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# On graded stable derived categories of isolated Gorenstein quotient singularities

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ABSTRACT

We show the existence of a full exceptional collection in the graded stable derived category of a Gorenstein isolated quotient singularity using a result of Orlov (2009) [Orl09]. We also show that the equivariant graded stable derived category of a Gorenstein Veronese subring of a polynomial ring with respect to an action of a finite group has a full strong exceptional collection, even if the corresponding quotient singularity is neither isolated nor Gorenstein.



## 1. Introduction

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded Noetherian ring over a field $k$. The ring $A$ is said to be *connected* if $A_0 = k$. A connected ring $A$ is *Gorenstein* with parameter $a$ if $A$ has finite injective dimension as a right module over itself and

$$\mathbb{R}\mathrm{Hom}_A(k, A) = k(a)[-n].$$

Here $\bullet(a)$ denotes the shift of $\mathbb{Z}$-grading and $\bullet[-n]$ is the shift in the derived category. The *graded stable derived category* is the quotient category

$$D^b_{\mathrm{sing}}(\mathrm{gr}\, A) = D^b(\mathrm{gr}\, A)/D^{\mathrm{perf}}(\mathrm{gr}\, A) \tag{1.1}$$

of the bounded derived category $D^b(\mathrm{gr}\, A)$ of finitely-generated $\mathbb{Z}$-graded right $A$-modules by the full triangulated subcategory $D^{\mathrm{perf}}(\mathrm{gr}\, A)$ consisting of perfect complexes. Here, an object of $D^b(\mathrm{gr}\, A)$ is *perfect* if it is quasi-isomorphic to a bounded complex of projective modules. Stable derived categories

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are introduced by Buchweitz [Buc87] motivated by the theory of *matrix factorizations* by Eisenbud [Eis80]. Stable derived categories are also known as *triangulated categories of singularities*, introduced by Orlov [Orl04] based on an idea of Kontsevich to study B-branes on Landau–Ginzburg models.

Let $R = k[x_1, \dots, x_{n+1}]$ be a polynomial ring in $n+1$ variables over a field $k$. We equip $R$ with a $\mathbb{Z}$-grading such that $\deg x_i = 1$ for all $i$. Let $G$ be a finite subgroup of $SL_{n+1}(k)$ whose order is not divisible by the characteristic of $k$. Assume that the natural action of $G$ on the affine space $\mathbb{A}^{n+1} = \operatorname{Spec} R$ is free outside of the origin. This assumption is equivalent to the condition that the invariant subring $A = R^G$ has an isolated singularity at the origin [IY08, Corollary 8.2]. Two examples of the stable derived categories of $A$ are studied by Iyama and Yoshino [IY08] and Keller, Murfet and Van den Bergh [KMVdB11]. The general case is studied by Iyama and Takahashi [IT].

Let $d \in \mathbb{N}$ be a divisor of $n+1$ and $B = \bigoplus_{i=0}^{\infty} A_{id}$ be the $d$-th Veronese subring of $A$. We prove the following in this paper:

**Theorem 1.1.** *The stable derived category $D^b_{\mathrm{sing}}(\operatorname{gr} B)$ has a full exceptional collection.*

The full exceptional collection given in Theorem 1.1 is strong when $d = n+1$. One the other hand, a result of Iyama and Takahashi [IT, Theorem 1.7] gives a full strong exceptional collection for $d = 1$. The proof of Theorem 1.1 is based on the existence of a full strong exceptional collection in the derived category of coherent sheaves on the stack $\operatorname{Proj} B = [(\operatorname{Spec} B \setminus \mathbf{0})/\mathbb{G}_m]$ and a result of Orlov [Orl09, Theorem 2.5(i)].

Next we discuss equivariant graded stable derived categories. Let $A$ be an $\mathbb{N}$-graded connected Gorenstein ring with parameter $a > 0$ and $G$ be a finite group acting on $A$ whose order is not divisible by the characteristic of $k$. The *crossed product algebra* $A \rtimes G$ is the vector space $A \otimes k[G]$ equipped with the ring structure

$$(a_1 \otimes g_1) \cdot (a_2 \otimes g_2) = a_1 \cdot g_1(a_2) \otimes g_1 \circ g_2,$$

where $k[G]$ is the group ring of $G$. A right $A \rtimes G$-module is often called a *$G$-equivariant $A$-module*. The crossed product algebra $A \rtimes G$ inherits a grading from $A$ so that the degree zero part is given by the group ring; $(A \rtimes G)_0 = k[G]$. This graded ring is not connected if $G$ is non-trivial.

Let $\operatorname{gr}^G A$ be the abelian category of finitely-generated $\mathbb{Z}$-graded right $A \rtimes G$-modules and $\operatorname{tor}^G A$ be its Serre subcategory consisting of finite-dimensional modules. The quotient abelian category is denoted by

$$\operatorname{qgr}^G A = \operatorname{gr}^G A / \operatorname{tor}^G A.$$

If $A$ is commutative, then $\operatorname{qgr}^G A$ is equivalent to the abelian category $\operatorname{coh}^G(\operatorname{Proj} A)$ of $G$-equivariant coherent sheaves of the stack $\operatorname{Proj} A = [(\operatorname{Spec} A \setminus \mathbf{0})/\mathbb{G}_m]$. Let $\operatorname{Irrep}(G) = \{\rho_0, \dots, \rho_r\}$ be the set of irreducible representations of $G$ where $\rho_0$ is the trivial representation. For any $k \in \mathbb{Z}$, the image of the graded $A \rtimes G$-module $A(k) \otimes \rho_i$ by the projection functor $\pi : \operatorname{gr} A \rtimes G \to \operatorname{qgr}^G A$ will be denoted by $\mathcal{O}(k) \otimes \rho_i$. The following is a straightforward generalization of [Orl09, Theorem 2.5(i)]:

**Theorem 1.2.** *There is a full and faithful functor $\Phi : D^b_{\mathrm{sing}}(\operatorname{gr}^G A) \to D^b(\operatorname{qgr}^G A)$ and a semiorthogonal decomposition*

$$\begin{aligned} D^b\bigl(\operatorname{qgr}^G A\bigr) = \bigl\langle &\mathcal{O} \otimes \rho_0, \dots, \mathcal{O} \otimes \rho_r, \mathcal{O}(1) \otimes \rho_0, \dots, \mathcal{O}(1) \otimes \rho_r, \dots, \\ &\mathcal{O}(a-1) \otimes \rho_0, \dots, \mathcal{O}(a-1) \otimes \rho_r, \Phi D^b_{\mathrm{sing}}\bigl(\operatorname{gr}^G A\bigr)\bigr\rangle. \end{aligned}$$

Let $R = k[x_1, \dots, x_{n+1}]$ be a polynomial ring in $n+1$ variables and $A = \bigoplus_{i=0}^{\infty} R_{id}$ be the $d$-th Veronese subring. We assume that $d$ is a divisor of $n+1$ so that $A$ is Gorenstein with parameter $a = (n+1)/d$. Let $G$ be any finite subgroup of $GL_{n+1}(k)$ whose order is not divisible by the characteristic of $k$. We have the following corollary of Theorem 1.2:

**Theorem 1.3.** *The stable derived category $D^b_{\mathrm{sing}}(\mathrm{gr}^G A)$ has a full strong exceptional collection.*

The organization of this paper is as follows: In Section 2, we study $\operatorname{Proj} B$ for the Veronese subring $B$ of the invariant ring and prove Theorem 1.1. We prove Theorem 1.2 in Section 3, which immediately gives Theorem 1.3. We discuss a few examples in Section 4.

## 2. Invariant subrings

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. The *right orthogonal* to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects $M$ such that $\operatorname{Hom}(N, M) = 0$ for any $N \in \mathcal{N}$. The left orthogonal ${}^\perp\mathcal{N}$ is defined similarly by $\operatorname{Hom}(M, N) = 0$ for any $N \in \mathcal{N}$. A full triangulated subcategory $\mathcal{N}$ of a triangulated category $\mathcal{D}$ is *left admissible* if any $X \in \mathcal{D}$ sits inside a distinguished triangle $N \to X \to M \xrightarrow{[1]} N$ such that $N \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Right admissible subcategories are defined similarly. A sequence $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ of full triangulated subcategories is a *weak semiorthogonal decomposition* if there is a sequence $\mathcal{N}_1 = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ of left admissible subcategories such that $\mathcal{N}_p$ is left orthogonal to $\mathcal{D}_{p-1}$ in $\mathcal{D}_p$. The decomposition is *orthogonal* if $\operatorname{Hom}(N, M) = 0$ for any $N \in \mathcal{N}_i$ and $M \in \mathcal{N}_j$ with $i \neq j$.

Let $k$ be a field and $\mathcal{D}$ be a $k$-linear triangulated category. An object $E$ of $\mathcal{D}$ is *exceptional* if $\operatorname{Hom}(E, E)$ is spanned by the identity morphism and $\operatorname{Ext}^i(E, E) = 0$ for $i \neq 0$. A sequence $(E_1, \ldots, E_r)$ of exceptional objects is an *exceptional collection* if $\operatorname{Ext}^i(E_j, E_\ell) = 0$ for any $i$ and any $1 \leqslant \ell < j \leqslant r$. An exceptional collection is *strong* if $\operatorname{Ext}^i(E_j, E_\ell) = 0$ for any $i \neq 0$ and any $1 \leqslant j \leqslant \ell \leqslant r$. An exceptional collection is *full* if the smallest full triangulated subcategory of $\mathcal{D}$ containing it is the whole of $\mathcal{D}$.

Let $G$ be a finite subgroup of $SL_{n+1}(k)$ acting freely on $\mathbb{A}^{n+1} \setminus \mathbf{0}$. We assume that the order of $G$ is not divisible by the characteristic of the base field $k$. The set of irreducible representations of $G$ will be denoted by $\operatorname{Irrep}(G) = \{\rho_0, \ldots, \rho_r\}$ where $\rho_0$ is the trivial representation. Let further $R = k[x_1, \ldots, x_{n+1}]$ be the coordinate ring of $\mathbb{A}^{n+1}$ and $A = R^G$ be the invariant subring. Equip $R$ with the $\mathbb{N}$-grading such that $\deg x_i = 1$ for all $i = 1, \ldots, n+1$, which induces an $\mathbb{N}$-grading on $A$. This defines a $\mathbb{G}_m$-action on $\operatorname{Spec} A$, and let

$$Y := \operatorname{Proj} A = \big[(\operatorname{Spec} A \setminus \mathbf{0})/\mathbb{G}_m\big] = \big[\big((\operatorname{Spec} R \setminus \mathbf{0})/G\big)/\mathbb{G}_m\big]$$

be the quotient stack. The abelian category $\operatorname{coh}^G \mathbb{P}^n$ of $G$-equivariant coherent sheaves on $\mathbb{P}^n$ is equivalent to the abelian category $\operatorname{coh} Y$ of coherent sheaves on $Y$, which in turn is equivalent to the quotient category

$$\operatorname{qgr} A = \operatorname{gr} A / \operatorname{tor} A$$

of the abelian category $\operatorname{gr} A$ of finitely-generated $\mathbb{Z}$-graded $A$-modules by the Serre subcategory consisting of finite-dimensional modules [Orl09, Proposition 2.17]. Note that $G$-action on $\mathbb{P}^n$ may not be free.

The following theorem is due to Beilinson:

**Theorem 2.1.** *(See Beĭlinson [Beĭ78].) $D^b \operatorname{coh} \mathbb{P}^n$ has a full strong exceptional collection*

$$\big(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)\big)$$

*consisting of line bundles.*

As an immediate corollary to Theorem 2.1, we have the following:

**Corollary 2.2.** *$D^b \operatorname{coh}^G \mathbb{P}^n$ has a full strong exceptional collection*

$$\big(\mathcal{O}_{\mathbb{P}^n} \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n} \otimes \rho_r, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_r, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_r\big).$$

Let $d$ be a divisor of $n+1$ and $B = \bigoplus_{i \in \mathbb{Z}} A_{id}$ be the $d$-th Veronese subring of $A = R^G$. Let further $G_d = G/T_d$ be the quotient of $G$ by the diagonal subgroup $T_d = \{\zeta \cdot \mathrm{id}_{\mathbb{A}^{n+1}} \in G \mid \zeta^d = 1\}$ consisting of $d$-th roots of unity. Then $B$ is the invariant subring $(R^{(d)})^{G_d}$ of the $d$-th Veronese subring $R^{(d)}$ of $R = k[x_1, \dots, x_{n+1}]$, and one has

$$X := \operatorname{Proj} B = \left[\left(\left(\operatorname{Spec} R^{(d)} \setminus \mathbf{0}\right)/G_d\right)/\mathbb{G}_m\right].$$

The group $T_d$ is a cyclic group whose order $e$ is a divisor of $d$. If $T_d$ is non-trivial, then $G_d$ is not a subgroup of $SL_{n+1}(k)$ but a subgroup of its quotient $SL_{n+1}(k)/T_d$, and the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ does not have a $G_d$-linearization. On the other hand, the line bundle $\mathcal{O}_{\mathbb{P}^n}(e)$ does have a $G_d$-linearization and descends to a line bundle $\mathcal{O}_X(e)$ on $X$.

Recall that the *root stack* $\sqrt[e]{\mathcal{L}/X}$ of a line bundle $\mathcal{L}$ on a stack $X$ is the stack whose object over $\varphi : T \to X$ is a line bundle $\mathcal{M}$ on $T$ together with an isomorphism $\mathcal{M}^{\otimes e} \xrightarrow{\sim} \varphi^* \mathcal{L}$ [AGV08,Cad07]. The morphism $G \to G_d$ of finite groups induces a morphism $p : Y \to X$ of quotient stacks, and the isomorphism

$$\phi : \mathcal{O}_Y(1)^{\otimes e} \xrightarrow{\sim} \pi^* \mathcal{O}_X(e)$$

of line bundles gives an identification of $Y$ with the root stack $\sqrt[e]{\mathcal{O}_X(e)/X}$. It follows that there is an orthogonal decomposition

$$D^b \operatorname{coh} Y = \left\langle p^* D^b \operatorname{coh} X,\ \mathcal{O}_Y(1) \otimes p^* D^b \operatorname{coh} X, \dots, \mathcal{O}_Y(e-1) \otimes p^* D^b \operatorname{coh} X \right\rangle \tag{2.1}$$

of the derived category [IU, Lemma 4.1].

The invariant ring $A$ is Gorenstein with parameter $\deg x_1 + \cdots + \deg x_{n+1} = n+1$ by Watanabe [Wat74, Theorem 1], and its Veronese subring $B$ is Gorenstein with parameter $a = (n+1)/d$ by Goto and Watanabe [GW78, Corollary 3.1.5]. The following theorem is due to Orlov:

**Theorem 2.3.** *(See [Orl09, Theorem 2.5(i)].) If $B$ is a Gorenstein ring with parameter $a > 0$, then there is a full and faithful functor $\Phi : D^b_{\mathrm{sing}}(\operatorname{gr} B) \to D^b(\operatorname{qgr} B)$ and a semiorthogonal decomposition*

$$D^b(\operatorname{qgr} B) = \left\langle \pi B, \dots, \pi\left(B(a-1)\right), \Phi D^b_{\mathrm{sing}}(\operatorname{gr} B)\right\rangle,$$

*where $\pi : \operatorname{gr} B \to \operatorname{qgr} B$ is the natural projection functor.*

Now we prove Theorem 1.1. First consider the case $d = 1$. Recall that the *right mutation* of an exceptional collection is given by

$$(E, F) \mapsto (F, R_F E)$$

where $R_F E$ is the mapping cone

$$R_F E = \left\{E \to \hom(E, F)^\vee \otimes F\right\}.$$

See [Rud90] and references therein for more about mutations of exceptional collections. Write $E_{i,j} = \mathcal{O}_Y(i) \otimes \rho_j$ and perform successive right mutations

$$\begin{aligned}
&(E_{0,0}, \dots, E_{0,r}, E_{1,0}, \dots, E_{1,r}, \dots, E_{n,0}, \dots, E_{n,r}) \\
&\quad \mapsto (E_{0,0}, \dots, E_{0,r-1}, E_{1,0}, R_{E_{1,0}} E_{0,r}, E_{1,1}, \dots, E_{1,r}, \dots, E_{n,0}, \dots, E_{n,r}) \\
&\quad \mapsto (E_{0,0}, \dots, E_{0,r-2}, E_{1,0}, R_{E_{1,0}} E_{0,r-1}, R_{E_{1,0}} E_{0,r}, E_{1,1}, \dots, E_{1,r}, \dots, E_{n,0}, \dots, E_{n,r})
\end{aligned}$$

$$\begin{aligned}
&\mapsto \cdots \\
&\mapsto (E_{0,0}, E_{1,0}, R_{E_{1,0}}E_{0,1}, \dots, R_{E_{1,0}}E_{0,r}, E_{1,1}, \dots, E_{1,r}, \dots, E_{n,0}, \dots, E_{n,r}) \\
&\mapsto \cdots \\
&\mapsto (E_{0,0}, E_{1,0}, E_{2,0}, R_{E_{2,0}}R_{E_{1,0}}E_{0,1}, \dots, R_{E_{2,0}}R_{E_{1,0}}E_{0,r}, R_{E_{2,0}}E_{1,1}, \dots, R_{E_{2,0}}E_{1,r}, E_{2,1}, \dots, E_{n,r}) \\
&\mapsto \cdots \\
&\mapsto (E_{0,0}, E_{1,0}, \dots, E_{n,0}, R_{E_{n,0}} \cdots R_{E_{1,0}}E_{0,1}, \dots, R_{E_{n,0}} \cdots R_{E_{1,0}}E_{0,r}, \\
&\quad R_{E_{n,0}} \cdots R_{E_{2,0}}E_{1,1}, \dots, R_{E_{n,0}}E_{n-1,r}, E_{n,1}, \dots, E_{n,r}) \\
&= (E_{0,0}, E_{1,0}, \dots, E_{n,0}, F_{0,1}, \dots, F_{0,r}, \dots, F_{n,1}, \dots, F_{n,r})
\end{aligned}$$

where

$$F_{i,j} = R_{E_{n,0}}R_{E_{n-1,0}} \cdots R_{E_{i+1,0}}E_{i,j}.$$

Since $\pi(A(i)) = \mathcal{O}_Y(i) \otimes \rho_0 = E_{i,0}$ for any $i \in \mathbb{Z}$, it follows that $D^b_{\mathrm{sing}}(\operatorname{gr} A)$ is equivalent to the full triangulated subcategory of $D^b(\operatorname{qgr} A)$ generated by the exceptional collection

$$(F_{0,1}, \dots, F_{0,r}, F_{1,1}, \dots, F_{1,r}, \dots, F_{n,1}, \dots, F_{n,r}).$$

This proves Theorem 1.1 in the case $d = 1$.

Now we discuss the case $d > 1$. Since an exceptional is indecomposable and the decomposition in (2.1) is not only semiorthogonal but orthogonal, each exceptional object in the full strong exceptional collection $(E_{0,0}, \dots, E_{n,r})$ on $Y$ belongs to one of orthogonal summands in (2.1). It follows that the exceptional collection in Corollary 2.2 is divided into $e$ copies of an exceptional collection, each of which is pulled-back from $X$ and tensored with $\mathcal{O}_Y(i)$ for $i = 0, \dots, e-1$. Let $(E_{i,j})_{(i,j) \in \Lambda}$ be the exceptional collection generating the summand $p^* D^b \operatorname{coh} X$ in the orthogonal decomposition in (2.1). Since $e$ divides $d$, the collection $(\mathcal{O}_Y, \mathcal{O}_Y(d), \dots, \mathcal{O}_Y((a-1)d)) = (p^*\mathcal{O}_X, p^*\mathcal{O}_X(d), \dots, p^*\mathcal{O}_X((a-1)d))$ is a part of this collection. On the other hand, one has $\pi(B(i)) = \mathcal{O}_X(di)$ for any $i \in \mathbb{Z}$ since $B$ is the $d$-th Veronese subring. Now one can move these objects to the left by mutation, and Theorem 1.1 follows from Theorem 2.3 just as in the $d = 1$ case.

When $d = n + 1$, then $B$ is Gorenstein with parameter 1, and one does not need any mutation, so that $D^b_{\mathrm{sing}}(\operatorname{gr} B)$ has a full strong exceptional collection.

One can generalize the story to the case with arbitrary weights $\deg x_i = a_i$ and a finite subgroup $G \subset SL_{n+1}(k)$ with a free action on $\mathbb{A}^{n+1} \setminus \mathbf{0}$ commuting with the $\mathbb{G}_m$-action. The category $\operatorname{qgr} A$ is equivalent to the category of coherent sheaves on the weighted projective space $\mathbb{P}(a_1, \dots, a_{n+1})$, the Beilinson collection is given by $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a_1 + \dots + a_{n+1}))$, and the Gorenstein parameter of the polynomial ring is $a = a_1 + \dots + a_{n+1}$. The case $d = a_1 + \dots + a_{n+1}$ and $G = 1$ is discussed in [Ued08].

## 3. Crossed product algebras

Let $A$ be an $\mathbb{N}$-graded connected Gorenstein ring with parameter $a > 0$ and $G$ be a finite group acting on $A$. We assume that the characteristic of the base field $k$ does not divide the order of $G$. The set of irreducible representations of $G$ will be denoted by $\operatorname{Irrep}(G) = \{\rho_0, \dots, \rho_r\}$ where $\rho_0$ is the trivial representation.

**Proof of Theorem 1.2.** We need to show the existence of a full and faithful functor $\Phi : D^b_{\mathrm{sing}}(\operatorname{gr} A \rtimes G) \to D^b(\operatorname{qgr} A \rtimes G)$ and a semiorthogonal decomposition

$$D^b(\text{qgr}\, A \rtimes G) = \big\langle \mathcal{O} \otimes \rho_0, \dots, \mathcal{O} \otimes \rho_r, \dots, \\ \mathcal{O}(a-1) \otimes \rho_0, \dots, \mathcal{O}(a-1) \otimes \rho_r, \Phi D^b_{\text{sing}}(\text{gr}\, A \rtimes G)\big\rangle.$$

Since $A$ is Gorenstein, $A$ has finite injective dimension as left and right module over itself. It follows that $A \rtimes G$ also has finite injective dimension as left and right module over itself, and one has mutually inverse equivalences

$$D = \mathbb{R}\text{Hom}_{A \rtimes G}\ (\bullet, A \rtimes G) : D^b\big(\text{gr}^G A\big)^\circ \to D^b\big(\text{gr}^G A^\circ\big),$$
$$D^\circ = \mathbb{R}\text{Hom}_{(A \rtimes G)^\circ}(\bullet, A \rtimes G) : D^b\big(\text{gr}^G A^\circ\big)^\circ \to D^b\big(\text{gr}^G A\big)$$

of triangulated categories, where $\bullet^\circ$ denotes the opposite rings and categories.

For an integer $i$, let $\mathcal{S}_{<i}$ be the full subcategory of $D^b(\text{gr}^G A)$ consisting of complexes of torsion modules concentrated in degrees less than $i$. In other words, it is the full triangulated subcategory of $D^b(\text{gr}^G A)$ generated by $k(e) \otimes \rho$ for $e < -i$ and $\rho \in \text{Irrep}(G)$, where $k(e) \otimes \rho$ is the $e$-shift of the $A \rtimes G$-module which is isomorphic to $\rho$ as a $G$-module and annihilated by $A_+ = \bigoplus_{i=1}^\infty A_i$. One can show just as in [Orl09, Lemma 2.3] that $\mathcal{S}_{<i}$ is left admissible in $D^b(\text{gr}^G A)$ and the left orthogonal is the derived category $D^b(\text{gr}^G A_{\geqslant i})$ of graded $G \rtimes G$ modules $M$ such that $M_p = 0$ for any $p < i$;

$$D^b\big(\text{gr}^G A\big) = \big\langle \mathcal{S}_{<i}, D^b\big(\text{gr}^G A_{\geqslant i}\big)\big\rangle. \tag{3.1}$$

Let further $\mathcal{P}_{<i}$ be the full subcategory of $D^b(\text{gr}^G A)$ generated by projective modules $A(m) \otimes \rho$ for $m > -i$ and $\rho \in \text{Irrep}(G)$. One can also show

$$D^b\big(\text{gr}^G A\big) = \big\langle D^b\big(\text{gr}^G A_{\geqslant i}\big), \mathcal{P}_{<i}\big\rangle \tag{3.2}$$

just as in [Orl09, Lemma 2.3]. The proof of [Orl09, Lemma 2.4] carries over verbatim to the $G$-equivariant case, and gives weak semiorthogonal decompositions

$$D^b\big(\text{gr}^G A_{\geqslant i}\big) = \langle \mathcal{D}_i, \mathcal{S}_{\geqslant i}\rangle, \tag{3.3}$$
$$D^b\big(\text{gr}^G A_{\geqslant i}\big) = \langle \mathcal{P}_{\geqslant i}, \mathcal{T}_i\rangle \tag{3.4}$$

where $\mathcal{D}_i$ and $\mathcal{T}_i$ are equivalent to $D^b(\text{qgr}^G A)$ and $D^b_{\text{sing}}(\text{gr}^G A)$ respectively. (3.1) and (3.3) show that $\mathcal{S}_{\geqslant i}$ is right admissible in $D^b(\text{gr}^G A)$. The functor $D$ takes the subcategory $\mathcal{S}_{\geqslant i}(A)$ to the subcategory $\mathcal{S}_{<-i-a+1}(A^\circ)$, so that the right orthogonal $\mathcal{S}^\perp_{\geqslant i}(A)$ is sent to the left orthogonal ${}^\perp\mathcal{S}_{<-i-a+1}(A^\circ)$. The latter subcategory coincides with the right orthogonal $\mathcal{P}^\perp_{<-i-a+1}(A^\circ)$ by (3.1) and (3.2). The functor $D^\circ$ takes the right orthogonal $\mathcal{P}^\perp_{<-i-a+1}(A^\circ)$ to the left orthogonal ${}^\perp\mathcal{P}_{\geqslant i+a}(A)$, so that one has an equality

$$\mathcal{S}^\perp_{\geqslant i} = {}^\perp\mathcal{P}_{\geqslant i+a} \tag{3.5}$$

of subcategories of $D^b(\text{gr}^G A)$. One has a weak semiorthogonal decomposition

$$D^b\big(\text{gr}^G A\big) = \langle \mathcal{S}_{<i}, \mathcal{D}_i, \mathcal{S}_{\geqslant i}\rangle$$

by (3.1) and (3.3), which gives

$$D^b\big(\text{gr}^G A\big) = \langle \mathcal{P}_{\geqslant i+a}, \mathcal{S}_{\geqslant i}, \mathcal{D}_i\rangle$$

by (3.5). Since Gorenstein parameter $a$ is positive, the subcategory $\mathcal{P}_{\geqslant i+a}$ is not only right orthogonal but also left orthogonal to $\mathcal{S}_{<i}$, and one obtains a weak semiorthogonal decomposition

$$D^b\big(\mathrm{gr}^G A\big) = \langle \mathcal{S}_{\geqslant i}, \mathcal{P}_{\geqslant i+a}, \mathcal{D}_i \rangle. \tag{3.6}$$

On the other hand, (3.1) and (3.4) give a weak semiorthogonal decomposition

$$D^b\big(\mathrm{gr}^G A\big) = \langle \mathcal{S}_{\geqslant i}, \mathcal{P}_{\geqslant i}, \mathcal{T}_i \rangle. \tag{3.7}$$

By combining (3.6), (3.7) and

$$\mathcal{P}_{\geqslant i} = \big\langle \mathcal{P}_{\geqslant i+a}, A(-i-a+1) \otimes \rho_0, \dots, A(-i-a+1) \otimes \rho_r, \dots, A(-i) \otimes \rho_0, \dots, A(-i) \otimes \rho_r \big\rangle,$$

one obtains

$$\mathcal{D}_i = \big\langle A(-i-a+1) \otimes \rho_0, \dots, A(-i-a+1) \otimes \rho_r, \dots, A(-i) \otimes \rho_0, \dots, A(-i) \otimes \rho_r, \mathcal{T}_i \big\rangle,$$

and Theorem 1.2 follows by setting $i = -a + 1$. $\square$

Let $A = \bigoplus_{i \in \mathbb{Z}} R_{id}$ be the $d$-th Veronese ring of $R = k[x_1, \dots, x_{n+1}]$ for a divisor $d$ of $n+1$, and $G$ be a finite subgroup $GL_{n+1}(k)$ whose order is not divisible by the characteristic of $k$. Theorem 1.3 is an immediate consequence of Theorem 1.2:

**Proof of Theorem 1.3.** The graded ring $A$ is Gorenstein with parameter $a = (n+1)/d$, and one has an equivalence

$$\mathrm{qgr}^G A \cong \mathrm{coh}^G \mathbb{P}^n$$

of abelian categories. The derived category $D^b \mathrm{coh}^G \mathbb{P}^n$ has a full strong exceptional collection

$$\big(\mathcal{O}_{\mathbb{P}^n} \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n} \otimes \rho_r, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_r, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_r\big).$$

Theorem 1.2 shows that the full subcategory of $D^b \mathrm{coh}^G \mathbb{P}^n$ generated by

$$\begin{aligned}&\big(\mathcal{O}_{\mathbb{P}^n}(a) \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n}(a) \otimes \rho_r, \mathcal{O}_{\mathbb{P}^n}(a+1) \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n}(a+1) \otimes \rho_r, \dots,\\ &\quad \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_0, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_r\big)\end{aligned}$$

is equivalent to $D^b_{\mathrm{sing}}(\mathrm{gr}^G A)$, and Theorem 1.3 is proved. $\square$

## 4. Examples

We discuss a few examples in this section. Let us first consider the case when $G \subset SL_2(\mathbb{C})$ is the binary dihedral group of type $D_4$. The invariant subring $A = \mathbb{C}[x_1, x_2]^G$ is generated by three elements $u$, $v$ and $w$ of degrees 4, 8 and 10 satisfying $u^5 + uv^2 + w^2 = 0$. One has $\mathrm{Irrep}(G) = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}$ and the quiver describing the total morphism algebra of the full strong exceptional collection $(\mathcal{O} \otimes \rho_0, \dots, \mathcal{O}(1) \otimes \rho_4)$ is given as follows:

$$\begin{tikzcd}
\mathcal{O} \otimes \rho_0 \arrow[drr] & \mathcal{O} \otimes \rho_1 \arrow[dr] & \mathcal{O} \otimes \rho_2 \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & \mathcal{O} \otimes \rho_3 \arrow[dl] & \mathcal{O} \otimes \rho_4 \arrow[dll] \\
\mathcal{O}(1) \otimes \rho_0 & \mathcal{O}(1) \otimes \rho_1 & \mathcal{O}(1) \otimes \rho_2 & \mathcal{O}(1) \otimes \rho_3 & \mathcal{O}(1) \otimes \rho_4
\end{tikzcd}$$

Since the Gorenstein parameter of $A$ is two, we have to remove $\mathcal{O} \otimes \rho_0$ and $\mathcal{O}(1) \otimes \rho_0$ from the left. The object $\mathcal{O} \otimes \rho_0$ can be removed without any mutation, and when we remove $\mathcal{O}(1) \otimes \rho_0$, only $\mathcal{O} \otimes \rho_2$ will be affected, which will be turned into

$$R_{\mathcal{O}\otimes\rho_2}\mathcal{O}(1) \otimes \rho_0 = \{\mathcal{O} \otimes \rho_2 \to \mathcal{O}(1) \otimes \rho_0\}.$$

The resulting quiver is given as follows:

$$\begin{tikzcd}
\mathcal{O} \otimes \rho_1 \arrow[dr] & R_{\mathcal{O}\otimes\rho_2}\mathcal{O}(1) \otimes \rho_0 \arrow[dl] \arrow[drr] \arrow[drrr] & \mathcal{O} \otimes \rho_3 \arrow[dl] & \mathcal{O} \otimes \rho_4 \arrow[dll] \\
\mathcal{O}(1) \otimes \rho_1 & \mathcal{O}(1) \otimes \rho_2 & \mathcal{O}(1) \otimes \rho_3 & \mathcal{O}(1) \otimes \rho_4
\end{tikzcd}$$

The resulting full exceptional collection is strong in this case, and the corresponding quiver is a disjoint union of two Dynkin quivers of type $D_4$.

Now let us take a Veronese subring of $A$. Since the Gorenstein parameter of $A$ is two, only the second Veronese subring $B = \bigoplus_{i\in\mathbb{Z}} A_{2i}$ is Gorenstein, which has Gorenstein parameter one. Since $A$ has no odd components, $B$ is isomorphic to $A$ as an algebra, and only the grading is changed. The stack $\operatorname{Proj} B = [(\operatorname{Spec} B \setminus \mathbf{0})/\mathbb{G}_m]$ is a weighted projective line $\mathbb{X}_{2,2,2}$ in the sense of Geigle and Lenzing [GL87] with three orbifold points of order 2, which is obtained from $\operatorname{Proj} A$ by the inverse root construction (i.e. by removing the generic stabilizer). It follows that $D^b \operatorname{qgr} A$ is equivalent to the direct sum of two copies of $D^b \operatorname{qgr} B$, and $D^b \operatorname{qgr} B$ is equivalent to the full subcategory of $D^b \operatorname{qgr} A$ generated by half of the full strong exceptional collection in $D^b \operatorname{qgr} A$ shown below:

$$\begin{tikzcd}
\mathcal{O} \otimes \rho_0 \arrow[drr] & \mathcal{O} \otimes \rho_1 \arrow[dr] & & \mathcal{O} \otimes \rho_3 \arrow[dl] & \mathcal{O} \otimes \rho_4 \arrow[dll] \\
& & \mathcal{O}(1) \otimes \rho_2 & &
\end{tikzcd}$$

Since the Gorenstein parameter of $B$ is one, $D^b_{\mathrm{sing}}(\operatorname{gr} B)$ is equivalent to the full subcategory of $D^b(\operatorname{qgr} B)$ generated by the exceptional collection obtained from the above collection by removing $\mathcal{O} \otimes \rho$, which gives a Dynkin quiver of type $D_4$:

$$\begin{tikzcd}
\mathcal{O} \otimes \rho_1 \arrow[dr] & \mathcal{O} \otimes \rho_3 \arrow[d] & \mathcal{O} \otimes \rho_4 \arrow[dl] \\
& \mathcal{O}(1) \otimes \rho_2 &
\end{tikzcd}$$

On the other hand, the crossed product algebra $R \rtimes G$ with $R = \mathbb{C}[x_1, x_2]$ is regular, so that $D^b_{\mathrm{sing}}(\operatorname{gr}^G R)$ is zero. The graded stable derived category $D^b_{\mathrm{sing}}(\operatorname{gr} R^{(2)})$ of the second Veronese subring $R^{(2)} \rtimes G$ is equivalent to the full subcategory of $D^b(\operatorname{qgr}^G R^{(2)}) \cong D^b \operatorname{coh}^G \mathbb{P}^1$ generated by the strong exceptional collection

$$\big(\mathcal{O}(1) \otimes \rho_0, \mathcal{O}(1) \otimes \rho_1, \dots, \mathcal{O}(1) \otimes \rho_4\big)$$

by Theorem 1.3, which is just the direct sum of five copies of the derived category of finite-dimensional vector spaces.

Next we consider the case when $G = \langle \exp(2\pi\sqrt{-1}/3) \cdot \mathrm{id}_{\mathbb{A}^3} \rangle$ is a cyclic subgroup of $SL_3(k)$ of order three. The total morphism algebra of the full strong exceptional collection $(\mathcal{O} \otimes \rho_0, \dots, \mathcal{O}(2) \otimes \rho_2)$ in $D^b \operatorname{coh}^G \mathbb{P}^2$ is given as follows:

$$\mathcal{O} \otimes \rho_0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(1) \otimes \rho_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(2) \otimes \rho_2$$

$$\mathcal{O} \otimes \rho_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(1) \otimes \rho_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(2) \otimes \rho_0$$

$$\mathcal{O} \otimes \rho_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(1) \otimes \rho_0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(2) \otimes \rho_1$$

Note that this is the disjoint union of three copies of the Beilinson quiver for $\mathbb{P}^2$. The full exceptional collection in $D^b_{\mathrm{sing}}(\operatorname{gr} A)$ is obtained from the above collection by removing $\mathcal{O} \otimes \rho_0$, $\mathcal{O}(1) \otimes \rho_0$ and $\mathcal{O}(2) \otimes \rho_0$. To remove the second and the third object, we can mutate the above collection as

$$\mathcal{O} \otimes \rho_0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(1) \otimes \rho_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(2) \otimes \rho_2$$

$$\mathcal{O}(2) \otimes \rho_0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(3) \otimes \rho_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(4) \otimes \rho_2$$

$$\mathcal{O}(1) \otimes \rho_0 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(2) \otimes \rho_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{O}(3) \otimes \rho_2$$

so that the three objects $\mathcal{O} \otimes \rho_0$, $\mathcal{O}(1) \otimes \rho_0$ and $\mathcal{O}(2) \otimes \rho_0$ can safely be removed from the left to obtain three copies of the generalized Kronecker quiver

$$\bullet \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bullet$$

with three arrows. On the other hand, the third Veronese subring $B = \bigoplus_{i \in \mathbb{Z}} A_{3i}$ is Gorenstein with parameter one and satisfies $\operatorname{Proj} B = \mathbb{P}^3$, so that $D^b_{\mathrm{sing}}(\operatorname{gr} B)$ is equivalent to the derived category of modules over the generalized Kronecker quiver with three arrows. These results are in complete agreement with the works of Iyama and Yoshino [IY08], Keller, Murfet and Van den Bergh [KMVdB11], and Iyama and Takahashi [IT]. The stable derived category of $R \rtimes G$ for the above $G$ and $R = \mathbb{C}[x_1, x_2, x_3]$ is zero again, and that of its third Veronese subring $R^{(3)} \rtimes G$ is equivalent to the full subcategory of $D^b \operatorname{qgr}^G R^{(3)} \cong D^b \operatorname{coh}^G \mathbb{P}^2$ generated by the strong exceptional collection

$$\big(\mathcal{O}(1) \otimes \rho_0, \mathcal{O}(1) \otimes \rho_1, \mathcal{O}(1) \otimes \rho_2, \mathcal{O}(2) \otimes \rho_0, \mathcal{O}(2) \otimes \rho_1, \mathcal{O}(2) \otimes \rho_2\big)$$

which happens to be equivalent to $D^b_{\mathrm{sing}}(\operatorname{gr} A)$ above.

Theorem 1.2 can be useful also in other contexts. An integer $n \times n$ matrix $(a_{ij})_{i,j=1}^n$ defines a polynomial

$$W = \sum_{i=1}^n x_1^{a_{i1}} \cdots x_n^{a_{in}},$$

which is called *invertible* if the origin is an isolated singularity. They play a pivotal role in *transposition mirror symmetry* of Berglund and Hübsch [BH93], which attracts much attention recently. See e.g. [Kra] and references therein for more on invertible polynomials and mirror symmetry.

Any invertible polynomial is weighted homogeneous, and the choice of a weight is unique up to multiplication by a constant. The quotient ring $A = k[x_1, \dots, x_n]/(W)$ is Gorenstein with parameter

$$a = \deg x_1 + \cdots + \deg x_n - \deg W.$$

If $a$ is positive, then for any group $G$ of symmetries of $W$, one has a semiorthogonal decomposition in Theorem 1.2. One can also prove an analogue of [Orl09, Theorem 2.5(ii), (iii)] for $a \leqslant 0$ just as in Theorem 1.2. A typical example is the case when $G$ is a subgroup of the group

$$G^{\max} = \left\{ (\alpha_1, \dots, \alpha_n) \in \left(k^\times\right)^n \;\middle|\; \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{n1}} \cdots \alpha_n^{a_{nn}} = 1 \right\}$$

of *maximal diagonal symmetries* of $W$, but one can also deal with other cases such as the action of the symmetric group $\mathfrak{S}_n$ on the Fermat polynomial $W = x_1^m + \cdots + x_n^m$.

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