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On the notion of Cohen–Macaulayness for non-Noetherian rings ☆

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A B S T R A C T

There exist many characterizations of Noetherian Cohen–Macaulay rings in the literature. These characterizations do not remain equivalent if we drop the Noetherian assumption. The aim of this paper is to provide some comparisons between some of these characterizations in non-Noetherian case. Toward solving a conjecture posed by Glaz, we give a generalization of the Hochster–Eagon result on Cohen–Macaulayness of invariant rings, in the context of non-Noetherian rings.

1. Introduction

Throughout this paper all rings are commutative, associative, with identity, and all modules are unital. The theory of Cohen–Macaulay rings is a keystone in commutative algebra. However, the study of such rings have mostly been restricted to the class of Noetherian rings. On the other hand, certain families of non-Noetherian rings and modules have achieved a great deal of significance in commutative algebra. For example, a surprising result of Hochster indicates that non-vanishing of a certain *Čech* cohomology module of the ring of absolute integral closure of a Noetherian domain implies the Directed Summand Conjecture, see [Ho2, Theorem 6.1]. While Noetherian Cohen–Macaulay modules are studied in several research papers, not so much is known about them in the non-Noetherian case. To the best of our knowledge, until 1992, there was not any idea for extending the concept

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of Cohen–Macaulayness to non-Noetherian rings. In that time Glaz [G3], considered the notion of Cohen–Macaulayness for not Noetherian rings and conjectured that invariant subrings of certain types of rings would be Cohen–Macaulay. Two years later, she [G4, p. 219] defined an R-module M to be Cohen–Macaulay (in the sense of Glaz) if for each prime ideal $\mathfrak{p}$ of R,

$$\operatorname{ht}_M(\mathfrak{p}) = \operatorname{p.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}),$$

where $\operatorname{p.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$ is the polynomial grade of $\mathfrak{p}R_\mathfrak{p}$ on $R_\mathfrak{p}$-module $M_\mathfrak{p}$. Unfortunately, coherent regular rings are not Cohen–Macaulay with this definition. Then, in the same paper, Glaz asked how one can define a non-Noetherian notion of Cohen–Macaulayness such that the definition coincides with the original one in the Noetherian case, and that coherent regular rings are Cohen–Macaulay, see [G4, p. 220]. In the following, we collect Glaz's desired properties of the notion of Cohen–Macaulayness for non-Noetherian rings.

Conjecture 1.1. *There exists a definition of the notion of non-Noetherian Cohen–Macaulay rings such that it satisfies the following three conditions.*

(i) *The definition coincides with the original definition in the Noetherian case.*
(ii) *Coherent regular rings are Cohen–Macaulay.*
(iii) *For a coherent regular ring R and a group G of automorphisms of R, assume that there exists a module retraction map $\rho : R \to R^G$ and that R is a finitely generated R^G-module. Then R^G is Cohen–Macaulay.*

Then, Hamilton [H1,H2,H3] has introduced the concept of weak Bourbaki (height) unmixed rings, as a first step towards non-Noetherian Cohen–Macaulay rings. Hamilton [H2] added the following two more properties that must be satisfied by non-Noetherian Cohen–Macaulay rings.

(H1) R is Cohen–Macaulay if and only if $R[X]$ is Cohen–Macaulay.
(H2) R is Cohen–Macaulay if and only if $R_\mathfrak{p}$ is Cohen–Macaulay for all prime ideals $\mathfrak{p}$ of R.

More recently, Hamilton and Marley [HM] introduced a definition for non-Noetherian Cohen–Macaulayness rings. If a ring R satisfies their definition, then we say that R is Cohen–Macaulay in the sense of Hamilton–Marley. They used the theory of *Čech* cohomology modules to show that Cohen–Macaulayness in the sense of Hamilton–Marley satisfies the assertions (i) and (ii) of Conjecture 1.1. Adopt the assumption of Conjecture 1.1(iii) and assume in addition that $\dim R \leqslant 2$ and G is finite such that its order is a unit in R. Then Hamilton and Marley proved the assertion (iii) of Conjecture 1.1. Also, they proved the if part of (H1) and (H2) by their definition.

Perhaps it is worth pointing out that there are many characterizations of Noetherian Cohen–Macaulay rings and modules. In the non-Noetherian case, these are not necessarily equivalent. All of these characterizations have been chosen as candidates for definition of non-Noetherian Cohen–Macaulay rings, see Definition 3.1. The aim of the present paper is to provide some comparisons between these definitions in not necessarily Noetherian case. Also, toward solving Conjecture 1.1, we will present a definition of the notion of Cohen–Macaulayness in not necessarily Noetherian case.

Let R be a ring and $\mathfrak{a}$ an ideal of R. The organization of this paper is as follows.

In Section 2, we deal with the notion of grade of ideals on modules. There are many definitions for the notion of grade of an ideal of a non-Noetherian ring. To make things easier, after recalling these definitions, for the convenience of the reader, we collect some of their properties. For our purpose, it seems to be better to use the Koszul grade. This notion of grade is based on the work [Ho1]. We denote the Koszul grade of an ideal $\mathfrak{a}$ on an R-module M by $\operatorname{K.grade}_R(\mathfrak{a}, M)$.

In Section 3, we explore interrelation between different definitions of non-Noetherian Cohen–Macaulay rings. These definitions include the Glaz and Hamilton–Marley definitions and the notion of weak Bourbaki unmixed rings. Assume that $\mathcal{A}$ is a non-empty subclass of the class of all ideals of a ring R. We give some connections between preceding modules and modules that are Cohen–

Macaulay modules in the sense of $\mathcal{A}$ (note that an R-module M is said to be Cohen–Macaulay in the sense of $\mathcal{A}$, if the equality

$$\mathrm{ht}_M(\mathfrak{a}) = \mathrm{K.grade}_R(\mathfrak{a}, M)$$

holds for all ideals $\mathfrak{a}$ in $\mathcal{A}$). These classes of ideals include the class of all finitely generated ideals, prime ideals, maximal ideals and the class of all ideals. Our work in this section is motivated by observing that the inequality

$$\mathrm{K.grade}_R(\mathfrak{a}, M) \leqslant \mathrm{ht}_M(\mathfrak{a})$$

holds for all ideals $\mathfrak{a}$ of R.

In Section 4, we construct three methods for introducing examples of non-Noetherian rings which are Cohen–Macaulay in the sense of any definition of Cohen–Macaulayness that appeared in the present paper. Our first example provides the Cohen–Macaulayness of the polynomial ring $R[X_1, X_2, \ldots]$, where R is Noetherian and Cohen–Macaulay. Our second example implies the Cohen–Macaulayness of absolute integral closure of Noetherian complete local domains of prime characteristic. Our third example concludes the Cohen–Macaulayness of the perfect closure of Noetherian regular local domains of prime characteristic.

In Section 5, we give another definition of Cohen–Macaulayness. We call it weakly Cohen–Macaulay, see Definition 5.1. Concerning Conjecture 1.1, we will present the following theorem.

Theorem 1.2. *The following assertions hold.*

(i) *A Noetherian ring is Cohen–Macaulay with original definition in Noetherian case if and only if it is weakly Cohen–Macaulay.*
(ii) *Coherent regular rings are weakly Cohen–Macaulay.*
(iii) *Let R be a weakly Cohen–Macaulay ring and G a finite group of automorphisms of R such that the order of G is a unit in R. Assume that R is finitely generated as an R^G-module. Then R^G is weakly Cohen–Macaulay.*
(iv) *Let R be a Noetherian Cohen–Macaulay ring. Then the polynomial ring $R[X_1, X_2, \ldots]$ is weakly Cohen–Macaulay.*
(v) *If $R_\mathfrak{p}$ is weakly Cohen–Macaulay for all prime ideals $\mathfrak{p}$ of R, then R is weakly Cohen–Macaulay.*

After proving Theorem 1.2, we continue our study of the behavior of rings of invariants of different types of non-Noetherian Cohen–Macaulay rings. In view of Definition 3.1, our list of the different definitions of Cohen–Macaulayness, includes Cohen–Macaulayness in the sense of (finitely generated) ideals, weak Bourbaki (height) unmixed.

2. Different types of the notion of grade

In this section $\mathfrak{a}$ is an ideal of a commutative ring R and M an R-module. We first give a general discussion on the notion of grade. There are many definitions for notion of grade of $\mathfrak{a}$ on M. Grade over not necessarily Noetherian rings was first defined by Barger [B] and Hochster [Ho1]. After them, Alfonsi [A] combined the grade notions of them into a more general notion of grade for non-Noetherian rings and modules. In this section, for the convenience of the reader, we collect some of their properties. To make things easier, we first recall them.

Definition 2.1. Let $\mathfrak{a}$ be an ideal of a ring R and M an R-module. Take Σ be the family of all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. Here, inf and sup are formed in $\mathbb{Z} \cup \{\pm\infty\}$ with the convention that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

(i) In order to give the definition of Koszul grade when $\mathfrak{a}$ is finitely generated by a generating set $\underline{x} := x_1, \ldots, x_r$, we first denote the Koszul complex related to $\underline{x}$ by $\mathbb{K}_\bullet(\underline{x})$. Koszul grade of $\mathfrak{a}$ on M is defined by

$$\text{K.grade}_R(\mathfrak{a}, M) := \inf\bigl\{i \in \mathbb{N} \cup \{0\} \bigm| H^i\bigl(\text{Hom}_R\bigl(\mathbb{K}_\bullet(\underline{x}), M\bigr)\bigr) \neq 0\bigr\}.$$

Note that by [BH, Corollary 1.6.22] and [BH, Proposition 1.6.10(d)], this does not depend on the choice of generating sets of $\mathfrak{a}$. For an ideal $\mathfrak{a}$ (not necessarily finitely generated), Koszul grade of $\mathfrak{a}$ on M can be defined by

$$\text{K.grade}_R(\mathfrak{a}, M) := \sup\bigl\{\text{K.grade}_R(\mathfrak{b}, M)\colon\ \mathfrak{b} \in \Sigma\bigr\}.$$

By using [BH, Proposition 9.1.2(f)], this definition coincides with the original definition for finitely generated ideals.

(ii) A finite sequence $\underline{x} := x_1, \dots, x_r$ of elements of R is called weak regular sequence on M if x_i is a nonzero-divisor on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, r$. If in addition $M \neq (\underline{x})M$, $\underline{x}$ is called regular sequence on M. The classical grade of $\mathfrak{a}$ on M, denoted by $\text{c.grade}_R(\mathfrak{a}, M)$, is defined to the supremum of the lengths of all weak regular sequences on M contained in $\mathfrak{a}$.

(iii) (See [N, p. 149].) The polynomial grade of $\mathfrak{a}$ on M is defined by

$$\text{p.grade}_R(\mathfrak{a}, M) := \lim_{m\to\infty} \text{c.grade}_{R[t_1,\dots,t_m]}\bigl(\mathfrak{a}R[t_1, \dots, t_m], R[t_1, \dots, t_m] \otimes_R M\bigr).$$

(iv) In the case that $\mathfrak{a}$ is finitely generated by generating set $\underline{x} := x_1, \dots, x_r$, the *Čech* grade of $\mathfrak{a}$ on M is defined by

$$\check{\text{C}}\text{.grade}_R(\mathfrak{a}, M) := \inf\bigl\{i \in \mathbb{N} \cup \{0\} \bigm| H^i_{\underline{x}}(M) \neq 0\bigr\},$$

where $H^i_{\underline{x}}(M)$ is denoted the i-th cohomology of *Čech* complex of M related to $\underline{x}$. [HM, Proposition 2.7] implies that

$$\inf\bigl\{i \in \mathbb{N} \cup \{0\} \bigm| H^i_{\underline{x}}(M) \neq 0\bigr\} = \text{K.grade}_R(\mathfrak{a}, M).$$

So $\check{\text{C}}\text{.grade}_R(\mathfrak{a}, M)$ does not depend on the choice of the generating sets of $\mathfrak{a}$. For not necessarily finitely generated ideal $\mathfrak{a}$ the *Čech* grade of $\mathfrak{a}$ on M is defined

$$\check{\text{C}}\text{.grade}_R(\mathfrak{a}, M) := \sup\bigl\{\check{\text{C}}\text{.grade}_R(\mathfrak{b}, M)\colon\ \mathfrak{b} \in \Sigma\bigr\}.$$

By the same argument as (i), this is well defined.

(v) (See [B].) The Ext grade of $\mathfrak{a}$ on M is defined by

$$\text{E.grade}_R(\mathfrak{a}, M) := \inf\bigl\{i \in \mathbb{N} \cup \{0\} \bigm| \text{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\bigr\}.$$

(vi) The local cohomology grade of $\mathfrak{a}$ on M is defined by

$$\text{H.grade}_R(\mathfrak{a}, M) := \inf\bigl\{i \in \mathbb{N} \cup \{0\} \bigm| H^i_{\mathfrak{a}}(M) := \varinjlim_n \text{Ext}^i_R\bigl(R/\mathfrak{a}^n, M\bigr) \neq 0\bigr\}.$$

(vii) Let M be a finitely presented R-module and N an R-module. By defining from [A], $\text{grade}_R(M, N) \geqslant n$ if and only if for every finite complex

$$\mathbf{P}_\bullet : P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

of finitely generated projective R-modules P_i, there exists a finite complex

$$\mathbf{Q}_\bullet : Q_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$$

of finitely generated projective modules Q_j, and a chain map $\mathbf{P}_\bullet \to \mathbf{Q}_\bullet$ over M such that the induced maps:

$$H^i\big(\operatorname{Hom}_R(\mathbf{Q}_\bullet, N)\big) \to H^i\big(\operatorname{Hom}_R(\mathbf{P}_\bullet, N)\big)$$

are zero maps for $0 \leqslant i < n$. $\operatorname{grade}_R(M, N)$ is equal to the largest integer n for which the above condition is satisfied. If no such integer n exists we put $\operatorname{grade}_R(M, N) = +\infty$.

We now recall the definition of $\operatorname{grade}_R(L, .)$ for a general R-module L. By definition, $\operatorname{grade}_R(L, N) \geqslant n$ if for every $\ell \in L$, $(0 :_R \ell)$ contains a finitely generated ideal I_ℓ satisfying $\operatorname{grade}_R(R/I_\ell, N) \geqslant n$. [G1, Theorem 7.1.10] implies that, if L is finitely presented, then two definitions of $\operatorname{grade}_R(L, N)$ coincide. We shall write $\text{A.grade}_R(\mathfrak{a}, N)$ instead of $\operatorname{grade}_R(R/\mathfrak{a}, N)$.

In the next two propositions, we recall some properties and relations between different types of the notion of grade that appeared in Definition 2.1. In what follows we will make use of them several times.

Proposition 2.2. *Let $\mathfrak{a}$ be an ideal of a ring R and M an R-module. Then the following hold.*

(i) *Let $\underline{y} := y_1, \ldots, y_t$ be a regular sequence of elements of $\mathfrak{a}$ on M. Then*

$$\text{p.grade}_R(\mathfrak{a}, M) = t + \text{p.grade}_R\left(\mathfrak{a}, \frac{M}{\underline{y}M}\right).$$

(ii) *Let $f : R \to S$ be a flat ring homomorphism. Then*

$$\text{K.grade}_R(\mathfrak{a}, M) \leqslant \text{K.grade}_S(\mathfrak{a}S, M \otimes_R S).$$

(iii) *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be a pair of ideals of R. Then*

$$\text{K.grade}_R(\mathfrak{a}, M) \leqslant \text{K.grade}_R(\mathfrak{b}, M).$$

(iv) *(Change of rings.) Let $f : R \to S$ be a ring homomorphism and N an S-module. Then*

$$\text{K.grade}_R(\mathfrak{a}, N) = \text{K.grade}_S(\mathfrak{a}S, N).$$

(v) *Let $f : R \to S$ be a faithfully flat ring homomorphism. Then*

$$\text{K.grade}_R(\mathfrak{a}, M) = \text{K.grade}_S(\mathfrak{a}S, M \otimes_R S).$$

(vi) $\text{p.grade}_R(\mathfrak{a}, M) = \text{p.grade}_R(\mathfrak{p}, M)$ *for some prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$.*

(vii) *If $\mathfrak{a}$ is finitely generated, then*

$$\text{A.grade}_R(\mathfrak{a}, M) = \inf\big\{\text{A.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \bigm| \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{Supp}_R M\big\}.$$

Proof. (i) This is Theorem 15 of Chapter 5 in [N].

(ii) First assume that $\mathfrak{a}$ is finitely generated by generating set $\underline{x} := x_1, \dots, x_n$. The symmetry of Koszul cohomology and Koszul homology says that

$$H_i\big(\mathbb{K}_\bullet(\underline{x}) \otimes_R M\big) \cong H^{n-i}\big(\operatorname{Hom}_R\big(\mathbb{K}_\bullet(\underline{x}), M\big)\big),$$

see [BH, Proposition 1.6.10(d)]. Thus the claim in this case follows from [BH, Proposition 9.1.2(c)]. The desired result for not necessarily finitely generated ideals follows from the first case.

(iii) In the case $\mathfrak{a} \subseteq \mathfrak{b}$ is a pair of finitely generated ideals of R, the claim is in [BH, Proposition 9.1.2(f)]. The claim in general case follows from this.

(iv) First assume that $\mathfrak{a}$ is finitely generated by generating set $\underline{x}$. The claim follows from the isomorphism

$$\operatorname{Hom}_R\big(\mathbb{K}_\bullet(\underline{x}), N\big) \cong \operatorname{Hom}_S\big(\mathbb{K}_\bullet(\underline{x}) \otimes_R S, N\big).$$

Now, assume that $\mathfrak{a}$ is a general ideal of R (not necessarily finitely generated). Then, by the former case, we have

$$\text{K.grade}_R(\mathfrak{a}, N) \leqslant \text{K.grade}_S(\mathfrak{a}S, N).$$

Now, let $\underline{y}$ be a finite sequence of elements of $\mathfrak{a}S$. Then there exists a finite sequence $\underline{x}$ of elements of $\mathfrak{a}$ such that $\underline{y}S \subseteq \underline{x}S$. Again, by the former case,

$$\text{K.grade}_S(\underline{y}S, N) \leqslant \text{K.grade}_S(\underline{x}S, N) = \text{K.grade}_R(\underline{x}R, N) \leqslant \text{K.grade}_R(\mathfrak{a}, N).$$

This completes the proof.

(v) This is in [G1, Lemma 7.1.7(2)].

(vi) This is Theorem 16 of Chapter 5 in [N].

(vii) This is in [G1, Theorem 7.1.11]. □

Proposition 2.3. *Let $\mathfrak{a}$ be an ideal of a ring R and M an R-module. Then the following hold.*

(i)
$$\begin{aligned}\text{c.grade}_R(\mathfrak{a}, M) &\leqslant \text{p.grade}_R(\mathfrak{a}, M)\\ &= \text{K.grade}_R(\mathfrak{a}, M)\\ &= \check{\text{C}}\text{.grade}_R(\mathfrak{a}, M)\\ &= \text{A.grade}_R(\mathfrak{a}, M).\end{aligned}$$

(ii) $\text{H.grade}_R(\mathfrak{a}, M) = \text{E.grade}_R(\mathfrak{a}, M)$.

(iii) *If $\mathfrak{a}$ is finitely generated, then* $\text{E.grade}_R(\mathfrak{a}, M) = \text{K.grade}_R(\mathfrak{a}, M)$.

Proof. (i) One can deduce easily, from Proposition 2.2(i) that

$$\text{c.grade}_R(\mathfrak{a}, M) \leqslant \text{p.grade}_R(\mathfrak{a}, M).$$

Assume that Σ runs through all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. In light of [N, Theorem 5.11] we see that

$$\text{p.grade}_R(\mathfrak{a}, M) = \sup\big\{\text{p.grade}_R(\mathfrak{b}, M)\colon\ \mathfrak{b} \in \Sigma\big\}.$$

In view of [HM, Proposition 2.7], one has

$$\mathrm{p.grade}_R(\mathfrak{b}, M) = \mathrm{K.grade}_R(\mathfrak{b}, M) = \check{\mathrm{C}}.\mathrm{grade}_R(\mathfrak{b}, M)$$

for all finitely generated ideals $\mathfrak{b}$ of R. This yields such equalities for all ideals $\mathfrak{a}$ of R.

On the other hand, equivalency (1) $\Leftrightarrow$ (4) of [G1, Theorem 7.1.8], says that the equality

$$\mathrm{A.grade}_R(\mathfrak{b}, M) = \mathrm{K.grade}_R(\mathfrak{b}, M)$$

holds for all finitely generated ideals $\mathfrak{b}$ of R. By definition, such equality holds for any ideals if one can shows that

$$\mathrm{A.grade}_R(\mathfrak{a}, M) = \sup\bigl\{\mathrm{A.grade}_R(\mathfrak{b}, M)\colon \ \mathfrak{b} \in \Sigma\bigr\}.$$

To see this, first assume that $\mathrm{A.grade}_R(\mathfrak{a}, M) \geqslant n$. Then one can find a finitely generated subideal J of $(\mathfrak{a} :_R 1) = \mathfrak{a}$ satisfying $\mathrm{A.grade}_R(J, M) \geqslant n$. So

$$n \leqslant \sup\bigl\{\mathrm{A.grade}_R(\mathfrak{b}, M)\colon \ \mathfrak{b} \in \Sigma\bigr\}.$$

Conversely, let n be an integer such that $\sup\{\mathrm{A.grade}_R(\mathfrak{b}, M)\colon \ \mathfrak{b} \in \Sigma\} \geqslant n$. Then there is a finitely generated subideal $\mathfrak{b}_0$ of $\mathfrak{a}$ such that $\mathrm{A.grade}_R(\mathfrak{b}_0, M) \geqslant n$. So, for any r in R we have $\mathfrak{b}_0 \subseteq (\mathfrak{a} :_R r)$ and $\mathrm{A.grade}_R(\mathfrak{b}_0, M) \geqslant n$. Hence $\mathrm{A.grade}_R(\mathfrak{a}, M) \geqslant n$.

(ii) This follows from [Str, Proposition 5.3.15].

(iii) This is in [Str, Proposition 6.1.6]. $\square$

The assumptions and results of Proposition 2.3 are sharp. To see an example consider the following.

Example 2.4. (i) In Proposition 2.3(iii) the finitely generated assumption on $\mathfrak{a}$ is really needed. To see this, let $R := \mathbb{F}[x_1, \ldots, x_n, \ldots]/(x_1^1, \ldots, x_n^n, \ldots)$, where $\mathbb{F}$ is a field. Set $\mathfrak{a} := (x_1, \ldots, x_n, \ldots)$. Then by [B, p. 367], one has

$$\mathrm{K.grade}_R(\mathfrak{a}, R) = 0 \neq \mathrm{E.grade}_R(\mathfrak{a}, R).$$

(ii) Adopt the notation of (i) and assume that Σ runs over all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. By Proposition 2.3(i), one has

$$\mathrm{E.grade}_R(\mathfrak{b}, R) = \mathrm{H.grade}_R(\mathfrak{b}, R) = \mathrm{K.grade}_R(\mathfrak{b}, R) = 0.$$

Therefore

$$\mathrm{E.grade}_R(\mathfrak{a}, M) \neq \sup\bigl\{\mathrm{E.grade}_R(\mathfrak{b}, M)\colon \ \mathfrak{b} \in \Sigma\bigr\},$$

and

$$\mathrm{H.grade}_R(\mathfrak{a}, M) \neq \sup\bigl\{\mathrm{H.grade}_R(\mathfrak{b}, M)\colon \ \mathfrak{b} \in \Sigma\bigr\}.$$

(iii) Let $R := \mathbb{F}[[X, Y]]$, where $\mathbb{F}$ is a field and set $M := \bigoplus_{0 \neq r \in (X,Y)} R/rR$. By inspection of [Str, p. 91], we find that $\mathrm{E.grade}_R(\mathfrak{m}, M) = 1$ and $\mathrm{c.grade}_R(\mathfrak{m}, M) = 0$. This shows that the inequality of Proposition 2.3(i) does not equality in general. However, if M is a finitely generated module over a Noetherian ring R, then [BH, Theorem 1.2.5] provides that $\mathrm{c.grade}_R(\mathfrak{a}, M) = \mathrm{E.grade}_R(\mathfrak{a}, M)$ for all ideals $\mathfrak{a}$ of R such that $M \neq \mathfrak{a}M$.

As an easy application of Proposition 2.3(ii), we give an elementary proof of a result of Foxby. He proved the following result as an immediate application of the New Intersection Theorem and it has an important role in [Fo].

Corollary 2.5. *(See [Fo, Corollary 1.5].) Let $(A, \mathfrak{m})$ be a Noetherian local ring and C an A-module which satisfies $C \neq \mathfrak{m}C$. Then* $\operatorname{E.grade}_A(\mathfrak{m}, C) \leqslant (\dim C \leqslant) \dim A$.

Proof. Note that $\operatorname{K.grade}_A(\mathfrak{m}, C) < \infty$, since $C \neq \mathfrak{m}C$. By Grothendieck's Vanishing Theorem, $H^i_{\mathfrak{m}}(C) = 0$ for all $i > \dim C$. Now, the claim follows by Proposition 2.3(ii) and (iii). $\square$

3. Relations between different definitions of Cohen–Macaulay rings

There are many characterizations of Noetherian Cohen–Macaulay modules in the literature. If we apply these characterizations to non-Noetherian modules, then they are not necessarily equivalent. The aim of this section is to provide some relations between these definitions, when we apply them to not necessarily Noetherian rings and modules.

3.1. The basic definitions

In this subsection we recall some candidates for the notion of Cohen–Macaulayness in the context of non-Noetherian rings and modules. In what follows we need the notion of weakly associated prime ideals of an R-module M. Recall that a prime ideal $\mathfrak{p}$ is weakly associated to M if $\mathfrak{p}$ is minimal over $(0 :_R m)$ for some $m \in M$. We denote the set of weakly associated primes of M by $\operatorname{wAss}_R M$. Also, in order to give the Hamilton and Marley definition of Cohen–Macaulayness, we need to recall the following definitions (a) and (b).

(a) (See [Sch, Definition 2.3].) Let $\underline{x} = x_1, \dots, x_r$ be a system of elements of R. For $m \geqslant n$ there exists a chain map

$$\varphi_n^m(\underline{x}) : \mathbb{K}_\bullet\big(\underline{x}^m\big) \to \mathbb{K}_\bullet\big(\underline{x}^n\big),$$

which induces by multiplication of $(\prod x_i)^{m-n}$. $\underline{x}$ is called weak proregular if for each $n > 0$ there exists an $m \geqslant n$ such that the maps

$$H_i\big(\varphi_n^m(\underline{x})\big) : H_i\big(\mathbb{K}_\bullet\big(\underline{x}^m\big)\big) \to H_i\big(\mathbb{K}_\bullet\big(\underline{x}^n\big)\big)$$

are zero for all $i \geqslant 1$.

(b) (See [HM, Definition 3.1].) A sequence $\underline{x} := x_1, \dots, x_\ell$ is called a parameter sequence on R, if:

(1) $\underline{x}$ is a weak proregular sequence;
(2) $(\underline{x})R \neq R$; and
(3) $H^\ell_{\underline{x}}(R)_{\mathfrak{p}} \neq 0$ for all prime ideals $\mathfrak{p} \in \mathrm{V}(\underline{x}R)$.

Also, $\underline{x}$ is called a strong parameter sequence on R if $x_1, \dots, x_i$ is a parameter sequence on R for all $1 \leqslant i \leqslant \ell$.

Let M be an R-module and $\mathfrak{q}$ a prime ideal of R. By $\operatorname{ht}_M(\mathfrak{q})$, we mean the Krull dimension of the $R_{\mathfrak{q}}$-module $M_{\mathfrak{q}}$. Also,

$$\operatorname{ht}_M(\mathfrak{a}) := \inf\big\{\operatorname{ht}_M(\mathfrak{q}) \bigm| \mathfrak{q} \in \operatorname{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a})\big\}.$$

Now, we are ready to recall the following definitions of the different types of Cohen–Macaulay rings.

Definition 3.1. Let R be a ring and M an R-module.

(i) (See [HM, Definition 4.1].) R is called Cohen–Macaulay in the sense of Hamilton–Marley, if each strong parameter sequence on R becomes a regular sequence on R. We denote this property by HM.

(ii) (See [G4, p. 219].) M is called Cohen–Macaulay in the sense of Glaz, if for each prime ideal $\mathfrak{p}$ of R,

$$\mathrm{ht}_M(\mathfrak{p}) = \mathrm{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$$

and denote this by Glaz.

(iii) (See [H2, Definitions 1 and 2].) Let $\mathfrak{a}$ be a finitely generated ideal of R. Set $\mu(\mathfrak{a})$, the minimal number of elements of R that need to generate $\mathfrak{a}$. Assume that for each ideal $\mathfrak{a}$ with the property $\mathrm{ht}\,\mathfrak{a} \geqslant \mu(\mathfrak{a})$, we have $\min(\mathfrak{a}) = \mathrm{wAss}_R(R/\mathfrak{a})$. A ring with such property is called weak Bourbaki unmixed. We denote this property by WB.

(iv) Let $\mathcal{A}$ be a non-empty subclass of the class of all ideals of a ring R. We say that M is Cohen–Macaulay in the sense of $\mathcal{A}$, if $\mathrm{ht}_M(\mathfrak{a}) = \mathrm{K.grade}_R(\mathfrak{a}, M)$ for all ideals $\mathfrak{a}$ in $\mathcal{A}$. We denote this property by $\mathcal{A}$. The classes we are interested in are $\mathrm{Supp}_R(M)$, $\mathrm{Supp}_R(M) \cap \max(R)$, the class of all ideals and the class of all finitely generated ideals. We denote them respectively by Spec, Max, ideals and f.g. ideals.

It is clear from the above definition that any zero-dimensional ring is Cohen–Macaulay in the sense of each part of Definition 3.1. Also, any one-dimensional integral domain is Cohen–Macaulay in the sense of each part of Definition 3.1.

3.2. Relations

The following diagram illustrates our work in this subsection:

$$\text{Max} \;\Leftarrow\; \text{Spec} \;\Leftrightarrow\; \text{ideals} \;\Rightarrow\; \text{Glaz} \;\Rightarrow\; \text{f.g. ideals} \;\Rightarrow\; \text{HM} \;\Leftarrow\; \text{WB}. \qquad (*)$$

Also, when the base ring is coherent, we show that Spec $\Rightarrow$ WB.

The key to the work in this subsection is given by the following elementary lemma.

Lemma 3.2. *Let $\mathfrak{a}$ be an ideal of a ring R and M a finitely generated R-module. Then*

$$\mathrm{K.grade}_R(\mathfrak{a}, M) \leqslant \mathrm{ht}_M(\mathfrak{a}).$$

Proof. If $M/\mathfrak{a}M = 0$, then $\mathrm{ht}_M(\mathfrak{a}) = +\infty$. Therefore, we can assume that $\mathrm{Supp}_R(\frac{M}{\mathfrak{a}M}) = V(\mathfrak{a}) \cap \mathrm{Supp}\, M \neq \emptyset$. Let $\mathfrak{q} \in V(\mathfrak{a}) \cap \mathrm{Supp}\, M$. By parts (ii) and (iii) of Proposition 2.2, one gets

$$\mathrm{K.grade}_R(\mathfrak{a}, M) \leqslant \mathrm{K.grade}_{R_\mathfrak{q}}(\mathfrak{a}R_\mathfrak{q}, M_\mathfrak{q}) \leqslant \mathrm{K.grade}_{R_\mathfrak{q}}(\mathfrak{q}R_\mathfrak{q}, M_\mathfrak{q}).$$

Thus, it is enough for us to show that if $(R, \mathfrak{m})$ is a quasi-local ring and M a finitely generated nonzero R-module, then $\mathrm{K.grade}_R(\mathfrak{m}, M) \leqslant \dim M$. Applying Proposition 2.2(iv) for the ring homomorphism $R \to R/\mathrm{Ann}\, M$, we may assume that M is a faithful R-module. So, $\dim M = \dim R$. If $\dim R = \infty$, we have nothing to prove. Hence we can assume that $\dim R < \infty$. [HM, Proposition 2.4] says that $H^i_{\underline{y}}(M) = 0$ for all $i > \dim R = \dim M$ and all finite sequences $\underline{y}$ of elements of R. On the other hand for a finite sequence $\underline{x}$ of elements of $\mathfrak{m}$, by Nakayama's Lemma, $M/\underline{x}M \neq 0$, and so $\mathrm{K.grade}_R(\underline{x}, M) < \infty$. Consequently, by using Proposition 2.3(i),

$$\mathrm{K.grade}_R(\mathfrak{m}, M) = \check{\mathrm{C}}.\mathrm{grade}_R(\mathfrak{m}, M) \leqslant \dim M. \qquad \square$$

The next result gives the proof of the following implications:

$$\text{Spec} \quad \Leftrightarrow \quad \text{ideals} \quad \Rightarrow \quad \text{Glaz} \quad \Rightarrow \quad \text{f.g. ideals}.$$

Theorem 3.3. *Let M be a finitely generated R-module. Consider the following conditions*:

(i) $\operatorname{ht}_M(\mathfrak{p}) = \text{K.grade}_R(\mathfrak{p}, M)$ *for all prime ideals* $\mathfrak{p}$ *of* R.
(ii) $\operatorname{ht}_M(\mathfrak{a}) = \text{K.grade}_R(\mathfrak{a}, M)$ *for all ideals* $\mathfrak{a}$ *of* R.
(iii) $\operatorname{ht}_M(\mathfrak{q}) = \text{K.grade}_{R_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}, M_\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}, \mathfrak{q}$ *in* $\operatorname{Supp}_R(M)$ *with* $\mathfrak{q} \subseteq \mathfrak{p}$.
(iv) $\operatorname{ht}_M(\mathfrak{p}) = \text{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}$ *in* $\operatorname{Supp}_R(M)$.
(v) $\operatorname{ht}_M(\mathfrak{a}) = \text{K.grade}_R(\mathfrak{a}, M)$ *for all finitely generated ideals* $\mathfrak{a}$ *of* R.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{a}$ be an ideal of R. By Proposition 2.2(vi) and Proposition 2.3(i), there exists a prime ideal $\mathfrak{p}$ of R containing $\mathfrak{a}$ such that $\text{K.grade}_R(\mathfrak{a}, M) = \text{K.grade}_R(\mathfrak{p}, M)$. In view of Lemma 3.2, one can find that

$$\text{K.grade}_R(\mathfrak{a}, M) = \text{K.grade}_R(\mathfrak{p}, M) = \operatorname{ht}_M(\mathfrak{p}) \geqslant \operatorname{ht}_M(\mathfrak{a}) \geqslant \text{K.grade}_R(\mathfrak{a}, M),$$

which completes the proof.

(ii) $\Rightarrow$ (i) This is trivial.

(ii) $\Rightarrow$ (iii) This follows from the following

$$\text{K.grade}_R(\mathfrak{q}, M) \leqslant \text{K.grade}_{R_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}, M_\mathfrak{p}) \leqslant \operatorname{ht}_{M_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}) = \operatorname{ht}_M(\mathfrak{q}),$$

where the last inequality follows from Lemma 3.2.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (v) Let $\mathfrak{a}$ be a finitely generated ideal of R. Then, Proposition 2.2(vii), Proposition 2.3(i) and our assumption, imply that

$$\begin{aligned}
\text{K.grade}_R(\mathfrak{a}, M) &= \inf\bigl\{\text{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \bigm| \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{Supp}_R M\bigr\} \\
&= \inf\bigl\{\operatorname{ht}_{M_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \bigm| \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{Supp}_R M\bigr\} \\
&= \operatorname{ht}_M(\mathfrak{a}),
\end{aligned}$$

which completes the proof. $\square$

Let R be a ring and let $\underline{x} := x_1, \ldots, x_\ell$ be a parameter sequence on R. Due to [HM, Proposition 3.6] we know that $\operatorname{ht}(\underline{x})R \geqslant \ell$. Then, in view of [HM, Proposition 4.10], any weak Bourbaki unmixed ring is Cohen–Macaulay in the sense of Hamilton–Marley. Thus, in order to complete the proof of all of desired implications of the diagram $(*)$, we need to state the following.

Theorem 3.4. *Let R be a Cohen–Macaulay ring in the sense of finitely generated ideals. Then R is Cohen–Macaulay in the sense of Hamilton–Marley.*

Proof. Let $\underline{x} := x_1, \ldots, x_\ell$ be a strong parameter sequence on R. By equivalency (a) $\Leftrightarrow$ (c) of [HM, Proposition 4.2], its enough to show that

$$\text{K.grade}_R(\underline{x}R, R) = \text{p.grade}_R(\underline{x}R, R) = \ell.$$

For a finite sequence $\underline{y} := y_1, \ldots, y_m$ of elements of R, [HM, Proposition 3.6] state that $\operatorname{ht}(\underline{y}R) \geqslant m$, if $\underline{y}$ is a parameter sequence on R. Since $\underline{x}R \neq R$, $\text{K.grade}_R(\underline{x}R, R) < \infty$. So $\text{K.grade}_R(\underline{x}R, R) \leqslant \ell$. Then, it turns out that

$$\text{K.grade}_R(\underline{x}R, R) \leqslant \ell \leqslant \operatorname{ht}(\underline{x}R) = \text{K.grade}_R(\underline{x}R, R).$$

Therefore, $\text{K.grade}_R(\underline{x}R, R) = \ell$, as claimed. $\square$

Theorem 3.10 is one of our main results in this subsection. To prove it, we need a couple of lemmas.

Lemma 3.5. *Let R be a Cohen–Macaulay ring in the sense of (finitely generated) ideals and x a regular element of R. Then R/xR is Cohen–Macaulay in the sense of (finitely generated) ideals. In particular, a ring A is Cohen Macaulay in the sense of (finitely generated) ideals, if either $A[[X]]$ or $A[X]$ is as well.*

Proof. Let $\mathfrak{b} := \mathfrak{a}/xR$ be an ideal (resp. finitely generated ideal) of R/xR. By parts (i) and (iv) of Proposition 2.2, one can find that

$$\text{K.grade}_{R/xR}(\mathfrak{b}, R/xR) = \text{K.grade}_R(\mathfrak{a}, R/xR) = \text{K.grade}_R(\mathfrak{a}, R) - 1.$$

Then it yields that:

$$\begin{aligned}
\text{K.grade}_R(\mathfrak{a}, R) - 1 &= \text{K.grade}_{R/xR}(\mathfrak{b}, R/xR) \\
&\leqslant \operatorname{ht}_{R/xR}(\mathfrak{b}) \\
&\leqslant \operatorname{ht}_R(\mathfrak{a}) - 1 \\
&= \text{K.grade}_R(\mathfrak{a}, R) - 1,
\end{aligned}$$

which completes the proof. $\square$

Remark 3.6.

(i) There exists an example of a quasi-local ring R such that it is Cohen–Macaulay in the sense of Hamilton–Marley but R/xR is not Cohen–Macaulay in the sense of Hamilton–Marley for some regular element x of R, see [HM, Example 4.9].
(ii) Assume that $(R, \mathfrak{m})$ is a quasi-local ring, which is equidimensional, semicatenary and weak Bourbaki unmixed. Let x be a regular element of R. [H3, Theorem D] shows that R/xR is weak Bourbaki unmixed.

Recall that a module is coherent if it is finitely generated and each of its finitely generated submodule is finitely presented. A ring is coherent if it is coherent as a module over itself. Noetherian rings are coherent. There are many examples of non-Noetherian coherent rings. For instance, any non-Noetherian valuation domain is a non-Noetherian coherent ring.

Lemma 3.7. *Let R be a coherent ring and $\underline{x} := x_1, \ldots, x_\ell$ a finite sequence of elements of R. Then $H^i(\operatorname{Hom}_R(\mathbb{K}_\bullet(\underline{x}), R))$ is finitely generated R-module for all i.*

Proof. Let $\mathbf{F}^\bullet : 0 \to F^0 \to \cdots \to F^i \xrightarrow{\varphi^i} F^{i+1} \to \cdots \to F^\ell \to 0$ be the Koszul complex of R related to $\underline{x}$. Let i be an integer between 0 and ℓ. By using the exact sequence

$$F^i \to F^{i+1} \to \operatorname{im}\varphi^i \to 0,$$

we find that $\operatorname{im}\varphi^i$ is finitely presented. Consider the exact sequence

$$0 \to \ker\varphi^i \to F^i \to \operatorname{im}\varphi^i \to 0,$$

in which the maps are the natural one. Keep in mind that R is coherent. Now, [G1, Theorem 2.5.1] yields that $\ker\varphi^i$ is finitely presented. From this the claim follows. $\square$

Remark 3.8.

(i) The coherent assumption on R in Lemma 3.7 is really needed. To see an example, let A be a $\mathbb{C}$-algebra generated by all degree two monomials of $\mathbb{C}[X_1, X_2, \ldots] := \bigcup_{n=1}^{\infty} \mathbb{C}[X_1, \ldots, X_n]$ and set $R := A/(X_1X_2)$. We use small letters to indicate the images in R. Then $(0 :_R x_1^2) = (x_2x_i\colon\ i \in \mathbb{N})$ is not finitely generated. So the first Koszul homology related to x_1^2 is not finitely generated (cf. [G2, Example 2]).

(ii) If Koszul (co)homology modules are finitely generated, then one can see that the vanishing of first Koszul (co)homology modules implies the exactness of Koszul complex. But there exists an example which does not satisfy this. In [K, Example 2], Kabele gives a quasi-local ring (S, N) for which $N := (n_1, \ldots, n_r)S$ is finitely generated and the first Koszul cohomology module of S on $(n_1, \ldots, n_r)$ is zero but some Koszul cohomology modules of S on $(n_1, \ldots, n_r)$ is not zero. In particular, Koszul (co)homology modules are not finitely generated in general.

Lemma 3.9. *Let R be a Cohen–Macaulay ring in the sense of ideals. Then* $\operatorname{wAss}_R(R) = \min(R)$, *where* $\min(R)$ *is the set of all minimal prime ideals of R.*

Proof. It is well known that $\min(R) \subseteq \operatorname{wAss}_R(R)$. Let $\mathfrak{p} \in \operatorname{wAss}_R(R)$. Then [HM, Lemma 2.8] state that $\text{p.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}) = 0$. By applying Proposition 2.3(i), one has $\text{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}) = 0$. The inequality

$$\text{K.grade}_R(\mathfrak{p}, R) \leqslant \text{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p})$$

shows that $\text{K.grade}_R(\mathfrak{p}, R) = 0$. Therefore, $\operatorname{ht}_R(\mathfrak{p}) = 0$, i.e., $\mathfrak{p} \in \min(R)$. $\square$

Now, we are ready in the position to present our next main result.

Theorem 3.10. *Let R be a coherent ring. If R is Cohen–Macaulay in the sense of ideals, then R is weak Bourbaki unmixed.*

Proof. By Theorem 3.3 and [G1, Theorem 2.4.2], $R_\mathfrak{p}$ is Cohen–Macaulay in the sense of ideals and it is coherent for all prime ideals $\mathfrak{p}$ of R. Also, if $R_\mathfrak{p}$ is weak Bourbaki unmixed for any $\mathfrak{p} \in \operatorname{Spec} R$, then by [H2, Theorem 3], R is weak Bourbaki unmixed. Thus, we may and do assume that R is quasi-local. Let $\mathfrak{a}$ be a proper finitely generated ideal of R with the property that $\operatorname{ht}\mathfrak{a} \geqslant \mu(\mathfrak{a})$. Then,

$$\text{K.grade}_R(\mathfrak{a}, R) \leqslant \mu(\mathfrak{a}) \leqslant \operatorname{ht}\mathfrak{a}.$$

So

$$\ell := \text{K.grade}_R(\mathfrak{a}, R) = \mu(\mathfrak{a}) = \operatorname{ht}\mathfrak{a},$$

since R is Cohen–Macaulay in the sense of ideals. Let $\underline{x} := x_1, \ldots, x_\ell$ be a generating set for $\mathfrak{a}$. Now, we show that $\underline{x}$ is a strong parameter sequence. Let $1 \leqslant i < \ell$ and set $\mathfrak{a}_i := (x_1, \ldots, x_i)R$. As the reader might have guessed, we consider the following long exact sequence of R-modules and R-homomorphisms

$$\cdots \to H^j\big(\operatorname{Hom}_R\big(\mathbb{K}_\bullet(x_1, \ldots, x_i), R\big)\big) \xrightarrow{x_{i+1}} H^j\big(\operatorname{Hom}_R\big(\mathbb{K}_\bullet(x_1, \ldots, x_i), R\big)\big)$$
$$\to H^{j+1}\big(\operatorname{Hom}_R\big(\mathbb{K}_\bullet(x_1, \ldots, x_{i+1}), R\big)\big) \to H^{j+1}\big(\operatorname{Hom}_R\big(\mathbb{K}_\bullet(x_1, \ldots, x_{i+1}), R\big)\big) \to \cdots.$$

By Lemma 3.7, $H^j(\operatorname{Hom}_R(\mathbb{K}_\bullet(x_1, \ldots, x_i), R))$ is finitely generated for all j. Also, x_{i+1} belongs to the Jacobson radical of R. By using Nakayama's Lemma, one can find that

$$\text{K.grade}_R(\mathfrak{a}_i + x_{i+1}R, R) \leqslant \text{K.grade}_R(\mathfrak{a}_i, R) + 1.$$

An easy induction shows that

$$\text{K.grade}_R\big(\mathfrak{a}_i + (x_{i+1}, \ldots, x_\ell), R\big) \leqslant \text{K.grade}_R(\mathfrak{a}_i, R) + (\ell - i).$$

On the other hand, $\text{K.grade}_R(\mathfrak{a}_i + (x_{i+1}, \ldots, x_\ell), R) = \ell$. Hence $\text{K.grade}_R(\mathfrak{a}_i, R) \geqslant i$. This implies that $\text{K.grade}_R(\mathfrak{a}_i, R) = i$, since $\mathfrak{a}_i$ can be generated by i's elements. So by [HM, Proposition 3.3(e)], $x_1, \ldots, x_i$ is a parameter sequence on R. Thus, $\underline{x}$ is a strong parameter sequence on R. In view of Theorem 3.4, R is Cohen–Macaulay in the sense of Hamilton–Marley. Therefore, $\underline{x}$ forms a weak regular sequence on R. So Lemma 3.5 implies that $R/\mathfrak{a}$ is Cohen–Macaulay in the sense of ideals. Now, let $\mathfrak{p} \in \operatorname{wAss}_R(R/\mathfrak{a})$. Then, Lemma 3.9 shows that $\operatorname{ht}_{R/\mathfrak{a}}(\mathfrak{p}/\mathfrak{a}) = 0$, i.e., $\mathfrak{p} \in \min(\mathfrak{a})$. $\square$

3.3. Examples

In this subsection, we provide some counter-examples to show that none of the following implications are valid:

$$\begin{array}{ccccccc} & & & \text{WB} & & \\ & & & \Uparrow & & \\ \text{f.g. ideals} & \Leftarrow & \text{Max} & \Leftrightarrow & \text{HM} & \Rightarrow & \text{f.g. ideals.} \end{array} \qquad (**)$$

One might ask whether the second statement of Theorem 3.3 is true, if $\operatorname{ht}_R(\mathfrak{m}) = \text{K.grade}_R(\mathfrak{m}, R)$ for all maximal ideals $\mathfrak{m}$ of R. This, would not be the case, as the next example shows.

Example 3.11. Let (R, m) be a Noetherian local Cohen–Macaulay ring of dimension $d > 1$. Let $X(d-1) := \{\mathfrak{p} \in \operatorname{Spec} R\colon \operatorname{ht}\mathfrak{p} \leqslant d-1\}$. Set $M_{d-1} := \bigoplus_{\mathfrak{p} \in X(d-1)} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and consider $S := R \ltimes M_{d-1}$, the trivial extension of R by M_{d-1}. Then S is a quasi-local ring with the unique maximal ideal $\mathfrak{n} := \mathfrak{m} \ltimes M_{d-1}$. By inspection of [HM, Example 2.10], we know that $\text{K.grade}_S(\mathfrak{n}, S) = \operatorname{ht}(\mathfrak{n})$. Thus, S is Cohen–Macaulay in the sense of maximal ideals. Again, in light of [HM, Example 2.10], we see that $\text{K.grade}_S(\mathfrak{a}, S) = 0$ for all ideals $\mathfrak{a}$ of S with the property that $\operatorname{rad}(\mathfrak{a}) \neq \mathfrak{n}$. Now, take a be in $\mathfrak{m}$ but not in $\bigcup\{\mathfrak{p}\colon \mathfrak{p} \in \min(R)\}$. One has $\operatorname{rad}((a, 0)S) \neq \mathfrak{n}$ and $\operatorname{ht}((a, 0)S) \neq 0$. This yields that S is not Cohen–Macaulay in the sense of finitely generated ideals. Also, by [HM, Example 4.3], S is not Cohen–Macaulay in the sense of Hamilton–Marley.

In view of [Ber], a ring is called regular if every finitely generated ideal has finite projective dimension. For example, valuation domains are coherent and regular. So they are Cohen–Macaulay in the sense of Hamilton–Marley, see [HM, Theorem 4.8]. Then, the next result completes our list of counter-examples to the diagram $(**)$.

Proposition 3.12. *Let $(R, \mathfrak{m})$ be a valuation domain. Then, the following are equivalent.*

(i) *R is Cohen–Macaulay in the sense of ideals.*
(ii) *R is Cohen–Macaulay in the sense of prime ideals.*
(iii) *R is Cohen–Macaulay in the sense of Glaz.*
(iv) *R is Cohen–Macaulay in the sense of finitely generated ideals.*
(v) $\dim R \leqslant 1$.
(vi) *R is weak Bourbaki unmixed.*
(vii) *R is Cohen–Macaulay in the sense of maximal ideals.*

Proof. Without loss of generality we can assume that R is not a field. Let $\underline{x}$ be a finite sequence of nonzero elements of $\mathfrak{m}$. Since R is a valuation domain, there is an element r such that $rR = (\underline{x})R$. Hence $\mathrm{K.grade}_R(\underline{x}R, R) \leqslant 1$. Thus $\mathrm{K.grade}_R(\underline{x}R, R) = 1$, because R is a domain. Therefore, we bring the following statement:

$$\mathrm{K.grade}(\mathfrak{a}, R) = 1 \quad \text{for all nonzero proper ideals } \mathfrak{a} \text{ of } R. \tag{$\star$}$$

The assertions (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) all hold by Theorem 3.3.

(iv) $\Rightarrow$ (v) For a contradiction assume that $\dim R > 1$. Since the ideals of R are linearly ordered by means of inclusion, R has only one prime ideal of height one, say $\mathfrak{p}$. Let $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then $\mathrm{ht}(xR) > 1$. So in view of $(\star)$, R is not Cohen–Macaulay in the sense of finitely generated ideals. This contradiction shows that $\dim R \leqslant 1$.

(v) $\Rightarrow$ (ii) This is obvious.

(i) $\Rightarrow$ (vi) Any finitely generated ideal of a valuation domain is principal. So valuation domains are coherent. Therefore, this implication follows by Theorem 3.10.

(vi) $\Rightarrow$ (v) It is enough to show that any valuation domain of dimension greater than 1 is not weak Bourbaki unmixed. Assume that R is of that type. Then there is the chain $0 \subsetneqq \mathfrak{p} \subsetneqq \mathfrak{q}$ of prime ideals of R such that $\mathrm{ht}(\mathfrak{p}) = 1$. Let $a \in \mathfrak{p} \setminus \{0\}$ and consider the ideal $\mathfrak{a} := aR$. Since ideals of R are linearly ordered by means of inclusion, $\min(\mathfrak{a}) = \{\mathfrak{p}\}$. Assume that $\min(\mathfrak{a}) = \mathrm{wAss}_R(R/\mathfrak{a})$. Let $b \in \mathfrak{q} \setminus \mathfrak{p}$. Then a, b is a weak R-sequence of length 2, which is a contradiction with $(\star)$. This shows that $\min(\mathfrak{a}) \neq \mathrm{wAss}_R(R/\mathfrak{a})$ and consequently R is not weak Bourbaki unmixed.

(ii) $\Rightarrow$ (vii) is trivial and the remainder implication (vii) $\Rightarrow$ (v) follows by $(\star)$. □

Remark 3.13. Let $(R, \mathfrak{m})$ be an unique factorization valuation domain which is not a field. By inspection of $(\star)$ in the proof of Proposition 3.12, one has $\dim R = 1$, and so R is Cohen–Macaulay in the sense of each part of Definition 3.1. Indeed, let $\mathfrak{p}$ be a prime ideal of R with height one. It is enough to show that $R/\mathfrak{p}$ is a field. One has $\mathfrak{p} = xR$ for some x in $\mathfrak{p}$, because R is an unique factorization domain. Let $\mathfrak{b} := \mathfrak{a}/xR$ be a nonzero proper ideal of R/xR, where $\mathfrak{a}$ is an ideal of R. Then by $(\star)$ in the proof of Proposition 3.12, we have $\mathrm{K.grade}(\mathfrak{b}, R/xR) = 1$ and $\mathrm{K.grade}(\mathfrak{a}, R) = 1$. In light of Proposition 2.2(i) one has

$$\mathrm{K.grade}_{R/xR}(\mathfrak{b}, R/xR) = \mathrm{K.grade}_R(\mathfrak{a}, R/xR) = \mathrm{K.grade}_R(\mathfrak{a}, R) - 1 = 0.$$

This contradiction shows that R/xR has no any nonzero proper ideal. Therefore, $R/\mathfrak{p}$ is a field as claimed.

4. Examples of Cohen–Macaulay rings

In this section we will construct some examples of non-Noetherian Cohen–Macaulay rings. Our first example provides the Cohen–Macaulayness of the ring

$$R[X_1, X_2, \ldots] := \bigcup_{i=1}^{\infty} R[X_1, \ldots, X_i],$$

when R is Noetherian and Cohen–Macaulay. Such result gives us that at least one of the Hamilton's conditions for an appropriate definition of non-Noetherian Cohen–Macaulay ring.

Theorem 4.1. *Let R be a Noetherian Cohen–Macaulay ring. Then the ring $R[X_1, X_2, \ldots]$ is Cohen–Macaulay in the sense of each part of Definition* 3.1.

Proof. First, we show that $R' := R[X_1, X_2, \ldots]$ is Cohen–Macaulay in the sense of prime ideals. Let $\mathfrak{p}$ be a prime ideal of R'. We need to show that the equality $\mathrm{K.grade}_{R'}(\mathfrak{p}, R') = \mathrm{ht}_{R'}(\mathfrak{p})$ holds. For any positive integer i, set $R_i := R[X_1, \ldots, X_i]$ and consider the prime ideal $\tilde{\mathfrak{p}}_i := \mathfrak{p} \cap R_i$. Then we have the following chain of subsets of R':

$$\tilde{\mathfrak{p}}_1 \subseteq \tilde{\mathfrak{p}}_2 \subseteq \cdots \subseteq \tilde{\mathfrak{p}}_i \subseteq \tilde{\mathfrak{p}}_{i+1} \subseteq \cdots.$$

Consider the following, only possible, cases (a) and (b).

(a) For infinitely many i's, the condition $\tilde{\mathfrak{p}}_i R_{i+1} \subsetneqq \tilde{\mathfrak{p}}_{i+1}$ is satisfied.
(b) Just only for finitely many i's, the condition $\tilde{\mathfrak{p}}_i R_{i+1} \subsetneqq \tilde{\mathfrak{p}}_{i+1}$ holds.

In the case (a), for infinitely many i's the inequality $\mathrm{ht}_{R_i}(\tilde{\mathfrak{p}}_i) < \mathrm{ht}_{R_{i+1}}(\tilde{\mathfrak{p}}_{i+1})$ is true, since $\mathrm{ht}_{R_i}(\tilde{\mathfrak{p}}_i) = \mathrm{ht}_{R_{i+1}}(\tilde{\mathfrak{p}}_i R_{i+1})$. Then for such i's, it turns out that

$$\begin{aligned} \mathrm{K.grade}_{R'}(\tilde{\mathfrak{p}}_i R', R') &= \mathrm{K.grade}_{R_i}(\tilde{\mathfrak{p}}_i, R_i) \\ &= \mathrm{ht}_{R_i}(\tilde{\mathfrak{p}}_i) \\ &< \mathrm{ht}_{R_{i+1}}(\tilde{\mathfrak{p}}_{i+1}) \\ &= \mathrm{K.grade}_{R_{i+1}}(\tilde{\mathfrak{p}}_{i+1}, R_{i+1}) \\ &= \mathrm{K.grade}_{R'}(\tilde{\mathfrak{p}}_{i+1} R', R'), \end{aligned}$$

where the first equality follows from Proposition 2.2(v) and second from the Cohen–Macaulayness of R_i. Hence $\mathrm{K.grade}_{R'}(\mathfrak{p}, R') = \infty$ and consequently $\mathrm{K.grade}_{R'}(\mathfrak{p}, R') = \mathrm{ht}_{R'}(\mathfrak{p})$.

In the case (b), there is an integer $k > 0$ such that $\tilde{\mathfrak{p}}_k R_{k+j} = \tilde{\mathfrak{p}}_{k+j}$ for all $j > 0$. So

$$\mathfrak{p} = \bigcup_{i \geqslant 1} \tilde{\mathfrak{p}}_i = (\tilde{\mathfrak{p}}_1 \cup \cdots \cup \tilde{\mathfrak{p}}_k) \cup \left(\bigcup_{j \geqslant 1} \tilde{\mathfrak{p}}_k R_{k+j} \right).$$

In particular, $\mathfrak{p}$ is finitely generated. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a generating set for $\mathfrak{p}$. Thus, there is a positive integer as m such that $\alpha_j \in R_m$ for all $1 \leqslant j \leqslant \ell$. One can see easily that

$$\big((\alpha_1, \ldots, \alpha_\ell)R_m\big)R' \cap R_m = (\alpha_1, \ldots, \alpha_\ell)R_m,$$

because R'/R_m is a faithfully flat ring extension. In particular, $(\alpha_1, \ldots, \alpha_\ell)R_m$ is a prime ideal of R_m. Now, by [H1, Lemma 4.1],

$$\mathrm{ht}_{R_m}\big((\alpha_1, \ldots, \alpha_\ell)R_m\big) = \mathrm{ht}_{R'}(\mathfrak{p}).$$

Therefore

$$\begin{aligned}\operatorname{ht}_{R'}(\mathfrak{p}) &= \operatorname{ht}_{R_m}\big((\alpha_1,\dots,\alpha_\ell)R_m\big)\\ &= \text{K.grade}_{R_m}\big((\alpha_1,\dots,\alpha_\ell)R_m, R_m\big)\\ &= \text{K.grade}_{R'}\big((\alpha_1,\dots,\alpha_\ell)R', R'\big)\\ &= \text{K.grade}_{R'}(\mathfrak{p}, R').\end{aligned}$$

So $R' = R[X_1, X_2, \dots]$ is Cohen–Macaulay in the sense of prime ideals. Due to Theorem 3.3 we know that R' is Cohen–Macaulay in the sense of ideals. Also, in view of Theorem 3.4, R' is Cohen–Macaulay in the sense of Hamilton–Marley. By [G1, Corollary 2.3.4], R' is coherent. Thus, Theorem 3.10 implies that R' is weak Bourbaki unmixed. □

Remark 4.2. Let R be a Noetherian Cohen–Macaulay ring and $\mathfrak{a}$ a finitely generated ideal of $R[X_1, X_2, \dots]$ with the property that $\operatorname{ht}\mathfrak{a} \geqslant \mu(\mathfrak{a})$. Then by [H1, Theorem 4.2], all of the weak associated primes of $\mathfrak{a}$ have the same height, i.e., $R[X_1, X_2, \dots]$ is weak Bourbaki height unmixed. In particular, $R[X_1, X_2, \dots]$ is weak Bourbaki unmixed.

Let $(R, \mathfrak{m})$ be a Noetherian local domain and let R^+ be the integral closure of R in the algebraic closure of its field of fractions. Theorem 4.5 provides the Cohen–Macaulayness of R^+. To deal with this, we establish the following lemma.

Lemma 4.3. *Let $f : R \to S$ be a flat and integral ring homomorphism. If R is Cohen–Macaulay in the sense of ideals, then S is also Cohen–Macaulay in the sense of ideals.*

Proof. Let $\mathfrak{q}$ be in $\operatorname{Spec} S$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. In view of Proposition 2.2(ii), we have

$$\begin{aligned}\operatorname{ht}\mathfrak{q} &\leqslant \operatorname{ht}\mathfrak{p}\\ &= \text{K.grade}_R(\mathfrak{p}, R)\\ &\leqslant \text{K.grade}_S(\mathfrak{p}S, S)\\ &\leqslant \text{K.grade}_S(\mathfrak{q}, S),\end{aligned}$$

and so Lemma 3.2 completes the proof. □

Note that by [AH, Theorem 4.5], R^+ is not coherent, when R is of dimension at least 3 and of positive characteristic. So in the next result we cannot apply Theorem 3.10 for it.

Theorem 4.4. *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then the following holds.*

(i) *If R is of prime characteristic p, then R^+ is Cohen–Macaulay in the sense of each part of Definition* 3.1.
(ii) *If $\dim R \geqslant 4$ and R is of mixed characteristic, then R^+ is not Cohen–Macaulay in the sense of finitely generated ideals.*
(iii) *If $\dim R < 3$, then R^+ is Cohen–Macaulay in the sense of each part of Definition* 3.1.
(iv) *If $\dim R \geqslant 3$ and R containing a field of characteristic* 0*, then R^+ is not Cohen–Macaulay in the sense of finitely generated ideals.*

Proof. By Cohen's Structure Theorem there exists a complete regular local subring $(A, \mathfrak{m}_A)$ of R such that R is a finitely generated A-module. Recall that $R^+ = A^+$. Then, without loss of generality we can assume that R is regular.

(i) First, we show that R^+ is Cohen–Macaulay in the sense of ideals. In view of [HH, Theorem 5.15], R^+ is a balanced big Cohen–Macaulay R-algebra, i.e., every system of parameters is regular on R^+.

Over regular local rings, [HH, 6.7, Flatness] state that any balanced big Cohen–Macaulay module is flat. Then, Lemma 4.3 yields that R^+ is Cohen–Macaulay in the sense of ideals.

Next, we show that R^+ is weak Bourbaki unmixed. Let $\mathfrak{a}$ be a finitely generated ideal of R^+ with the property that $\operatorname{ht}\mathfrak{a} \geqslant \mu(\mathfrak{a})$. Then,

$$\mathrm{K.grade}_{R^+}\big(\mathfrak{a}, R^+\big) \leqslant \mu(\mathfrak{a}) \leqslant \operatorname{ht}\mathfrak{a}.$$

So

$$n := \mathrm{K.grade}_{R^+}\big(\mathfrak{a}, R^+\big) = \mu(\mathfrak{a}) = \operatorname{ht}\mathfrak{a},$$

since R^+ is Cohen–Macaulay in the sense of ideals. Let $\{a_1, \ldots, a_n\}$ be a generating set for $\mathfrak{a}$. The ring R^+ is a direct union of module finite ring extensions of R. Such ring extensions are Noetherian, local and complete, since R is local and complete. Let A be one of them, which contains R and a_i for all $1 \leqslant i \leqslant n$. In view of $A^+ = R^+$, we can assume that $a_i \in R$ for all $1 \leqslant i \leqslant n$. Set $\mathfrak{b} := a_1R + \cdots + a_nR$. Then $\mathfrak{b}R^+ = \mathfrak{a}$. Because R^+ is an integral extension of R, we have

$$n = \operatorname{ht}\mathfrak{a} \leqslant \operatorname{ht}\mathfrak{b} \leqslant n.$$

So $n := \mu(\mathfrak{b}) = \operatorname{ht}\mathfrak{b}$. This implies that $\{a_1, \ldots, a_n\}$ is a part of a system of parameter for R. Keep in mind that R^+ is a balanced big Cohen–Macaulay R-algebra. This say's that $\{a_1, \ldots, a_n\}$ is a regular sequence on R^+. It follows from Lemma 3.5 and Lemma 3.9 that $\mathrm{wAss}_{R^+}(R^+/\mathfrak{a}) = \min(\mathfrak{a})$.

(ii) For a contradiction assume that R^+ is Cohen–Macaulay in the sense of finitely generated ideals. Then by Theorem 3.4, R^+ is Cohen–Macaulay in the sense of Hamilton–Marley. Also, [AH, Proposition 3.6] state that R^+ is not a balanced big Cohen–Macaulay algebra for R. Thus, there exists a system of parameters of R as $\underline{x} := x_1, \ldots, x_\ell$ such that $\underline{x}$ is not regular sequence on R^+. For any $1 \leqslant i \leqslant \ell$ set $\underline{x}_i := x_1, \ldots, x_i$. Then $\operatorname{ht}(\underline{x}_iR) = i$, because R is Cohen–Macaulay. [Mat, Theorem 19.4] says that regular rings are normal. In particular, going down theorem holds for the integral extension R^+/R. By applying this, one can find that $\operatorname{ht}(\underline{x}_iR^+) = i$. So $\mathrm{K.grade}_{R^+}(\underline{x}_iR^+, R^+) = i$, because R^+ is Cohen–Macaulay in the sense of finitely generated ideals. By using [HM, Proposition 3.3(e)], one can find that $\underline{x}_i$ is a parameter sequence on R^+. Therefore, $\underline{x}$ is a strong parameter sequence on R^+. Then $\underline{x}$ is a regular sequence on R^+, since R^+ is Cohen–Macaulay in the sense of Hamilton–Marley. This is a contradiction.

(iii) Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension less than 3. One can see easily that R^+ is a balanced big Cohen–Macaulay R-algebra. Thus by a same reason as (i), R^+ is Cohen–Macaulay in the sense of each part of Definition 3.1.

(iv) By our assumptions, one can see that R^+ is not a balanced big Cohen–Macaulay R-algebra, see e.g. [R, p. 617]. Then by a same method as (ii), R^+ is not Cohen–Macaulay in the sense of finitely generated ideals. □

Let R be a domain containing a field of characteristic $p > 0$. We let R_∞ denote the perfect closure of R, that is, R_∞ is the ring obtained by adjoining to R the p^n-th roots of all its elements for all n. The next result gives the Cohen–Macaulayness of R_∞.

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of prime characteristic p. Then R_∞ is Cohen–Macaulay in the sense of each part of Definition* 3.1.

Proof. For each positive integer n, set $R_n := \{x \in R_\infty \mid x^{p^n} \in R\}$. By using of [BH, Corollary 8.2.8], one can find that the R-algebra R_n is flat. Since $R_\infty := \varinjlim_n R_n$, so R_∞ is flat R-algebra. Therefore by Lemma 4.3, R_∞ is Cohen–Macaulay in the sense of ideals.

Let $\mathfrak{a}$ be a finitely generated ideal of R_∞ with the property that $\operatorname{ht}\mathfrak{a} \geqslant \mu(\mathfrak{a})$. Then

$$m := \mu(\mathfrak{a}) = \operatorname{ht}\mathfrak{a} = \operatorname{K.grade}_{R_\infty}(\mathfrak{a}, R_\infty).$$

Let $\{a_1, \dots, a_m\}$ be a generating set for $\mathfrak{a}$. There is an integer ℓ such that $a_i \in R_\ell$ for all $1 \leqslant i \leqslant m$. Set $\mathfrak{b} := a_1R_\ell + \dots + a_mR_\ell$. In order to pass from R to R_ℓ assume that R is d-dimensional. So $\mathfrak{m}$ can be generated by d elements, namely $x_1, \dots, x_d$. The ring R_ℓ is local with the maximal ideal $(x_1^{1/p^\ell}, \dots, x_d^{1/p^\ell})R_\ell$. In particular, R_ℓ is regular. Hence we can replace R by R_ℓ. Also, $\mathfrak{b}R_\infty = \mathfrak{a}$ and $m := \mu(\mathfrak{b}) = \operatorname{ht}\mathfrak{b}$. In view of the equality $\mu(\mathfrak{b}) = \operatorname{K.grade}_R(\mathfrak{b}, R)$ and by [BH, Exercise 1.2.21], one can generated $\mathfrak{b}$ by an R-regular sequence $\underline{b} := b_1, \dots, b_m$. Keep in mind that R_∞ is a flat R-algebra. Then $\underline{b}$ forms a regular sequence on R_∞. From this, Lemma 3.5 and Lemma 3.9 we get that $\min(\mathfrak{a}) = \operatorname{wAss}_{R_\infty}(R_\infty/\mathfrak{a})$. Therefore, R_∞ is weak Bourbaki unmixed. □

The argument of the next result involves the concept of Generalized Principal Ideal Theorem. By definition, a ring R satisfies GPIT (for Generalized Principal Ideal Theorem) if $\operatorname{ht}(\mathfrak{p}) \leqslant n$ for each prime ideal $\mathfrak{p}$ of R which is minimal over an n-generated ideal of R. Rings, with this property are denoted by GPIT. For more details on this, see e.g. [ADEH]. To see an easy example of non-GPIT ring, let $(V, \mathfrak{m})$ be an infinite-dimensional valuation domain. Then, for any positive integer n one can find an element x_n such that $\operatorname{ht}(x_nV) = n$.

Corollary 4.6. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic p. Then the following assertions hold.*

(i) *If R is complete, then R^+ is weak Bourbaki height unmixed.*
(ii) *If R is regular, then R_∞ is weak Bourbaki height unmixed.*

Proof. The proof of (ii) is similar as (i). Thus, we give only the proof of (i). To do this, first note that by [H1, Theorem 3.3] over GPIT, weak Bourbaki height unmixed follows by weak Bourbaki unmixed. Thus, in view of Theorem 4.4(i), the claim follows by showing that R^+ is GPIT. Due to [ADEH, Corollary 2.3] we know that any ring which is integral over a Noetherian domain is GPIT. Therefore R^+ is GPIT. □

5. Cohen–Macaulayness of rings of invariants

Let R be a commutative ring and G a finite group of automorphisms of R. The subring of invariants defined by

$$R^G := \{x \in R\colon \sigma(x) = x \text{ for all } \sigma \in G\}.$$

Assume that the order of G is a unit in R. Then by a famous result of Hochster and Eagon [HE, Proposition 13], we know that if R is Noetherian and Cohen–Macaulay, then R^G is as well. Our main aim of the present section can be regarded as a non-Noetherian version of this result. First, we give the proof of Theorem 1.2. To do this, we need a new definition for the notion of Cohen–Macaulayness for arbitrary commutative rings as desired in Theorem 1.2.

Definition 5.1. Let $\underline{x} := x_1, \dots, x_\ell$ be a finite sequence of elements of a ring R.

(i) For an R-module L set $\mathbb{K}_\bullet(\underline{x}; L) := \mathbb{K}_\bullet(\underline{x}) \otimes_R L$. Recall that for a pair of integers $m \geqslant n$, there exists a chain map

$$\varphi_n^m(\underline{x}; L) : \mathbb{K}_\bullet(\underline{x}^m; L) \to \mathbb{K}_\bullet(\underline{x}^n; L)$$

which induces by multiplication of $(\prod x_i)^{m-n}$. We call $\underline{x}$ a generalized proregular sequence on R if for each positive integer n and any finitely generated R-module M, there exists an integer $m \geqslant n$ such that the maps

$$H_i(\varphi_n^m(\underline{x}; M)) : H_i(\mathbb{K}_\bullet(\underline{x}^m; M)) \to H_i(\mathbb{K}_\bullet(\underline{x}^n; M))$$

are zero for all $i \geqslant 1$.

(ii) We say that $\underline{x}$ is a generalized parameter sequence on R, if:
  (1) $\underline{x}$ is a generalized proregular sequence,
  (2) $(\underline{x})R \neq R$, and
  (3) $H^\ell_{\underline{x}}(R)_\mathfrak{p} \neq 0$ for all prime ideals $\mathfrak{p} \in \mathrm{V}(\underline{x}R)$.

(iii) We call $\underline{x}$ a generalized strong parameter sequence on R, if $x_1, \ldots, x_i$ is a parameter sequence on R for all $1 \leqslant i \leqslant \ell$.

(iv) We say that R is weakly Cohen–Macaulay, if each generalized strong parameter sequence on R is a regular sequence on R.

Remark 5.2. (i) Assume that R is a Noetherian ring. Let $\underline{x} := x_1, \ldots, x_\ell$ be a finite sequence of elements of R and $m \geqslant n$ a pair of positive integers. [Str, Lemma 4.3.3] says that the morphisms

$$H_i(\varphi_n^m(\underline{x}; R)) : H_i(\mathbb{K}_\bullet(\underline{x}^m; R)) \to H_i(\mathbb{K}_\bullet(\underline{x}^n; R))$$

are eventually null. Now, let M be a finitely generated R-module. By making straightforward modification of [Str, Lemma 4.3.3], one can see that the following homomorphisms

$$H_i(\varphi_n^m(\underline{x}; M)) : H_i(\mathbb{K}_\bullet(\underline{x}^m; M)) \to H_i(\mathbb{K}_\bullet(\underline{x}^n; M))$$

are eventually null. Then any finite sequence of elements of R is a generalized proregular sequence.

(ii) Generalized parameter sequence does not coincide with (partial) systems of parameters if the ring is Noetherian and local. To see an example, let $\mathbb{F}$ be a field and consider the ring $R := \mathbb{F}[[X, Y, Z]]/(X) \cap (Y, Z)$. We use small letters to indicate the images in R. As was shown by [Mat, Theorem 14.1(ii)], y is a partial systems of parameter. Note that $\min(yR) = \mathfrak{p} := (y, z)$, and so $\operatorname{ht}\mathfrak{p} = 0$. By using Grothendieck Vanishing Theorem, $H^1_y(R)_\mathfrak{p} = 0$. Therefore, y is not a generalized parameter sequence.

(iii) If $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring, then by [Mat, Theorem 14.1(ii)], there exists a choice $\underline{x} := x_1, \ldots, x_d$ of system of parameters such that $\operatorname{ht}(x_1, \ldots, x_i) = i$ for all $1 \leqslant i \leqslant d$. Then $\operatorname{ht}(\mathfrak{p}) = i$ for all $\mathfrak{p} \in \min(x_1, \ldots, x_i)$ and by applying Grothendieck non-vanishing theorem, $H^i_{x_1,\ldots,x_i}(R)_\mathfrak{p} \neq 0$. This yields that $\underline{x}$ is a generalized strong parameter sequence.

(iv) For convention, the ideal generated by the empty sequence is the zero ideal and the empty sequence is a regular sequence of length zero over any ring.

Lemma 5.3. *Let R be a ring. Then the following assertions hold.*

(i) *Assume that R is Noetherian and Cohen–Macaulay. Then the ring $R[X_1, X_2, \ldots]$ is weakly Cohen–Macaulay.*

(ii) *If $R_\mathfrak{p}$ is weakly Cohen–Macaulay for all prime ideals $\mathfrak{p}$ of R, then R is weakly Cohen–Macaulay.*

Proof. (i) Note that if a ring is Cohen–Macaulay in the sense of Hamilton–Marley, then it is weakly Cohen–Macaulay. So (i) follows from Theorem 4.1.

(ii) Let $\underline{x}$ be a generalized strong parameter sequence on R and $\mathfrak{p}$ a prime ideal containing $\underline{x}$. Let N be a finitely generated $R_\mathfrak{p}$-module. One can find a finitely generated R-module as M such that $M_\mathfrak{p} \cong N$. Since $\underline{x}$ is a generalized proregular sequence on R for each positive integer n there exists an $m \geqslant n$ such that the maps

$$H_i\bigl(\varphi_n^m(\underline{x};M)\bigr):H_i\bigl(\mathbb{K}_\bullet\bigl(\underline{x}^m;M\bigr)\bigr)\to H_i\bigl(\mathbb{K}_\bullet\bigl(\underline{x}^n;M\bigr)\bigr)$$

are zero for all $i \geqslant 1$. On the other hand localization commutes with homology functors. Therefore, $\underline{x}$ is a generalized proregular sequence on $R_\mathfrak{p}$. By [HM, Proposition 3.3(c)], $\underline{x}$ is a strong parameter sequence on $R_\mathfrak{p}$. Hence, $\underline{x}$ is a generalized strong parameter sequence on $R_\mathfrak{p}$. So, $\underline{x}$ is a regular sequence on $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. In particular, $\underline{x}$ is a regular sequence on $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ containing $\underline{x}R$. Therefore, $\underline{x}$ is a regular sequence on R. □

The preparation of Theorem 1.2 in the introduction is finished. Now, we proceed to the proof of it. We repeat Theorem 1.2 to give its proof.

Theorem 5.4. *The following assertions hold.*

(i) *A Noetherian ring is Cohen–Macaulay with original definition in Noetherian case if and only if it is weakly Cohen–Macaulay.*
(ii) *Coherent regular rings are weakly Cohen–Macaulay.*
(iii) *Let R be a weakly Cohen–Macaulay ring and G a finite group of automorphisms of R such that the order of G is a unit in R. Assume that R is finitely generated as an R^G-module. Then R^G is weakly Cohen–Macaulay.*
(iv) *Let R be a Noetherian Cohen–Macaulay ring. Then the polynomial ring $R[X_1, X_2, \ldots]$ is weakly Cohen–Macaulay.*
(v) *If $R_\mathfrak{p}$ is weakly Cohen–Macaulay for all prime ideals $\mathfrak{p}$ of R, then R is weakly Cohen–Macaulay.*

Proof. (i) Let R be a Noetherian ring. Note that, in view of Remark 5.2(i), any finite sequence of elements of R is a generalized proregular sequence. First, assume that R is Cohen–Macaulay with original definition in Noetherian case. Then by Lemma 5.3(ii), we may and do assume that $(R,\mathfrak{m})$ is local. Let $\underline{x} := x_1,\ldots,x_\ell$ be a strong generalized parameter sequence for R. Due to [HM, Remark 3.2] we know that $\mathrm{ht}(\underline{x}R)=\ell$. In particular, $\underline{x}$ is a (partial) systems of parameters. So $\underline{x}$ is a regular sequence on R. This shows that R is weakly Cohen–Macaulay.

Now, assume that R is weakly Cohen–Macaulay. Let $\mathfrak{a}$ be an ideal of R of height ℓ. In view of [BH, Theorem A.2, p. 412], one can find a sequence $\underline{x} := x_1,\ldots,x_\ell$ of elements of $\mathfrak{a}$ such that $\mathrm{ht}(x_1,\ldots,x_i)=i$ for all $1 \leqslant i \leqslant \ell$ and $\mathrm{ht}(\underline{x}R)=\mathrm{ht}(\mathfrak{a})$. Then by using [HM, Remark 3.2], $\underline{x}$ is a generalized strong parameter sequence on R. Thus, $\underline{x}$ is a regular sequence on R, and so $\mathrm{c.grade}_R(\mathfrak{a},R) \geqslant \mathrm{ht}(\mathfrak{a})$. Therefore, R is Cohen–Macaulay with original definition in Noetherian case.

(ii) [HM, Theorem 4.8] says that any coherent regular ring is locally Cohen–Macaulay in the sense of Hamilton–Marley, and so locally weakly Cohen–Macaulay. Therefore, Lemma 5.3(ii) implies (ii).

(iii) Let $\underline{x} := x_1,\ldots,x_\ell$ be a generalized parameter sequence on R^G. In order to show that $\underline{x}$ is a generalized parameter sequence on R, we need to show that the following three assertions hold:

(a) $\underline{x}$ is a generalized proregular sequence on R,
(b) $(\underline{x})R \neq R$, and
(c) $H_{\underline{x}}^\ell(R)_\mathfrak{q} \neq 0$ for all prime ideals $\mathfrak{q} \in \mathrm{V}(\underline{x}R)$.

Let M be a finitely generated R-module. Since R is a finitely generated R^G-module, we get that M is also finitely generated as an R^G-module. From this one can find easily that $\underline{x}$ is a generalized proregular sequence on R. Hence (a) is satisfied.

The assertion (b) trivially holds. In order to show (c), assume for a contradiction that $H_{\underline{x}}^\ell(R)_\mathfrak{q}=0$ for some prime ideal $\mathfrak{q} \in \mathrm{V}(\underline{x}R)$. It follows from [Bk, Proposition 23, p. 324] that $S^{-1}(R^G)=(S^{-1}R)^G$ for any multiplicative closed subset S of R^G. Set $\mathfrak{p} := \mathfrak{q} \cap R^G$ and $S = R^G \setminus \mathfrak{p}$. So $(R^G)_\mathfrak{p} \cong (R_\mathfrak{p})^G$ and $\mathfrak{p} \in \mathrm{V}(\underline{x}R^G)$. Since $\underline{x}$ is a parameter sequence on R^G, we have

$$0 \neq \bigl(H_{\underline{x}}^\ell\bigl(R^G\bigr)\bigr)_\mathfrak{p} \cong H_{\underline{x}}^\ell\bigl(\bigl(R^G\bigr)_\mathfrak{p}\bigr) \cong H_{\underline{x}}^\ell\bigl((R_\mathfrak{p})^G\bigr).$$

Also, $H^{\ell}_{\underline{x}}(R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} \cong H^{\ell}_{\underline{x}}(R_{\mathfrak{q}})$. Then, to simplify the notation, after replacing R by $R_{\mathfrak{p}}$ and R^G by $(R^G)_{\mathfrak{p}}$, we can assume that $(R^G, \mathfrak{m})$ is a quasi-local ring with the following properties; $H^{\ell}_{\underline{x}}(R^G) \neq 0$, $\mathfrak{q} \cap R^G = \mathfrak{m}$ and $H^{\ell}_{\underline{x}}(R)_{\mathfrak{q}} = 0$.

Let $\sigma : R \to R$ be an element of G and $y \in R^G$. Then the assignment $r/y^n \mapsto \sigma(r)/y^n$ induces an R^G-algebra isomorphism $\sigma_y : R_y \to R_y$. This gives an R^G-isomorphism of the *Čech* complexes $\sigma_1 : \check{\mathbf{C}}_{\bullet}(\underline{x}, R) \to \check{\mathbf{C}}_{\bullet}(\underline{x}, R)$. Let $1 \leqslant i \leqslant \ell$. Thus we have an R^G-isomorphisms of the *Čech* cohomology modules

$$\sigma_2^i : H^i\big(\check{\mathbf{C}}_{\bullet}(\underline{x}, R)\big) \to H^i\big(\check{\mathbf{C}}_{\bullet}(\underline{x}, R)\big).$$

Note that $\sigma_2^i(tm) = \sigma(t)\sigma_2^i(m)$ for $t \in R$ and $m \in H^i_{\underline{x}}(R)$. From this one can find that the assignment $m/s \mapsto \sigma_2^i(m)/\sigma(s)$ for $s \in R \setminus \mathfrak{q}$ and $m \in H^i_{\underline{x}}(R)$, induces the following R^G-isomorphisms

$$\sigma_3^i : H^i_{\underline{x}}(R)_{\mathfrak{q}} \to H^i_{\underline{x}}(R)_{\sigma(\mathfrak{q})}.$$

Assume that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of R lying over $\mathfrak{m}$. In view of [Bk, Theorem 2(i), p. 331], one can find an element σ in G such that $\sigma(\mathfrak{q}_1) = \mathfrak{q}_2$. Also, any maximal ideals of R contracted to $\mathfrak{m}$. Thus, from the definition of σ_3^i, we have $H^{\ell}_{\underline{x}}(R)_{\sigma(\mathfrak{n})} = 0$ for all $\mathfrak{n} \in \max(R)$ and consequently $H^{\ell}_{\underline{x}}(R) = 0$. Consider the Reynolds operator $\rho : R \to R^G$. It sends $r \in R$ to $\frac{1}{|G|}\sum_{g \in G} gr$. This follows that R^G is a direct summand of R as R^G-module. So $H^{\ell}_{\underline{x}}(R^G) = 0$, a contradiction. This completes the proof of (c).

Now, assume that $\underline{x}$ is a generalized strong parameter sequence on R^G. The same reason as above, shows that $\underline{x}$ is a generalized strong parameter sequence on R. Since R is weakly Cohen–Macaulay, we get that $\underline{x}$ is a regular sequence on R. By applying [BH, Proposition 6.4.4(c)], we find that $\underline{x}$ is a regular sequence on R^G. This completes the proof of (iii).

(iv) and (v) are proved in Lemma 5.3. □

In the proof of the next result, we use the method of the proof of Lemma 3.2(ii) and Lemma 4.1 in [TZ]. Recall that, a group G is said to be locally finite if for every $x \in R$ the orbit of x has finite cardinality.

Lemma 5.5. *Let R be a ring and G a group of automorphisms of R.*

(i) *Let $\mathfrak{a}$ be an ideal of R and S a pure extension of R. Then* $\mathrm{K.grade}_R(\mathfrak{a}, R) \geqslant \mathrm{K.grade}_S(\mathfrak{a}S, S)$.
(ii) *Let $\mathfrak{a}$ be an ideal of R^G. Assume that there is a Reynolds operator for the extension R/R^G. Then* $\mathrm{K.grade}_{R^G}(\mathfrak{a}, R^G) \geqslant \mathrm{K.grade}_R(\mathfrak{a}R, R)$.
(iii) *Let $\mathfrak{q}$ be a prime ideal of R and G a locally finite group of automorphisms of R such that the cardinality of orbit of x is a unit in R for every $x \in R$. Then* $\mathrm{ht}(\mathfrak{q}) \leqslant \mathrm{ht}(\mathfrak{q} \cap R^G)$. *The equality holds if G is finite.*

Proof. (i) Let $\underline{y} := y_1, \ldots, y_s$ be a finite sequence of elements of $\mathfrak{a}S$. Then there exists a finite subset $\underline{x} := x_1, \ldots, x_\ell$ of elements of $\mathfrak{a}$ such that $\underline{y}S \subseteq \underline{x}S$. In view of [BH, Exercise 10.3.31(a)], one can find that the natural map

$$H_i\big(\mathbb{K}_{\bullet}(\underline{x})\big) \to H_i\big(\mathbb{K}_{\bullet}(\underline{x}) \otimes_R S\big)$$

is injective for all i. Then, by symmetry of Koszul cohomology and Koszul homology, one has $\mathrm{K.grade}_R(\underline{x}R, R) \geqslant \mathrm{K.grade}_R(\underline{x}R, S)$. Now, by Proposition 2.2(iii) and (iv), we find that

$$\begin{aligned}\mathrm{K.grade}_R(\mathfrak{a}, R) &\geqslant \mathrm{K.grade}_R(\underline{x}R, R)\\ &\geqslant \mathrm{K.grade}_R(\underline{x}R, S)\\ &= \mathrm{K.grade}_S(\underline{x}S, S)\\ &\geqslant \mathrm{K.grade}_S(\underline{y}S, S).\end{aligned}$$

So the claim follows from definition.

(ii) By using Reynolds operator, one can find that R is a pure extension of R^G. So (ii) follows from (i).

(iii) Since G is locally finite, so by [Bk, Proposition 22, p. 323], the ring extension R/R^G is integral. The first claim follows from this. Let

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{q} \cap R^G$$

be a chain of prime ideals of R^G. By lying over theorem, there exists $\mathfrak{q}_0 \in \mathrm{Spec}(R)$ such that $\mathfrak{q}_0 \cap R^G = \mathfrak{p}_0$. Thus by going up theorem, there is a chain of prime ideals of R as $\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap R^G = \mathfrak{p}_i$. In view of [Bk, Theorem 2(i), p. 331], there exists an automorphism σ in G such that $\sigma(\mathfrak{q}_n) = \mathfrak{q}$. It is clear that

$$\sigma(\mathfrak{q}_0) \subsetneqq \sigma(\mathfrak{q}_1) \subsetneqq \cdots \subsetneqq \sigma(\mathfrak{q}_n) = \mathfrak{q}$$

is a chain of prime ideals of R and so $\mathrm{ht}\,\mathfrak{q} \geqslant \mathrm{ht}(\mathfrak{q} \cap R^G)$. $\square$

We now apply Lemma 5.5 to obtain the following result on the Cohen–Macaulayness of rings of invariants in the sense of (finitely generated) ideals.

Theorem 5.6. *Let R be a Cohen–Macaulay ring in the sense of (finitely generated) ideals and G a finite group of automorphisms of R such that the order of G is a unit in R. Let $\mathfrak{a}$ be a (finitely generated) ideal of R^G. Then* $\mathrm{K.grade}_{R^G}(\mathfrak{a}, R^G) = \mathrm{K.grade}_R(\mathfrak{a}R, R)$ *and* $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}R)$. *In particular, R^G is Cohen–Macaulay in the sense of (finitely generated) ideals.*

Proof. Let $\mathfrak{a}$ be a (finitely generated) ideal of R^G and $\mathfrak{q} \in \mathrm{Spec}\,R$ be such that $\mathrm{ht}(\mathfrak{a}R) = \mathrm{ht}\,\mathfrak{q}$. Thus, by Lemma 5.5(iii), $\mathrm{ht}(\mathfrak{a}R) = \mathrm{ht}(\mathfrak{q} \cap R^G)$. Therefore, Lemma 3.2 and Lemma 5.5(ii) yield that

$$\mathrm{ht}\,\mathfrak{a} \geqslant \mathrm{K.grade}_{R^G}\big(\mathfrak{a}, R^G\big) \geqslant \mathrm{K.grade}_R(\mathfrak{a}R, R) = \mathrm{ht}(\mathfrak{a}R) = \mathrm{ht}\big(\mathfrak{q} \cap R^G\big) \geqslant \mathrm{ht}\,\mathfrak{a},$$

which completes the proof. $\square$

To complete our desired list of the behavior of rings of invariants, on the different types of Cohen–Macaulay rings, we need to state the following result. A consequence of this is given by Corollary 5.8.

Proposition 5.7. *Let R be a weak Bourbaki (height) unmixed ring and G a finite group of automorphisms of R such that the order of G is a unit in R. Then R^G is weak Bourbaki (height) unmixed.*

Proof. The proof of weak Bourbaki height unmixed case is similar as weak Bourbaki unmixed case. So we give only the proof of weak Bourbaki unmixed case. Let $\mathfrak{a}$ be a finitely generated ideal of R^G with the property that $\mathrm{ht}\,\mathfrak{a} \geqslant \mu(\mathfrak{a})$. Assume that $\mathfrak{p}$ belongs to $\mathrm{wAss}_{R^G}(R^G/\mathfrak{a})$. Then there exists an element r in R^G such that $\mathfrak{p} \in \min((\mathfrak{a} :_{R^G} r))$. Let $\mathfrak{q}$ be any prime ideal of R lying over $\mathfrak{p}$. First, we show that $\mathfrak{q} \in \mathrm{wAss}_R(R/\mathfrak{a}R)$. To do this, let $\mathfrak{q}'$ be a prime ideal of R such that $(\mathfrak{a}R :_R r) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$. By contraction of this to R^G we get that $\mathfrak{q}' \cap R^G = \mathfrak{q} \cap R^G$, because $\mathfrak{a}R \cap R^G = \mathfrak{a}$. So $\mathfrak{q}' = \mathfrak{q}$, i.e., $\mathfrak{q} \in \mathrm{wAss}_R(R/\mathfrak{a}R)$. Let $\mathfrak{q}_0$ be a prime ideal of R such that $\mathrm{ht}(\mathfrak{a}R) = \mathrm{ht}(\mathfrak{q}_0)$. Then, in view of Lemma 5.5(iii),

$$\mathrm{ht}(\mathfrak{a}R) = \mathrm{ht}(\mathfrak{q}_0) = \mathrm{ht}\big(\mathfrak{q}_0 \cap R^G\big) \geqslant \mathrm{ht}(\mathfrak{a}) \geqslant \mu(\mathfrak{a}) \geqslant \mu(\mathfrak{a}R).$$

This implies that $\mathfrak{q} \in \min(\mathfrak{a}R)$.

Now, we show that $\mathfrak{p} \in \min(\mathfrak{a})$. To see this, let $\mathfrak{p}'$ be a prime ideal of R^G and assume that $\mathfrak{a} \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$. By lying over theorem, there exists $\mathfrak{q}' \in \mathrm{Spec}(R)$ such that $\mathfrak{q}' \cap R^G = \mathfrak{p}'$. By applying the going up theorem to this, we find a prime ideal $\mathfrak{q}''$ of R such that $\mathfrak{q}' \subseteq \mathfrak{q}''$ and $\mathfrak{q}'' \cap R^G = \mathfrak{p}$. As we saw, one has $\mathfrak{q}'' \in \mathrm{wAss}_R(R/\mathfrak{a}R) = \min(\mathfrak{a}R)$. This implies that $\mathfrak{p}' = \mathfrak{p}$ and consequently $\mathfrak{p} \in \min(\mathfrak{a})$. □

The statement of the next result involves a non-Noetherian version of the concept of veronese subrings in polynomial ring $R := \mathbb{C}[X_1, X_2, \ldots]$. Let $f := X_{i_1}^{j_1} \cdots X_{i_\ell}^{j_\ell}$ be a monomial in R. The degree of f is defined by $d(f) := \sum_{k=1}^{\ell} j_k$. Let n be a positive integer. We call the $\mathbb{C}$-algebra generated by all monomials of degree n, the n-th veronese subring of R. We denoted it by R_n.

Corollary 5.8. *Let n be a positive integer and let R_n be the n-th veronese subring of $R := \mathbb{C}[X_1, X_2, \ldots]$. Then R_n is Cohen–Macaulay in the sense of each part of Definition* 3.1.

Proof. In light of Theorem 4.1 we see that $\mathbb{C}[X_1, X_2, \ldots]$ is Cohen–Macaulay in the sense of each part of Definition 3.1. Since $\mathbb{C}$ is an algebraically closed field, then for each positive integer n, $\mathbb{C} \setminus \{0\}$ has a multiplicative subgroup G of order n. Let g be in G. The assignment $X_i \mapsto gX_i$ induces an action of G on R. Assume that f is a monomial in R. Then f belongs to R^G if and only if $g^{d(f)} = 1$ for all $g \in G$. On the other hand by [Ha, V. Theorem 5.3], G is cyclic. So f belongs to R^G if and only if $d(f) = \ell n$ for some $\ell \in \mathbb{N} \cup \{0\}$. From this we have $R^G = R_n = \mathbb{C}[f\colon\ d(f) \in n\mathbb{N}]$. Due to Theorem 5.6 and Proposition 5.7 we know that R_n is Cohen–Macaulay in the sense of ideals and weak Bourbaki unmixed. Now, the claim follows by Theorem 3.3. □

It is noteworthy to remark that the converse of the previous results of this section are not true and their assumptions are really needed.

Remark 5.9. (i) Let $\mathbb{F}$ be a perfect field of characteristic 2. In [Ber], Bertin presented an action of a finite group G of order 4 on $R := \mathbb{F}[X, Y, Z, W]$ such that R^G is Noetherian but not Cohen–Macaulay. Thus, in Theorem 5.6 and Proposition 5.7 the unit assumption on $|G|$ is really needed, even if R is Noetherian and regular.

(ii) Let A be a Noetherian normal domain which is not Cohen–Macaulay. In particular, A is a Krull domain. A beautiful result of Bergman [Be, Proposition 5.2] state that there is a principal ideal domain R and an infinite cycle group G such that $R^G = A$. So, in Theorem 5.6 the finite assumption on G is really needed, even if R^G is Noetherian and regular.

(iii) Let $\mathbb{F}$ be a field and set $R := \mathbb{F}[[X, Y]]/(XY, Y^2)$. Then R is not Cohen–Macaulay. The assignments $X \mapsto X$ and $Y \mapsto -Y$ induce an isomorphism call it g. Consider the group of automorphisms generated by g and denote it by $G := \langle g \rangle$. Then $|G| = 2$ and $R^G = \mathbb{F}[[X]]$ (cf. [F2, p. 448]). Therefore, the converse part of Proposition 5.7 is not true, even if R^G is Noetherian and regular.

(iv) Fogarty [F2] presented a wild action of a cyclic group G on a local Noetherian ring R such that R^G is Noetherian and $\operatorname{depth} R - \operatorname{depth} R^G$ can be arbitrarily large. Thus the assumptions of G in Lemma 5.5(ii) is really needed.

(v) Nagata constructed a zero-dimensional Noetherian ring R and a finite group G of automorphisms of R such that R^G is non-Noetherian, see e.g. the introduction of [F1]. The ring extension R/R^G is integral, because G is finite. Since R is zero-dimensional, so R^G is zero-dimensional. This is clear that any zero-dimensional ring is Cohen–Macaulay in the sense of each part of Definition 3.1. Thus, R^G is as well. Therefore, it is possible R^G becomes Cohen–Macaulay without the unit assumption on $|G|$.

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