Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 320 (2008) 980–986
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The size of the commutator subgroup of finite groups

Marcel Herzog [a,b], Gil Kaplan [a,b,*], Arieh Lev [a,b]

[a] *School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel*

[b] *School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 2 Rabenu Yeruham st., Tel-Aviv 61083, Israel*

Received 24 October 2005

Available online 3 June 2008

Communicated by Gernot Stroth

Abstract

We prove that every finite non-abelian group G such that $\Phi(G) = 1$ satisfies the inequality $|G'| > [G : Z(G)]^{1/2}$.

Keywords: Commutator subgroup; Centre; Frattini subgroup

1. Introduction

All groups in this paper are finite. We use the standard notation $Z(G)$, $\Phi(G)$ for the center and the Frattini subgroup of G. We use further the notation $F(G)$ for the Fitting subgroup of G and $U(G)$ for the nilpotent residual of G (i.e., $U(G)$ is the smallest normal subgroup of G such that the respective quotient is nilpotent).

Our main result is the following theorem, which was conjectured in [6] for a solvable group $G \neq 1$ such that $\Phi(G) = Z(G) = 1$.

Theorem A. *Let G be any non-abelian group such that $\Phi(G) = 1$. Then*

$$|G'| > \left[G : Z(G)\right]^{1/2}.$$

* Corresponding author at: School of Computer Sciences, The Academic College of Tel-Aviv-Yaffo, 4 Antokolsky st., Tel-Aviv 64044, Israel.

E-mail address: gilk@mta.ac.il (G. Kaplan).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.12.036

Notice that the family of Frobenius groups of order $p(p-1)$, where p is a prime, shows that the inequality in Theorem A cannot be improved to $|G'| > [G : Z(G)]^{(1/2)+\epsilon}$, for any constant $\epsilon > 0$. Furthermore, as is shown in [6], Example A3 (see also [2]), the assumption $\Phi(G) = 1$ cannot be omitted in Theorem A. We note that for proving Theorem A we needed to use the well-known $k(GV)$-conjecture, whose proof was recently completed by Gluck, Magaard, Riese and Schmid [9]. We note further that our proof includes a result on the solvable residual of a group with a trivial Fitting subgroup (see Proposition 2.4).

In [6] we proved the inequality $|G'| > |G|^{1/3}$ (see Corollary A1 there) for a solvable group $G \neq 1$ such that $\Phi(G) = Z(G) = 1$. Moreover, as explained in [10], this result can be sharpened (by an easy modification of the proof) by considering $U(G)$ (which is, evidently, a subgroup of G'):

Theorem 1.1. *(See [10], end of Section 1.) Let $G \neq 1$ be a solvable group such that $\Phi(G) = Z(G) = 1$. Then $|U(G)| > |G|^{1/3}$.*

The following example shows that the inequality in Theorem 1.1 cannot be improved to $|U(G)| > |G|^{1/2}$.

Example 1.2. Let H be the Sylow 2-subgroup of $GL_2(3)$. Then H is a non-abelian group of order 16. Let V be the space of all row vectors of length 2 over the field $\mathbf{Z}_3$. Then H acts on V by right multiplication and we consider the respective semi-direct product $G = VH$. Then $|G| = 16 \cdot 9 = 144$, $\Phi(G) = Z(G) = 1$ and $|U(G)| = |V| = 9 < |G|^{1/2}$.

However, we still have the following theorem with respect to $U(G)$. We denote by $\mathcal{M}$ and $\mathcal{F}$ the sets of Mersenne primes and Fermat primes, respectively.

Theorem B. *Let G be a non-abelian group of order $p^\alpha q^\beta$, where p and q are primes, $p < q$, $\alpha, \beta \geqslant 1$. Assume $\Phi(G) = 1$. Then*

$$\left|U(G)\right| \geqslant 2^{1/2} \cdot \left[G : Z(G)\right]^{1/2},$$

provided that $(p, q) \notin (2, \mathcal{M}) \cup (2, \mathcal{F})$.

Notice that Example 1.2 above shows that certain restrictions on the primes p, q in Theorem B cannot be omitted. In view of Theorem B, we pose the following conjecture.

Conjecture 1.3. *Let G be a non-abelian group of odd order such that $\Phi(G) = 1$. Then $|U(G)| > [G : Z(G)]^{1/2}$ (maybe even $|U(G)| \geqslant 2^{1/2} \cdot [G : Z(G)]^{1/2}$).*

Note added in proof

Conjecture 1.3 was verified in the paper [5].

We conclude this section by mentioning briefly some other recent results on the size of the commutator subgroup. It is proved in [11] (see Theorem A there), that any non-abelian group G with all Sylow subgroups abelian (no condition on $\Phi(G)$) satisfies the inequality $|G'| > [G : Z(G)]^{1/2}$. Notice that Example A3 in [6] shows that this is not valid for $\{p, q\}$-groups, when one Sylow subgroup is abelian and the other is metabelian. Finally, in [10] there are results (Theorems A and B there) ensuring that any non-nilpotent group G (no condition on $\Phi(G)$) has certain factors K/M, where K is subnormal in G and M is nilpotent, such that $|U(K/M)| > |K/M|^{1/2}$.

2. Proof of Theorem A

In our proof we use the following result (see [9,12]). Throughout this section, $k(G)$ denotes the number of conjugacy classes of G.

The $k(GV)$-Theorem. *Let p be a prime. Let G be a p'-group acting faithfully on an elementary abelian p-group V. Then $k(GV) \leqslant |V|$ (where GV is the natural semi-direct product).*

We shall use the following immediate corollary.

Corollary 2.1. *Let p be a prime. Let G be a p'-group acting faithfully on a non-trivial elementary abelian p-group V. Then $k(G) < |V|$.*

Proof. It is enough to show that $k(G) < k(GV)$. Notice first that if $x, y \in G$ are not conjugate in G, then they are not conjugate in GV. Indeed, assume on the contrary $x^{gv} = y$ for $g \in G$, $v \in V$. Then $[x^g, v] = (x^g)^{-1}y$, implying $(x^g)^{-1}y \in G \cap V = 1$ and $x^g = y$, a contradiction. Thus $k(G)$ is equal to the number of GV-conjugacy classes which intersect G. Since GV has also a conjugacy class outside G, we have $k(G) < k(GV)$. □

The following two lemmas are essential for the proof of Theorem A.

Lemma 2.2. *Let G be a group with a minimal normal subgroup N, where N is a p-group for a prime p. Suppose $N \leqslant C_G(U(G))$. Then $k(G/C_G(N)) < |N|$.*

Proof. Since $C_G(N) \geqslant U(G)$, the group $G/C_G(N)$ is nilpotent. Moreover by [15, Exercise 654(ii)], $O_p(G/C_G(N)) = 1$. Thus $G/C_G(N)$ is a p'-group acting faithfully on the elementary abelian p-group N, and the result follows by Corollary 2.1. □

For the proof of Theorem A the following lemma will be needed only in the case $\Phi(G) = 1$. However, for the sake of completeness, we provide a more general result.

Lemma 2.3. *Let $G \neq 1$ be a meta-nilpotent (i.e., nilpotent by nilpotent) group. Then*

(i) $k(G/F(G)) < |F(G)/\Phi(G)|$.
(ii) $|F(G)| > \frac{|G|^{1/2} \cdot |\Phi(G)|^{1/2}}{|(G/F(G))'|^{1/2}}$.

Proof. Assume first that $\Phi(G) = 1$. Then $F(G)$ is abelian. Moreover, by a result of Gaschütz [7, Kapitel III, Satz 4.5], $F(G) = Dr_{1 \leqslant i \leqslant m} N_i$, a direct product of minimal normal subgroups of G. We have $N_i \leqslant C_G(F(G))$. Since G is meta-nilpotent we have $U(G) \leqslant F(G)$, and thus $N_i \leqslant C_G(U(G))$. Hence, by Lemma 2.2, $k(G/C_G(N_i)) < |N_i|$ for each i.

Denote $L_1 = N_1, L_2 = N_1N_2, \ldots, L_m = N_1N_2 \cdots N_m = F(G)$. We prove by induction that $k(G/C_G(L_i)) < |L_i|$ for all $1 \leqslant i \leqslant m$. This is already known for $i = 1$. Suppose it is proved for j and we now obtain it for $j + 1$. We have

$$k\big(G/C_G(L_j)\big) < |L_j|. \tag{1}$$

Notice that $G/C_G(L_{j+1}) = G/(C_G(L_j N_{j+1})) = G/(C_G(L_j) \cap C_G(N_{j+1}))$. Now $C_G(L_j)/(C_G(L_j) \cap C_G(N_{j+1}))$ is a normal subgroup of $G/(C_G(L_j) \cap C_G(N_{j+1}))$ and the respective quotient is isomorphic to $G/C_G(L_j)$. Thus, by (3) in [8], we have

$$k\big(G/C_G(L_{j+1})\big) \leqslant k\big(G/C_G(L_j)\big) \cdot k\big(C_G(L_j)/\big(C_G(L_j) \cap C_G(N_{j+1})\big)\big). \tag{2}$$

Notice further that $C_G(L_j)/(C_G(L_j) \cap C_G(N_{j+1}))$ is isomorphic to $C_G(L_j)C_G(N_{j+1})/C_G(N_{j+1})$. This last group acts coprimely (recall the proof of Lemma 2.2) and faithfully on N_{j+1}, thus $k(C_G(L_j)C_G(N_{j+1})/C_G(N_{j+1})) < |N_{j+1}|$ by Corollary 2.1. Using this, we obtain from (1) and (2) that $k(G/C_G(L_{j+1})) < |L_j| \cdot |N_{j+1}| = |L_{j+1}|$, which completes the inductive step. Now, since $C_G(L_m) = C_G(F(G)) = F(G)$, part (i) of the theorem has been proved for the case $\Phi(G) = 1$.

Suppose now that $\Phi(G) > 1$ and denote $G^* = G/\Phi(G)$. Since $F(G^*) = F(G)/\Phi(G)$ [14, 5.2.15(ii)], we obtain that $G^*/F(G^*)$ is isomorphic to $G/F(G)$. Hence, by applying part (i) to G^* (which has a trivial Frattini subgroup), we obtain part (i) for G.

It remains to prove part (ii). Since $[G : G'] \leqslant k(G)$ for any group G, part (i) implies $[(G/F(G)) : (G/F(G))'] < [F(G) : \Phi(G)]$. Thus $|F(G)|^2 > \frac{|G|\cdot|\Phi(G)|}{|(G/F(G))'|}$ and the result holds. □

The main part of the proof of Theorem A is absorbed in the following proposition.

Proposition A. *Let $G \neq 1$ be any group such that $\Phi(G) = Z(G) = 1$. Then*

$$|G'| > |G|^{1/2}.$$

Proof. We apply induction on $|G|$ (there are certain similarities between the current proof and the proof of [6, Theorem A]). We consider first the case $F := F(G) > 1$. We put

$$F_0/F = \Phi(G/F), \qquad F_1/F_0 = Z(G/F_0).$$

Since $\Phi(G/F_0) = 1$, we obtain by Lemma 7 in [6] that

$$Z(G/F_1) = \Phi(G/F_1) = 1.$$

Now F_1 is meta-nilpotent since $(F_1/F)/\Phi(G/F)$ is abelian (the details are given in the proof of [6, Theorem A]). Furthermore F is abelian and (by the mentioned above result of Gaschütz) $F = Dr N_i$, where each N_i is minimal normal in G. Since $Z(G) = 1$, we have $N_i \leqslant G'$ for each i and so $F \leqslant G'$.

Suppose first that $F_1 = G$. Then G is meta-nilpotent and Lemma 2.3 is applicable to G. Thus $|F| > \frac{|G|^{1/2}}{[G':F]^{1/2}}$, which implies $|G'|^{1/2} \cdot |F|^{1/2} > |G|^{1/2}$. From $F \leqslant G'$ it follows now that $|G'| > |G|^{1/2}$, as required.

Thus we may assume from now on that $F_1 < G$. Applying the inductive hypothesis to the group G/F_1, we obtain

$$[G' : G' \cap F_1] = |G'F_1/F_1| = \big|(G/F_1)'\big| > |G/F_1|^{1/2}. \tag{3}$$

Furthermore,

$$|G' \cap F_1| \geqslant \left|F_1' F\right| = |F| \cdot \left|F_1' F/F\right| = |F| \cdot \left|(F_1/F)'\right|. \tag{4}$$

Now since $F = F(F_1)$, $\Phi(F_1) = 1$ and F_1 is meta-nilpotent, we deduce by Lemma 2.3 that $|F| > \frac{|F_1|^{1/2}}{|(F_1/F)'|^{1/2}}$. Combining this with (4) we have

$$|G' \cap F_1| > \frac{|F_1|^{1/2}}{|(F_1/F)'|^{1/2}} \cdot \left|(F_1/F)'\right| = |F_1|^{1/2} \cdot \left|(F_1/F)'\right|^{1/2} \geqslant |F_1|^{1/2}. \tag{5}$$

From (3) and (5) follows now $|G'| > |G|^{1/2}$, as required. This completes the inductive argument in the case $F(G) > 1$. Hence, the proof of Proposition A will be completed by proving the following result. □

Proposition 2.4. *Let $G \neq 1$ be a group such that $F(G) = 1$. Let $Res(G)$ denote the solvable residual of G (i.e., the smallest normal subgroup of G such that the respective quotient is solvable; this is the minimal term in the derived series of G). Then*

$$\left|Res(G)\right| > |G|^{1/2}.$$

Proof. Apply induction on $|G|$. Let N be a minimal normal subgroup of G and let N_1/N be the maximal normal solvable subgroup of G/N. Then $F(G/N_1) = 1$. Suppose first that $N_1 < G$. Then from $F(N_1) \leqslant F(G) = 1$ it follows by induction that

$$\left|Res(G) \cap N_1\right| \geqslant \left|Res(N_1)\right| > |N_1|^{1/2}. \tag{6}$$

By applying further the inductive hypothesis to G/N_1, we obtain

$$\left[Res(G) : Res(G) \cap N_1\right] = \left|Res(G)N_1/N_1\right| = \left|Res(G/N_1)\right| > |G/N_1|^{1/2}. \tag{7}$$

Now from (6) and (7) follows $|Res(G)| > |G|^{1/2}$, as claimed.

Thus we may assume from now on that $N_1 = G$, i.e., G/N is solvable and $Res(G) = N = T^{\alpha}$, a direct product, where T is a simple non-abelian group (recall that $F(G) = 1$) and $\alpha \geqslant 1$ is an integer. We notice that N is the unique minimal normal subgroup of G. Indeed, suppose on the contrary there exists another minimal normal subgroup, say M, of G. Then $M \cap N = 1$ and M is embedded in the solvable group G/N, contradicting $F(G) = 1$.

We deduce that $C_G(N) = 1$ and $N \leqslant G \leqslant Aut(N) = Aut(T)\ wr\ S_{\alpha}$ (see [15, Lemma 9.24]). Thus G/N is a solvable group embedded in $Out(T)\ wr\ S_{\alpha}$. Any element of G/N has the form (b, σ), where b belongs to the base subgroup of $Out(T)\ wr\ S_{\alpha}$ and $\sigma \in S_{\alpha}$. Then the function $(b, \sigma) \mapsto \sigma$ is a homomorphism from G/N into S_{α}. Denote the image of this homomorphism by D. Then D is a solvable subgroup of S_{α} and thus, by [4], Theorem 5.8B, $|D| \leqslant f(\alpha) := 24^{(\alpha-1)/3}$. Since $|G/N| \leqslant |Out(T)|^{\alpha} \cdot |D|$, it follows that $|G/N| \leqslant |Out(T)|^{\alpha} \cdot f(\alpha) < |Out(T)|^{\alpha} \cdot 3^{\alpha}$. Since $Res(G) = N$, we want to show that $|G/N| < |N| = |T|^{\alpha}$. For that, it suffices to check that $|Out(T)| \leqslant |T|/3$ for each simple non-abelian group T. Indeed, for $T = A_n$, $n \geqslant 5$, and for T sporadic we have $|Out(T)| \leqslant 4$ (see [3, remark on p. ix and Table 1]). For the Chevalley groups the inequality $|Out(T)| \leqslant |T|/3$ is verified by [3, Tables 5 and 6]. Thus the proof is completed. □

We are ready now for the final step.

Proof of Theorem A. Since $\Phi(G)=1$, the non-trivial group $G/Z(G)$ satisfies $\Phi(G/Z(G))=Z(G/Z(G))=1$ by [6, Lemma 7]. Hence, by Proposition A, we have $|(G/Z(G))'|>|G/Z(G)|^{1/2}$. Since $|G'|\geqslant[G'Z(G):Z(G)]=|(G/Z(G))'|$, the theorem is now obtained. $\square$

3. Proof of Theorem B

Like Lemma 2.3, the following lemma will be applied only in the case $\Phi(G)=1$. In its proof, we use a result on coprime actions, in the case that both groups involved are of prime power order [13, Theorem 3.3(b)].

Lemma 3.1. *Let $G\neq 1$ be a meta-nilpotent group of order $p^\alpha q^\beta$, where p and q are primes, $p<q$, $\alpha,\beta\geqslant 1$. Then $|F(G)|\geqslant 2^{1/2}\cdot|G|^{1/2}\cdot|\Phi(G)|^{1/2}$, provided that $(p,q)\notin(2,\mathcal{M})\cup(2,\mathcal{F})$.*

Proof. Suppose first that $\Phi(G)=1$. Then $F:=F(G)$ is abelian and $F=Dr\,N_i$, where each N_i is minimal normal in G. Since G is meta-nilpotent, $G/C_G(N_i)$ is nilpotent. It follows by [15, Exercise 654(ii)], that $G/C_G(N_i)$ acts coprimely and faithfully on N_i, for each i. Hence, for each i, one of these groups is a p-group, while the other is a q-group.

Using now [13, Theorem 3.3(b)] and our assumptions on (p,q), we obtain $|G/C_G(N_i)|\leqslant|N_i|/2$, and so $2|G|\leqslant|N_i|\cdot|C_G(N_i)|$ for each i. Now, since $F=Dr\,N_i$, by repeated use of [6, Lemma 4] (actually we deal here with a slightly different version of that lemma, since the inequality is weak; however the reader can check that this version of the lemma is proved like the original version) we have $2|G|\leqslant|F|\cdot|C_G(F)|=|F|^2$ and $2^{1/2}\cdot|G|^{1/2}\leqslant|F|$, completing the proof in the case $\Phi(G)=1$. The result for a general group G is easily obtained by using $F(G/\Phi(G))=F/\Phi(G)$. $\square$

Similarly to the proof of Theorem A, the main part of the current proof is contained in the following claim.

Proposition B. *Let G be a group of order $p^\alpha q^\beta$, where p and q are primes, $p<q$, $\alpha,\beta\geqslant 1$. Assume $\Phi(G)=Z(G)=1$. Then*

$$\left|U(G)\right|\geqslant 2^{1/2}\cdot|G|^{1/2},$$

provided that $(p,q)\notin(2,\mathcal{M})\cup(2,\mathcal{F})$.

Proof. Note first that G is solvable by Burnside's theorem. We apply induction on $|G|$ and proceed similarly to the proof of [6], Theorem A and the proof of Proposition A in the current paper. Let $F:=F(G)$ and put

$$F_0/F=\Phi(G/F),\qquad F_1/F_0=Z(G/F_0).$$

Then (see the proof of Theorem A)

$$Z(G/F_1)=\Phi(G/F_1)=1,$$

and F_1 is meta-nilpotent. Hence by Lemma 3.1

$$|F| = \left|F(F_1)\right| \geqslant 2^{1/2} \cdot |F_1|^{1/2}. \tag{8}$$

As $\Phi(G) = 1$, we have that F is abelian and $F = Dr N_i$, where each N_i is minimal normal in G. Since $Z(G) = 1$, $[G, N_i] = N_i$ for each i, and thus $N_i \leqslant U(G)$ for each i (see [1, (9) on p. 36]). Hence $F \leqslant U(G)$. If $G = F_1$ then the proposition follows by (8). Thus, we may suppose that $G > F_1$ and apply the inductive hypothesis to the group G/F_1. Then

$$\left[U(G) : U(G) \cap F_1\right] = \left|U(G)F_1/F_1\right| = \left|U(G/F_1)\right| \geqslant 2^{1/2} \cdot |G/F_1|^{1/2}. \tag{9}$$

Furthermore by (8) we have

$$\left|U(G) \cap F_1\right| \geqslant |F| \geqslant 2^{1/2} \cdot |F_1|^{1/2}. \tag{10}$$

By (9) and (10) we obtain $|U(G)| \geqslant 2^{1/2} \cdot |G|^{1/2}$. $\square$

The final step is done similarly to Section 1.

Proof of Theorem B. Since $\Phi(G) = 1$, the non-trivial group $G/Z(G)$ satisfies $\Phi(G/Z(G)) = Z(G/Z(G)) = 1$ by [6, Lemma 7]. Hence, $|G/Z(G)|$ is divisible by *both* primes p and q and Proposition B is applicable to $G/Z(G)$. Consequently, $|U(G/Z(G))| \geqslant 2^{1/2} \cdot |G/Z(G)|^{1/2}$. Since $|U(G)| \geqslant [U(G)Z(G) : Z(G)] = |U(G/Z(G))|$, the theorem is proved. $\square$

References

[1] R. Baer, Group elements of prime power index, Trans. Amer. Math. Soc. 75 (1953) 20–47.

[2] M. Bianchi, A. Gillio, H. Heineken, L. Verardi, Groups with big centralizers, Instituto Lombardo (Rend. Sc.) A 130 (1996) 25–42.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, An ATLAS of Finite Groups, Oxford University Press, 1985.

[4] J.D. Dixon, B. Mortimer, Permutation Groups, Grad. Texts in Math., vol. 163, Springer-Verlag, New York, 1996.

[5] S. Dolfi, et al., J. Group Theory 10 (2007) 299–305.

[6] M. Herzog, G. Kaplan, A. Lev, On the commutator and the center of finite groups, J. Algebra 278 (2004) 494–501.

[7] B. Huppert, Endliche Gruppen, vol. I, Springer-Verlag, 1967.

[8] P.X. Gallagher, The number of conjugacy classes in a finite group, Math. Z. 118 (1970) 175–179.

[9] D. Gluck, K. Magaard, U. Riese, P. Schmid, The solution of the $k(GV)$-problem, J. Algebra 279 (2004) 694–719.

[10] G. Kaplan, A. Lev, The existence of large commutator subgroups in factors and subgroups of non-nilpotent groups, Arch. Math. 85 (2005) 197–202.

[11] G. Kaplan, A. Lev, On groups satisfying $|G'| > [G : Z(G)]^{1/2}$, Beitrage Algebra Geom. 47 (2006) 271–274.

[12] R. Knörr, On the number of characters in a p-block of a p-solvable group, Illinois J. Math. 28 (1984) 181–210.

[13] O. Manz, T. Wolf, Representations of Solvable Groups, Cambridge University Press, Cambridge, 1993.

[14] D.J.S. Robinson, A Course in the Theory of Groups, Springer-Verlag, Berlin, 1982.

[15] J.S. Rose, A Course on Group Theory, Dover Publications Inc., Cambridge University Press, New York, 1978.