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The computation of the logarithmic cohomology for plane curves

Francisco-Jesús Castro-Jiménez [a], Nobuki Takayama [b,*]

[a] *Department of Mathematics, University of Sevilla, Sevilla, Spain*
[b] *Department of Mathematics, Kobe University, Rokko 1-1, Kobe, Japan*

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A B S T R A C T

We will give algorithms of computing bases of logarithmic cohomology groups for square-free polynomials in two variables.

1. Introduction

Let us denote by $R = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ the polynomial ring, by $A_n = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ the complex Weyl algebra of order n and by $(\Omega_R^\bullet, d)$ (or simply $(\Omega^\bullet, d)$) the complex of polynomial (or regular) differential forms (i.e. the complex of differential forms with polynomial coefficients) where d is the exterior derivative.

The elements of A_n are called linear differential operators with polynomial coefficients. An element $P(x, \partial)$ in A_n can be written as a finite sum $P(x, \partial) = \sum_\alpha a_\alpha(x)\partial^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $a_\alpha(x) \in R$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Here ∂_i stands for the partial derivative $\frac{\partial}{\partial x_i}$.

For a non-zero polynomial $f \in R$ we denote by R_f the ring of rational functions

$$R_f = \left\{ \frac{g}{f^m} \;\middle|\; g \in R,\ m \in \mathbb{N} \right\}$$

* Corresponding author.
E-mail addresses: castro@us.es (F.-J. Castro-Jiménez), takayama@math.kobe-u.ac.jp (N. Takayama).

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and by $(\Omega^\bullet_f, d) := (R_f \otimes_R \Omega^\bullet_R, d)$ the complex of rational differential forms with coefficients in R_f where d is the corresponding exterior derivative.

Let us denote by $Der_{\mathbb{C}}(R)$ the free R-module of polynomial vector fields (or equivalently of $\mathbb{C}$-linear derivations of R). Following K. Saito [19] we will denote by $Der_R(-\log f)$ (or simply $Der(-\log f)$ if no confusion is possible) the R-module of logarithmic vector fields with respect to f, i.e.

$$Der_R(-\log f) = \left\{ \delta = \sum_{i=1}^n a_i(x)\partial_i \in Der_{\mathbb{C}}(R) \;\middle|\; \delta(f) \in R \cdot f \right\}.$$

$Der_R(-\log f)$ is canonically isomorphic to the R-module $Syz_R(\partial_1(f), \dots, \partial_n(f), f)$ of syzygies among $(\partial_1(f), \dots, \partial_n(f), f)$. This isomorphism associates the logarithmic vector field $\delta = \sum_i a_i(x)\partial_i$ with the syzygy $(a_1(x), \dots, a_n(x), -\frac{\delta(f)}{f})$.

If f is a non-zero constant, then $Der(-\log f) = Der_{\mathbb{C}}(R)$. So we will assume from now that f is a non-constant polynomial in R.

It is clear that

$$f Der_{\mathbb{C}}(R) \subset Der(-\log f) \subset Der_{\mathbb{C}}(R)$$

and then $Der(-\log f)$ has rank n as R-module. The R-module $Der(-\log f)$ does not depend on the polynomial f but only on the hypersurface $D = \mathcal{V}(f) := \{a \in \mathbb{C}^n \mid f(a) = 0\} \subset \mathbb{C}^n$.

Assume f is reduced (i.e. f is square-free). According to K. Saito [19] a rational differential p-form $\omega \in \Omega^p_f$ is said to be logarithmic with respect to f (or with respect to the hypersurface $D = \mathcal{V}(f) \subset \mathbb{C}^n$) if both $f\omega$ and $f\,d\omega$ are regular (i.e. $f\omega \in \Omega^p_R$ and $f\,d\omega \in \Omega^{p+1}_R$). We denote by $\Omega^p_R(\log f)$ (or simply $\Omega^p(\log f)$) the R-module of logarithmic differential p-forms with respect to f. K. Saito [19, Corollary 1.6] proved that $Der(-\log f)$ is a reflexive R-module whose dual is $\Omega^1(\log f)$. We denote by $(\Omega^\bullet(\log f), d)$ the complex

$$0 \to \Omega^0(\log f) \xrightarrow{d} \Omega^1(\log f) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\log f) \to 0$$

which will be called the logarithmic de Rham complex and is also, for simple notation, denoted by $\Omega^\bullet(\log f)$ if no confusion arises.

Algorithms of computing dimensions and bases of the de Rham cohomology groups $H^i(\Omega^\bullet_f)$ are given by T. Oaku and N. Takayama [15,17] and U. Walther [22]. Here, f is any non-zero polynomial in n variables. The purpose of this paper is to give algorithms of computing dimensions and bases of the logarithmic de Rham cohomology groups $H^i(\Omega^\bullet(\log f))$ as $\mathbb{C}$-vector spaces in the case of two variables.

1.1. Logarithmic comparison theorem

The rings R and R_f have natural structures of left A_n-module where ∂_i acts on a polynomial g and on a rational function $\frac{g}{f^m}$ as the partial derivative with respect to x_i.

The de Rham complex of a left A_n-module M, denoted by $DR(M)$, is by definition the complex of $\mathbb{C}$-vector spaces $(M \otimes_R \Omega^\bullet_R, \nabla^\bullet)$ where

$$\nabla^p : M \otimes_R \Omega^p_R \to M \otimes_R \Omega^{p+1}_R$$

is defined, for $p \geqslant 1$, by $\nabla^p(m \otimes \omega) = \nabla^0(m) \wedge \omega + m \otimes d\omega$ and $\nabla^0(m) = \sum_i \partial_i(m) \otimes dx_i$. Notice that $am \otimes \omega = m \otimes a\omega$ for $m \in M$, $\omega \in \Omega^p$ and $a \in R$. The complexes $\Omega^\bullet_f$ and $DR(R_f)$ are naturally isomorphic.

For any non-zero $f \in R$, the inclusion i_f is a natural morphism of complexes

$$i_f : \Omega^\bullet(\log f) \to \Omega^\bullet_f.$$

We say (see [3]) that f satisfies the (global) logarithmic comparison theorem if the morphism i_f is a quasi-isomorphism (i.e. if i_f induces an isomorphism $H^p(\Omega^\bullet(\log f)) \to H^p(\Omega^\bullet_f)$ for any p).

If $n = 2$ and f is a quasi-homogeneous polynomial such that the origin is the only singular point of the plane curve defined by f, then i_f is a quasi-isomorphism ([3, Corollary 2.7] and [2, Theorem 1.3]).

1.2. The case $n = 2$. Bases for $Der(-\log f)$

If $n = 2$, any finitely generated reflexive R-module is projective and then, by the Quillen–Suslin theorem, this R-module is free. So, if $n = 2$, the R-module $Der(-\log f)$ is free of rank 2. In this case, we would like to compute a basis of $Der(-\log f)$ by taking the polynomial $f = f(x_1, x_2)$ as input. By using the isomorphism

$$Der(-\log f) \simeq Syz_R\big(\partial_1(f), \partial_2(f), f\big)$$

and using Gröbner basis computation, a system of generators of $Der(-\log f)$ can be calculated. Then we can apply Quillen–Suslin algorithm (as presented for example in [9] and implemented in [7]) to compute such a basis. Known Quillen–Suslin algorithms use Gröbner bases computation. Nevertheless, in some cases, for a big family of polynomials $f(x_1, x_2)$ we will use an easier way to compute a basis of $Der(-\log f)$.

First of all, we can assume f to be a reduced polynomial since $Der(-\log f)$ depends only on the affine plane curve $D = \mathcal{V}(f) = \{(a_1, a_2) \in \mathbb{C}^2 \mid f(a_1, a_2) = 0\} \subset \mathbb{C}^2$.

Assume the plane curve $D = \mathcal{V}(f)$ is not smooth. The singular points of the plane curve $D = \mathcal{V}(f)$ (i.e. the affine algebraic set

$$Sing(D) := \mathcal{V}(f, f_1, f_2) = \big\{\underline{a} = (a_1, a_2) \in \mathbb{C}^2 \;\big|\; f(\underline{a}) = f_1(\underline{a}) = f_2(\underline{a}) = 0\big\}),$$

where $f_1 = \partial_1(f)$, $f_2 = \partial_2(f)$ – consists of a finite number of points (and it is not the empty set).

We will consider the affine plane $\mathbb{C}^2$ as a Zariski open subset of the projective plane $\mathbb{P}_2(\mathbb{C})$, the affine point (a_1, a_2) is mapped into the point with homogeneous coordinates $(1 : a_1 : a_2)$. Coordinates in $\mathbb{P}_2(\mathbb{C})$ will be denoted by $(x_0 : x_1 : x_2)$ and then the line at infinity is defined by $x_0 = 0$.

Let us denote $h = H(f), h_1 = H(f_1)$ and $h_2 = H(f_2)$ where $H(-)$ denotes homogenization with respect to the variable x_0. Denote by $S = \mathbb{C}[x_0, x_1, x_2]$ the polynomial ring graded by the degree of the polynomials. If $J = (h, h_1, h_2)$ denotes the ideal in S generated by h, h_1, h_2 then the quotient ring S/J has Krull dimension 1. Let us denote by S_+ the irrelevant ideal in S, i.e. the ideal generated by x_0, x_1, x_2. The following result is well known (see e.g. [10, Theorem 17.6, p. 136]).

Proposition 1.1. *The graded ring S/J is Cohen–Macaulay if and only if S_+ is not an embedded prime associated with J.*

If S/J is Cohen–Macaulay then the projective dimension of S/J is 2 and J satisfies the Hilbert–Burch theorem [6], i.e. there exists an exact sequence

$$0 \to S^2 \xrightarrow{\phi_2} S^3 \xrightarrow{\phi_1} J \to 0,$$

where $\phi_1(g_0, g_1, g_2) = g_0 h + g_1 h_1 + g_2 h_2$ and ϕ_2 is defined by a syzygy matrix of ϕ_1. In particular, since $\ker(\phi_1) = Syz_S(h, h_1, h_2)$ is a graded free S-module of rank 2 we can compute $\{s^{(1)} = (s_{10}, s_{11}, s_{12}),\; s^{(2)} = (s_{20}, s_{21}, s_{22})\}$ a minimal system of generators and this system is in fact a basis of $\ker(\phi_1)$. By dehomogenization (i.e. by setting $x_0 = 1$), we obtain a system $\{s^{(1)}|_{x_0=1}, s^{(2)}|_{x_0=1}\}$

of generators of $Syz_R(f, f_1, f_2) \simeq Der(-\log f)$ and since this R-module is free of rank 2, this last system is in fact a basis.

If S/J is not Cohen–Macaulay we cannot apply, in general, the Hilbert–Burch theorem and the previous procedure fails to compute a basis of $Der(-\log f)$.

Example 1.2. (a) Consider the polynomial $f = (x^3 + y^4 + xy^3)(x^2 - y^2)$. With the notations as before (and writing $x_1 = x$, $x_2 = y$, $x_0 = t$) we can use Macaulay 2 to prove that the corresponding S/J is Cohen–Macaulay and to compute a minimal system of generators of $Syz_S(h, h_1, h_2)$ and then a basis of $Der(-\log f)$.

```
Macaulay 2, version 1.1 with packages: Classic, Core, Elimination,
IntegralClosure, LLLBases, Parsing,
               PrimaryDecomposition, SchurRings, TangentCone

i1 : R=QQ[t,x,y];
i2 : f=(x^3+y^4+x*y^3)*(x^2-y^2);
i3 : f1=diff(x,f),f2=diff(y,f),h=homogenize(f,t),h1=homogenize(f1,t),h2=homogenize(f2,t);
i4 : Jf=ideal(h,h1,h2);
i5 : pdim coker gens Jf
o5 = 2
i6 : Syzf=kernel matrix({{h1,h2,h}});
i7 : mingens Syzf
o7 = {5} | 3x3+x2y-4xy2   -tx2+4txy+3x2y+4xy2-y3 |
     {5} | 2x2y+xy2-3y3   tx2-txy+3ty2+2xy2+4y3  |
     {6} | -15x2-5xy+18y2 5tx-18ty-15xy-23y2     |

               3       2
o7 : Matrix R  <--- R
```

Then a basis of $Der(-\log f)$ is

$$\left\{\left(x^3 + \frac{1}{3}x^2y - \frac{4}{3}xy^2\right)\partial_x + \left(\frac{2}{3}x^2y + \frac{1}{3}xy^2 - y^3\right)\partial_y,\right.$$
$$\left.\left(-x^2 + 4xy + 3x^2y + 4xy^2 - y^3\right)\partial_x + \left(x^2 - xy + 3y^2 + 2xy^2 + 4y^3\right)\partial_y\right\}.$$

(b) Consider the polynomial $g = (x^3 + y^4 + xy^3)(x^2 + y^2)$. With the notations as before (and writing $x_1 = x$, $x_2 = y$, $x_0 = t$) we can use Macaulay 2 to prove that the corresponding S/J is not Cohen–Macaulay and the minimal number of generators of $Syz_S(h, h_1, h_2)$ is 3. We can continue the last Macaulay 2 session:

```
i8 : g=(x^3+y^4+x*y^3)*(x^2+y^2);
i9 : g1=diff(x,g),g2=diff(y,g),h=homogenize(g,t),h1=homogenize(g1,t),h2=homogenize(g2,t);
i10 : Jg=ideal(h,h1,h2);
i11 : pdim coker gens Jg
o11 = 3
i12 : Syzg=kernel matrix({{h1,h2,h}});
i13 : mingens Syzg
o13 =
{5} | 3tx2-15x3-12txy-20x2y-6xy2-5y3 3x4+4x3y+3x2y2+4xy3    3tx3-3tx2y+12x3y+12txy2+16x2y2+6xy3+4y4 |
{5} | 3tx2+3txy-10x2y-9ty2-15xy2-y3  2x3y+3x2y2+2xy3+3y4    -3txy2+8x2y2+9ty3+12xy3+2y4             |
{6} | -15tx+75x2+54ty+100xy+11y2     -15x3-20x2y-13xy2-18y3 -15tx2+15txy-60x2y-54ty2-80xy2-16y3     |
             3       3
o13 : Matrix R  <--- R
```

We will revisit this example in Example 4.1.

2. Logarithmic A_n-modules

Let us denote by $M^{\log f}$ the quotient A_n-module $M^{\log f} = \frac{A_n}{A_n Der(-\log f)}$. Moreover, we denote by $\widetilde{Der}(-\log f)$ the set

$$\widetilde{Der}(-\log f) = \left\{ \delta + \frac{\delta(f)}{f} \;\Big|\; \delta \in Der(-\log f) \right\}$$

and by $\widetilde{M}^{\log f}$ the quotient A_n-module

$$\widetilde{M}^{\log f} = \frac{A_n}{A_n \widetilde{Der}(-\log f)}.$$

As quoted in Section 1.2, for $n = 2$ the R-module $Der(-\log f)$ (and hence $\Omega^1(\log f)$) is free of rank 2. Moreover, by [19, 1.8] there exists an R-basis $\{\delta_1, \delta_2\}$ of $Der(-\log f)$ satisfying $\det(A) = f$ where

$$\delta_i = a_{i1}\partial_1 + a_{i2}\partial_2, \quad i = 1, 2,$$

and A is the matrix (a_{ij}). Then the dual basis of $\{\delta_1, \delta_2\}$ is $\{\omega_1, \omega_2\}$ with

$$\omega_1 = \frac{1}{f}(a_{22}\,dx_1 - a_{21}\,dx_2), \qquad \omega_2 = \frac{1}{f}(-a_{12}\,dx_1 + a_{11}\,dx_2).$$

The R-module $\Omega^2(\log f)$ is free of rank 1 and $\omega_1 \wedge \omega_2$ is a basis of it. Moreover we have $\omega_1 \wedge \omega_2 = \frac{dx_1 \wedge dx_2}{f}$.

Theorem 2.1. *(See [1, Theorem 4.2.1].) Let $f \in R = \mathbb{C}[x, y]$ be a non-zero reduced polynomial. There exists a natural isomorphism in the corresponding derived category*

$$\Omega^\bullet(\log f) \xrightarrow{\simeq} \mathbf{R}Hom_{A_2}\big(M^{\log f}, R\big),$$

where the last complex is the solution complex of $M^{\log f}$ with values in R.

Remark 2.2. This theorem is proved in [1] in the setting of analytic $\mathcal{D}$-modules, using the notion of V_0-module and the *logarithmic Spencer resolution* of $M^{\log f}$. In our case, once a basis $\{\delta_1, \delta_2\}$ is fixed in $Der(-\log f)$, the logarithmic Spencer resolution of $M^{\log f}$ is nothing but

$$0 \to A \xrightarrow{\epsilon_2} A^2 \xrightarrow{\epsilon_1} A \xrightarrow{\pi} M^{\log f} \to 0, \tag{1}$$

where A stands for A_2, the morphism π is the natural projection, the A-module morphism ϵ_1 is defined by $\epsilon_1(P_1, P_2) = P_1\delta_1 + P_2\delta_2$ (for $P_i \in A$) and ϵ_2 is defined by $\epsilon_2(Q) = Q(-\delta_2 - b_1, \delta_1 - b_2)$ for $Q \in A$ where the polynomials b_i are defined by the equality $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 = b_1\delta_1 + b_2\delta_2$.

Using the previous free resolution the solution complex $\mathbf{R}Hom_A(M^{\log f}, R)$ is represented by the complex of $\mathbb{C}$-vector spaces

$$0 \to R \xrightarrow{\epsilon_1^*} R^2 \xrightarrow{\epsilon_2^*} R \to 0,$$

where $\epsilon_1^*(g) = (\delta_1(g), \delta_2(g))$ for $g \in R$ and $\epsilon_2^*(h_1, h_2) = \delta_1(h_2) - \delta_2(h_1) - b_1h_1 - b_2h_2$ for $h_i \in R$.

The natural morphism of Theorem 2.1 is described as the morphism of complexes

$$\begin{array}{ccccc} \Omega^0(\log f) = R & \xrightarrow{d} & \Omega^1(\log f) & \xrightarrow{d} & \Omega^2(\log f) \\ \eta_0 \downarrow & & \eta_1 \downarrow & & \eta_2 \downarrow \\ R & \xrightarrow{\epsilon_1^*} & R^2 & \xrightarrow{\epsilon_2^*} & R \end{array}$$

where $\eta_0 = id$, $\eta_1(h_1\omega_1 + h_2\omega_2) = (h_1, h_2)$ and $\eta_2(g\omega_1 \wedge \omega_2) = g$ for $h_1, h_2, g \in R$ and where $\{\omega_1, \omega_2\}$ is the dual basis in $\Omega^1(\log f)$ of the basis $\{\delta_1, \delta_2\}$ in $Der(-\log f)$. It is easy to check by computation that this morphism $\eta_\bullet$ of complexes of vector spaces is in fact an isomorphism of complexes.

To each finitely generated left A_n-module M we associate the complex of finitely generated right A_n-modules $\mathbf{R}Hom_{A_n}(M, A_n)$. To this one we associate the complex of finitely generated left A_n-modules $Hom_R(\Omega^n_R, \mathbf{R}Hom_{A_n}(M, A_n))$ which is by definition the dual M^* of the left A_n-module M (see e.g. [11, Déf. 4.1.6]).

If M is holonomic (i.e. if the dimension of the characteristic variety is n) then it can be shown that $Ext^i_{A_n}(M, A_n) = 0$ for $i \neq n$ and then M^* is the left holonomic A_n-module $Hom_R(\Omega^n_R, Ext^n_{A_n}(M, A_n))$ (see e.g. [11, p. 41]). Assume $Ext^n_{A_n}(M, A_n) = \frac{A_n}{J}$ for some right ideal $J \subset A_n$. Then $Hom_R(\Omega^n_R, A_n/J)$ is naturally isomorphic to the left A_n-module $\frac{A_n}{J^T}$ where J^T is the left ideal $J^T = \{P^T \mid P \in J\}$ and P^T is the formal adjoint of the operator P.

If N_1, N_2 are finitely generated left A_n-modules there exists a natural isomorphism of complexes in the corresponding derived category (see e.g. [11, pp. 40–41])

$$\mathbf{R}Hom_{A_n}(N_1, N_2) \to \mathbf{R}Hom_{A_n}\big(\mathbf{R}Hom_{A_n}(N_2, A_n), \mathbf{R}Hom_{A_n}(N_1, A_n)\big)$$

and then a natural isomorphism

$$\mathbf{R}Hom_{A_n}(N_1, N_2) \to \mathbf{R}Hom_{A_n}\big(N_2^*, N_1^*\big).$$

In particular, if $N_2 = R = \mathbb{C}[x_1, \ldots, x_n]$ then there exists a natural isomorphism from $\mathbf{R}Hom_{A_n}(N_1, R)$ (i.e. the solution complex of N_1) to

$$\mathbf{R}Hom_{A_n}\big(R^*, N_1^*\big).$$

As the complex $\mathbf{R}Hom_{A_n}(R, A_n)$ is naturally isomorphic to Ω^n_R we can identify R and R^* and then we have a natural isomorphism

$$\mathbf{R}Hom_{A_n}(N_1, R) \xrightarrow{\simeq} \mathbf{R}Hom_{A_n}\big(R, N_1^*\big) \xrightarrow{\simeq} DR\big(N_1^*\big). \tag{2}$$

The following theorem will be used later.

Theorem 2.3. *(See [4, Theorem 3.1].) Let $f \in \mathbb{C}[x, y]$ be a non-zero reduced polynomial. Then there exists a natural isomorphism $(M^{\log f})^* \simeq \widetilde{M}^{\log f}$.*

The following corollary is an obvious consequence of Theorems 2.1 and 2.3 using also isomorphism (2).

Corollary 2.4. *For any non-zero reduced polynomial $f \in \mathbb{C}[x, y]$, the complexes $\Omega^\bullet(\log f)$ and $DR(\widetilde{M}^{\log f})$ are naturally isomorphic in the derived category.*

As a consequence of Corollary 2.4 and by [15,17,22], the cohomology of the complex $\Omega^\bullet(\log f)$ can be computed from a given polynomial f, since a system of generators of the R-module $\widetilde{Der}(-\log f)$ can be computed by using the R-syzygies of $(\partial_1(f), \partial_2(f), f)$ and we have algorithms of computing $DR(M)$ for holonomic modules M. We note that $\widetilde{M}^{\log f}$ is shown to be holonomic [1, Corollary 4.2.2].

In order to compute bases of $H^i(\Omega^\bullet(\log f))$, we give the explicit form

$$\tau^\bullet : \Omega^\bullet(\log f) \to DR(\widetilde{M})$$

of the quasi-isomorphism of complexes (regarded as objects in the abelian category) given in Corollary 2.4. In general, this quasi-isomorphism is not an isomorphism between the chosen representatives of the two complexes of vector spaces. This morphism is given as follows.

$\tau^0 : R = \Omega^0(\log f) \to \widetilde{M}$ is defined by $\tau^0(g) = \overline{gf}$ where $\overline{()}$ means the equivalence class modulo the ideal $A_2\widetilde{Der}(-\log f)$.

$\tau^1 : \Omega^1(\log f) \to \widetilde{M} \otimes_R \Omega^1_R$ is defined by

$$\tau^1(g_1\omega_1 + g_2\omega_2) = \sum_i \overline{g_i} \otimes f\omega_i.$$

$\tau^2 : \Omega^2(\log f) \to \widetilde{M} \otimes_R \Omega^2_R$ is defined by $\tau^2(g\omega_1 \wedge \omega_2) = \overline{g} \otimes f\omega_1 \wedge \omega_2$.

It is derived by a diagram chase of the double complex constructed from the Spencer resolution and the Koszul resolution of R (see [21, Example 3.1] and [4]).

3. Algorithm

Let us summarize our algorithm of computing logarithmic cohomology groups in the two-dimensional case. Most tensor products $\otimes$ in the sequel are over A_2. If we omit the subscript A_2 for $\otimes$, it means that the tensor product is over A_2.

Algorithm 3.1.

Input: a non-zero reduced polynomial $f(x, y)$

Output: dimensions and bases of $H^i(\Omega^\bullet(\log f))$.

1. Compute a free basis $\{s = (s_0, s_1, s_2), t = (t_0, t_1, t_2)\}$ of the syzygy module of f, f_x, f_y over the polynomial ring $\mathbb{C}[x, y]$. This step can be performed by the following way.
   (a) Compute the minimal syzygy of $h(f)$, $h(f_x)$, $h(f_y)$. Here, $h(g)$ is the homogenization of g. If the number of generators is 2, then the dehomogenizations of these generators are s and t.
   (b) If we fail on the first step, apply an algorithm for the Quillen–Suslin theorem to obtain s and t (call the procedure Quillen–Suslin).
2. Define a left ideal in A_2 by
$$I = A_2 \cdot \{-s_0 + s_1\partial_x + s_2\partial_y, -t_0 + t_1\partial_x + t_2\partial_y\}. \tag{3}$$
   Compute the dimensions and bases of the de Rham cohomology groups for $\widetilde{M} = A_2/I$ with the algorithm in [15,17]. In other words, replace the A_2-module $\mathbb{C}[x, y, 1/f]$ by A_2/I of (3) in Algorithm 1.2 in [15].
3. The bases of the previous step are given in $A_2/(\partial_x A_2 + \partial_y A_2) \otimes \widetilde{M}^\bullet$ where $\widetilde{M}^\bullet$ is a $(1, 1, -1, -1)$-adapted free resolution of $\widetilde{M}$. Here, $\widetilde{M}^\bullet = (A_2^{n_i}, d_i)$ is called a $(1, 1, -1, -1)$-adapted free resolution when $(\mathrm{gr}\, A_2^{n_i}, \mathrm{gr}\, d_i)$ is exact with a suitable degree shift where $\mathrm{gr}\, A_2^{n_i}$ is the graded module with the grading $\deg x = \deg y = 1$ and $\deg \partial_x = \deg \partial_y = -1$ (see [17,18] as to details). Bases of de Rham cohomology groups in $\Omega^\bullet \otimes \widetilde{M} \simeq_{q.is} DR(\widetilde{M})$ are determined by the transfer algorithm of U. Walther [22, Theorem 2.5 (Transfer Theorem)]. Here, $\Omega^\bullet$ is the Koszul resolution of the right A_2-module $A_2/(\partial_x A_2 + \partial_y A_2)$.

4. Bases of cohomology groups in $\Omega^\bullet(\log f)$ is obtained by computing the preimage of τ^i given in Corollary 2.4; $\Omega^\bullet(\log f) \simeq^\tau_{q.is} DR(\widetilde{M})$. A procedure of computing the preimage is given at the end of this section.

In the first step, we should firstly try to find the minimal syzygy. Because, usually it is faster than applying implementations and algorithms for the Quillen–Suslin theorem.

The following example will illustrate how our algorithm works.

Example 3.2. We consider the case of $f = xy(x-y)$. Two canonical generators of $I = A_2\widetilde{Der}(-\log f)$ are

$$\ell_1 = 3 + x\partial_x + y\partial_y, \qquad \ell_2 = -(2x-y) + \left(-x^2 + xy\right)\partial_x.$$

The associated canonical logarithmic forms are

$$\omega_1 = \frac{1}{f}x(x-y)\,dy, \qquad \omega_2 = \frac{1}{f}(-y\,dx + x\,dy).$$

Let us proceed on the step 2. We apply the procedure of computing the de Rham cohomology groups [15,18] for A_2/I. The maximal integral root of the b function for $I = A_2 \cdot \{\ell_1, \ell_2\}$ with respect to the weight $(1,1,-1,-1)$ is 1. The dehomogenization of the $(1,1,-1,-1)$-minimal filtered free resolution of A_2/I, which is adapted, is

$$C^\bullet\colon \quad A_2[0] \xrightarrow{a^{-2}} A_2[1] \oplus A_2[0] \xrightarrow{a^{-1}} A_2[1], \tag{4}$$

where

$$a^{-2}(c) = c(-\ell_2, \ell_1 - 1) \quad \text{for } c \in A_2,$$
$$a^{-1}(c,d) = (c,d)\begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix} \quad \text{for } (c,d) \in A_2[1] \oplus A_2[0].$$

Put $\Omega(2) = A_2/(\partial_x A_2 + \partial_y A_2)$ which is the right A_2-module and is isomorphic to Ω^2_R. Following [15, Procedure 1.8], we truncate the complex $\Omega(2) \otimes_{A_2} C^\bullet$ to the forms of $(1,1,-1,-1)$-degree at most 1 since the maximal integral root of the b-function is 1. The truncated complex is the following complex of finite-dimensional vector spaces

$$\mathbb{C} \xrightarrow{\bar{a}^{-2}} (\mathbb{C} + \mathbb{C}x + \mathbb{C}y) \oplus \mathbb{C} \xrightarrow{\bar{a}^{-1}} (\mathbb{C} + \mathbb{C}x + \mathbb{C}y) \xrightarrow{\bar{a}^{0}} 0. \tag{5}$$

Here,

$$\bar{a}^{-2}(1) = (-\ell_2, \ell_1 - 1) \mod \partial_x A_2 + \partial_y A_2 = (0,0),$$
$$\bar{a}^{-1}(a + bx + cy, d) = (a + bx + cy)\ell_1 + d\ell_2 \mod \partial_x A_2 + \partial_y A_2 = a.$$

Therefore, the cohomology groups are $H^0(\Omega(2) \otimes C^\bullet) = \operatorname{Ker} \bar{a}^{-2} = \mathbb{C}$, $H^1(\Omega(2) \otimes C^\bullet) = \operatorname{Ker} \bar{a}^{-1} / \operatorname{Im} \bar{a}^{-2} = (\mathbb{C}x + \mathbb{C}y) \oplus \mathbb{C}$, $H^2(\Omega(2) \otimes C^\bullet) = \operatorname{Ker} \bar{a}^0 / \operatorname{Im} \bar{a}^{-1} = \mathbb{C}x + \mathbb{C}y$.

Finally, we perform the step 3 and 4. Put $\widetilde{M} = A_2/I$. In order to give bases of the cohomology groups in $\widetilde{M} \otimes_R \Omega^i_R$, we apply the transfer theorem (algorithm) of Uli Walther [22]. We consider the double complex $\Omega^\bullet \otimes C^\bullet$ constructed from $\Omega(2) \otimes C^\bullet$ and $\Omega^\bullet \otimes \tilde{M}$. The transfer algorithm translates cohomology classes in the complex $\Omega(2) \otimes C^\bullet$ into those in the complex $\Omega^\bullet \otimes \tilde{M}$. This translation can be performed by a diagram chase in the double complex with Gröbner basis computations. See 2.4 of [22].

Two cohomology classes $1 \otimes x$ and $1 \otimes y$ in $H^2(\Omega(2) \otimes C^\bullet)$ are transfered to $1 \otimes x$ and $1 \otimes y$ in $H^2(\Omega^\bullet \otimes \tilde{M})$ respectively. It follows from the definition of τ^2, $x\omega_1 \wedge \omega_2$ and $y\omega_1 \wedge \omega_2$ is the basis of $H^2(\Omega^\bullet(\log f))$.

Let us compute transfers of bases of $H^1(\Omega(2) \otimes C^\bullet)$. The cohomology class $1 \otimes (x, 0)$ in $H^1(\Omega(2) \otimes C^\bullet)$ is transfered to $(xy\,dx - x^2\,dy) \otimes 1$ in $H^1(\Omega^\bullet \otimes \tilde{M})$. Let us compute the preimage by τ^1. Solving $c_1 f\omega_1 + c_2 f\omega_2 = xy\,dx - x^2\,dy$ (as cohomology class), we obtain $c_1 = 0, c_2 = -x$. Therefore, $1 \otimes (x, 0)$ stands for $-x\omega_2$. Analogously, $1 \otimes (y, 0)$ is transfered to $-y^2\,dx + xy\,dy$ and stands for $-y\omega_2$ and $1 \otimes (0, 1)$ is transfered to $x(y - x)\,dy$ and stands for ω_1. In summary,

$$H^1\big(\Omega^\bullet(\log f)\big) = \mathbb{C}(-x)\omega_2 + \mathbb{C}(-y)\omega_2 + \mathbb{C}\omega_1.$$

The base $1 \otimes 1$ in $H^0(\Omega(2) \otimes C^\bullet)$ is transfered to $xy(x - y) \otimes 1 \in H^0(\Omega^\bullet \otimes \widetilde{M})$ which is equal to $\tau^0(1)$. Hence, $H^0(\Omega^\bullet(\log f)) = \mathbb{C} \cdot 1$.

Before presenting implementations and larger examples, we explain about a procedure (step 4) to find a preimage of τ^i in general. The transfer algorithm gives an element in $\Omega^i \otimes_{A_2} \widetilde{M}$ where $\Omega^\bullet$ is the Koszul resolution of $\Omega(2) \simeq \Omega_R^2$ as the right A_2-module. This element can be identified with a differential form with coefficients in $\widetilde{M}$. We need to find the preimage of it by τ^i which lies in $\Omega^i(\log f)$. This can be performed by the method of undetermined coefficients.

Consider the case of τ^1. Take an element $c_1\omega_1 + c_2\omega_2$ in $\Omega^1(\log f)$ where $c_i \in R$. We have seen in Corollary 2.4 that

$$\tau^1(c_1\omega_1 + c_2\omega_2) = f\bar{\omega}_1 \otimes_{A_2} \bar{c}_1 + f\bar{\omega}_2 \otimes_{A_2} \bar{c}_2 \in \begin{pmatrix} A_2 \\ \oplus \\ A_2 \end{pmatrix} \otimes_{A_2} \widetilde{M} \mod d\widetilde{M}. \tag{6}$$

Here, we identify $\binom{1}{0} \otimes_{A_2} m_1$ with $m_1 \otimes_R dx$ and $\binom{0}{1} \otimes_{A_2} m_2$ with $m_2 \otimes_R dy$, $m_i \in \widetilde{M}$ and when $\omega_i = a_i\,dx + b_i\,dy$, we denote $\binom{a_i}{b_i}$ by $\bar{\omega}_i$. We are given an element of the form $m_1\,dx + m_2\,dy$, $m_i \in \widetilde{M}$ as the output of the transfer algorithm. We regard m_i as an element in A_2 in the sequel. We rewrite $f\omega_i$ as $f\omega_1 = A\,dx + B\,dy$ and $f\omega_2 = C\,dx + D\,dy$. Assume I is generated by ℓ_1 and ℓ_2. Then, it follows from the definition of τ^1 and Corollary 2.4 that there exist $c_i \in R$, d_i^j, $e \in A_2$ such that

$$Ac_1 + Cc_2 = m_1 + \sum_{j=1}^{2} d_1^j \ell_j + \partial_x e, \tag{7}$$

$$Bc_1 + Dc_2 = m_2 + \sum_{j=1}^{2} d_2^j \ell_j + \partial_y e. \tag{8}$$

To find the preimage $c_1\omega_1 + c_2\omega_2$ by τ^1, we may solve (7) and (8). Fix a degree bound m for c_i, d_i^j, e and determine these elements by the method of unknown coefficients. The identities (7) and (8) induce a system of linear equations over $\mathbb{C}$ for the coefficients. Increasing the degree bound and solving the system, we will be able to obtain c_1 and c_2 in finite steps by virtue of the existence claim.

Consider the case of τ^2. Since our basis in $H^2(\Omega^\bullet \otimes \tilde{M})$ is given in terms of x and y and $f\omega_1 \wedge \omega_2 = dx \wedge dy$, we need no computation to find the preimage by τ^2.

Let us consider the case of τ^0. Let m be an output of the transfer algorithm. It lies in A_2 in general. Finding the preimage g of τ^0 can be done by solving $gf = m + \sum_{j=1}^{2} d_j\ell_j$ where $g \in R$ and $d_j \in A_2$.

4. Implementation and examples

The second and third steps of Algorithm 3.1 can be performed with the help of the D-module package on Macaulay2 1.1 or later. Our code is merged to the D-module package [8] with the command name `logCohomology`. Unfortunately, this implementation has not installed an efficient algorithm of computing b-function by Noro [12] to get the truncated complex in [15,17]. Then, only relatively small examples are feasible. Example 4.1 is computed by our Macaulay2 program. Example 4.2 is computed by our implementation on kan/k0 (`logc2.k`, `http://www.openxm.org`) and Risa/Asir with an implementation of [12] (the transfer algorithm has not been implemented yet for kan/k0). Our implementation does not contain that for the Quillen–Suslin theorem. We utilize the implementation by A. Fabianska on Maple [7] when the step 1(a) fails.

Example 4.1 *(Continued from Example 1.2(b)).* We will determine bases of $H^i(\Omega^\bullet(\log f))$ where $f = (x^3+y^4+xy^3)(x^2+y^2)$. We firstly use Fabianska's program for the Quillen–Suslin theorem to find the 2 free generators of the syzygies of f, f_x, f_y. The two rows of the matrix $S = \left(\begin{smallmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23}\end{smallmatrix}\right)$ are generators where

$$S_{11} = \left(\frac{115}{6}y - 5/2\right)x - 6y^3 - \frac{43}{6}y^2 + 9y,$$

$$S_{12} = \left(-\frac{23}{6}y + 1/2\right)x^2 + \left(y^3 + y^2 - 2y\right)x - \frac{5}{6}y^3,$$

$$S_{13} = \left(\frac{1}{3}y + 1/2\right)x^2 + \left(-3y^2 + \frac{1}{2}y\right)x + y^4 + \frac{4}{3}y^3 - \frac{3}{2}y^2,$$

$$S_{21} = \frac{46}{15}x^2 + \left(-\frac{24}{25}y^2 + \frac{22}{75}y\right)x + \frac{12}{5}y^2,$$

$$S_{22} = -\frac{46}{75}x^3 + \left(\frac{4}{25}y^2 - \frac{2}{25}y\right)x^2 - \frac{8}{15}y^2x,$$

$$S_{23} = \frac{4}{75}x^3 - \frac{12}{25}yx^2 + \left(\frac{4}{25}y^3 - \frac{2}{75}y^2\right)x - \frac{2}{5}y^3.$$

The determinant of $\left(\begin{smallmatrix} S_{12} & S_{13} \\ S_{22} & S_{23}\end{smallmatrix}\right)$ is $\frac{1}{3}f$. We put $\omega_1 = \frac{1}{f}(S_{23}\,dx - S_{22}\,dy)$ and $\omega_2 = \frac{1}{f}(-S_{13}\,dx + S_{12}\,dy)$. ($\sqrt{3}\omega_i$ agrees with the ω_i in Theorem 2.4.)

We apply the integration algorithm and the transfer algorithm for $\widetilde{M}$. We obtain the following result. (1) $H^0(DR(\widetilde{M}))$ is spanned by $1 \otimes f$ and then we have $H^0(\Omega^\bullet(\log f)) \simeq \mathbb{C} \cdot 1$. (2) $H^2(DR(\widetilde{M}))$ is spanned by $1 \otimes a$ where a runs over

```
                                 3          3          2       3        4
o9 = {{1}, {-x}, {y }, {-x*y }, {x*y }, {x y}, {y }}
```

(We have pasted the output of our Macaulay 2 program.) Then, we have

$$H^2\big(\Omega^\bullet(\log f)\big) \simeq \big(\mathbb{C} \cdot 1 + \mathbb{C} \cdot (-x) + \cdots + \mathbb{C} \cdot y^4\big)\omega_1 \wedge \omega_2.$$

(3) $H^1(DR(\widetilde{M}))$ is spanned by 3 differential forms $m_1\,dx + m_2\,dy$ where m_1, m_2 are elements in A_2, of which explicit expressions are a little lengthy. We solve the identities (7) and (8) to find c_1 and c_2. In other words, we need to compute preimages of $m_1\,dx + m_2\,dy$ by τ^1. As we explained, this can be done by the method of undetermined coefficients degree by degree. We can find solutions when the

Table 1

p	q	Dimensions	Timing in seconds
10	11	(8, 1, 1)	3.5
10	12	(9, 1, 1)	4.6
10	13	(10, 1, 1)	6.9
10	14	(11, 1, 1)	9.4
10	20	(17, 1, 1)	55.0
10	21	(18, 1, 1)	86.8

degree of c_i, d_i^j, e with respect to x, y is 6 and that with respect to ∂_x, ∂_y is 0. Here is a basis of the 3-dimensional vector space $H^1(\Omega^\bullet(\log f))$ obtained by this method.

- $-yx\omega_1 - \frac{4}{25}x^2\omega_2$,
- $\left(\left(\frac{215}{28}y - \frac{1101}{280}\right)x - \frac{367}{56}y^2\right)\omega_1 + \left(\frac{43}{35}x^2 - \frac{367}{350}yx\right)\omega_2$,
- $\left(\left(y - \frac{11}{30}\right)x - \frac{28}{9}y^3 - \frac{13}{6}y^2 + \frac{14}{3}y\right)\omega_1 + \left(\frac{4}{25}x^2 + \left(-\frac{112}{225}y^2 + \frac{2}{5}y\right)x + \frac{56}{45}y^2\right)\omega_2$.

All programs and session logs to find this answer are obtainable from `http://www.math.kobe-u.ac.jp/OpenXM/Math/LogCohomology/readme.html`. The logarithmic comparison theorem does not hold for this example. In fact, the dimensions of the de Rham cohomology groups $H^i(\Omega_f^\bullet)$ $(i = 2, 1, 0)$ are 5, 3, 1 respectively.

Example 4.2. We apply a part of our algorithm to compute the dimensions of the cohomology groups $H^i(\Omega^\bullet(\log f))$ for $f = x^p + y^q + xy^{q-1}$. Here is Table 1 of p, q and the dimensions of H^2, H^1, H^0 and timing data. The program is executed on a machine with 2G RAM and Pentium III (1G Hz).

The homogenization of f, f_x, f_y generates an ideal that is Cohen–Macaulay. These examples do not need to call the subprocedure Quillen–Suslin. However, the logarithmic comparison theorem does not hold for these examples (see [2]). Computation of de Rham cohomology groups is not feasible by our implementation.

5. Another algorithm

In the previous sections, we have presented an algorithm of computing a basis of the logarithmic cohomology groups for plane curves with respect to the canonical basis w_1 and w_2. However, this algorithm relies on algorithms for the Quillen–Suslin theorem and they are sometimes slow. We will present another algorithm, which is independent of the Quillen–Suslin theorem, but it does not give a basis with respect to the canonical basis.

Let f be a reduced polynomial in two variables. We denote by Ω_f^k the set of k-forms with coefficients in $\mathbb{C}[x, y, 1/f]$. The k form $\omega \in \Omega_f^k$ is called *logarithmic* k-form iff both of $f\omega$ and $df \wedge \omega$ have polynomial coefficients. The space of logarithmic k-forms is denoted by $\Omega^k(\log f)$. It is easy to see that $\Omega^2(\log f) = \frac{\mathbb{C}[x,y]\,dx\wedge dy}{f}$. Let us determine all the logarithmic 1-forms. Let (p, q, r) a triple of polynomials such that

$$f_y p - f_x q + fr = 0 \quad \text{(syzygy equation)}. \tag{9}$$

Note that $(0, f, f_x)$, $(f, 0, -f_y)$, $(f_x, f_y, 0)$ are trivial solutions of the syzygy equation. For a solution (p, q, r) of the syzygy equation, $\omega = \frac{p\,dx + q\,dy}{f}$ belongs to $\Omega^1(\log f)$. Conversely, any logarithmic

1-form can be expressed in this way. In fact, the condition that $df \wedge \omega$ has a polynomial coefficient is equivalent to that $f_y p - f_x q$ is a multiple of f.

Put $\omega = \frac{p\,dx + q\,dy}{f}$. Let $e(x, y)$ be any polynomial. Then, $d(e\omega) = (Le)\frac{dx\wedge dy}{f}$ where

$$L = q\partial_x - p\partial_y + q_x - p_y + \frac{f_y p - f_x q}{f}.$$

We denote the Weyl algebra A_2 by D for simplicity in the sequel. Suppose that L_i $(i = 1, \ldots, m)$, stands for a set of generators of the solution space of the syzygy equation, which is a $\mathbb{C}[x, y]$-module. Then $d\Omega^1(\log f) = \sum L_i \mathbb{C}[x, y]\,dx \wedge dy/f$. Therefore, the computation of $H^2(\Omega^\bullet(\log f))$ is nothing but the computation of $\mathbb{C}[x, y]/\sum_{i=1}^m L_i \bullet \mathbb{C}[x, y]$ (see Oaku's book [14] on the same question in the one variable case). Put $I^* = D \cdot \{L_1^*, \ldots, L_m^*\}$, which is a left D ideal. We denote by F_k the $\mathbb{C}$-subvector space of $D \cap \mathbb{C}[x, y]$ of which $(1, 1, -1, -1)$-order is less than or equal to k [20, pp. 14, 203]. Here, $(1, 1)$ is the weight for (x, y) and $(-1, -1)$ is that for (∂_x, ∂_y).

Algorithm 5.1. $H^2(\Omega^\bullet(\log f))$.

Step 1. Find generators of the syzygy equation and obtain explicit expressions of L_i.
Step 2. Compute a $(1, 1, -1, -1)$-Gröbner basis (standard basis) of I. We denote the elements of the Gröbner basis by $L_i{}^*$ (renaming).
Step 3. Find the monic generator $b(-\partial_x x - \partial_y y)$ of $\mathrm{in}_{(1,1,-1,-1)}(I) \cap \mathbb{C}[-\partial_x x - \partial_y y]$.
Step 4. Let k_0 be the maximal non-negative root of $b(s) = 0$. Then, return $\mathbb{C}$-vector space basis $\{c_i\}$ of

$$F_{k_0}/\sum_i L_i \cdot F_{k_0 - \mathrm{ord}_{(1,1,-1,-1)}(L_i)}.$$

$\{c_i\,dx \wedge dy/f\}$ is a basis of H^2.

The steps 2–4 can also be done by computing $D/(I^* + \partial_x D + \partial_y D)$ (0th integral module) where I^* is the formal adjoint of I. (As to details for the steps 2–4, see [16].)

The left ideal generated by L_i^* is nothing but $D \cdot \widetilde{Der}(-\log f)$.

Theorem 5.2. *If* $\dim V(f, f_x, f_y) \leqslant 0$, $\dim V(f, f_x) \leqslant 1$, $\dim V(f, f_y) \leqslant 1$, *then Algorithm* 5.1 *is correct.*

We note that when f is reduced, the assumption of the correctness holds.

Proof. Let I be the left ideal in D generated by $L_1, \ldots, L_m$. We may assume that I contains $f\partial_x$, $f\partial_y$ and $f_y\partial_x - f_x\partial_y$. Therefore, the characteristic variety of I is contained in $V(f(x, y)\xi, f(x, y)\eta, f_y(x, y)\xi - f_x(x, y)\eta)$, of which dimension is less than or equal to 2 from the assumption. In fact, assume $(a, b) \in V(f, f_x, f_y)$. Then, ξ and η are free and then the dimension of the characteristic variety is less than or equal to 2. Assume $(a, b) \in V(f, f_x) \setminus V(f, f_x, f_y)$. Then, we have $f(a, b) = 0$, $f_x(a, b) = 0$ and $f_y(a, b) \neq 0$. Then, η is free and $\xi = 0$, then the dimension of the characteristic variety is less than or equal to 2. The rest cases can be shown analogously. Therefore, D/I is a holonomic D-module and hence a non-trivial b exists [20, Chapter 5, Theorem 5.1.2]. The rest of the correctness proof is analogous with that of the 0th integration algorithm of D-modules [13], [20, Chapter 5, Theorems 5.2.6 and 5.5.1]. $\square$

Example 5.3. For $f = (x^3 + y^4 + xy^3)(x^2 + y^2)$, we have $\dim H^2(\Omega^\bullet(\log f)) = 7$ with Algorithm 5.1. The execution time is 1.9 s.

Let $\tilde{\omega}_1, \ldots, \tilde{\omega}_m$ be a set of generators of $\mathbb{C}[x, y]$-module $\Omega^1(\log f)$. Define $\tilde{\tau}_0(g) = \overline{gf}$, $\tilde{\tau}_1(\sum_{i=1}^m g_i\tilde{\omega}_i) = \sum_{i=1}^m \bar{g}_i \otimes f\tilde{\omega}_i$, and $\tilde{\tau}_2(g\frac{dx\wedge dy}{f}) = \bar{g} \otimes dx \wedge dy$. Then, it is easy to show that $\tau_i = \tilde{\tau}_i$

by expressing $\tilde{\omega}_i$ in terms of the canonical basis ω_1, ω_2. Therefore, $\tilde{\tau}_i$ gives the quasi-isomorphism of Corollary 2.4. Hence, we have the following algorithm.

Algorithm 5.4.

Input: a non-zero reduced polynomial f.

Output: the dimension and bases of $H^i(\Omega^\bullet(\log f))$, $i = 0, 1$.

Replace the free basis in step 1 of Algorithm 3.1 by a set of generators of the syzygy module and perform the steps 2 and 3.

In step 4, use $\tilde{\tau}_0, \tilde{\tau}_1$ to find preimages.

We close this paper with a remark for future study. Comparison theorems for logarithmic cohomology groups in the n variable case are studied under some conditions (see, e.g., [1,5] and their references). It is an interesting problem to generalize our result to the n-variable case based on these theorems and the algorithm given by Tsai and Walther [21].

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