in \Sigma_T$. Therefore, $X' \in \operatorname{add} T$ and the given short exact sequence splits, a contradiction which establishes our claim.

By [1, (1.8)] we get an A-module V such that $U = T \oplus V$. However, T itself is a tilting module. The definition of U and Bongartz' lemma [1, (2.6)] imply $U = T$. The proof is now complete. □

2.2. Let A be an Artin algebra, and M be an A-module. It follows from 1.3 and 1.4 that, for $X \in \operatorname{Gen} M$, there exists a short exact sequence

$$0 \to K \to M_0 \xrightarrow{f_0} X \to 0$$

such that $\operatorname{Hom}_A(-, f_0) : \operatorname{Hom}_A(-, M_0) \to \operatorname{Hom}_A(-, X)$ is a projective cover in $\mathcal{F}_M$. We call such a sequence an $\operatorname{add} M$*-approximating sequence for* X. In the following technical proposition we collect some properties of approximating sequences.

Proposition. *Let A be an Artin algebra, $M = T \oplus N$ be an A-module, $X \in \operatorname{Gen} M$ and $0 \to K \to M_0 \to X \to 0$ be an $\operatorname{add} M$-approximating sequence for X.*

(a) *If $\operatorname{Ext}^1_A(T, M) = 0$, then $\operatorname{Ext}^1_A(T, K) = 0$.*
(b) *If T_A is a tilting module and $N \in \mathcal{T}(T)$, then $K \in \mathcal{T}(T)$.*
(c) *If $N = 0$ and $M = T$ is a tilting module then, for every indecomposable summand K' of K, we have $\operatorname{Hom}_A(K', T) \neq 0$.*
(d) *If $N = 0$ and $M = T$ is a convex tilting module, then $K \in \operatorname{add} M$.*
(e) *If $N = DA$ and K' is an indecomposable summand of K such that $\operatorname{id} K' \leqslant 1$, then $\operatorname{Hom}_A(K', T) \neq 0$.*
(f) *If $N = DA$ and T is a convex tilting module, then $K \in \operatorname{add} T$.*

Proof. (a) By hypothesis, the given approximating sequence is of the form

$$0 \to K \to T_0 \oplus N_0 \to X \to 0$$

with $T_0 \in \operatorname{add} T$, $N_0 \in \operatorname{add} N$. Applying $\operatorname{Hom}_A(T, -)$ yields an exact sequence

$$0 \to \operatorname{Ext}^1_A(T, K) \to \operatorname{Ext}^1_A(T, T_0 \oplus N_0).$$

Since the assumption implies that $\operatorname{Ext}^1_A(T, T_0 \oplus N_0) = 0$, the statement follows.

(b) Clearly, $N \in \mathcal{T}(T)$ implies $M \in \mathcal{T}(T)$ and so $\mathcal{T}(T) = \operatorname{Gen} M$. Since T is a tilting module, it follows from (a) that $\operatorname{Ext}^1_A(T, K) = 0$. Hence $K \in \mathcal{T}(T)$.

(c) This is trivial.

(d) Let K' be any indecomposable summand of K. Since, by (b), $K \in \mathcal{T}(T)$, we have $\operatorname{Hom}_A(T, K') \neq 0$. Also, by (c), $\operatorname{Hom}_A(K', T) \neq 0$. Convexity yields $K' \in \operatorname{add} T$. Thus $K \in \operatorname{add} T$.

(e) We may write the given approximating sequence in the form

$$0 \to K \xrightarrow{\left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right]} T_0 \oplus I_0 \xrightarrow{[f_1 f_2]} X \to 0$$

with $T_0 \in \operatorname{add} T$ and I_0 injective. Assume K' is an indecomposable summand of K such that $\operatorname{id} K' \leqslant 1$. Suppose also that $\operatorname{Hom}_A(K', T) = 0$. Thus, in particular, $g_1(K') = 0$. Hence we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r, "g_2|K'"] \arrow[d, "i"] & I_0 \arrow[r, "f'"] \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] & I' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & K \arrow[r, "\left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right]"] & T_0 \oplus I_0 \arrow[r, "{[f_1 f_2]}"] & X \arrow[r] & 0 \end{tikzcd} \tag{$*$}$$

where $i : K' \to K$ is the inclusion map and h is induced by passing to cokernels. Since $\operatorname{id} K' \leqslant 1$, the module I' is injective. Since $f_2 = hf'$, we have a commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd} & T_0 \oplus I' \arrow[dr, "{[f_1 h]}"] & \\ T_0 \oplus I_0 \arrow[ur, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right]"] \arrow[rr, "{[f_1 f_2]}"] & & X \end{tikzcd}$$

which induces a commutative diagram in $\mathcal{F}_M$,

$$\begin{tikzcd} & \operatorname{Hom}_A(-, T_0 \oplus I') \arrow[dr, "{\operatorname{Hom}_A(-,[f_1 h])}"] & \\ \operatorname{Hom}_A(-, T_0 \oplus I_0) \arrow[ur, "{\operatorname{Hom}_A(-,\left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right])}"] \arrow[rr, "{\operatorname{Hom}_A(-,[f_1 f_2])}"] & & \operatorname{Hom}_A(-, X) \end{tikzcd}$$

Since I' is injective, $T_0 \oplus I' \in \operatorname{add} M$, so that $\operatorname{Hom}_A(-, T_0 \oplus I')$ is projective. By 1.5, $\operatorname{Hom}_A(-, \left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right])$ is a section. Since $T_0 \oplus I_0$, $T_0 \oplus I' \in \operatorname{add} M$, then $\left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right]$ is a section. In particular, f' is injective. But this implies $K' = 0$, an absurdity.

(f) Let K' be any indecomposable summand of K. By (b), the module K' is in $\mathcal{T}(T)$, so that $\operatorname{Hom}(T, K')$ is nonzero, and $\operatorname{id} K'$ is less than or equal to one. Thus, by (e), $\operatorname{Hom}_A(K', T) \neq 0$. Convexity yields $K' \in \operatorname{add} T$. Hence $K \in \operatorname{add} T$. $\square$

2.3. We are able to prove our first main theorem.

Theorem. *Let A be a tilted algebra, Σ be a complete slice in* $\operatorname{mod} A$, $T = \bigoplus_{U \in \Sigma} U$ *and* $M = A \oplus T \oplus DA$. *Then* $\operatorname{gl.dim.End}(M) \leqslant 3$. *In particular,* $\operatorname{rep.dim.} A \leqslant 3$.

Proof. By 1.2 it suffices to find, for each indecomposable A-module X, a short exact sequence

$$0 \to M_1 \to M_0 \to X \to 0$$

with $M_0, M_1 \in \operatorname{add} M$, such that the induced sequence

$$0 \to \operatorname{Hom}_A(-, M_1) \to \operatorname{Hom}_A(-, M_0) \to \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$.

Assume first that $X \in \mathcal{F}(T)$. Then $\operatorname{pd} X \leqslant 1$. Let $0 \to P_1 \to P_0 \xrightarrow{f_0} X \to 0$ be a projective resolution of X. Since $T \oplus DA \in \mathcal{T}(T)$ and $X \in \mathcal{F}(T)$, we have $\operatorname{Hom}_A(T \oplus DA, X) = 0$. Therefore $\operatorname{Hom}_A(M, X) = \operatorname{Hom}_A(A, X)$ and

$$\operatorname{Hom}_A(-, P_0) \xrightarrow{\operatorname{Hom}_A(-, f_0)} \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$.

Let now $X \in \mathcal{T}(T)$. Since $X \in \mathcal{T}(T) = \operatorname{Gen}(T) = \operatorname{Gen}(T \oplus DA)$, there exists, by 1.4, an $\operatorname{add}(T \oplus DA)$-approximation of X

$$0 \to K \to T_0 \oplus I_0 \to X \to 0$$

with $T_0 \in \operatorname{add}(T)$ and I_0 injective. Since, by 2.1, T is a convex tilting module, it follows from 2.2(f) that $K \in \operatorname{add} M$. Since $\operatorname{Hom}_A(-, f_0)$ is a projective cover in $\mathcal{F}_{T \oplus DA}$, invoking 1.2 concludes the proof. $\square$

3. Quasi-tilted algebras

3.1. We refer to [16] for the original definition of quasi-tilted algebras. We use the following equivalent one: an Artin algebra A is *quasi-tilted* if $\operatorname{gl.dim.} A \leqslant 2$ and, for every $X \in \operatorname{ind} A$ we have $\operatorname{pd} X \leqslant 1$ or $\operatorname{id} X \leqslant 1$, see [16]. Another characterisation is useful: let $\mathcal{L}_A$ (or $\mathcal{R}_A$) be the full subcategory of $\operatorname{ind} A$ having as objects all the modules X such that, whenever there exists a path $Y \to \cdots \to X$ (or a path $X \to \cdots \to Y$) in $\operatorname{ind} A$, then $\operatorname{pd} Y \leqslant 1$ (or $\operatorname{id} Y \leqslant 1$, respectively). Then A is quasi-tilted if and only if $A_A \in \operatorname{add} \mathcal{L}_A$, or if and only if $DA_A \in \operatorname{add} \mathcal{R}_A$ (see [16, (II.1.4)]). Moreover, $\mathcal{L}_A \cup \mathcal{R}_A = \operatorname{ind} A$, see [16, (II.1.13)]. We conjecture that, if A is quasi-tilted, then $\operatorname{rep.dim.} A \leqslant 3$. A first step in this direction is the following proposition.

Proposition. *Let A be a quasi-tilted algebra which is not tilted, and let $M = A \oplus DA$. Then* $\operatorname{gl.dim.End}_A(M) \leqslant 4$. *In particular,* $\operatorname{rep.dim.} A \leqslant 4$.

Proof. It suffices to show that, for any indecomposable module X, we have $\operatorname{pd} \operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 2$.

Assume first $X \in \mathcal{L}_A$. Then $\operatorname{pd} X \leqslant 1$. Let $0 \to P_1 \to P_0 \xrightarrow{f_0} X \to 0$ be a projective resolution. Suppose $\operatorname{Hom}_A(DA, X) \neq 0$. Since $X \in \mathcal{L}_A$ and $\mathcal{L}_A$ is closed under predecessors, there exists an injective in $\mathcal{L}_A$. But then A is tilted, by [16, (II.3.4)]. Hence $\operatorname{Hom}_A(DA, X) = 0$, from which we deduce that $\operatorname{Hom}_A(-, P_0) \xrightarrow{\operatorname{Hom}_A(-, f_0)} \operatorname{Hom}_A(-, X) \to 0$ is exact in $\mathcal{F}_M$. Therefore $\operatorname{pd} \operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 1$.

If now $X \notin \mathcal{L}_A$, then $X \in \mathcal{R}_A$. Consider an add M-approximating sequence

$$0 \to K \to P_0 \oplus I_0 \to X \to 0$$

with P_0 projective and I_0 injective. If $K \in \operatorname{add} \mathcal{L}_A$ then, by the first case considered above, we have $\operatorname{pd} \operatorname{Hom}_A(M, K)_{\operatorname{End}_A(M)} \leqslant 1$. Therefore $\operatorname{pd} \operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 2$ and we have finished. Assume thus that K has an indecomposable summand K' lying in $\mathcal{R}_A \setminus \mathcal{L}_A$. Since $P_0 \in \operatorname{add} \mathcal{L}_A$ and $\mathcal{L}_A$ is closed under predecessors, we have $\operatorname{Hom}_A(K', P_0) = 0$. But then 2.2(e) yields $\operatorname{id} K' \geqslant 2$, a contradiction which completes the proof. $\square$

3.2. We notice that, if A is quasi-tilted but not tilted, then $\operatorname{Hom}_A(DA, A) = 0$, hence $\operatorname{End}_A(A \oplus DA) \simeq \left(\begin{smallmatrix} A & 0 \\ DA & A \end{smallmatrix}\right)$, where the algebra structure is induced from the bimodule structure of DA. This is a (finite dimensional) quotient of the repetitive algebra of A, known as the *duplicated algebra* $\bar{A}$ of A (see, for instance, [5,6,17]). It is shown in [5, (1.1)] that, for any Artin algebra A, we have

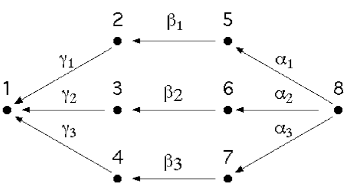
$$\text{gl.dim. } A + 1 \leqslant \text{gl.dim. } \bar{A} \leqslant 2 \, \text{gl.dim. } A + 1.$$

Thus, if A is quasi-tilted but not hereditary, then

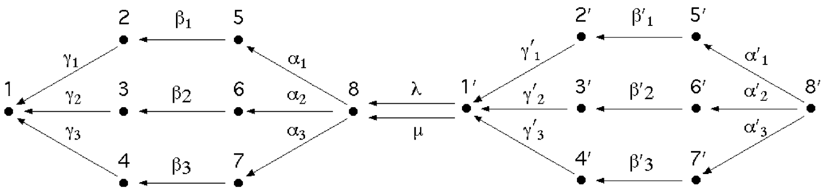
$$3 \leqslant \text{gl.dim. } \bar{A} \leqslant 5.$$

The preceding proposition improves the upper bound of the preceding inequality, thus answering the question in [6, (5.2)]. We give an example of a (tame) quasi-tilted algebra A which is not tilted and such that gl.dim. $\bar{A} = 4$.

Example. Let k be a field, and A be the finite-dimensional k-algebra given by the quiver



bound by the relations $\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 = 0$. This is a tubular algebra (see [21, p. 268]). In this case the quiver of $\bar{A}$, constructed as shown in [17, (2.4)] is



and

$$\bar{A}_{\bar{A}} = 1 \oplus \begin{matrix}2\\1\end{matrix} \oplus \begin{matrix}3\\1\end{matrix} \oplus \begin{matrix}4\\1\end{matrix} \oplus \begin{matrix}5\\2\\1\end{matrix} \oplus \begin{matrix}6\\3\\1\end{matrix} \oplus \begin{matrix}7\\4\\1\end{matrix} \oplus \begin{matrix} & 8 & \\ 5 & 6 & 7\\ 2 & 3 & 4\\ 1 & & 1\end{matrix} \oplus \begin{matrix} & 1' & \\ 8 & & 8\\ 5 & 6 & 7\\ 2 & 3 & 4\\ & 1 & \end{matrix} \oplus$$

$$\oplus \begin{matrix}2'\\1'\\8\\5\\2\end{matrix} \oplus \begin{matrix}3'\\1'\\8\\6\\3\end{matrix} \oplus \begin{matrix}4'\\1'\\8\\7\\4\end{matrix} \oplus \begin{matrix}5'\\2'\\1\\8\\5\end{matrix} \oplus \begin{matrix}6'\\3'\\1'\\8\\6\end{matrix} \oplus \begin{matrix}7'\\4'\\1'\\8\\7\end{matrix} \oplus \begin{matrix} & 8' & \\ 5' & 6' & 7'\\ 2' & 3' & 4'\\ 1' & & 1'\\ & 8 & \end{matrix}$$

(where indecomposable projectives are represented by their Loewy series). It is easy to see that, if S is the simple $\bar{A}$-module corresponding to the point $8'$, then $\operatorname{pd} S_{\bar{A}} = 4$. Since by 3.1, gl.dim. $\bar{A} \leqslant 4$, we infer that gl.dim. $\bar{A} = 4$.

4. Laura algebras

4.1. An Artin algebra is a *laura algebra* if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$, and it is a *strict laura algebra* if it is laura but not quasi-tilted. We refer to [2–4,20,24] for properties of laura algebras. We recall that, if A is a strict laura algebra, then it is *left* and *right supported* [3, (4.4)]. In other words, if E (or F) denotes the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-injectives in $\operatorname{add} \mathcal{L}_A$ (or Ext-projectives in $\operatorname{add} \mathcal{R}_A$, respectively), then $\operatorname{add} \mathcal{L}_A = \operatorname{Cogen} E$ and $\operatorname{add} \mathcal{R}_A = \operatorname{Gen} F$. Moreover, if A_λ is the endomorphism algebra of the direct sum of all indecomposable projectives in $\mathcal{L}_A$, then A_λ is the direct product of tilted algebras, and the restriction of E to each of the directed components of A_λ is a convex tilting module. One defines dually A_ρ, which is also a direct product of tilted algebras, and the restriction of F to each of the connected components of A_ρ is a convex tilting module [3, (4.2), (5.1)].

Here, we let A be a strict laura algebra, and we let N be the direct sum of all indecomposable A-modules not lying in $\mathcal{L}_A \cup \mathcal{R}_A$ (this sum is finite, because A is laura).

We may now prove the main result of this section.

Theorem. *Let A be a strict laura algebra, and let $M = A \oplus E \oplus N \oplus F \oplus DA$, with E, F, N as above. Then* $\mathrm{gl.dim.End}_A(M) \leqslant 3$. *In particular,* $\text{rep.dim.}\, A \leqslant 3$.

Proof. As in the proof of 2.3 it suffices to show that, for any indecomposable A-module X, we have $\operatorname{pd}\operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 1$.

Suppose first that $X \in \mathcal{L}_A \setminus \mathcal{R}_A$. Clearly, we may assume that $X \notin \operatorname{add} M$ and consider a projective resolution $0 \to P_1 \to P_0 \to X \to 0$. If $\operatorname{Hom}_A(DA, X) \neq 0$, then $X \in \operatorname{add} E$, by [3, (3.1)], so $X \in \operatorname{add} M$, a contradiction. Therefore $\operatorname{Hom}_A(DA, X) = 0$. Moreover, $\operatorname{Hom}_A(N, X) = 0$, since $N \in \operatorname{add}(\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A))$, with $X \in \mathcal{L}_A$, and $\mathcal{L}_A$ is closed under predecessors. On the other hand, $\operatorname{Hom}_A(F, X) \neq 0$ implies $X \in \mathcal{R}_A$, contradicting our assumption. This shows that $\operatorname{Hom}_A(F, X) = 0$. Thus the sequence

$$0 \to \operatorname{Hom}_A(-, P_1) \to \operatorname{Hom}_A(-, P_0) \to \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$, and so $\operatorname{pd}\operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 1$.

If $X \in \operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$, then $X \in \operatorname{add} N \subseteq \operatorname{add} M$ and there is nothing to show.

Finally, let $X \in \mathcal{R}_A$. Then X is an A_ρ-module. Moreover, X is generated by F. Therefore, by 1.4, there exists an $\operatorname{add}(F \oplus DA_\rho)$-approximating sequence

$$0 \to F_1 \to F_0 \oplus I_0 \xrightarrow{f_0} X \to 0$$

with $F_0 \in \operatorname{add} F$ and I_0 injective, so that

$$\operatorname{Hom}_A(-, f_0) : \operatorname{Hom}_A(-, F_0 \oplus I_0) \to \operatorname{Hom}_A(-, X)$$

is a projective cover in $\mathcal{F}_{F \oplus DA_\rho}$. Since F is a convex tilting A_ρ-module, it follows from 2.2(f) that $F_1 \in \operatorname{add} F$. Moreover, F is a slice module in $\operatorname{mod} A_\rho$ and $\operatorname{add}\mathcal{R}_A = \operatorname{Gen} F$, so that any morphism from a module in $\operatorname{ind} A \setminus \mathcal{R}_A$ to X factors through F, and hence through $F_0 \oplus I_0$. Therefore the sequence

$$0 \to \operatorname{Hom}_A(-, F_1) \to \operatorname{Hom}_A(-, F_0 \oplus I_0) \xrightarrow{\operatorname{Hom}_A(-, f_0)} \operatorname{Hom}_A(-, X) \to 0$$

is exact in $\mathcal{F}_M$, and so $\operatorname{pd}\operatorname{Hom}_A(M, X)_{\operatorname{End}_A(M)} \leqslant 1$. The proof is now complete. $\square$

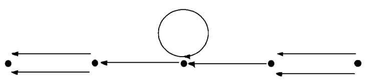
As a direct consequence of this theorem, if A is a strict shod algebra [12], or a strict weakly shod algebra [13], then $\text{rep.dim.}\, A \leqslant 3$.

4.2. We recall that the *weak representation dimension* $\text{w.rep.dim.}\, A$ of an Artin algebra A is the infimum of the global dimensions of the endomorphism algebras of the generators of $\operatorname{mod} A$. Clearly, $\text{w.rep.dim.}\, A \leqslant \text{rep.dim.}\, A$ and also $\text{w.rep.dim.}\, A \leqslant \text{gl.dim.}\, A$. Thus, the next corollary follows immediately from our theorem in 4.1 and the fact that quasi-tilted algebras have global dimension at most 2.

Corollary. *Let A be a laura algebra, then* $\text{w.rep.dim.}\, A \leqslant 3$.

Remark. In the case where A is a strict laura algebra, we can be more precise: let $M = A \oplus N \oplus E \oplus F$, where E, F and N are as above. Then gl.dim.End M is at most 3. Indeed, we may repeat in this case the proof of 4.1, since the existence of an add F-approximating sequence $0 \to F_1 \to F_0 \to X \to 0$ for $X \in \mathcal{R}_A$, with $F_0, F_1 \in \operatorname{add} F$, is granted by 1.4 and 2.2(d).

4.3. We recall that the global dimension of a laura algebra may be infinite and, even, such an algebra may have infinitely many isomorphism classes of indecomposable modules with infinite projective dimension, as is shown by the following example of [2, (2.3)]. Let k be a field, and A be the radical square zero k-algebra given by the quiver



It was shown in [18] that, if an Artin algebra A verifies rep.dim. $A \leqslant 3$ (or even w.rep.dim. $A \leqslant 3$) then its finitistic dimension fin.dim. A is finite. We thus obtain the following corollary.

Corollary. *Let A be a laura algebra, then* fin.dim. $A < \infty$.

If, for instance, A is the radical square zero algebra above, then it is easily seen that fin.dim. $A \leqslant 2$ (for instance, by computing the Auslander–Reiten quiver of A, see [2, (2.3)]).

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