Journal of Algebra 321 (2009) 230–247

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A geometric Schur–Weyl duality for quotients of affine Hecke algebras

G. Pouchin

École Normale Supérieure, Département de Mathématiques et Applications, 45 rue d'Ulm, 75005 Paris, France

ARTICLE INFO

Article history:
Received 19 March 2008
Available online 18 October 2008
Communicated by Peter Littelmann

Keywords:
Representation theory
Affine Hecke algebra
Schur–Weyl duality

ABSTRACT

After establishing a geometric Schur–Weyl duality in a general setting, we recall this duality in type A in the finite and affine case. We extend the duality in the affine case to positive parts of the affine algebras. The positive parts have nice ideals coming from geometry, allowing duality for quotients. Some of the quotients of the positive affine Hecke algebra are then identified to some cyclotomic Hecke algebras and the geometric setting allows the construction of canonical bases.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

The so-called Schur–Weyl duality is a bicommutant theorem which classically holds between $GL_d(\mathbb{C})$ and the symmetric group $\mathfrak{S}_n$. The first group acts naturally on $V = \mathbb{C}^d$ and diagonally on $V^{\otimes n}$. The symmetric group acts on $V^{\otimes n}$ by permuting the tensors. The theorem says that the algebras of these groups are the commutant of each other inside $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ when $d \geqslant n$. More precisely we have canonical morphisms:

$$\phi : \mathbb{C}\big[GL_d(\mathbb{C})\big] \to \mathrm{End}_{\mathbb{C}}\big(V^{\otimes n}\big), \qquad \psi : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}_{\mathbb{C}}\big(V^{\otimes n}\big)$$

such that

$$\phi\big(\mathbb{C}\big[GL_d(\mathbb{C})\big]\big) = \mathrm{End}_{\mathfrak{S}_n}\big(V^{\otimes n}\big)$$

and

$$\psi\big(\mathbb{C}[\mathfrak{S}_n]\big) = \mathrm{End}_{GL_d(\mathbb{C})}\big(V^{\otimes n}\big).$$

E-mail address: pouchin@dma.ens.fr.

0021-8693/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2008.09.018

The image of the first map is called the Schur algebra, and the second map is injective when $d \geqslant n$.

The case of a base field of arbitrary characteristic has been studied by numerous autors [CL,Gr].

This theorem has many other versions, for affine algebras and for quantum ones (see [J]).

The aim of this article is to establish geometrically a Schur–Weyl duality between some quotients of a "half" of affine q-Schur algebras (which are themselves quotients of the affine quantum enveloping algebra of GL_d) and some quotients of a "half" of the affine Hecke algebra $\widehat{\mathcal{H}}_n$ (of type A), when $d \geqslant n$.

The Schur–Weyl duality between affine q-Schur algebras and affine Hecke algebras is already known for some time and can be expressed nicely by considering convolution algebras on some flag varieties (see [VV]). We are considering quotients of subalgebras which arise naturally in this geometric interpretation: the ideals which we use to define our quotients are just the functions whose support lies in some closed subvariety.

In fact some of the quotients of Hecke algebras defined this way are particular cases of cyclotomic Hecke algebras (where all parameters are equal to zero). One interesting outcome of our construction is the existence of canonical bases for such algebras. These are simply defined as the restriction of certain simple perverse sheaves to the open subvarieties we are considering. The Schur–Weyl duality also provides a strong link with affine q-Schur algebras.

The article is organized as follows: the first part is the geometric setting needed to have a Schur–Weyl duality. The second part is the application of the first part in type A for the finite and affine case. In the third part we show that the geometric Schur–Weyl duality remains when we restrict ourselves to some subalgebras verifying some conditions. We apply this in the next part to the affine case by taking positive parts of our affine algebras. These positive algebras have interesting two-sided ideals coming from geometry, so the fifth part is dedicated to quotients by such ideals, in particular we establish the Schur–Weyl duality for our quotient algebras. The quotients are also identified. The sixth part deals with the construction of canonical bases of our quotients, using their construction in terms of intersection complexes.

The last part of the article is the study of the case $d < n$, where only one half of the bicommutant holds. This answers a question of Green [G] in the affine case.

2. Schur–Weyl duality in a general setting

Let G be a group acting on two sets X and Y. Let us assume that we have the following data:

- a decomposition $Y = \bigsqcup_{i\in I} Y_i$, where I is a finite set,
- for each $i \in I$, a surjective G-equivariant map, $\phi_i : X \to Y_i$, which has finite fibers of constant cardinal m_i,
- an element $\omega \in I$ for which the map ϕ_ω is bijective.

We equip the products $X \times Y$, $X \times X$ and $Y \times Y$ with the diagonal G-action.

Let $A = \mathbb{C}_G(Y \times Y)$ be the set of G-invariant functions which take non-zero values on a finite number of G-orbits, and define $B = \mathbb{C}_G(X \times X)$ in the same way. These are equipped with the convolution product

$$f * g(L, L'') = \sum_{L'} f(L, L')g(L', L'').$$

The space $C = \mathbb{C}_G(Y \times X)$ is endowed with a natural action by convolution of A (resp. B) on the left (resp. on the right).

Theorem 2.1 *(Bicommutant Theorem). We have*:

$$\mathrm{End}_B(C) = A,$$

$$\mathrm{End}_A(C) = B.$$

Proof. Let us prove the first assertion. Let $P \in \mathrm{End}_B(C)$.

From the decomposition $Y = \bigsqcup_{i\in I} Y_i$, we can split C as a direct sum of vector spaces:

$$C = \bigoplus_{i\in I} \mathbb{C}_G(Y_i \times X)$$

and hence $\mathrm{End}(C)$ as:

$$\mathrm{End}(C) = \bigoplus_{i,j} \mathrm{Hom}\big(\mathbb{C}_G(Y_i \times X), \mathbb{C}_G(Y_j \times X)\big). \tag{1}$$

The (i, j)-component $P'_{(i,j)} = P'$ of P with respect to (1) is the morphism defined by $P'(f) = 1_{\mathcal{O}_{\Delta(j)}} * P(1_{\mathcal{O}_{\Delta(i)}} * f)$, where $\Delta(l)$ is the diagonal of $Y_l \times Y_l$.

The G-equivariant surjective map $\phi_i : X \to Y_i$ gives rise to the G-equivariant maps:

$$Id \times \phi_i : Y_j \times X \to Y_j \times Y_i,$$

$$\phi_i \times Id : X \times X \to Y_i \times X.$$

From these surjective maps we canonically build the injections:

$$\psi_i : \mathbb{C}_G(Y_j \times Y_i) \hookrightarrow \mathbb{C}_G(Y_j \times X),$$

$$\chi_j : \mathbb{C}_G(Y_j \times X) \hookrightarrow \mathbb{C}_G(X \times X).$$

For every f in $\mathbb{C}_G(Y_i \times X)$ we have:

$$P'(f) = P'\big(m_i^{-1} 1_{\Delta(Y_i\times X)} * \chi_i(f)\big) = m_i^{-1} P'(1_{\Delta(Y_i\times X)}) * \chi_i(f)$$

where $P'(1_{\Delta(Y_i\times X)}) \in \mathbb{C}_G(Y_j \times X)$.

We will now prove that $P'(1_{\Delta(Y_i\times X)})$ belongs to the image of $\mathbb{C}_G(Y_j \times Y_i)$ under ψ_i.

By definition the image of ψ_i in $\mathbb{C}_G(Y_j \times X)$ is the set of functions taking the same values on two orbits of $Y_j \times X$ which have the same image in $Y_j \times Y_i$. We introduce:

$$Z_i = \big\{(L, L') \in X \times X,\ \phi_i(L) = \phi_i(L')\big\}.$$

Observe that $P'(1_{\Delta(Y_i\times X)}) * 1_{Z_i} = P'(1_{\Delta(Y_i\times X)} * 1_{Z_i}) = m_i P'(1_{\Delta(Y_i\times X)})$. The following lemma implies that $P'(1_{\Delta(Y_i\times X)})$ belongs to $\mathrm{Im}(\psi_i)$.

Lemma 2.2.

$$\mathrm{Im}(\psi_i) = \big\{h \in \mathbb{C}_G(Y_j \times X),\ h * 1_{Z_i} = m_i h\big\}.$$

Proof. For the inclusion of the left-hand side in the right-hand side, we can write for $h \in \mathrm{Im}(\psi_i)$ and $(L, L') \in Y_j \times X$:

$$\begin{aligned} h * 1_{Z_i}(L, L') &= \sum_{L''} h(L, L'') 1_{Z_i}(L'', L') \\ &= \sum_{\phi_i(L'')=\phi_i(L')} h(L, L'') = m_i h(L, L'). \end{aligned}$$

For the other inclusion, let $h \in \mathbb{C}_G(Y_j \times X)$ be such that $h * 1_{Z_i} = m_i h$. Take (L, M) and (L, N) in $Y_i \times X$ such that $\phi_j(M) = \phi_j(N)$. Then we have $1_{Z_i}(L', M) = 1_{Z_i}(L', N)$ for every L' in X. Then:

$$\begin{aligned} m_i h(L, M) = h * 1_{Z_i}(L, M) &= \sum_{L'} h(L, L') 1_{Z_i}(L', M) \\ &= \sum_{L'} h(L, L') 1_{Z_i}(L', N) = h * 1_{Z_i}(L, N) = m_i h(L, N) \end{aligned}$$

and so $h \in \mathrm{Im}(\psi_i)$. □

Let $g := \psi_i^{-1}(P'(1_{\Delta(Y_i \times X)})) \in \mathbb{C}_G(Y_j \times Y_i)$.

So we have for $(L, M) \in Y_j \times X$, $g(L, \phi_i(M)) = P'(1_{\Delta(Y_i \times X)})(L, M)$. We can now prove that:

$$\forall f \in \mathbb{C}(Y_i \times X), \quad P'(f) = g * f.$$

Indeed we have seen that $P'(f) = m_i^{-1} P'(1_{\Delta(Y_i \times X)}) * \chi_i(f)$. But we have:

$$\begin{aligned} m_i^{-1} P'(1_{\Delta(Y_i \times X)}) * \chi(f)(L, M) &= m_i^{-1} \sum_{N \in X} P'(1_{\Delta(Y_i \times X)})(L, N) \chi_i(f)(N, M) \\ &= m_i^{-1} \sum_{N \in X} g\big(L, \phi_i(N)\big) f\big(\phi_i(N), M\big) \\ &= \sum_{N' \in Y_i} g(L, N') f(N', M) \\ &= g * f(L, M). \end{aligned}$$

So we have the result for P'.

To have it for P, it suffices to sum on the orthogonal idempotents. For every $(i, j) \in I^2$ we have built $g_{(i,j)} \in \mathbb{C}_G(Y_j \times Y_i)$ such that for every f in $\mathbb{C}_G(Y_i \times X)$, $P'_{(i,j)}(f) = g_{(i,j)} * f$. Let $g = \sum_{i,j} g_{(i,j)}$.

Then for $f = \bigoplus_i f_i \in \mathbb{C}_G(Y \times X) = \bigoplus_i \mathbb{C}_G(Y_i \times X)$, we have:

$$P(f) = \sum_{i,j \in I} 1_{\mathcal{O}_{\Delta(j)}} * P(1_{\mathcal{O}_{\Delta(i)}} * f) = \sum_{i,j \in I} P'_{(i,j)}(f_i) = \sum_{i,j \in I} g_{(i,j)} * f_i = g * f.$$

We now turn to the second assertion.

Take $P \in \mathrm{End}_A(C)$.

The projector on $\mathbb{C}_G(Y_i \times X)$ parallel to the rest of the sum is the convolution on the left by the function $1_{\mathcal{O}_{\Delta_i}}$, where Δ_i is the diagonal of $Y_i \times Y_i$. But P commutes with the action of A, so these subspaces are stable.

The next lemma will allow us to focus on one such subspace.

Lemma 2.3. *The A-module C is generated by $\mathbb{C}_G(Y_\omega \times X)$.*

Proof. It is sufficient to verify that for every f in $\mathbb{C}_G(Y_i \times X)$, we have the following formula:

$$f = \psi_\omega^{-1}(f) * 1_{\Delta(Y_\omega \times X)}$$

where ψ_ω is the isomorphism deduced from ϕ_ω:

$$\psi_\omega : \mathbb{C}_G(Y_i \times Y_\omega) \to \mathbb{C}_G(Y_i \times X). \qquad \square$$

As C is generated as an A-module by $\mathbb{C}_G(Y_\omega \times X)$, the endomorphism P is entirely determined by its restriction P' to $\mathbb{C}_G(Y_\omega \times X)$. Then we can consider $P' \in \mathrm{End}_{\mathbb{C}_G(Y_\omega \times Y_\omega)}(\mathbb{C}_G(Y_\omega \times X))$.

But the canonical isomorphism $\phi_\omega : Y_\omega \to X$ allows us to identify $B = \mathbb{C}_G(X \times X)$ with $\mathbb{C}_G(Y_\omega \times X)$ and $\mathbb{C}_G(Y_\omega \times Y_\omega)$. This way we can see P' as an element of $\mathrm{End}_B(B) = B$. □

3. Applications

3.1. The linear group

Let q be a power of a prime number p and $\mathbb{F}_q$ the finite field with q elements. We note $G = GL_n(\mathbb{F}_q)$. In the following everything takes place in a vector space V on $\mathbb{F}_q$ of dimension n. We fix an integer $d \geqslant n$.

The complete flag manifold X is:

$$X = \big\{(L_i)_{1\leqslant i\leqslant n} \,\big|\, L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n = V,\ \dim L_i = i\big\}.$$

The partial flag manifold Y of length d is:

$$Y = \big\{(L_i)_{1\leqslant i\leqslant d} \,\big|\, L_1 \subseteq L_2 \subseteq \cdots \subseteq L_d \subseteq V\big\}.$$

The group G acts canonically on the varieties X and Y.

A composition of n of length d is a sequence of integers $\underline{d} = (d_1, \ldots, d_d)$ which have a sum equal to n. Let $\Lambda(n, d)$ be the set of compositions of n of length d.

For each composition $\underline{d}$ of n we have a connected component $Y_{\underline{d}}$ of Y defined by $Y_{\underline{d}} = \{(L_\bullet) \in Y,\ \forall i \dim(L_{i+1}/L_i) = d_i\}$ and the decomposition:

$$Y = \bigsqcup_{\underline{d}\in\Lambda(n,d)} Y_{\underline{d}}.$$

Also we have a canonical surjective G-equivariant map $\phi_{\underline{d}} : X \to Y_{\underline{d}}$.

As $d \geqslant n$, the element $\omega = (\underbrace{1, \ldots, 1}_{n}, 0, \ldots, 0)$ belongs to $\Lambda(n, d)$. The canonical morphism ϕ_ω is then an isomorphism.

The hypotheses are verified so we can apply the theorem in Section 1. In this case the algebras constructed are well known:

Proposition 3.1. *The convolution algebra $\mathbb{C}_G(X \times X)$ is isomorphic to the Hecke algebra $\mathcal{H}_n$ with parameter $q = v^{-2}$.*

Proposition 3.2. *The convolution algebra $\mathbb{C}_G(Y \times Y)$ is isomorphic to the q-Schur algebra $S_q(n, d)$.*

Thus we obtain the standard Schur–Weyl duality.

3.2. The affine case

Let us write $\mathcal{M}$ for the set of $\mathbb{F}_q[\![z]\!]$-submodules of $(\mathbb{F}_q((z)))^n$ which are free of rank n.

The complete affine flag variety $\widehat{X}$ is:

$$\widehat{X} = \big\{(L_i)_{i\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}} \,\big|\, \forall i\ L_i \subseteq L_{i+1},\ L_{i+n} = z^{-1}L_i,\ \dim_{\mathbb{F}_q} L_i/L_{i-1} = 1\big\}.$$

The affine partial flag variety $\widehat{Y}$ of length d is:

$$\widehat{Y} = \big\{(L_i)_{i\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}} \,\big|\, \forall i\ L_i \subseteq L_{i+1},\ L_{i+d} = z^{-1}L_i\big\}.$$

Let G be $GL_n(\mathbb{F}_q((z)))$. The varieties defined above are equipped with a canonical action of G.
We still have a decomposition of $\widehat{Y}$:

$$\widehat{Y} = \bigsqcup_{\underline{d}\in\Lambda(n,d)} \widehat{Y}_{\underline{d}}$$

where $\widehat{Y}_{\underline{d}}$ is the subvariety of Y defined by:

$$\widehat{Y}_{\underline{d}} = \left\{(L_\bullet) \in \widehat{Y},\ \forall i \in \{1,\dots,d\} \dim_{\mathbb{F}_q}(L_i/L_{i-1}) = d_i\right\}.$$

For each element $\underline{d} \in \Lambda(n,d)$, we have a G-equivariant surjective map:

$$\phi_{\underline{d}} : \widehat{X} \to \widehat{Y}_{\underline{d}}$$

defined by $\phi_{\underline{d}}(L_\bullet)_0 = L_0$ and $\phi_{\underline{d}}(L_\bullet)_i = L_{\sum_{k=1}^{i} d_k}$ for $i \in \{1,\dots,d\}$.
We can identify these algebras as we did in the previous paragraph:

Proposition 3.3. *(See [IM].) The algebra $\mathbb{C}_G(\widehat{X}\times\widehat{X})$ is isomorphic to the affine Hecke algebra $\widehat{\mathcal{H}}_n$ with parameter $q = v^{-2}$.*

Proposition 3.4. *(See [VV].) The algebra $\mathbb{C}_G(\widehat{Y}\times\widehat{Y})$ is isomorphic to the affine q-Schur algebra $\widehat{S}_q(n,d)$.*

4. Subalgebras deduced from subvarieties

We will now see that the bicommutant theorem remains true, under some additional hypothesis, for a subspace of $\mathbb{C}_G(Y\times X)$ and subalgebras of $\mathbb{C}_G(Y\times Y)$ and $\mathbb{C}_G(X\times X)$.

Let X, Y, Y_i be as in Section 1.

Suppose we are given G-subvarieties $\mathcal{Z} \subseteq Y\times X$, $\mathcal{X} \subseteq X\times X$, $\mathcal{Y} \subseteq Y\times Y$ satisfying the following conditions:

- for every $i, j \in I$, when we write $\mathcal{Z}_i = \mathcal{Z} \cap (Y_i \times X)$ and $\mathcal{Y}_{i,j} = \mathcal{Y} \cap (Y_i \times Y_j)$, then

$$(\phi_i \times Id_X)(\mathcal{X}) = \mathcal{Z}_i$$

and

$$(\phi_i \times \phi_j)(\mathcal{X}) = \mathcal{Y}_{(i,j)},$$

- $\Delta_X \subseteq \mathcal{X}$ and $\mathbb{C}_G(\mathcal{X})$ is a subalgebra of $\mathbb{C}_G(X\times X)$.

From the above assumptions it follows that:

- $\mathbb{C}_G(\mathcal{Z})$ is stable for the action of $\mathbb{C}_G(\mathcal{X})$ on $\mathbb{C}_G(Y\times X)$.
- $\mathbb{C}_G(\mathcal{Y})$ is a subalgebra of $\mathbb{C}_G(Y\times Y)$, and $\mathbb{C}_G(\mathcal{Z})$ is stable for the action of $\mathbb{C}_G(Y\times X)$.
- The spaces $\mathbb{C}_G(\mathcal{X})$, $\mathbb{C}_G(\mathcal{Y})$ and $\mathbb{C}_G(\mathcal{Z})$ contain the characteristic functions of the diagonals $1_{\mathcal{O}_{\Delta(X\times X)}}$, $1_{\mathcal{O}_{\Delta(Y_i\times X)}}$ and $1_{\mathcal{O}_{\Delta(Y_i\times Y_j)}}$ (for every i, j in I).
- The subspace $\mathbb{C}_G(\mathcal{Z}_\omega)$ generates $\mathbb{C}_G(\mathcal{Z})$ as a $\mathbb{C}_G(\mathcal{Y})$-module.
- The diagonal function $1_{\mathcal{O}_{\Delta(Y_i\times X)}}$ generates $\mathbb{C}_G(\mathcal{Z}_i)$ as a $\mathbb{C}_G(\mathcal{X})$-module.
- We have isomorphisms deduced from ψ and χ:

$$\mathbb{C}_G(\mathcal{Y}_{(\omega,\omega)}) \simeq \mathbb{C}_G(\mathcal{Z}_\omega) \simeq \mathbb{C}_G(\mathcal{X}).$$

Theorem 4.1. *Under the previous conditions, the following bicommutant theorem holds*:

$$\mathrm{End}_{\mathbb{C}_G(\mathcal{X})}\big(\mathbb{C}_G(\mathcal{Z})\big) = \mathbb{C}_G(\mathcal{Y}),$$
$$\mathrm{End}_{\mathbb{C}_G(\mathcal{Y})}\big(\mathbb{C}_G(\mathcal{Z})\big) = \mathbb{C}_G(\mathcal{X}).$$

The proofs are the same as in the case of the whole space.

5. The positive part of the affine Hecke algebra

We get back to the setting of Section 2.2. Thus $\widehat{X}$ and $\widehat{Y}$ are resp. the complete affine flag variety and the partial affine flag variety. We recall that we take $d \geqslant n$, where n is the rank of the free modules and d is the periodicity in the partial affine flag variety. Consider the subvarieties:

- $\mathcal{X} = (\widehat{X} \times \widehat{X})^+ = \{(L_\bullet, L'_\bullet) \in \widehat{X} \times \widehat{X}, L'_0 \subseteq L_0\}$,
- $\mathcal{Y} = (\widehat{Y} \times \widehat{Y})^+ = \{(L_\bullet, L'_\bullet) \in \widehat{Y} \times \widehat{Y}, L'_0 \subseteq L_0\}$,
- $\mathcal{Z} = (\widehat{Y} \times \widehat{X})^+ = \{(L_\bullet, L'_\bullet) \in \widehat{Y} \times \widehat{X}, L'_0 \subseteq L_0\}$

which give rise to the convolution algebras

- $A_+ = \mathbb{C}_G((\widehat{Y} \times \widehat{Y})^+)$,
- $B_+ = \mathbb{C}_G((\widehat{X} \times \widehat{X})^+)$.

The subspace

$$C_+ = \mathbb{C}_G\big((\widehat{Y} \times \widehat{X})^+\big)$$

is an (A_+, B_+)-bimodule. It is easy to check that the hypothesis of Theorem 4.1 are verified, so that the bicommutant theorem still holds:

Proposition 5.1. *We have*:

$$\mathrm{End}_{A_+}(C_+) = B_+,$$
$$\mathrm{End}_{B_+}(C_+) = A_+.$$

Proof. It is a direct application of Theorem 4.1. □

Our immediate aim is to identify precisely the algebra B_+. For this, we need to recall in more details the structure of affine Weyl group in type A.

5.1. The extended affine Weyl group in type A

Let us first recall first the definition of the extended affine Weyl group in the general case of a connected reductive group G over $\mathbb{C}$. We write T for a maximal torus of G, $W_0 = N_G(T)/T$ is the Weyl group of G. The group W_0 acts on the character group $X = \mathrm{Hom}(T, \mathbb{C}^*)$, which allows us to consider the semidirect product $W = W_0 \rtimes X$, which is called the extended affine Weyl group of G. The root system R of G generates a sublattice of X, noted Y. The semidirect product $W' = W_0 \rtimes Y$ is called the affine Weyl group of G. It is a Coxeter group, unlike the extended affine Weyl group. It is also a normal subgroup of W.

There is an abelian subgroup Ω of W such that $\omega^{-1} S \omega = S$ for every $\omega \in \Omega$ and $W = \Omega \rtimes W'$.

In the case of $G = GL_n(\mathbb{C})$, the Weyl group W_0 is isomorphic to the symmetric group $\mathfrak{S}_n$. We write $S = \{s_1, \ldots, s_{n-1}\}$ for its simple reflections. The group W' is still a Coxeter group, which is generated

by the simple reflections of W_0 and an additional elementary reflection s_0. The group Ω is isomorphic to $\mathbb{Z}$, and is generated by an element ρ which verifies $\rho^{-1}s_i\rho = s_{i+1}$ for every $i = 1, \dots, n-1$, where we write s_n for s_0.

The group W' is then the group generated by the elements $s_0, \dots, s_{n-1}$, with the following relations:

(1) $s_i^2 = 1$ for every $i = 1, \dots, n-1$,
(2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ where the indices are taken modulo n.

We have $W = \Omega \rtimes W'$, where the group Ω is isomorphic to $\mathbb{Z}$, generated by an element ρ which verifies:

$$\rho^{-1} s_i \rho = s_{i-1}.$$

The character group of a torus of $GL_n(\mathbb{C})$ is naturally isomorphic to $\mathbb{Z}^n$. Thus we have $W = \mathfrak{S}_n \ltimes \mathbb{Z}^n$, where the group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permutation.

The group $W = \mathfrak{S}_n \ltimes \mathbb{Z}^n$ can also be considered as a subgroup of the group of the automorphisms of $\mathbb{Z}$ in the following way: to each $(\sigma, (\lambda_i)) \in \mathfrak{S}_n \ltimes \mathbb{Z}^n$, we associate the element $\tilde{\sigma} \in \mathrm{Aut}(\mathbb{Z})$ defined by:

$$\tilde{\sigma}(i) = \sigma(r) + kn + \lambda_r n$$

where $i = kn + r$ is the Euclidian division of i by n, taking the rest between 1 and n.

In fact if we write τ for the element of $\mathrm{Aut}(\mathbb{Z})$ defined by:

$$\tau : i \mapsto i + n$$

and set $\mathrm{Aut}_n(\mathbb{Z}) = \{\sigma \in \mathrm{Aut}(\mathbb{Z}),\ \sigma\tau = \tau\sigma\}$. Then we obtain the following isomorphism:

Lemma 5.2. *The previous map provides an isomorphism of groups*:

$$\mathfrak{S}_n \ltimes \mathbb{Z}^n \simeq \mathrm{Aut}_n(\mathbb{Z}).$$

Under this isomorphism, the element s_i $(0 \leqslant i \leqslant n-1)$ is mapped to $\tilde{s}_i$ defined by:

$$\begin{cases} \tilde{s}_i(j) = j & \text{if } j \neq i, i+1 \bmod(n), \\ \tilde{s}_i(j) = j+1 & \text{if } j = i \bmod(n), \\ \tilde{s}_i(j) = j-1 & \text{if } \quad j = i+1 \bmod(n). \end{cases}$$

The element ρ is mapped to $\tilde{\rho}$ defined by $\tilde{\rho}(i) = i + 1$.

The orbits of the action of G on $\widehat{X} \times \widehat{X}$ are parametrized by the elements of the extended affine Weyl group $\widehat{\mathfrak{S}}_n$. Then we can write $\mathcal{O}_w$ for an orbit, with w in $\widehat{\mathfrak{S}}_n$. This can be done explicitly in the following way. A couple of flags $L_\bullet$ and $L'_\bullet$ are in the orbit w if there is a base $e_1, \dots, e_n$ of the $\mathbb{F}_q((z))$-module $\mathbb{F}_q((z))^n$ such that

$$L_i = \prod_{w(j) \leqslant i} \mathbb{F}_q e_j \quad \text{and} \quad L'_i = \prod_{j \leqslant i} \mathbb{F}_q e_j$$

where we define e_i for all $i \in \mathbb{Z}$ by the condition $e_{i+kn} = z^{-k} e_i$ for all $k \in \mathbb{Z}$.

Theorem 5.3. *(See [IM].) The algebra $\mathbb{C}_G(\widehat{X}\times\widehat{X})$ is isomorphic to the affine Hecke algebra $\widehat{\mathcal{H}}_n$ specialized at $v^{-2}=q$ and the isomorphism is given by*:

$$\phi: 1_{\mathcal{O}_w}\mapsto T_w$$

for every $w\in\widehat{\mathfrak{S}}_n$.

To identify the positive part of the affine Hecke algebra, it is necessary to recall its different presentations.

5.2. The affine Hecke algebra

Definition 1 *(The affine Hecke algebra $\widehat{\mathcal{H}}_n$).* The affine Hecke algebra $\widehat{\mathcal{H}}_n$ is a $\mathbb{C}[v,v^{-1}]$-algebra which may be defined by generators and relations in either of the following ways:

1. The generators are the T_w, for $w\in\widehat{\mathfrak{S}}_n=\mathfrak{S}_n\ltimes\mathbb{Z}^n$. The relations are:
   (1) $T_wT_{w'}=T_{ww'}$ if $l(ww')=l(w)+l(w')$, where $l(w)$ is the length of w,
   (2) $(T_{s_i}+1)(T_{s_i}-v^{-2})=0$ for $s_i=(i,i+1)$.
2. The generators are $T_i^{\pm1}$, $i=1\dots n-1$ and $X_j^{\pm1}$, $j=1\dots n$. The relations are:
   (1) $T_iT_j=T_jT_i$ if $|i-j|>1$,
   (2) $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$,
   (3) $T_iT_i^{-1}=T_i^{-1}T_i=1$,
   (4) $(T_i+1)(T_i-v^{-2})=0$,
   (5) $X_iX_i^{-1}=X_i^{-1}X_i=1$,
   (6) $X_iT_j=T_jX_i$ if $i\neq j,j+1$,
   (7) $T_iX_iT_i=v^{-2}X_{i+1}$,
   (8) $X_iX_j=X_jX_i$.

The isomorphism ψ between these two presentations is uniquely defined by the following conditions:

$$\psi(T_{s_i})=T_i,$$
$$\psi\big(\tilde{T}^{-1}_{(\lambda_1,\dots,\lambda_n)}\big)=X_1^{\lambda_1}\cdots X_n^{\lambda_n}$$

if λ is dominant, which means $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$, and if we write $\tilde{T}_w=v^{-l(w)}T_w$.

One checks that:

$$T_\rho\mapsto v^{1-n}X_1^{-1}T_1\cdots T_{n-1}.$$

The multiplication map defines an isomorphism of $\mathbb{C}$-vector spaces

$$\widehat{\mathcal{H}}_n\simeq\mathbb{C}[\mathfrak{S}_n]\otimes_{\mathbb{C}}\mathbb{C}\big[v,v^{-1}\big]\big[X_1^{\pm1},\dots,X_n^{\pm1}\big].$$

5.3. The positive subalgebra

We are now ready to describe the convolution algebra B^+ of G-invariant functions on the positive part $\mathcal{X}$ of the product variety $\widehat{X}\times\widehat{X}$.

Theorem 5.4. *The algebra $\mathbb{C}_G(\mathcal{X})$ is isomorphic to the subalgebra $\widehat{\mathcal{H}}_n^+$ of $\widehat{\mathcal{H}}_n$ generated by $\mathcal{H}_n$ and the elements X_i. This means that as a vector space, we have*:

$$\mathbb{C}_G(\mathcal{X})\simeq\mathbb{C}[\mathfrak{S}_n]\otimes_{\mathbb{C}}\mathbb{C}\big[v,v^{-1}\big][X_1,\dots,X_n].$$

Proof. The first observation is that every element T_i is in $\mathbb{C}_G(\mathcal{X})$.

The element X_1 is in $\mathbb{C}_G(\mathcal{X})$ because, by the isomorphism ψ introduced Section 5.2, we have $X_1 = v^{1-n}T_{n-1}\cdots T_1 T_\rho^{-1}$. But the element T_ρ^{-1} is the characteristic function of an orbit in $\mathcal{X}$. As X_1 is in $\mathbb{C}_G(\mathcal{X})$, the relations (7) prove that the X_is are in $\mathbb{C}_G(\mathcal{X})$ as well.

We will now prove that the algebra $\mathbb{C}_G(\mathcal{X})$ is generated by the elements T_ρ^{-1}, $T_1, \ldots, T_{n-1}$. For this purpose we first get back to the groups.

Lemma 5.5. *The subsemigroup $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ of $\mathfrak{S}_n \ltimes \mathbb{Z}^n$ is generated by the elements $\rho^{-1}, s_1, \ldots, s_{n-1}$.*

Proof. First it is clear that the elements $s_1, \ldots, s_{n-1}$ are in $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$, because they are in $\mathfrak{S}_n$. The element ρ^{-1} can be written in $\mathfrak{S}_n \ltimes \mathbb{Z}^n$ as $((n\cdots 21), (0, \ldots, 0, -1))$ (to see it, use the bijection $\mathfrak{S}_n \ltimes \mathbb{Z}^n \simeq \mathrm{Aut}_n(\mathbb{Z})$). This element belongs to $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$.

We now prove that every element w of $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ can be written as a product of elements among $\rho^{-1}, s_1, \ldots, s_{n-1}$.

We define the degree of an element $w = (\sigma, (\lambda_i)) \in \mathfrak{S}_n \ltimes \mathbb{Z}^n$ by $d = \sum_{i=1}^n \lambda_i$. Let us prove the result by induction on the degree d of $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$.

If $d = 0$, the result is true because w is an element in $\mathfrak{S}_n$.

For $d < 0$, let us consider $w' = \rho w$. The degree of w' is $d+1$, so we have the result by induction if $w' \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. Only the case where $w' \notin \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ remains.

Under the isomorphism $\mathfrak{S}_n \ltimes \mathbb{Z}^n \simeq \mathrm{Aut}_n(\mathbb{Z})$, the subset $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ is mapped to $\{\tilde{s} \in \mathrm{Aut}_n(\mathbb{Z}),\ \forall i = 1, \ldots, n,\ \tilde{s}(i) \leqslant n\}$. The conditions $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ and $w' = \rho w \notin \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ give in $\mathrm{Aut}_n(\mathbb{Z})$: for every $i = 1, \ldots, n$, $\tilde{w}(i) \leqslant n$ and $\exists j,\ 1 \leqslant j \leqslant n$, $\tilde{w}'(j) = \tilde{w}(j) + 1 \geqslant n+1$. For this j we have: $\tilde{w}(j) + 1 = n+1$ hence $\tilde{w}(j) = n$, which is equivalent to $\sigma(j) = n$ and $\lambda_j = 0$.

Besides, we have $d < 0$, so there is a k such that $\lambda_k < 0$. We write $t \in \mathfrak{S}_n$ for the transposition (kj) and we consider $w'' = wt$. Then $w'' = (\sigma t, (\lambda_i)_1^n)$ and so for every $i = 1, \ldots, n$, $\tilde{w}''(i) < n$. We just saw that this is equivalent to $\rho w'' \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. So we can apply our induction to $x = \rho w''$, which is of degree $d+1$, to obtain that x is in the subsemigroup generated by $\rho^{-1}, s_1, \ldots, s_{n-1}$. As $w = \rho^{-1} x t$, this is also true for w. □

To lift this result to the Hecke algebra, we need a little more: we have to prove that every element of $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ has a reduced decomposition as a product of $s_1, \ldots, s_{n-1}, \rho^{-1}$.

Lemma 5.6. *Every element of $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ has a reduced decomposition which involves only the elements $s_1, \ldots, s_{n-1}, \rho^{-1}$.*

Proof. For $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$, we write k for its length and d for its degree ($d \leqslant 0$). We now proceed by induction on $k - d$.

If $k = 0$, the element w is of length 0 so it is a power of ρ, which is negative because $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. We are done.

If $d = 0$, the element w is of degree 0 in $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$ so it is an element of $\mathfrak{S}_n$. Then we are done because an element of $\mathfrak{S}_n$ has a minimal decomposition which uses only $s_1, \ldots, s_{n-1}$.

The last case is when $d < 0$ and $k > 0$. We know that w has a minimal decomposition of the form:

$$w = \rho^l s_{i_1} \cdots s_{i_k}$$

where $0 \leqslant i_r \leqslant n-1$.

We now split the proof in two cases:

If $s_{i_k} \neq s_0$, then we can apply the induction to ws_{i_k}, which has the same degree as w and whose length is $l(w) - 1$. We deduce from this a minimal writing of ws_{i_k} which involves only the elements $s_1, \ldots, s_{n-1}, \rho^{-1}$, then a minimal writing of w using only these elements, by multiplying by s_{i_k}.

The remaining case is when $s_{i_k} = s_0$. In this case we have $l(ws_0) < l(w)$. We know at this point by using the isomorphism $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n \simeq \mathrm{Aut}_n(\mathbb{Z})$ (cf. [S, Cor. 4.2.3], or [G, Cor. 1.3.3]) that $l(ws_0) < l(w)$

implies that $\tilde{w}(0) > \tilde{w}(1)$. Take the element $w' = ws_0\rho$, and let us show that it belongs to $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. We need to show that for every i such that $1 \leqslant i \leqslant n$, we have $\tilde{w}'(i) \leqslant n$.

By definition $w'(i) = ws_0(i+1)$. So if $1 \leqslant i \leqslant n-2$, we have that $\tilde{w}'(i) = \tilde{w}(i+1) \leqslant n$ because $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. We have $\tilde{w}'(n) = \tilde{w}(s_0(n+1)) = \tilde{w}(n) \leqslant n$ too because $w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$. We can deduce that $\tilde{w}'(n-1) = \tilde{w}(n+1) = n + \tilde{w}(1) < n + \tilde{w}(0) = \tilde{w}(n) \leqslant n$, from which it follows that $w' \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n$.

As w' has a length equal to $k-1$ and degree $d+1$ and is in the semigroup $\mathfrak{S}_n \ltimes \mathbb{Z}_-^n$, we can apply the induction hypothesis: w' has a minimal writing which involves only $s_1, \ldots, s_{n-1}, \rho^{-1}$. But as $w = w'\rho^{-1}s_0 = w's_1\rho^{-1}$, we have a minimal writing of w using the $s_1, \ldots, s_{n-1}, \rho^{-1}$. □

Observe that by construction, we have:

$$\mathbb{C}_G(\mathcal{X}) = \bigoplus_{w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n} \mathbb{C}\big[v, v^{-1}\big] 1_{\mathcal{O}_w} = \bigoplus_{w \in \mathfrak{S}_n \ltimes \mathbb{Z}_-^n} \mathbb{C}\big[v, v^{-1}\big] T_w.$$

By Lemma 5.6, any T_w may be written as a product of elements $T_\rho^{-1}, T_1, \ldots, T_{n-1}$. We easily check that the algebra generated by these elements is precisely $\mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}} \mathbb{C}[v, v^{-1}][X_1, \ldots, X_n]$. □

6. Quotients

Now that we have at our disposal a bicommutant theorem for the positive parts of the Hecke algebra and the Schur algebra, we can try to find subvarieties whose corresponding subalgebras are two-sided ideals of these algebras, which allows us to take quotients and hope to still have a bicommutant theorem.

Let $\lambda = (\lambda_i)_{i=1}^n \in \mathbb{N}^n$ be a dominant partition (i.e. $\lambda_1 \geqslant \cdots \geqslant \lambda_n$). For every $(L_\bullet, L'_\bullet) \in \mathcal{X}$, as $L'_0 \subseteq L_0$ are two free $\mathbb{F}_q[\![z]\!]$-modules of rank n, the quotient L_0/L'_0 is a torsion $\mathbb{F}_q[\![z]\!]$-module of rank at most n. We know that the isomorphism classes of torsion $\mathbb{F}_q[\![z]\!]$-modules of rank at most n are parametrized by the dominant n-weights $\mu_1, \ldots, \mu_n$, with $\mu_1 \geqslant \cdots \geqslant \mu_n$.

We define:

$$\mathcal{X}_\lambda = \big\{ (L_\bullet, L'_\bullet) \in \mathcal{X},\ L_0/L'_0 \text{ of type } \mu \,\big|\, \forall i = 1, \ldots, d,\ \lambda_i \leqslant \mu_i \big\}.$$

Lemma 6.1. *The set $\mathbb{C}_G(\mathcal{X}_\lambda)$ is a two-sided ideal of $\mathbb{C}_G(\mathcal{X})$.*

Proof. We have to check that if f and g are in $\mathbb{C}_G(\mathcal{X})$ with f supported on $\mathcal{X}_\lambda$, then $f * g$ and $g * f$ are supported on $\mathcal{X}_\lambda$.

But if $(L_\bullet, L'_\bullet) \in \mathcal{X}$ and $(L'_\bullet, L''_\bullet) \in \mathcal{X}_\lambda$, we have that $L''_0 \subseteq L'_0 \subseteq L_0$ and L'_0/L''_0 is of type μ, with $\forall i = 1, \ldots, n$, $\lambda_i \leqslant \mu_i$. From the inclusion $L'_0/L''_0 \subseteq L_0/L''_0$ we deduce that L_0/L''_0 is of type ν with $\nu_i \geqslant \mu_i$ $\forall i$. So $\nu_i \geqslant \lambda_i$ and $(L_\cdot, L''_\cdot)$ belongs to $\mathcal{X}_\lambda$. Finally $f * g$ is supported on $\mathcal{X}_\lambda$.

If $(L_\bullet, L'_\bullet) \in \mathcal{X}_\lambda$ and $(L'_\bullet, L''_\bullet) \in \mathcal{X}$ then we have $L''_0 \subseteq L'_0 \subseteq L_0$ and L_0/L'_0 is of type μ with $\mu_i \geqslant \lambda_i$ $\forall i = 1 \cdots n$. As L_0/L'_0 is a quotient of L_0/L''_0, the type ν of L_0/L''_0 verifies $\nu_i \geqslant \mu_i$. So $\nu_i \geqslant \lambda_i$ and $(L_\bullet, L''_\bullet) \in \mathcal{X}_\lambda$, which gives that $g * f$ is supported on $\mathcal{X}_\lambda$. □

We now define for i, j in I:

$$\mathcal{Z}_{\lambda,i} = (\phi_i \times Id)(\mathcal{X}_\lambda) \subseteq \mathcal{Z}_i,$$

$$\mathcal{Y}_{\lambda,i,j} = (\phi_i \times \phi_j)(\mathcal{X}_\lambda) \subseteq \mathcal{Y}_{i,j},$$

$$\mathcal{Z}_\lambda = \bigsqcup_{i \in I} \mathcal{Z}_{\lambda,i},$$

$$\mathcal{Y}_\lambda = \bigsqcup_{i,j \in I} \mathcal{Y}_{\lambda,i,j}.$$

In the same way $\mathbb{C}_G(\mathcal{X}_\lambda)$ is a two-sided ideal of $\mathbb{C}_G(\mathcal{X})$, we have the following statement:

Lemma 6.2. *The set $\mathbb{C}_G(\mathcal{Y}_\lambda)$ is a two-sided ideal of $\mathbb{C}_G(\mathcal{Y})$.*

The actions of $\mathbb{C}_G(\mathcal{X}_\lambda)$ and $\mathbb{C}_G(\mathcal{Y}_\lambda)$ map the space $\mathbb{C}_G(\mathcal{Z})$ to $\mathbb{C}_G(\mathcal{Z}_\lambda)$. Now write $A_\lambda = \mathbb{C}_G(\mathcal{Y})/\mathbb{C}_G(\mathcal{Y}_\lambda)$, $B_\lambda = \mathbb{C}_G(\mathcal{X})/\mathbb{C}_G(\mathcal{X}_\lambda)$ and $C_\lambda = \mathbb{C}_G(\mathcal{Z})/\mathbb{C}_G(\mathcal{Z}_\lambda)$. The quotient space C_λ is then an (A_λ, B_λ)-bimodule. We will write $\widehat{\mathcal{H}}_{n,\lambda}$ for B_λ in the next part.

We can now state:

Theorem 6.3 *(Bicommutant of the quotient). We have*

$$\operatorname{End}_{A_\lambda}(C_\lambda) = B_\lambda,$$

$$\operatorname{End}_{B_\lambda}(C_\lambda) = A_\lambda.$$

Proof. We prove the first assertion.

Let $P \in \operatorname{End}_{A_\lambda}(C_\lambda)$. As in Theorem 2.1, the fact that the endomorphism P commutes with the action of A_λ implies that it commutes with the action of the projectors on $C_{\lambda,i}$ where $C_{\lambda,i} = \mathbb{C}_G[\mathcal{Z}_i]/\mathbb{C}_G[\mathcal{Z}_{\lambda,i}]$, so the subspaces $C_{\lambda,i}$ are stable by P.

We also know that as an A_λ-module, C_λ is generated by $C_{\lambda,\omega}$. So we only have to study the restriction of P to $C_{\lambda,\omega}$, where P commutes with the action of $A_{\lambda,\omega,\omega}$.

But there are canonical isomorphisms $A_{\lambda,\omega,\omega} \simeq C_{\lambda,\omega} \simeq B_\lambda$. Then we can consider that P belongs to $\operatorname{End}_{B_\lambda}(B_\lambda)$ which is equal to B_λ. This proves the result.

Now we prove the second point of the theorem.

Let P be in $\operatorname{End}_{B_\lambda}(C_\lambda)$, and let P_i be its restriction to $C_{\lambda,i}$, so that $P_i \in \operatorname{Hom}_{B_\lambda}(C_{\lambda,i}, C_\lambda)$.

The canonical morphisms given in Theorem 2.1 go through to the positive parts to give for every $i \in I$ an injection:

$$\chi_i : C_i^+ \hookrightarrow B^+$$

whose left inverse is given by the left multiplication by $\frac{1}{m_i} 1_{\Delta_i}$, where Δ_i is the diagonal in $(\widehat{Y}_i \times \widehat{X})^+$.

We write α for the map from C_λ to C_+ which associates to a function in C_λ its unique representative in $\mathbb{C}_G(\mathcal{Z})$ whose restriction to $\mathcal{Z}_\lambda$ is zero.

Now define $P'_i \in \operatorname{Hom}_{B_+}(C_{+,i}, C_+)$ by the formula:

$$P'_i(f) = \alpha\left(P_i\left(\frac{1}{m_i} 1_{\Delta_i}\right)\right) * \chi_i(f),$$

and $P' = \bigoplus_{i\in I} P_i$.

We easily check that P' commutes with the action of B_λ and secondly that through the canonical map $\operatorname{End}_{B_+}(C_+) \to \operatorname{End}_{B_\lambda}(C_\lambda)$ the morphism P' maps to P.

We have lifted P and got $P' \in \operatorname{End}_{B_+}(C_+)$. The bicommutant theorem for the positive parts gives us the fact that $P' \in A_+$, hence $P \in A_\lambda$ as desired. We are done. □

We can now identify the two-sided ideal in question.

Proposition 6.4. *The two-sided ideal $\mathbb{C}_G(\mathcal{X}_\lambda)$ is generated by the element $X^{\lambda'} = \prod_{i=1}^n X_i^{\lambda_{n-i}}$.*

Proof. From the definition of $\mathbb{C}_G(\mathcal{X}_\lambda)$, it is obvious that as a vector space we can write:

$$\mathbb{C}_G(\mathcal{X}_\lambda) = \bigoplus_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{dom}(\mu)\geqslant\lambda}} \mathbb{C}T_{(\sigma,-\mu)}$$

where $\mathrm{dom}(\mu)$ is the partition deduced from μ by reordering, and where the partial order between compositions is given by $\lambda \geqslant \mu \Leftrightarrow \forall i = 1, \ldots, n,\ \lambda_i \geqslant \mu_i$. In particular the element $\prod_{i=1}^{n} X_i^{\lambda_{n-i}} = v^{-l(\lambda)} T_{(Id,-\lambda')}$ belongs to $\mathbb{C}_G(\mathcal{X}_\lambda)$, and thus $\widehat{\mathcal{H}}_n^+ X^{\lambda'} \widehat{\mathcal{H}}_n^+ \subseteq \mathbb{C}_G(\mathcal{X}_\lambda)$.

The inclusion of the right-hand side in the left-hand side is done.

For the other inclusion, prove first:

Lemma 6.5. *For every dominant composition ν the following holds*:

$$\mathcal{H}_n T_{(Id,\nu)} \mathcal{H}_n = \bigoplus_{\sigma,\sigma' \in \mathfrak{S}_n} \mathbb{C} T_{(\sigma,\nu^{\sigma'})}.$$

Proof. As the element $T_{(Id,\nu)}$ belongs to the sum on the right-hand side and this space is stable by the action of $\mathcal{H}_n$ on the right and on the left, the inclusion of the left-hand side in the right-hand side is clear.

We show by induction on the length of σ' that every element in the right-hand side belong to $\mathcal{H}_n T_{(Id,\nu)} \mathcal{H}_n$.

For $\sigma' = Id$: as we have $l(\sigma, \nu) = l(\sigma, 0) + l(Id, \nu)$ because ν is dominant, the equation $T_{(Id,\nu)}.T_{(\sigma,0)} = T_{(\sigma,\nu)}$ holds, which implies that $T_{(\sigma,\nu)} \in \mathcal{H}_n T_{(Id,\nu)} \mathcal{H}_n$.

For $l(\sigma') > 0$: we write $\sigma' = ts$, where $s \in S$ and $l(t) = l(\sigma') - 1$. By induction we know that for every $u \in \mathfrak{S}_n$, we have $T_{(u,\nu^t)} \in \mathcal{H}_n T_{(Id,\sigma)} \mathcal{H}_n$. The following holds:

$$T_{(u,\nu^t)}.T_{(s,0)} = \begin{cases} T_{(us,\nu^{\sigma'})} & \text{if } (us, \nu^{\sigma'}) = l(u, \nu^t) + 1, \\ (1-q) T_{(u,\nu^t)} + q T_{(us,\nu^{\sigma'})} & \text{if } l(us, \nu^{\sigma'}) = l(u, \nu^t) - 1. \end{cases}$$

As the left-hand side term and $T_{(u,\nu^t)}$ belong to $\mathcal{H}_n T_{(Id,\nu)} \mathcal{H}_n$ for every $u \in \mathfrak{S}_n$, we have also $T_{(\sigma,\nu^{\sigma'})} \in \mathcal{H}_n T_{(Id,\nu)} \mathcal{H}_n$ for every $\sigma \in \mathfrak{S}_n$. □

To prove the proposition, we first remark that we have the equality $T_{(Id,-\mathrm{dom}(\mu)')} = T_{(Id,-\lambda')}.T_{(Id,-\mathrm{dom}(\mu)'+\lambda')}$ for each composition μ such that $\mathrm{dom}(\mu) \geqslant \lambda$, which implies that $T_{(Id,-\mathrm{dom}(\mu)')} \in \mathbb{C}_G(\mathcal{X}_\lambda)$. By applying the lemma to $-\mathrm{dom}(\mu)'$ for each μ such that $\mathrm{dom}(\mu) \geqslant \lambda$ we obtain the inclusion:

$$\mathbb{C}_G(\mathcal{X}_\lambda) \subseteq \bigoplus_{\mathrm{dom}(\mu) \geqslant \lambda} \mathcal{H}_n X^{\mathrm{dom}(\mu)'} \mathcal{H}_n = \widehat{\mathcal{H}}_n^+ X^{\lambda'} \widehat{\mathcal{H}}_n^+$$

which gives the equality. □

So far we have defined for each partition λ a closed subset $\mathcal{X}_\lambda$ and a two sided ideal $I_\lambda = \mathbb{C}_G(\mathcal{X}_\lambda)$ generated by $X^{\lambda'}$. We can ask if every two sided ideal comes this way.

The first remark to make is that we can associate to every finite set of partition $\underline{\lambda} = (\lambda^{(i)})_i$ the closed subset $\bigcup_i \mathcal{X}_{\lambda^{(i)}}$. The corresponding two-sided ideal is the sum $I_{\underline{\lambda}} = \sum_i I_{\lambda^{(i)}}$. It is easy to see that the bicommutant theorem holds in this case too (the proofs are the same).

Theorem 6.6. *Every G-stable closed subset $\mathcal{F}$ of $\mathcal{X}$ such that $\mathbb{C}_G(\mathcal{F})$ is a two-sided ideal of $\mathbb{C}_G(\mathcal{X})$ is of the form $\mathcal{X}_{\underline{\lambda}}$, for a finite set of partition $\underline{\lambda} = (\lambda^{(i)})_i$.*

Proof. As $\mathbb{C}$-vector spaces, we have

$$\mathbb{C}_G(\mathcal{F}) = \bigoplus_{\mathcal{O}_w \subseteq \mathcal{F}} \mathbb{C} T_w. \tag{2}$$

The next lemma is a refinement of Lemma 6.5.

Lemma 6.7. *For every $w \in \widehat{\mathfrak{S}}_n$ we have*:

$$\mathcal{H}_n T_w \mathcal{H}_n = \bigoplus_{\sigma,\sigma' \in \mathfrak{S}_n} \mathbb{C} T_{\sigma w \sigma'}.$$

We use the lemma to rewrite the sum (2) as:

$$\mathbb{C}_G(\mathcal{F}) = \sum_{\mathcal{O}_{(Id,\lambda)} \subseteq \mathcal{F}} \mathcal{H}_n T_{(Id,\lambda)} \mathcal{H}_n \tag{3}$$

where the sum is over the dominant partitions λ. Indeed, every $w \in \widehat{\mathfrak{S}}_n$ belongs to a class $\mathfrak{S}_n(Id,\lambda)\mathfrak{S}_n$ for a dominant λ.

We have the usual partial order on the partitions λ, and the set of minimal partitions $(\lambda^{(i)}) = \underline{\lambda}$ is finite. Using that $\mathbb{C}_G(\mathcal{F})$ is in fact a $(\widehat{\mathcal{H}}_n^+, \widehat{\mathcal{H}}_n^+)$-bimodule, the equality (3) gives:

$$\mathbb{C}_G(\mathcal{F}) = \sum_i \widehat{\mathcal{H}}_n^+ T_{(Id,\lambda^{(i)})} \widehat{\mathcal{H}}_n^+ = \mathbb{C}_G(\mathcal{X}_{\underline{\lambda}}).$$

Then $\mathcal{F} = \mathcal{X}_{\underline{\lambda}}$. □

Remark 6.8. There is an established Schur–Weyl duality between cyclotomic Hecke algebras and the so-called cyclotomic q-Schur algebras (cf. [SS,A]), but only in the semisimple case. Our quotients are cyclotomic Hecke algebras (with all parameters equal to zero) when we take the partition $(d, 0, \ldots, 0)$ and the semisimplicity is obviously not verified in this case.

7. Canonical basis of $\widehat{\mathcal{H}}_{n,\lambda}$

The geometric construction of our algebras allows us to construct canonical bases for them. Such bases, which are also called Kazhdan–Lusztig bases for Hecke algebras, were introduced for quantum enveloping algebras by Kashiwara and Lusztig (see [L1]). These bases have several important properties, which include positivity of the structure constants and compatibility with bases of representations (see [A2,LLT,VV]).

Write $\zeta : \widehat{\mathcal{H}}_n^+ \to \widehat{\mathcal{H}}_{n,\lambda}$ for the quotient map.

Let us call B the canonical basis of the affine Hecke algebra $\widehat{\mathcal{H}}_n$. This basis $B = (b_{\mathcal{O}})$ is defined by the formula:

$$b_{\mathcal{O}} = \sum_{i,\mathcal{O}'} v^{-i+\dim \mathcal{O}} \dim \mathcal{H}^i_{\mathcal{O}'}(IC_{\mathcal{O}}) 1_{\mathcal{O}'}$$

where $\mathcal{H}^i_{\mathcal{O}'}(IC_{\mathcal{O}})$ is the fiber at any point in $\mathcal{O}'$ of the cohomology sheaf of the intersection complex of $\mathcal{O}$.

As $\mathcal{X}$ is a closed subset of $\widehat{X} \times \widehat{X}$, the subset B^+ of B defined by $B^+ = \{b \in B, b \in \mathbb{C}_G(\mathcal{X})\}$ is a basis of $\mathbb{C}_G(\mathcal{X})$.

Theorem 7.1. *The set of elements*:

$$B' = \left\{ \zeta(b),\ b \in B^+ \,\middle|\, \zeta(b) \neq 0 \right\}$$

form a basis of $\widehat{\mathcal{H}}_{n,\lambda}$.

Proof. It suffices to see that:

$$\mathrm{Ker}(\zeta) = \bigoplus_{\phi(b)=0} \mathbb{C}b.$$

As $\mathrm{Ker}(\zeta)$ is the set of functions supported on the closed subset $\mathcal{X}_\lambda$, the elements $b_{\mathcal{O}}$, where $\mathcal{O} \subseteq \mathcal{X}_\lambda$ form a basis of $\mathrm{Ker}(\zeta)$. The theorem follows. □

8. The case $d < n$

The previous bicommutant theorems are true only in the case $d \geqslant n$. In the case $d < n$, one half of the result still holds.

Theorem 8.1. *If $d < n$ the map*:

$$\mathbb{C}_G(X \times X) \to \mathrm{End}_{\mathbb{C}_G(Y\times Y)}\big(\mathbb{C}_G(Y \times X)\big)$$

is surjective, when X is the complete (resp. affine) flag variety and Y the (resp. affine) flag variety of length d.

Proof. Let $d < n$. We associate as before to each composition $\underline{d}$ of n of length d with a connected component $Y_{\underline{d}}$ of the affine flag variety of length d. To a composition of n of length d we associate a subset of $S = \{0, \ldots, n-1\}$ of order at least $n - d$, in the way that the composition of n give the sequence dimensions of successive factors in the flag while the set I give which step in a complete flag are forgotten. We have a bijection between these subsets and isomorphism classes of connected components in Y.

We then write Y_I for a connected component in the corresponding class.

Let W be the extended affine Weyl group of GL_n. We recall that it is the semidirect product $W' \rtimes \Omega$, where W' is the affine Weyl group of GL_n and Ω is isomorphic to $\mathbb{Z}$, generated by ρ. The group W' is a Coxeter group, which is equipped with the length function l. Each element w of W can be uniquely written $w'\rho^z$, where w' is an element of W' and $z \in \mathbb{Z}$. We define the length $l(w)$ of w by $l(w')$ and its height $h(w) = |z|$.

For each $I \subseteq S$ there is a G-invariant surjective map:

$$\phi_I : X \twoheadrightarrow Y_I$$

and for each $I \subseteq J$ there is also a surjective G-morphism:

$$\phi_{I,J} : Y_I \twoheadrightarrow Y_J.$$

From these we deduce the maps:

$$\theta_I : \mathbb{C}_G(X \times X) \to \mathbb{C}_G(Y_I \times X),$$

$$\theta_{I,J} : \mathbb{C}_G(Y_I \times X) \to \mathbb{C}_G(Y_J \times X)$$

given by:

$$\theta_I(f)(L, L') = \sum_{L'' \in \phi_I^{-1}(L)} f(L'', L') = 1_{\Delta(Y_I \times X)} * f$$

and

$$\theta_{I,J}(g)(L,L') = \sum_{L''\in\phi_{I,J}^{-1}(L)} g_I(L'',L) = 1_{\Delta(Y_J\times Y_I)} * g_I.$$

They commute with the right action of $\mathbb{C}_G(X\times X)$ by convolution.

By summing over the $I\subseteq S$ of order greater or equal to $n-d$, we define a map:

$$\theta : \mathbb{C}_G(X\times X) \to \mathbb{C}_G(Y\times X).$$

Lemma 8.2. *The image of the map θ is the set*:

$$\left\{ f = \sum_{|I|\geqslant n-d} f_I \in \mathbb{C}_G(Y\times X) \;\middle|\; \forall I\subseteq J,\ f_J = \theta_{I,J}(f_I) \right\}.$$

Proof. The inclusion of the image of θ in this set comes from the equality $\theta_{I,J}\circ\theta_I = \theta_J$.

Let's prove the other inclusion. We must find, given a family of functions $(f_I)_{|I|\geqslant n-d}$ verifying for each $I\subseteq J$ the equality $\theta_{I,J}(f_I) = f_J$, a function f in $\mathbb{C}_G(X\times X)$ such that $f_I = \theta_I(f)$ for each I.

In order to give a function in $\mathbb{C}_G(X\times X)$, we have to give a value for each orbit $\mathcal{O}_w$, with $w\in W$. By abuse of notation, we will denote that value by $f(\mathcal{O}_w)$. We proceed in two steps.

Consider the set M of all $w\in W$ such that for each I of order at least $n-d$, the orbit $\mathcal{O}_w$ is not open in the fiber $(\phi_I\times Id)^{-1}(\mathcal{O}_{W_I w})$, where $\mathcal{O}_{W_I w}$ is the image of $\mathcal{O}_w$ in $Y_I\times X$. It is equivalent to say that for each I of order at least $n-d$ (and strictly less than n), the element w is not of maximal length in the class $W_I w$ (seen as a subset of W), where W_I is the Young subgroup of W generated by the elements s_i with $i\in I$ (it is a finite group because $|I|<n$).

The functions f_I have a compact support, so we can choose k such that for each $w\in W$ of length or height greater or equal to k, f_I vanishes on $\mathcal{O}_{W_I w}$ for every I.

Fix an integer $l>k$. For each element w of M of length less or equal to l we assign an arbitrary value to $f(\mathcal{O}_w)$, and for the element of length or height greater or equal to l we set $f(\mathcal{O}_w)=0$.

Now we have to define $f(\mathcal{O}_w)$ for $w\in W-M$. By definition, for such an element w there exists a set I for which the orbit $\mathcal{O}_w$ is dense in the fiber $(\phi_I\times Id)^{-1}(\mathcal{O}_{W_I w})$.

We know that w is of maximal length in $W_I w$ if and only if for each $i\in I$ we have $l(s_i w) < l(w)$. We can deduce from this that there is a maximal set $I(w)$ such that w is of maximal length in $W_{I(w)}w$.

Now we define $f(\mathcal{O}_w)$ for $w\in W-M$, proceeding by induction on the length of w. To that purpose we use the following equation, where we write simply I for $I(w)$:

$$m_I(w) f(\mathcal{O}_w) = f_I(\mathcal{O}_{W_I w}) - \sum_{\substack{w'\in W_I w\\ w'\neq w}} m_I(w') f(\mathcal{O}_{w'})$$

where $m_I(w') = |\{L''_\bullet\in\phi_I^{-1}(L_\bullet),\ (L''_\bullet, L'_\bullet)\in\mathcal{O}_{w'}\}|$ for any $L_\bullet\in Y_I$ such that $(L_\bullet, L'_\bullet)\in\mathcal{O}_{W_I w}$.

This determines the values $f(\mathcal{O}_w)$ because each element in the sum has a length strictly smaller than $l(w)$, so that $f(\mathcal{O}_{w'})$ is already defined.

It remains to show that beyond a fixed length or height the obtained values $f(\mathcal{O}_w)$ are zero (for the function to be compactly supported), and that the function given by this method is a solution to our problem.

We start with the first point.

We prove first that for each element w of length greater or equal to $l+\frac{n(n-1)}{2}|I(w)|$, $f(\mathcal{O}_w)$ is zero.

If w is not maximal in any of its classes $W_I w$, then we have taken $f(\mathcal{O}_w) = 0$. If not, for $I = I(w)$, the element w is maximal in its class $W_I w$, and $f(\mathcal{O}_w)$ is given by:

$$m_I(w) f(\mathcal{O}_w) = f_I(\mathcal{O}_{W_I w}) - \sum_{\substack{w' \in W_I w \\ w' \neq w}} m_I(w') f(\mathcal{O}_{w'}).$$

In the sum the elements w' have a length greater or equal to $l + \frac{n(n-1)}{2}|I(w)| - \frac{n(n-1)}{2}$. But as they are not maximal in $W_I w$, if they are in M they satisfy $|I(w')| < |I(w)|$. Then if w' is in M it has a length greater or equal to $l + \frac{n(n-1)}{2}|I(w')|$, and if w' is not in M we have $f(\mathcal{O}_{w'}) = 0$. By induction on $|I(w')|$, we easily see that each $f(\mathcal{O}_{w'})$ in the right sum is zero, and $f_I(\mathcal{O}_{W_I w})$ too, so $f(\mathcal{O}_w)$ is zero. We also know that $f(\mathcal{O}_w)$ is zero for each element w of length greater or equal to l. Therefore f is non-zero only on a finite set.

It remains to check that the function just built is a solution to our problem. We have to show that for each I and for each $w \in W$, the following holds:

$$f_I(\mathcal{O}_{W_I w}) = \sum_{w' \in W_I w} m_I(w') f(\mathcal{O}_{w'}).$$

Obviously, it is sufficient to check this equation when w is of maximal length in its class $W_I w$. If $I = I(w) =: J$, then the equation is true by construction of $f(\mathcal{O}_w)$. If not we have $I \subseteq J$. We proceed by induction: suppose that the equality is true for each x such that $l(x) < l(w)$.

By construction of $f(\mathcal{O}_w)$ we have:

$$f_J(\mathcal{O}_{W_J w}) = \sum_{w'' \in W_J w} m_J(w'') f(\mathcal{O}_{w''}).$$

If we decompose the sum along the elements of the class $W_I \setminus W_J w$ we obtain:

$$f_J(\mathcal{O}_{W_J w}) = \sum_{x \in W_I \setminus W_J w} \sum_{z \in W_I x} m_J(z) f(\mathcal{O}_z),$$

$$f_J(\mathcal{O}_{W_J w}) = \sum_{\substack{x \in W_I \setminus W_J w \\ x \neq W_I w}} \sum_{z \in W_I x} m_J(z) f(\mathcal{O}_z) + \sum_{y \in W_I w} m_J(y) f(\mathcal{O}_y).$$

For $w' \in W$ we define $m_{I,J}(W_I w') = |\{L''_\bullet \in \phi_{I,J}^{-1}(L_\bullet),\ (L''_\bullet, L'_\bullet) \in \mathcal{O}_{W_I w'}\}|$ for any $L_\bullet \in Y_J$ such that $(L_\bullet, L'_\bullet) \in \mathcal{O}_{W_J w}$. From the identity $\phi_J = \phi_{I,J}\phi_I$ we deduce $m_J(w') = m_{I,J}(W_I w') m_I(w')$. So we can write:

$$f_J(\mathcal{O}_{W_J w}) = \sum_{\substack{x \in W_I \setminus W_J w \\ x \neq W_I w}} m_{I,J}(W_I x) \sum_{z \in W_I x} m_J(z) f(\mathcal{O}_z) + m_{I,J}(W_I w) \sum_{y \in W_I w} m_I(y) f(\mathcal{O}_y)$$

(where $W_I \setminus W_J w$ is seen as a subset of the quotient $W_I \setminus W$).

But each $z \in W_I x$ for $x \neq W_I w$ is of length less than $l(w)$. Then we can use

$$\sum_{z \in W_I x} m_I(z) f(\mathcal{O}_z) = f_I(\mathcal{O}_{W_I x}).$$

So we have:

$$f_J(\mathcal{O}_{W_J w}) = \sum_{\substack{x\in W_I\setminus W_J w\\ x\neq W_I w}} m_{I,J}(W_I x) f_I(\mathcal{O}_{W_I x}) + \sum_{y\in W_I w} m_I(y) f(\mathcal{O}_y).$$

But from the equality $f_J = \theta_{I,J}(f_I)$ the following holds:

$$f_J(\mathcal{O}_{W_J w}) = \sum_{\substack{x\in W_I\setminus W_J w\\ x\neq W_I w}} m_{I,J}(W_I x) f_I(\mathcal{O}_{W_I x}) + m_{I,J}(W_I w) f_I(\mathcal{O}_{W_I w}).$$

We finally obtain:

$$f_I(\mathcal{O}_{W_I w}) = \sum_{x\in W_I w} m_I(x) f(\mathcal{O}_x). \qquad \square$$

Let P be an element of $\mathrm{End}_{\mathbb{C}_G(Y\times Y)}(\mathbb{C}_G(Y\times X))$. As P commutes with the action of the characteristic functions of the diagonals of the components $Y_I\times X$, the subspaces $\mathbb{C}_G(Y_I\times X)$ are stable for the endomorphism P. So we can write $P=\bigoplus_I P_I$, where $P_I\in \mathrm{End}_{\mathbb{C}_G(Y_I\times Y_I)}(Y_I\times X)$.

Write $\psi_I:\mathbb{C}_G(Y_I\times Y_I)\hookrightarrow \mathbb{C}_G(Y_I\times X)$ for the injection deduced from the surjection $\phi_I: X\to Y_I$. Write $f_I := P_I(1_{\Delta(Y_I\times X)})$. For each $g\in\mathbb{C}_G(Y_I\times X)$ the following equalities hold:

$$P_I(g) = P_I\big(\psi_I(g) * 1_{\Delta(Y_I\times X)}\big) = \psi_I(g) * P_I(1_{\Delta(Y_I\times X)}) = \psi_I(g) * f_I.$$

The next step is to apply Lemma 8.2 to lift the f_Is to some $f\in\mathbb{C}_G(X\times X)$.

For this we have to check that the functions $f_I = P_I(1_{\Delta(Y_I\times X)})$ satisfy the hypothesis of the lemma.

But for $I\subseteq J$, we have $1_{\Delta(Y_J\times Y_I)} * 1_{\Delta(Y_I\times X)} = 1_{\Delta(Y_J\times X)}$, thus $f_J=\theta_{I,J}(f_I)$.

We deduce that there exists $f\in\mathbb{C}_G(X\times X)$ such that for each I of order greater or equal to $n-d$ we have $f_I=\theta_I(f)$. But by construction for $g\in\mathbb{C}_G(Y_I\times X)$ the product $\psi_I(g)*f_I$ is equal to $g*f$. We have shown that $P(g)=g*f,\ \forall g\in\mathbb{C}_G(Y\times X)$. The theorem follows. $\square$

Acknowledgment

This article is part of my PhD thesis, and I deeply thank my supervisor O. Schiffmann for his useful help, comments and his availability.

References

[A] S. Ariki, Cyclotomic q-Schur algebras as quotient of quantum algebra, J. Reine Angew. Math. 513 (1999) 53–69.
[A2] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m,1,n)$, J. Math. Kyoto Univ. 36 (1996) 789–808.
[CL] R.W. Carter, G. Lusztig, On the modular representations of general linear and symmetric groups, Math. Z. 136 (1974) 193–242.
[G] R.M. Green, The affine q-Schur algebra, J. Algebra 215 (1999) 379–411.
[Gr] J.A. Green, Polynomial Representations of GL_n, Lecture Notes in Math., vol. 830, Springer-Verlag, 1980.
[IM] N. Iwahori, H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups, Publ. Math. Inst. Hautes Etudes Sci. 25 (1965) 5–48.
[J] M. Jimbo, A q-analogue of $U(gl(N+1))$, Hecke algebras and the Yang–Baxter equation, Lett. Math. Phys. 11 (3) (1986) 247–252.
[L1] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, 1992.
[LLT] A. Lascoux, B. Leclerc, J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996) 205–263.
[S] J.-Y. Shi, The Kazhdan–Lusztig Cells in Certain Affine Weyl Groups, Lecture Notes in Math., vol. 1179, Springer-Verlag, 1986.
[SS] M. Sakamoto, T. Shoji, Schur–Weyl reciprocity for Ariki–Koike algebras, J. Algebra 221 (2) (1999) 293–314.
[VV] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J. 100 (2) (1999) 267–297.