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Primitive permutation groups with a solvable 2-transitive subconstituent

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ABSTRACT

For a permutation group G acting on a finite set Ω and a point $\alpha \in \Omega$, a *suborbit* $\Delta(\alpha)$ is an orbit of the point stabilizer G_α on Ω. The permutation group induced by G_α on $\Delta(\alpha)$ is called a *subconstituent* of G. Moreover, G is said to be *uniprimitive* if G is primitive but not 2-transitive. In this paper we investigate uniprimitive permutation groups which have a solvable 2-transitive subconstituent. We determine all such groups G which have a simple socle. The affine case, that is G has an elementary abelian socle, are also discussed and an infinite family of affine primitive groups with non-self-paired 2-transitive subconstituents are presented.

1. Introduction

Let G be a primitive permutation group acting on a finite set Ω. Consider the action of G on $\Omega \times \Omega$. If $\Delta \neq \{(\alpha,\alpha) \mid \alpha \in \Omega\}$ is a non-trivial orbit of this action then, for a point $\alpha \in \Omega$, $\Delta(\alpha) = \{\beta \mid (\alpha,\beta) \in \Delta\}$ is an orbit of the point stabilizer G_α on $\Omega \setminus \{\alpha\}$, which is called a *suborbit* of G. A *subconstituent* $G_\alpha^{\Delta(\alpha)}$ of G is the permutation group on $\Delta(\alpha)$ induced by G_α. The subconstituent is said to be *faithful* if $G_\alpha^{\Delta(\alpha)} \cong G_\alpha$. Throughout this paper we assume that G is *uniprimitive*, which means that G is primitive but not 2-transitive on Ω. So G has at least two suborbits.

For an orbit Δ of G on $\Omega \times \Omega$, we can define an *orbital graph* Γ with vertex set $V(\Gamma) = \Omega$ and edge set $E(\Gamma) = \Delta$. Define $\Delta' = \{(\beta,\alpha) \mid (\alpha,\beta) \in \Delta\}$, which is called the orbit *paired to* Δ. Then it is clear that Γ is an indirected graph if and only if $\Delta' = \Delta$. In this case Δ is said to be *self-paired*. The corresponding suborbit $\Delta(\alpha)$ and subconstituent $G_\alpha^{\Delta(\alpha)}$ are also said to be self-paired. From the point of view of the orbital graph Γ, the subconstituent $G_\alpha^{\Delta(\alpha)}$ is the local action of G_α on the set of vertices

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adjacent to α. However, if $G_\alpha^{\Delta(\alpha)}$ satisfies certain properties, the structure of G_α, or even G itself, will be strongly restricted. For example, if $|\Delta(\alpha)| = 2$ then G is a Frobenius group of prime degree [44, 18.7]. Primitive permutation groups with a suborbit of lengths 3 and 4 are also determined (see [45,37,41]). Since 1970's, Cameron, Knapp and Praeger intensively studied the structure of G_α under various assumptions on $G_\alpha^{\Delta(\alpha)}$ [2–4,21–24,33,34]. One of these assumptions is $G_\alpha^{\Delta(\alpha)}$ being 2-transitive on $\Delta(\alpha)$.

For a finite group G, its *socle* is the product of all minimal normal subgroups of G and denoted by $\mathrm{soc}(G)$. If G is a primitive permutation group, $\mathrm{soc}(G)$ is a direct product of some isomorphic simple groups. The O'Nan–Scott Theorem asserts that one of the following holds (see [26] for details):

(1) *Affine type*: $\mathrm{soc}(G)$ is abelian;
(2) *Almost simple type*: $T = \mathrm{soc}(G) \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for some non-abelian simple group T;
(3) $\mathrm{soc}(G) = T^k$ for some non-abelian simple group T and $k > 1$. In this case G can be further classified into three subcases:
  (a) simple diagonal action;
  (b) product action;
  (c) twisted wreath action.

In [35] Praeger proved that, if G has a 2-transitive subconstituent, then it cannot be of simple diagonal action (a) or product action (b).

The purpose of this paper is to investigate uniprimitive permutation groups which have a solvable 2-transitive subconstituent. In this case, $G_\alpha^{\Delta(\alpha)}$ is solvable and, by [44, 18.3], G_α itself is solvable. Thus G cannot be of twisted wreath action (c) because otherwise G_α is unsolvable (see [26, p. 391]). If G is of affine type, it has a regular normal subgroup. Then, by [38, Lemma 9], $G_\alpha^{\Delta(\alpha)}$ is faithful. In [15], Ivanov and Praeger classified the primitive permutation groups of affine type with a *self-paired* 2-transitive subconstituent. In this paper we give an infinite family of affine primitive groups with *non-self-paired* solvable 2-transitive subconstituents (see Example 3.2). Some further discussion on the affine type will be given in Section 3 as well.

If G is an almost simple group, then $M = G_\alpha$ is a solvable maximal subgroup of G. Almost all such subgroups were known since the early 1990's because at that time all maximal local subgroups of G were known (see for example [1,5,27]), except for the maximal 2-local subgroups of the sporadic simple groups Monster and Baby Monster. Ten years later, in [30], Meierfrankenfeld and Shpectorov proved that the lists of the maximal 2-local subgroups of these two groups in the ATLAS [7] are complete. In a recent paper [25], among other results, Li and Zhang determined all solvable maximal subgroups of almost simple groups. All pairs (G_0, M_0) are explicitly listed, where $G_0 \lhd G$ is minimal such that $M_0 = M \cap G_0$ is maximal in G_0 [25, Theorem 1 and Tables 14–20].

Let G be a uniprimitive permutation group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree d. By Huppert's classification of solvable 2-transitive groups [13], $d = p^a$ for some prime p and integer $a \geqslant 1$. Furthermore,

$$G_\alpha^{\Delta(\alpha)} \leqslant \Gamma\big(p^a\big) = \big\{x \mapsto ax^\alpha + b \;\big|\; a \neq 0,\ \alpha \in \mathrm{Aut}\big(GF\big(p^a\big)\big)\big\}$$

is a subgroup of all semilinear transformations of $GF(p^a)$, except for $d = 3^2, 5^2, 7^2, 11^2, 23^2$ or 3^4.

If $d \leqslant 4$, all primitive groups with a suborbit of length d were determined (see [44, 18.7] for $d = 2$, [45] for $d = 3$, [37] and [41] for $d = 4$). If $d = p$ is prime and G has a solvable 2-transitive subconstituent, then $G_\alpha^{\Delta(\alpha)}$ is sharply 2-transitive. All such groups of almost simple type have also been determined [42]. Therefore, for almost simple type, we may assume that $d = p^a \geqslant 8$, $a > 1$ and $G_\alpha^{\Delta(\alpha)}$ is **not** sharply 2-transitive on $\Delta(\alpha)$. Our main result is the following

Theorem 1.1. *Let G be a uniprimitive permutation group with non-abelian simple socle. If G has a solvable non-sharply 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $d = p^a \geqslant 8$ and $a > 1$, then G, G_α and d are one of the entries in Tables 1 or 2.*

Table 1
soc(G) is a classical simple group.

G	G_α	d	Remark
$PSL(3,4).2_2$	$3^2{:}Q_8.2$	9	
$PSL(3,4).2_3$	$3^2{:}Q_8.2$	9	
$PSL(3,4).2^2$	$3^2{:}Q_8.2 \times 2$	9	
$PSL(3,4).3.2_2$	$3^2{:}2S_4$	9	
$PSL(3,4).3.2_3$	$3^2{:}2S_4$	9	
$PSL(3,4).6$	$3^2{:}2A_4 \times 2$	9	
$\mathrm{Aut}(PSL(3,4))$	$3^2{:}2S_4 \times 2$	9	
$PSL(3,t).2$	$3^2{:}Q_8.2$	9	prime $t \equiv 4, 7 \pmod 9$
$PGL(3,t)$	$3^2{:}Q_8.3$	9	as above
$\mathrm{Aut}(PSL(3,t))$	$3^2{:}Q_8.S_3$	9	$t \equiv 4, 7 \pmod 9$ and $t \equiv 1 \pmod 4$
$PSL(3,t)$	$3^2{:}2A_4$	9	prime $t \equiv 1 \pmod 9$
$PSL(3,t).2$	$3^2{:}Q_8.S_3$	9	$t \equiv 1 \pmod 9$ and $t \equiv 1 \pmod 4$
$PSU(3,8^2)$	$3^2{:}2A_4$	9	three representations
$PSU(3,8^2).2$	$3^2{:}2S_4$	9	
$PSU(3,2^{2r}).2$	$3^2{:}Q_8.2$	9	prime $r > 3$
$PGU(3,2^{2r})$	$3^2{:}Q_8.3$	9	as above
$PSU(3,2^{2r}).S_3$	$3^2{:}Q_8.S_3$	9	as above
$PSU(3,t^2).2$	$3^2{:}Q_8.2$	9	prime $5 < t \equiv -4, -7 \pmod 9$
$PGU(3,t^2)$	$3^2{:}Q_8.3$	9	as above
$\mathrm{Aut}(PSU(3,t^2))$	$3^2{:}Q_8.S_3$	9	$t \equiv -4, -7 \pmod 9$ and $t \equiv -1 \pmod 4$
$PSU(3,t^2)$	$3^2{:}2A_4$	9	prime $t \equiv -1 \pmod 9$
$PSU(3,t^2).2$	$3^2{:}Q_8.S_3$	9	$t \equiv -1 \pmod 9$ and $t \equiv -1 \pmod 4$
$P\Omega^+(8,2).3$	$5^2{:}4A_4$	25	
$P\Omega^+(8,2).S_3$	$5^2{:}4S_4$	25	

Table 2
soc(G) is an alternating, sporadic or exceptional simple group.

G	G_α	d
J_1	$2^3{:}7{:}3$	8
He	$5^2{:}4A_4$	25
$He.2$	$5^2{:}4S_4$	25
ON	$3^4{:}2_-^{1+4}.D_{10}$	81
$ON.2$	$3^4{:}2_-^{1+4}.D_{10}.2$	81
Th	$7^2{:}(3 \times 2S_4)$	49[a]
$F_4(2).2$	$7^2{:}(3 \times 2S_4)$	49

[a] The existence of the subconstituent of degree 49 has not been determined.

Conversely, in Tables 1 and 2, each group G has indeed a solvable 2-transitive subconstituent of degree d, except for the Thompson sporadic simple group Th.

Remark 1. We are unable to determine whether Th has a 2-transitive subconstituent of degree 49.

Remark 2. In Table 1, the group $PSU(3,8^2)$ has three conjugacy classes of maximal subgroups $3^2{:}2A_4$, which yield three inequivalent permutation representations.

The paper is organized as follows. Some notation and preliminaries are collected in Section 2. In particular, we give a series of lemmas describing the possible structure of G_α and $G_\alpha^{\Delta(\alpha)}$. These tools enable us to check whether or not an almost simple group G has a required 2-transitive subconstituent of degree d. The affine case is discussed in Section 3. In Section 4 we treat the case where $\mathrm{soc}(G)$ is an alternating group or a sporadic simple group. Section 5 is devoted to treating the case where $\mathrm{soc}(G)$ is a classical simple group while Section 6 deals with the case of exceptional groups of Lie type.

Many computations are done by using the computer package GAP [11]. The permutation or matrix representations of the almost simple groups mentioned in this paper are taken from *ATLAS of Group Representations*, version 3 (http://brauer.maths.qmul.ac.uk/Atlas/v3/).

2. Notation and preliminaries

The notation and terminology used in this paper are standard (see, for example, [7,20,44]). For two groups K and H, $K.H$ is an arbitrary extension of K by H while $K{:}H$ stands for a split one. $K \circ H$ is the central product of K and H. For a prime r and a positive e, denote r^e as an elementary abelian group of that order and r^{1+2e} as an extra-special r-group. In particular, if r is odd, denote r_+^{1+2e} as the extra-special r-group with exponent r and r_-^{1+2e} as that of exponent r^2. For $r=2$, the notation 2_+^{1+2e} stands for a central product of even number of Q_8 while 2_-^{1+2e} for a central product of odd number of Q_8 together with a D_8. For a positive integer g, the symbol $[g]$ denotes an arbitrary group of order g, while Z_g stands for a cyclic group of that order. Sometimes, a single g is also used to denote a cyclic group of that order. For a group H, we use $\pi(H)$ to denote the set of all prime divisors of $|H|$. A *section* of H is the quotient group A/B for some $B \lhd A \leqslant H$. For a prime r, the notation $r^e \top n$ means that r^e exactly divides n.

For a group H and a prime r, the maximal normal r-subgroup and the maximal normal r'-subgroup of H are denoted by $O_r(H)$ and $O_{r'}(H)$ respectively. A group H is called *strongly r-constrained* if $C_H(O_r(H)) \leqslant O_r(H)$. In particular, $O_{r'}(H)=1$ if H is strongly q-constrained.

For a G_α-orbit $\Delta(\alpha)$, we use $\Delta'(\alpha)$ to denote the orbit of G_α *paired with* $\Delta(\alpha)$ (see [44, §16] for details). In particular, a suborbit $\Delta(\alpha)$ is said to be *self-paired* if $\Delta'(\alpha)=\Delta(\alpha)$. Furthermore, denote the kernel of G_α on $\Delta(\alpha)$ as $K(\alpha)$, namely $K(\alpha)=G_{\{\alpha\}\cup\Delta(\alpha)}$. Similarly, denote $K'(\alpha)=G_{\{\alpha\}\cup\Delta'(\alpha)}$.

Next we give some lemmas about the structure of G_α and $G_\alpha^{\Delta(\alpha)}$. Recall that $|\Delta(\alpha)|=d=p^a$.

Lemma 2.1. *Let G be a primitive group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $d \geqslant 3$. Suppose that $|G_\alpha^{\Delta(\alpha)}|=d(d-1)l$ and $K=K(\alpha)$ is the kernel of G_α on $\Delta(\alpha)$. Then:*

(1) *If $K \neq 1$, then there exists a subgroup $E \lhd K$ such that K/E is isomorphic to a non-trivial normal subgroup of $G_{\alpha\beta}^{\Delta(\alpha)}$ for $\beta \in \Delta(\alpha)$.*
(2) *If $E \neq 1$ then E is a q-group for some prime $q \mid d-1$. Furthermore, G_α, G_β and $G_{\alpha\beta}$ are all strongly q-constrained.*
(3) *$|G_\alpha{:}E|$ divides $d(d-1)^2l^2$.*

Proof. For $\beta \in \Delta(\alpha)$, by [21, 3.2], $G_\beta^{\Delta'(\beta)} \cong G_\beta^{\Delta(\beta)} \cong G_\alpha^{\Delta(\alpha)}$. It is clear that $\alpha \in \Delta'(\beta)$ and K acts on $\Delta'(\beta)$ since $K < G_{\alpha\beta} < G_\beta$ and $\Delta'(\beta)$ is a G_β-orbit. Now $G_\beta^{\Delta(\beta)}$ is also solvable 2-transitive. So by [21, 4.10] $K'(\beta)=K(\beta)$. The kernel of K on $\Delta'(\beta)$ is $E=K\cap K'(\beta)=K\cap K(\beta)$. Hence $K/E=K^{\Delta'(\beta)}$ is a normal subgroup of $G_{\alpha\beta}^{\Delta'(\beta)} \cong G_{\alpha\beta}^{\Delta(\alpha)}$. By [21, 4.11], E is a proper subgroup of K. Thus $K/E \neq 1$ and (1) is proved. Statement (2) follows from [21, 4.12].

If $|G_\alpha^{\Delta(\alpha)}|=|G_\alpha{:}K|=d(d-1)l$, then $|G_{\alpha\beta}^{\Delta(\alpha)}|=(d-1)l$ which is divisible by $|K/E|$. Hence statement (3) holds. $\square$

Lemma 2.2. *Let G be a primitive permutation group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $p^a \geqslant 8$, where p is a prime. If $|G_\alpha|=2^b\cdot 3^c$, then $p^a=9$ and $c \leqslant 4$.*

Proof. Clearly now $p=2$ or 3 because $p^a(p^a-1) \mid |G_\alpha|$. Since $p^a \geqslant 8$, we have $a \geqslant 2$ when $p=3$ and $a \geqslant 3$ when $p=2$. If $a>2$ and $p^a \neq 2^6$, then by a result of Zsigmondy [47], there exists a prime divisor r of p^a-1 such that $r \nmid p^i-1$ for $0<i<a$. In particular $r \geqslant 5$, contradicting the fact that $p^a-1 \mid |G_\alpha| = 2^b \cdot 3^c$. If $p^a = 2^6$, then 7 is a divisor of p^a-1, a contradiction. Hence we have $a=2$ and $p=3$. By [13], $|G_{\alpha\beta}^{\Delta(\alpha)}| \mid 2^4 \cdot 3$. Following from Lemma 2.1, G_α contains a 2-subgroup E such that $|G_\alpha{:}E|$ divides $|G_\alpha^{\Delta(\alpha)}| \cdot |G_{\alpha\beta}^{\Delta(\alpha)}|$, which divides $2^8 \cdot 3^4$. $\square$

Lemma 2.3. *Let G be a primitive group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree p^a and $K=K(\alpha)$ the kernel of G_α on $\Delta(\alpha)$. If $p \nmid a$ then either $p \nmid |K|$ or $p^a = 3^2$.*

Proof. Suppose that $p \mid |K|$. By Lemma 2.1, if the kernel E of K on $\Delta'(\beta)$ is non-trivial, then E is a q-group with $q \mid p^a-1$. It follows from Lemma 2.1 that p divides $|G_{\alpha\beta}^{\Delta(\alpha)}| = (p^a-1)l$ for some integer l. Thus $p \mid l$. Now $G_\alpha^{\Delta(\alpha)}$ is a solvable 2-transitive group. If $G_\alpha^{\Delta(\alpha)} \leqslant \Gamma(p^a)$, then $l \mid a$ and hence $p \mid a$, a contradiction. Therefore $G_\alpha^{\Delta(\alpha)}$ is one of the exceptional groups determined by Huppert [13]. It is easy to check that the only possibility is $p^a = 3^2$. $\square$

Lemma 2.4. *Let p be a prime and $a>1$ an integer. If $p^a \geqslant 8$ and $p^a-1 \mid 2^b \cdot 3^c$, then $a=2$.*

Proof. It is clear that $p^a \neq 2^6$. So if $a>2$, then by [47] there exists a prime divisor r of p^a-1 such that $r \nmid p^i-1$ for $0<i<a$. Clearly $r=2$ or 3. If $p=2$ then $r=3$. But $3 \mid 2^2-1$, a contradiction. If p is an odd prime, then $r=3$ for otherwise $2 \mid p-1$. Therefore $p \not\equiv 1 \pmod 3$. We also have $p \neq 3$ because $r=3$ is a divisor of p^a-1. It follows that $p \equiv -1 \pmod 3$ and hence $3 \mid p^2-1$, contradicting the assumption that $a>2$. $\square$

Lemma 2.5. *(See [41, Lemma 2.6].) Let $G_0 \lhd G$ and M a maximal subgroup of G. Suppose that $M_0 = M \cap G_0$ is maximal in G_0. If there exists an element $x \in G$ such that $|M{:}M \cap M^x| = d > 1$, then there exists an element $h \in G_0$ such that $1 \neq |M_0{:}M_0 \cap M_0^h|$ divides d.*

The following theorem gives a set of criterions for a primitive permutation group G to have a primitive non-regular subconstituent of degree d.

Theorem 2.6. *(See [36, Theorem 2.6].) Let G be a primitive permutation group acting on a finite set Ω. Suppose that, for $\alpha \in \Omega$, the point stabilizer G_α has a maximal subgroup H of index d which is not normal in G_α. Then the following hold:*

- (i) *G has a self-paired suborbit $\Delta(\alpha)$ of length d such that $G_{\alpha\beta} = H$ for some $\beta \in \Delta(\alpha)$ if and only if $|N_G(H){:}H|$ is even.*
- (ii) *If $1 < |N_G(H){:}H|$ is odd, then G has a non-self-paired suborbit $\Delta(\alpha)$ of length d with $G_{\alpha\beta} = H$ for some $\beta \in \Delta(\alpha)$.*
- (iii) *If $N_G(H) = H$, then G has a non-self-paired suborbit $\Delta(\alpha)$ of length d with $G_{\alpha\beta} = H$ for some $\beta \in \Delta(\alpha)$ if and only if there exists an element $x \in G$ such that $H^x < G_\alpha$ but H^x and H are not conjugate in G_α.* $\square$

Corollary 2.7. *Let G, G_α, H and d be as in Theorem 2.6. Suppose that G has a normal subgroup T of index 2. Denote $G_1 = G_\alpha \cap T$ and $H_1 = H \cap T$. If $N_{G_1}(H_1) = H_1$ and $|N_T(H_1){:}H_1| = 2$, then $N_G(H) > H$. In particular, G has a suborbit $\Delta(\alpha)$ of length d such that $G_{\alpha\beta} = H$ for some $\beta \in \Delta(\alpha)$.*

Proof. It is clear that $N_{G_\alpha}(H) = H$ for H is maximal and not normal in G_α. Since $|N_T(H_1){:}H_1| = 2$, there exists $t \in T \setminus G_1$ such that $H_1^t = H_1$. Suppose $H = \langle H_1, a\rangle$ for some $a \in H$. Then $t^a \in T$ and $H_1^{t^a} = H_1$ because $H_1 \lhd H$. It follows that $t^a \in N_T(H_1) = H_1 \cup H_1 t$. It is evident that $t^a \notin H_1$. So we have $a^{-1}ta = xt$ for some $x \in H_1$, which implies that $tat^{-1} = ax \in H$. Therefore $t \in N_G(H) \setminus H$ and thus $N_G(H) > H$. It follows from Theorem 2.6 that G has a suborbit $\Delta(\alpha)$ of length d with $G_{\alpha\beta} = H$. $\square$

Lemma 2.8. *Let G be a primitive group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $d = p^a$ and $K = K(\alpha)$ the kernel of G_α on $\Delta(\alpha)$. For any normal subgroup $Q \lhd G_\alpha$, either $Q \leqslant K$, or $p^a \mid |Q|$. Furthermore, if $Q \nleqslant K$, then $\overline{Q} = QK/K \cong p^a$ when one of the following holds.*

(1) *Q is a q-group for some prime q. In this case $p = q$.*
(2) *Q is abelian.*

Proof. Denote $\overline{P} = \mathrm{soc}(G_\alpha^{\Delta(\alpha)}) \cong p^a$. Then if $Q \nleqslant K$, $\overline{Q} = QK/K$ must contain $\overline{P}$, which implies that $p^a \mid |Q|$. Furthermore, if (1) holds, then $\overline{Q}$ is a q-group. So $p = q$. Now $1 \neq Z(\overline{Q}) \trianglelefteq G_\alpha^{\Delta(\alpha)}$. It follows that $\overline{P} \leqslant Z(\overline{Q})$ because $\overline{P}$ is the unique minimal normal subgroup of $G_\alpha^{\Delta(\alpha)}$. Hence $\overline{Q} \leqslant C_{G_\alpha^{\Delta(\alpha)}}(\overline{P}) = \overline{P} \cong p^a$. If (2) holds, then $\overline{Q}$ can be written as the direct product of its Hall p and p'-subgroups $\overline{Q} = \overline{Q}_p \times \overline{Q}_{p'}$. It follows that $\overline{Q}_p$ is a normal p-subgroup of $G_\alpha^{\Delta(\alpha)}$, hence $\overline{Q}_p = \overline{P}$ by the same argument in (1). Therefore, $\overline{Q}_{p'} \leqslant C_{G_\alpha^{\Delta(\alpha)}}(\overline{P}) = \overline{P}$, which implies that $\overline{Q}_{p'} = 1$. $\square$

Lemma 2.9. *Let G be a primitive group with a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $d = p^a$ and $K = K(\alpha)$ the kernel of G_α on $\Delta(\alpha)$. Suppose that $G_\alpha = Q{:}H$ where Q is the direct product of s cyclic groups Z_l and $QK/K \neq 1$. Then $QK/K \cong p^a$ with $a = s$ and $p \top l$.*

Proof. Suppose that $l = p^u \cdot m$, $(p, m) = 1$. If $u > 1$, then there exists an $x \in Q$ of order p^u. By Lemma 2.8, $QK/K \cong p^a$. Consider $\bar{x} \in QK/K$, which has order at most p. It follows that $1 \neq x^p \in K$. Write Q as the direct product of its Hall p and p'-subgroups $Q = Q_p \times Q_{p'}$. Then $Q_p \,\mathrm{char}\, Q \lhd G_\alpha$ and $Q_p \cap K \neq 1$ is the Sylow p-subgroup of K, which implies that $Q_p \cap K \,\mathrm{char}\, K$. By Lemma 2.1, there is a normal subgroup $E \lhd K$ such that K/E is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Delta(\alpha)}$. If $E \neq 1$ then E is an r-group with $r \mid p^a - 1$. Hence

$$Q_p \cap K \cong (Q_p \cap K)E/E \,\mathrm{char}\, K/E \lhd G_{\alpha\beta}^{\Delta'(\beta)} \cong G_{\alpha\beta}^{\Delta(\alpha)}.$$

Thus $G_{\alpha\beta}^{\Delta(\alpha)}$ has a normal p-subgroup, contradicting [14, II 3.2]. So we have $u = 1$. It is clear that $a \leqslant s$. If $a < s$, then the same method can be used to prove that $Q_p \cap K \neq 1$, a contradiction again. $\square$

3. Affine case

In this section we discuss the case of affine type. Let G be a uniprimitive permutation group with an elementary abelian socle Z_r^n for some prime r and integer $n \geqslant 1$. Then $G = Z_r^n{:}G_\alpha$, where $G_\alpha \leqslant GL(n, r)$ is the point stabilizer of $\alpha = 0 \in Z_r^n$. In this case, G_α is an irreducible subgroup of $GL(n, r)$. In addition, if G_α has a solvable 2-transitive quotient group $G_\alpha^{\Delta(\alpha)}$ then, by [38, Lemma 9], G_α itself is solvable and acts faithfully on $\Delta(\alpha)$. It follows that G is a solvable primitive permutation group. Moreover, $|\Delta(\alpha)| = d = p^a$ with prime $p \neq r$ and G_α has a unique minimal normal subgroup $K = Z_p^a$. Write $G_\alpha = K \rtimes L$. Then $d - 1 \mid |L|$ since $G_\alpha \cong G_\alpha^{\Delta(\alpha)}$ is 2-transitive of degree d.

It is well-known that there is a 1-1 correspondence between solvable primitive permutation groups G of degree r^n and the irreducible solvable subgroups G_α of $GL(n, r)$. Therefore, if we can determine all irreducible solvable subgroups of $GL(n, r)$, we will be able to check which of them has a 2-transitive action on some set. However, the determination of irreducible solvable subgroups of $GL(n, r)$ is essentially recursive: it depends on the determination of subgroups of $GL(m, r)$ for all divisors $m \mid n$ and solvable transitive subgroups of symmetric group $S_{n/m}$. Thus we can hardly give a uniform description of the irreducible solvable subgroups of $GL(n, r)$ for general n. On the other hand, in [39], among many other results, Suprunenko determined all maximal irreducible solvable subgroups of $GL(q, r)$, where the dimension q is a prime. This enables us to exclude the case for some small n. Since all such groups G are determined for $d \leqslant 5$ (see [45,37,41,43]), we assume $d \geqslant 7$ in the following proposition.

Proposition 3.1. *Let G be a uniprimitive permutation group such that $\mathrm{soc}(G) = Z_r^n$. If G has a solvable 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of length $d \geqslant 7$ then $n \geqslant 4$.*

Proof. If $n = 1$, $G_\alpha \leqslant GL(1,r) = Z_{r-1}$ is cyclic. Thus G_α cannot act 2-transitively on any set. If $n = 2$ then, by [39, Theorem 21.6], three cases should be considered. Recall that $G_\alpha = K{:}L$, where $K = Z_p^a$ and $|\Delta(\alpha)| = d = p^a$.

(1) $G_\alpha \leqslant Z_{r-1}^2{:}2$. If $K \cap Z_{r-1}^2 = 1$ then $|K| \mid 2$, a contradiction. Thus $1 \neq K \cap Z_{r-1}^2 \lhd G_\alpha$. The minimality of K implies that $K \leqslant Z_{r-1}^2$. Further, if $G_\alpha \cap Z_{r-1}^2 > K$ then $C_{G_\alpha}(K) > K$, contradicting [14, II 3.2]. So we have $G_\alpha \cap Z_{r-1}^2 = K$, which implies that

$$L \cong G_\alpha/K = G_\alpha/\big(G_\alpha \cap Z_{r-1}^2\big) \leqslant Z_2,$$

a contradiction.

(2) $G_\alpha \leqslant Z_{r^2-1}{:}2$. The same argument can be applied to exclude this case.

(3) $G_\alpha \leqslant (Z_{r-1} \circ Q_8).S_3$. In this case, $Z_{r-1} = Z(GL(2,r))$ is the subgroup of scalers. If $K \cap Z_{r-1} = 1$ then $|K| \mid 24$. The only possibility is $|K| = d = 8$, and thus $K = Z_2^3$. On the other hand, $K \cong KZ_{r-1}/Z_{r-1}$ is isomorphic to a subgroup of $(Z_{r-1} \circ Q_8).S_3/Z_{r-1} \cong Z_2^2.S_3$, which contains no subgroup isomorphic to Z_2^3. So we have $K = K \cap Z_{r-1}$, which implies that $K \leqslant Z(GL(2,r))$, a contradiction.

Similarly, if $n = 3$ then, by [39, Theorem 21.6], three cases should be considered.

(1) $G_\alpha \leqslant Z_{r-1}^3{:}S_3$. The same argument as in case of $n = 2$ can be used to prove that $G_\alpha \cap Z_{r-1}^3 = K$, which implies that L is isomorphic to a subgroup of S_3. Since $d - 1 = p^a - 1 \mid |L|$, we have $d = 7$. Thus $K = Z_7$ and, as a point stabilizer of 2-transitive group of degree 7, L must be Z_6, a contradiction.

(2) $G_\alpha \leqslant Z_{r^3-1}{:}3$. Similarly we can yield a contradiction as in case (1).

(3) $G_\alpha \leqslant (Z_{r-1} \circ E).2A_4$, where E is an extra-special group of order 3^3 with exponent 3 and $r \equiv 1 \pmod 3$. Moreover, $Z_{r-1} = Z(GL(3,r))$. Therefore $K \cap Z_{r-1} = 1$. So we have $G_\alpha \cap Z_{r-1} = 1$ because otherwise $C_{G_\alpha}(K) > K$. It follows that G_α is isomorphic to a subgroup of $G_\alpha Z_{r-1}/Z_{r-1} \leqslant 3^2{:}2A_4$. Thus $d = 3^2$ and $G_\alpha = 3^2{:}Q_8$ or $3^2{:}2A_4$. Write $M = Z_{r-1}G_\alpha$. If $G_\alpha = 3^2{:}2A_4$ then $M = Z_{r-1} \times G_\alpha = (Z_{r-1} \circ E).2A_4$, a contradiction. If $G_\alpha = 3^2{:}Q_8$ then the Sylow 3-subgroup of M must be elementary abelian since Z_{r-1} is the center of the group and $Z_{r-1} \cap G_\alpha = 1$. On the other hand, $M = (Z_{r-1} \circ E).Q_8$ has extra-special Sylow 3-subgroup E, a contradiction. $\square$

For $n \geqslant 4$, there exist many examples of the affine primitive groups with a solvable 2-transitive subconstituent. The GAP package IRREDSOL [12] provides a library of all irreducible solvable subgroups of $GL(n,r)$, up to conjugacy, for $r^n < 2^{16}$ and the library of the corresponding affine primitive solvable groups. It enables us to go through these groups and to look for examples of G which has solvable 2-transitive subconstituent. Most of the examples we found satisfy $G = Z_r^{p-1} \rtimes (Z_p{:}Z_{p-1})$ for some prime p with suborbit of length p. Next we give the construction of the infinite family of such groups.

Example 3.2. Let $p \geqslant 5$ be a prime and group

$$H = \big\langle a, b \bigm| a^p = b^{p-1} = 1,\ b^{-1}ab = a^s \big\rangle,$$

where $s^{p-1} \equiv 1 \pmod p$ and $s^i \not\equiv 1 \pmod p$ for $1 \leqslant i < p-1$. So $H = Z_p{:}Z_{p-1}$. For any prime r satisfying $r \equiv 1 \pmod{p(p-1)}$, $F = GF(r)$ is a splitting field for H. Therefore, H has $p-1$ linear representations and a unique faithful irreducible representation T of degree $p-1$ over F (see, for example, [9, §47] and [31]). Let ζ be a primitive p-th root of unit in F. Then T can be written as

$$T(a)=\begin{pmatrix}\zeta &&&&\\ &\zeta^{s}&&&\\ &&\zeta^{s^2}&&\\ &&&\ddots&\\ &&&&\zeta^{s^{p-2}}\end{pmatrix},\qquad T(b)=\begin{pmatrix}0&0&\dots&0&1\\ 1&0&\dots&0&0\\ &1&&&\\ &&\ddots&&\\ 0&\dots&&1&0\end{pmatrix}.$$

Denote $V=V(p-1,r)$ as the $(p-1)$-dimensional vector space over F. Then $G=V\rtimes T(H)$ is a primitive group of degree r^p. For the zero vector $0\in V$, the stabilizer $G_0=T(H)$. Let $v=(1,1,\dots,1)\in V$. Then it is easy to verify that $\Delta=v^{T(H)}$ has length p and the action of $T(H)$ on Δ is 2-transitive.

Furthermore, for any prime $r\neq p$, H is p-solvable. It follows from the Fong–Swan–Rukolaine Theorem [10, §22] that H has a unique faithful irreducible representation of degree $p-1$ over field $GF(r)$. Therefore we obtain an infinite family of primitive permutation groups $\mathscr{F}=\{Z_r^{p-1}\rtimes(Z_p{:}Z_{p-1})\}$ with degree r^{p-1} and the point stabilizer $G_0\cong Z_p{:}Z_{p-1}$. G has a suborbit Δ of length p and the corresponding subconstituent G_0^{Δ} is sharply 2-transitive. In addition, if $r>2$, then $|N_G(Z_{p-1}){:}Z_{p-1}|=r$ is odd. It follows from Theorem 2.6(ii) that the suborbit is not self-paired. In fact, G has exactly $r-1$ suborbits of length p, forming $(r-1)/2$ pairs of mutually paired suborbits.

Remark 3. If $r=2$, then $G=Z_2^{p-1}\rtimes T(H)$ is contained in the automorphism group of the *folded p-cube* $\square_p$ (see [15] for details). The full automorphism group of $\square_p$ is $Z_2^{p-1}\rtimes S_p$. In this case, the unique suborbit of G with length p is self-paired.

Example 3.3. The group $GL(7,3)$ contains an irreducible solvable subgroup $H\cong Z_2^3{:}Z_7{:}Z_3$. Thus we obtain a primitive permutation group $G=Z_3^7\rtimes H$ of degree 3^7 with point stabilizer $G_0=H$. Computation shows that G has 2 mutually paired suborbits of length 8 and, on each of them, the action of H is faithful and 2-transitive.

In the remainder of the paper we will treat the case that G is almost simple. So in what follows we always assume the following hypothesis:

G is a uniprimitive permutation group with a solvable and non-sharply 2-transitive subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree $d=p^a\geqslant 8$ $(a>1)$. Its socle $\mathrm{soc}(G)=T$ is a non-abelian simple group. $(*)$

In this case, G has a solvable maximal subgroup $M\cong G_\alpha$. Suppose $G_0\lhd G$ is minimal such that $M_0=M\cap G_0$ is maximal in G_0. Then all such pairs (G_0,M_0) are listed in [25, Theorem 1 and Tables 14–20]. In order to prove our Theorem 1.1, we will treat these pairs and their overgroups (G,M) by the methods developed in Section 2, to determine if they have a non-sharply 2-transitive suborbit on the right cosets of M in G.

4. Alternating and sporadic groups

In this section, we first assume that $T=\mathrm{soc}(G)=A_n$ $(n\geqslant 5)$ is an alternating group. In [36], all primitive groups with a solvable 2-transitive subconstituent are determined if its socle is an alternating group (see also [25, Table 14]). It follows that

Proposition 4.1. *If G satisfies hypothesis $(*)$, then* $\mathrm{soc}(G)\neq A_n$. □

Next we consider the sporadic case. Suppose that $T\trianglelefteqslant G\leqslant \mathrm{Aut}(T)$ where T is a sporadic simple group and G contains a maximal subgroup $M=G_\alpha$ which has a solvable and non-sharply 2-transitive constituent $G_\alpha^{\Delta(\alpha)}$ on the set of all cosets of M in G. It follows that (G,M) must be one of the entries of [25, Table 15] (see also [7,19,28,30]).

Table 3

G	$M = G_\alpha$	d
He	$5^2{:}4A_4$	25
$He.2$	$5^2{:}4S_4$	25
J_1	$2^3{:}7{:}3$	8
ON	$3^4{:}2_-^{1+4}D_{10}$	81
$ON.2$	$3^4{:}2_-^{1+4}D_{10}.2$	81
Th	$7^2{:}(3 \times 2S_4)$	49

Proposition 4.2. *Let G be a uniprimitive permutation group satisfying hypothesis $(*)$. If $T = \mathrm{soc}(G)$ is a sporadic simple group, then G, G_α and d are one of the entries in Table 3. Conversely, all these M have indeed a solvable 2-transitive constituent except for $(G, M) = (Th, 7^2{:}(3 \times 2S_4))$.*

Proof. Recall that $K = K(\alpha)$ is the kernel of M acting on $\Delta(\alpha)$ and E is the kernel of K acting on $\Delta'(\beta)$ for $\beta \in \Delta(\alpha)$ (cf. Lemma 2.1).

If $(G, M) = (He, 5^2{:}4A_4)$, let $L = 4A_4 < M$, then computation shows that $|N_G(L)| = 96$ while $|N_M(L)| = 48$. Hence by Theorem 2.6, G has indeed a subconstituent $G_\alpha^{\Delta(\alpha)}$ of degree 25. It is easy to show that this subconstituent is 2-transitive.

If $G = He.2$, write $T = He$ and $M_1 = 5^2{:}L_1 < M = 5^2{:}L$, where $L_1 = 4A_4$ and $L = 4S_4$. Computation shows that $N_{M_1}(L_1) = L_1$ and $|N_T(L_1){:}L_1| = 2$. It follows from Corollary 2.7 that G has indeed a suborbit of length 25. It is clear that $G_\alpha^{\Delta(\alpha)}$ is faithful and hence non-sharply 2-transitive.

Computation shows that $G = J_1$ has indeed a 2-transitive subconstituent of degree 8, with $G_\alpha = 2^3{:}7{:}3$.

For the case $(G, M) = (ON, 3^4{:}2_-^{1+4}D_{10})$, take $L = 2_-^{1+4}.D_{10}$ and z the unique involution in the center of L. Since all involutions of ON are conjugate, we may assume that $z = x^2$ where x is in the conjugacy class $4A$ by the notation of ATLAS [7]. By [32, Lemma 4.8],

$$Z_4.PSL(3,4) = C_G(x) \lhd C_G(z) = Z_4.PSL(3,4).2.$$

Hence $\langle x\rangle = Z(C_G(x))\,\mathrm{char}\,C_G(x) \lhd C_G(z)$, which implies that $N_G(L)$ normalizes $\langle x\rangle$ since $N_G(L) \leqslant C_G(z)$. It follows that, for any $y \in L$, $x^y = x^i$ for some $i = 1$ or 3. Thus

$$x^{-1}y^{-1}x = x^{i-1}y^{-1} = y^{-1} \quad \text{or} \quad x^2y^{-1} \in L$$

as $x^2 = z \in L$. This shows that $x \in N_G(L)$. If $N_M(L) > L$, then there exists a 3-subgroup P such that $N_M(L) = L \times P$. However, by [46, Lemma 2.6], L contains an element of order 4 which acts fixed point freely on $3^4 \geqslant P$, a contradiction. Therefore $N_M(L) = L$. If $x \in L$ then L contains $\langle x\rangle$ as a normal subgroup of order 4, a contradiction. It follows that $x \in N_G(L) \setminus N_M(L)$. Hence by Theorem 2.6, ON has a suborbit of length 81. It is not hard to show that the corresponding subconstituent is faithful and non-sharply 2-transitive. The existence of 2-transitive subconstituent for case $ON.2$ can be proved by the same argument.

If $(G, M) = (Th, 7^2{:}(3 \times 2S_4))$, then by Lemmas 2.1 and 2.2, we get $d = 49$. However, the existence of a 2-transitive subconstituent of degree 49 is unsettled.

If $(G, M) = (B, (2^2 \times 7^2{:}(3 \times 2A_4)).2)$ and $d = 49$, then $2^2 \leqslant K$ since it acts trivially on 7^2. Therefore, by [13], $G_{\alpha\beta}^{\Delta(\alpha)} = (3 \times Q_8){:}S_3$ and hence $K = 2^2$. On the other hand, if E is a 2-group, then by Lemma 2.1, G_α is strongly 2-constrained, contradicting the fact that $O_7(G_\alpha) \neq 1$. Thus $E = 1$ and $2^2 \cong K/E$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Delta(\alpha)}$, a contradiction.

If $(G, M) = (Co_3, 2^2.[2^7.3^2].S_3)$ then, by Lemma 2.2, $d = 9$. It follows from [13] that $G_\alpha^{\Delta(\alpha)} \cong 3^2{:}2A_4$ or $3^2{:}2S_4$. Hence the kernel K has order 2^6 or 2^7. Computation shows that M has no such normal subgroup.

If $(G, M) = (J_4, 11_+^{1+2}{:}(5 \times 2S_4))$ then $p = 2$ or 11. If $p = 2$ then $d = 8$ or 16. However, by Lemma 2.1 $|G_{\alpha\beta}^{\Delta(\alpha)}|$ must be divisible by 11^3, a contradiction. If $p = 11$ then $d = 11^2$ and by [13], $G_\alpha^{\Delta(\alpha)} = 11^2{:}(5 \times 2S_4)$. It follows from Lemma 2.1 that $G_{\alpha\beta}^{\Delta(\alpha)}$ has a normal subgroup of order 11, contradicting [14, II 3.2].

Similarly, by applying Lemmas 2.1 and 2.2 and [13] and combining with computation, one can exclude all other entries of [25, Table 15]. □

5. Classical groups

In this section we deal with the case where $T = \mathrm{soc}(G)$ is a simple classical group. Let G be a uniprimitive permutation group satisfying hypothesis $(*)$. Assume that $T = \mathrm{soc}(G)$ is defined over the finite field $F = GF(q)$ and V is the natural projective $GF(q)$-module for T of dimension n. Suppose that $M = G_\alpha$ is maximal in G. Then M is a maximal local subgroup of G and thus, following from Aschbacher [1], belongs to one of the eight subgroup collections $\mathscr{C}_i$ of G. The detailed description of these eight collections can be found in [1] and [20]. Furthermore, denote $G_0 \lhd G$ as the minimal normal subgroup of G such that $M_0 = M \cap G_0$ is maximal in G_0. Then all such pairs (G_0, M_0) are listed in [25, Tables 16–19].

We first give some examples of G which has a solvable non-sharply 2-transitive subconstituent. Then we prove that there exists no other group satisfying hypothesis $(*)$. We always assume that $K = K(\alpha)$ is the kernel of G_α acting on $\Delta(\alpha)$ and $E \trianglelefteq K$ is the kernel of K acting on $\Delta'(\beta)$ for a fixed $\beta \in \Delta(\alpha)$ (cf. Lemma 2.1).

Lemma 5.1. *Suppose $T = PSL(3, t)$ for some prime $t \equiv 1 \pmod 3$. Let M be a maximal subgroup of G such that*

$$M \cap T = \begin{cases} 3^2{:}Q_8, & \text{if } t \equiv 4, 7 \pmod 9, \\ 3^2.2A_4, & \text{otherwise.} \end{cases}$$

Then G has a non-sharply 2-transitive subconstituent of degree 9 with $G_\alpha \cong M$, if and only if

(1) $t \equiv 4, 7 \pmod 9$, $(G, M) = (PSL(3, t).2, 3^2{:}Q_8.2)$ *or* $(PGL(3, t), 3^2{:}Q_8.3)$;
(2) $t \equiv 4, 7 \pmod 9$ *and* $t \equiv 1 \pmod 4$, $G = \mathrm{Aut}(PSL(3, t))$, $M = 3^2{:}Q_8.S_3$;
(3) $t \equiv 1 \pmod 9$, $G = PSL(3, t)$, $M = 3^2{:}2A_4$; *or*
(4) $t \equiv 1 \pmod 9$ *and* $t \equiv 1 \pmod 4$, $G = PSL(3, t).2$, $M = 3^2{:}Q_8.S_3$.

Proof. It follows from [1] that M is now the normalizer of an extra-special 3-group. Denote the preimage of M in $GL(3, t)$ by $\overline{M}$. Then $\overline{M} = 3^2.Sp(2, 3)$.

It is well-known that, if $t \equiv 1 \pmod 4$ and $2^s \top t - 1$, then the Sylow 2-subgroup of $PSL(3, t)$ is a wreath product $Z_{2^s} \wr Z_2$. And if $t \equiv 3 \pmod 4$ and $2^s \top t^2 - 1$, then the Sylow 2-subgroup is a semi-dihedral group of order 2^{s+1}. It follows that $PSL(3, t)$ has only one conjugacy class of subgroups isomorphic to Q_8. Let ζ be a primitive element of $GF(t)$ and η an element of order 3 satisfying $1 + \eta + \eta^2 = 0$. Define matrices

$$X = \frac{\eta - 1}{3}\begin{pmatrix} \eta^2 & \eta & 1 \\ \eta & \eta & \eta \\ 1 & \eta & \eta^2 \end{pmatrix}, \qquad Y = \frac{1 - \eta}{3}\begin{pmatrix} 1 & \eta^2 & \eta^2 \\ 1 & \eta & 1 \\ \eta^2 & \eta^2 & 1 \end{pmatrix}.$$

It is clear that $X, Y \in SL(3, t)$ and one can check that $\langle X, Y\rangle C/C \cong Q_8 < T$ where C is the center of $GL(3, t)$. Furthermore, define matrix

$$Z = \begin{pmatrix} \frac{1}{2}(\zeta + \zeta^{-2}) & 0 & \frac{1}{2}(\zeta - \zeta^{-2}) \\ 0 & \zeta & 0 \\ \frac{1}{2}(\zeta - \zeta^{-2}) & 0 & \frac{1}{2}(\zeta + \zeta^{-2}) \end{pmatrix}.$$

Then $Z \in SL(3,t)$ has order $t-1$. It is not difficult to verify that $\langle Z\rangle C/C \cong Z_{(t-1)/3} = C_T(Q_8)$ and $N_T(Q_8)/C_T(Q_8) \cong \mathrm{Aut}(Q_8) = S_4$. It follows that $N_T(Q_8) = Z_{(t-1)/3}.S_4$ and its center $Z(N_T(Q_8)) = Z_{(t-1)/3}$. For any matrix $S \in GL(3,t)$, denote γ as the inverse transpose mapping $\gamma : S \mapsto (S')^{-1}$. Then we have

$$\langle X, Y\rangle^\gamma = \langle X, Y\rangle \quad \text{and} \quad Z^\gamma = Z^{-1}. \tag{1}$$

Notice that now $\mathrm{Out}(T) = S_3$. If $t \equiv 1 \pmod 9$ then $G = T$, $T.2$, $T.3$ or $\mathrm{Aut}(T)$ with $G_\alpha = 3^2{:}2A_4$, $3^2{:}Q_8.S_3$, $3^2{:}2A_4.3$ or $3^2{:}2A_4.S_3$ respectively. In the latter two cases, $|G_\alpha|$ is divisible by 3^4. It follows from [13] and Lemma 2.1 that $G_\alpha^{\Delta(\alpha)} = 3^2{:}Q_8.3$ or $3^2{:}Q_8.S_3$. However, by Lemma 2.1, this implies that $G_{\alpha\beta}^{\Delta(\alpha)} = 2A_4.3$ or $2A_4.S_3$ has a normal subgroup of order 3, a contradiction.

The former two cases are listed in Table 1. Write $G_\alpha = 3^2{:}L$. Then, in both cases, it is clear that $Q_8 = O_2(L)$ is characteristic in L. Thus we have $Q_8 \lhd N_G(L)$ which leads to $L \leqslant N_G(L) \leqslant N_G(Q_8)$. If $G = T$ and $L = Q_8.3 < N_T(Q_8)$, then any element $x \in Z(N_T(Q_8)) = Z_{(t-1)/3}$ with odd order belongs to $N_G(L) \setminus L$, which implies the existence of a subconstituent of degree 9 by Theorem 2.6.

If $G = T.2$, then it is an extension of T by an outer automorphism of order 2 which can be induced by γ. It follows from (1) that there exists $g \in G \setminus T$ such that $g^2 \in T$ and $g \in N_G(Q_8)$. Notice that now $N_T(Q_8) = Z_{(t-1)/3}.S_4$, $|N_G(Q_8){:}N_T(Q_8)| = 2$ and $N_G(Q_8)/C_G(Q_8) \cong S_4$. So we have $N_G(Q_8) = \langle N_T(Q_8), g\rangle$ and, by (1), $C_G(Q_8)$ is a dihedral group of order $2(t-1)/3$. On the other hand, we still have

$$Q_8.S_3 = L \leqslant N_G(L) \leqslant N_G(Q_8).$$

Let $L_0 = L \cap T = Q_8.3 \leqslant N_T(Q_8) = Z_{(t-1)/3}.S_4$. Then we can assume $L = \langle L_0, g\rangle$ such that $L_0^g = L_0$ and $g^2 \in L_0$. Moreover, let z be the generator of the cyclic normal subgroup of order $(t-1)/3$ in $C_G(Q_8)$. Then $z^g = z^{-1}$. We will prove that $N_G(L) > L$ if and only if $t \equiv 1 \pmod 4$. Notice that $L_0 \cap \langle z\rangle = \langle z^{(t-1)/6}\rangle$ has order 2. If $4 \mid o(z)$ then $\langle z\rangle$ contains an element z_1 of order 4. It is clear that $L_0^{z_1} = L_0$ and $z_1^g = z_1^{-1}$, which leads to $z_1^{-1}g^{-1}z_1 = z_1^{-2}g^{-1} \in L$. This shows that $z_1 \in N_G(L) \setminus L$. Conversely, assume $o(z) = 2k$ for some odd k. If $z^i \in N_G(L) \setminus L$ for some $1 \leqslant i < k$, then $g^{-1}z^ig = z^{-i}$, which implies that $gz^{2i} = g^{z^i} \in L = L_0 \cup L_0g$. It follows that the odd order element z^{2i} or $z^{4i} \in L_0$, contradicting the fact that $|L_0 \cap \langle z\rangle| = 2$. Thus we have $N_G(L) \cap \langle z\rangle = \langle z^k\rangle \in L_0$, which implies that $N_G(L) = \langle L_0, g\rangle = L$. Therefore we have proved that, when $G = T.2$, the 2-transitive subconstituent of degree 9 exists if and only if $t \equiv 1 \pmod 4$.

If $t \equiv 4, 7 \pmod 9$, then $G > T$ since $G_\alpha^{\Delta(\alpha)}$ is non-sharply 2-transitive. It follows that $G = T.2$, $T.3$ or $\mathrm{Aut}(T)$ with $G_\alpha = 3^2{:}Q_8.2$, $3^2{:}Q_8.3$ or $3^2{:}Q_8.S_3$ respectively. All of them are listed in Table 1. Moreover, if $G = T.2$ and $G_\alpha = 3^2{:}Q_8.2$, then take $L = Q_8.2 < G_\alpha$. Now the Sylow 2-subgroup of G has order > 16, so $|N_G(L)| > 16 = |N_{G_\alpha}(L)|$. Thus by Theorem 2.6, G has indeed a suborbit of length 9. It is evident that $G_\alpha^{\Delta(\alpha)}$ is faithful and non-sharply 2-transitive. If $G = T.3 = PGL(3,t)$ and $L = Q_8.3 < G_\alpha$ then, as in the case that $t \equiv 1 \pmod 9$ and $G = T$, one can prove that $Z(N_G(Q_8)) = Z_{t-1}$ which implies that $N_G(L) > L$. The existence of a subconstituent of degree 9 follows from Theorem 2.6. Finally, if $G = \mathrm{Aut}(T) = PGL(3,t).2$ then, as in the case that $t \equiv 1 \pmod 9$ and $G = T.2$, the same argument can be used to prove that G has a 2-transitive subconstituent of degree 9 if and only if $t \equiv 1 \pmod 4$. □

Lemma 5.2. (1) *Suppose $PSL(3,4) = T \trianglelefteq G \leqslant \mathrm{Aut}(PSL(3,4))$ and M is a maximal subgroup of G such that $M \cap T = 3^2{:}Q_8$. Then all these groups have a solvable non-sharply 2-transitive subconstituent of degree 9 except for $G = PSL(3,4).2$, in which case $G_\alpha^{\Delta(\alpha)}$ is sharply 2-transitive.*

(2) *Suppose $PSU(3,8^2) = T \trianglelefteq G \leqslant \mathrm{Aut}(PSU(3,8^2))$ and M is a maximal subgroup of G such that $M \cap T = 3^2{:}2A_4$. Then G has a solvable non-sharply 2-transitive subconstituent of degree 9 if and only if $(G,M) = (T, 3^2{:}2A_4)$ and $(T.2, 3^2{:}2S_4)$.*

Proof. (1) It follows from ATLAS [7] and computation.

(2) Computation shows that, for $(G, M) = (T, 3^2{:}2A_4)$ and $(T.2, 3^2{:}2S_4)$, G has indeed a solvable non-sharply 2-transitive subconstituent of degree 9. For the case that $G = T.3$ or $T.6$, computation shows that G has no suborbit of length 9. $\square$

Lemma 5.3. *Let $T = PSU(3, 2^{2r})$ for some prime $r > 3$ and M be a maximal subgroup of G such that $M \cap T \cong 3^2{:}Q_8$. Then G has a non-sharply 2-transitive subconstituent of degree 9 with $G_\alpha \cong M$, if and only if $T < G$.*

Proof. Now $G = T$, $T.2$, $T.3$ or $T.S_3$ and the corresponding maximal subgroup $M = 3^2{:}L$ where $L = Q_8$, $Q_8.2$, $Q_8.3$ or $Q_8.S_3$ respectively. Let t be the unique involution contained in $Q_8 \leqslant L$. Then

$$\langle t\rangle \operatorname{char} Q_8 = O_2(L) \operatorname{char} L \trianglelefteq N_G(L),$$

which implies that $N_G(L) \leqslant C_G(t)$. Assume that S is a Sylow 2-subgroup of T containing Q_8. It is well-known that S is a Suzuki 2-group of order q^3 which has the property that $Z(S) = S' = \Omega_1(S)$ has order q. Moreover, for $t \in S$, it is not hard to show that $C_G(t) = S.Z_{(q+1)/3}$, $S.D_{2(q+1)/3}$, $S.Z_{q+1}$ and $S.D_{2(q+1)}$ when $G = T$, $T.2$, $T.3$ and $T.S_3$ respectively (see for example [14, II 10.12] and [6]). In all these cases, $Z(C_G(t)) = Z(S)$. Recall that $N_G(L) \leqslant C_G(t)$. So any involution $u \in Z(S) = Z(C_G(t))$ other than t belongs to $N_G(L) \setminus L$. It follows from Theorem 2.6 that G has indeed a 2-transitive subconstituent of degree 9. However, when $G = T$, the subconstituent is sharply 2-transitive while the other three cases are listed in Table 1. $\square$

Lemma 5.4. *Suppose $T = PSU(3, t^2)$ for some prime $t \equiv -1 \pmod 3$. Let M be a maximal subgroup of G such that*

$$M \cap T = \begin{cases} 3^2{:}Q_8, & \text{if } t \equiv -4, -7 \pmod 9, \\ 3^2.2A_4, & \text{otherwise.} \end{cases}$$

Then G has a non-sharply 2-transitive subconstituent of degree 9 with $G_\alpha \cong M$, if and only if

(1) $5 < t \equiv -4, -7 \pmod 9$, $(G, M) = (PSU(3, t^2).2, 3^2{:}Q_8.2)$ *or* $(PGU(3, t^2), 3^2{:}Q_8.3)$;
(2) $t \equiv -4, -7 \pmod 9$ *and* $t \equiv -1 \pmod 4$, $G = \mathrm{Aut}(PSU(3, t^2))$, $M = 3^2{:}Q_8.S_3$;
(3) $t \equiv -1 \pmod 9$, $G = PSU(3, t^2)$, $M = 3^2{:}2A_4$; *or*
(4) $t \equiv -1 \pmod 9$ *and* $t \equiv -1 \pmod 4$, $G = PSU(3, t^2).2$, $M = 3^2{:}Q_8.S_3$.

Proof. In this case, for $t \equiv \pm 1 \pmod 4$, the Sylow 2-subgroups of T is the same as that of $PSL(3, t)$ for $t \equiv \mp 1 \pmod 4$. So T also has only one conjugacy class of subgroups isomorphic to Q_8. Let ζ be an element of order $t + 1$ in $GF(t^2)^\times$ and η an element of order 3 satisfying $1 + \eta + \eta^2 = 0$. Define matrices

$$X = \frac{\eta - 1}{3}\begin{pmatrix} \eta^2 & \eta & 1 \\ \eta & \eta & \eta \\ 1 & \eta & \eta^2 \end{pmatrix}, \qquad Y = \frac{1 - \eta}{3}\begin{pmatrix} 1 & \eta^2 & \eta^2 \\ 1 & \eta & 1 \\ \eta^2 & \eta^2 & 1 \end{pmatrix}.$$

It is clear that $X, Y \in SU(3, t^2)$ and one can check that $\langle X, Y\rangle C/C \cong Q_8 < T$ where C is the center of $GU(3, t^2)$. Furthermore, define matrix

$$Z = \begin{pmatrix} \frac{1}{2}(\zeta + \zeta^{-2}) & 0 & \frac{1}{2}(\zeta - \zeta^{-2}) \\ 0 & \zeta & 0 \\ \frac{1}{2}(\zeta - \zeta^{-2}) & 0 & \frac{1}{2}(\zeta + \zeta^{-2}) \end{pmatrix}.$$

Then $Z \in SU(3,t^2)$ has order $t+1$. It is not difficult to verify that $\langle Z\rangle C/C \cong Z_{(t+1)/3} = C_T(Q_8)$ and $N_T(Q_8)/C_T(Q_8) \cong \mathrm{Aut}(Q_8) = S_4$. It follows that $N_T(Q_8) = Z_{(t+1)/3}.S_4$ and its center $Z(N_T(Q_8)) = Z_{(t+1)/3}$.

For any $x \in GF(t^2)$, let $\tau : x \mapsto x^t$ be the field automorphism of order 2. For any matrix $A = (a_{ij}) \in GU(3,t^2)$, denote $A^\tau = (a_{ij}^\tau) = (a_{ij}^t)$. Then τ becomes an automorphism of $GU(3,t^2)$ and we have

$$\langle X, Y\rangle^\tau = \langle X, Y\rangle \quad \text{and} \quad Z^\tau = Z^{-1}. \tag{2}$$

As in the case of Lemma 5.1, if $t \equiv -1 \pmod 9$, then $G = T$ or $T.2$, and if $t \equiv -4, -7 \pmod 9$, then $G = T.2$, $T.3 = PGU(3,t^2)$ or $\mathrm{Aut}(T)$ with $G_\alpha = 3^2{:}Q_8.2$, $3^2{:}Q_8.3$ or $3^2{:}Q_8.S_3$ respectively. All of them are listed in Table 1.

If $t \equiv -1 \pmod 9$, then write $G_\alpha = 3^2{:}L$ for $G = T$ or $T.2$. Then, as in Lemma 5.1, we still have $N_G(L) \leqslant N_G(Q_8)$. For $G = T$ and $L = Q_8.3 < G_\alpha$, any element $x \in Z(N_T(Q_8)) = Z_{(t+1)/3}$ with odd order belongs to $N_G(L) \setminus L$, which implies the existence of a subconstituent of degree 9 by Theorem 2.6.

If $G = T.2$, then it is an extension of T by an outer automorphism of order 2 which can be induced by τ. It follows from (2) that there exists $g \in G \setminus T$ such that $g^2 \in T$ and $g \in N_G(Q_8)$. Similar to the linear case in Lemma 5.1, now $N_T(Q_8) = Z_{(t+1)/3}.S_4$, $|N_G(Q_8){:}N_T(Q_8)| = 2$ and $N_G(Q_8)/C_G(Q_8) \cong S_4$. So we have $N_G(Q_8) = \langle N_T(Q_8), g\rangle$ and, by (2), $C_G(Q_8)$ is a dihedral group of order $2(t+1)/3$. The same argument in the proof of Lemma 5.1 can be applied to prove that G has a 2-transitive subconstituent of degree 9 if and only if $t \equiv -1 \pmod 4$.

If $t \equiv 4, 7 \pmod 9$ and $G = T.2$ then $G_\alpha = 3^2{:}Q_8.2$. Write $L = Q_8.2 < G_\alpha$. Now the Sylow 2-subgroup of G has order > 16 which implies that $N_G(L) > L$. The existence of a 2-transitive subconstituent of degree 9 follows from Theorem 2.6. If $G = T.3 = PGL(3,t)$ and $L = Q_8.3 < G_\alpha$ then, as in the case that $t \equiv -1 \pmod 9$ and $G = T$, one can prove that $Z(N_G(Q_8)) = Z_{t+1}$ which implies that $N_G(L) > L$. Finally, if $G = \mathrm{Aut}(T) = PGL(3,t).2$ then, as in the case that $t \equiv -1 \pmod 9$ and $G = T.2$, the same argument can be used to prove that G has a 2-transitive subconstituent of degree 9 if and only if $t \equiv -1 \pmod 4$. $\square$

Next we consider the case where $T = \mathrm{soc}(G) = P\Omega^+(8,q)$ and G contains a graph automorphism of order 3.

Lemma 5.5. *Suppose $T = P\Omega^+(8,q)$ and G contains a graph automorphism of order 3. Then G satisfies hypothesis $(*)$ if and only if $G = P\Omega^+(8,2).3$ or $P\Omega^+(8,2).S_3$ and $M = 5^2{:}4A_4$ or $5^2{:}4S_4$ respectively, with a subconstituent of degree 25.*

Proof. In this case all maximal subgroups of G are determined by Kleidman [16] (see also [25, Table 19]). The solvability of M leads to either $q = 2$ or 3, or

$$|M \cap T| = \frac{16}{(2,q-1)^2}\left(q^2+1\right)^2 \quad \text{or} \quad \frac{192}{(2,q-1)^2}(q \pm 1)^4$$

(see [16, 4.2.1 and Table III]).

If $q = 2$ then, by [7], we get $G = P\Omega^+(8,2).3$ or $P\Omega^+(8,2).S_3$ and $M = 5^2{:}4A_4$ or $5^2{:}4S_4$. All of them are listed in Table 1. Moreover, write $M = 5^2{:}L$. Then computation shows that $|N_G(L){:}N_{G_\alpha}(L)| = 2$. Thus by Theorem 2.6, G has indeed a suborbit of length 25 and the corresponding subconstituent is non-sharply 2-transitive.

If $q = 3$, we get $G = P\Omega^+(8,3).A_4$ or $P\Omega^+(8,3).S_4$ with $M = 10^2{:}4A_4$ or $10^2{:}4S_4$. We constructed a permutation representation of degree 3360 for $G = P\Omega^+(8,3).S_4$ and $G_\alpha = 10^2{:}4S_4$. Denote L as 5-complement of G_α. Computation shows that $N_G(L) = L$ and all subgroups of order $|L| = 384$ in G_α are conjugate. It follows from Theorem 2.6 that G has no subconstituent of degree 25. For $G = P\Omega^+(8,3).A_4$ and $G_\alpha = 10^2{:}4A_4$, the same result was obtained by computation. Thus it is also excluded.

Next suppose that $q \geqslant 4$. If $|M \cap T| = 16(q^2+1)^2/((2,q-1)^2)$, then by [16] $M \cap T \cong (D_{2h} \times D_{2h}).2^2 = Z_h^2.[4].2^2$ where $h = (q^2+1)/(2,q-1)$. Notice that

$$\mathrm{Out}(T) = \begin{cases} Z_f \times S_3, & \text{if } q \text{ is even,} \\ Z_f \times S_4, & \text{if } q \text{ is odd.} \end{cases}$$

First suppose that $Z_h^2 \nleqslant K$. It follows from Lemmas 2.8 and 2.9 that $Z_h^2K/K \cong p^2$. If q is even then we get

$$\left(4^f+1\right)^2 \leqslant 16^2 \cdot 6^2 \cdot f^2.$$

Hence $f \leqslant 4$. Similarly, if q is odd then

$$\left(t^{2f}+1\right)^2 \leqslant 16^2 \cdot 24^2 \cdot 4f,$$

where $q = t^f$ for some prime t. Elementary calculation shows that, in both cases, G does not satisfy hypothesis $(*)$.

Next suppose that $Z_h^2 \leqslant K$ but $Z_h^2.[4].2^2 \nleqslant K$. It follows from Lemma 2.8 that $(Z_h^2.[4].2^2)K/K \cong p^a$. Hence $p=2$ and $a=3$ or 4. However, $[4].2^2$ is non-abelian, which forces $d = p^a = 8$. By Lemma 2.1, now E is not a 2-group and hence 2 divides $|K/E|$. This contradicting the fact that $|G_{\alpha\beta}^{\Delta(\alpha)}| = 21$.

Finally suppose that $Z_h^2.[4].2^2 \leqslant K$. Then $G_\alpha^{\Delta(\alpha)}$ is a section of $\mathrm{Out}(T)$. It is easy to show that $G_\alpha^{\Delta(\alpha)}$ now cannot be a non-sharply 2-transitive group of degree $d \geqslant 8$.

If $|M \cap T| = 192(q \pm 1)^4/((2,q-1)^2)$ then, by the similar argument as above, one can prove that G cannot satisfy hypothesis $(*)$. □

In order to complete the investigation for classical groups, we prove the following proposition.

Proposition 5.6. *There is no other classical group satisfying hypothesis* $(*)$ *except for those in Lemmas* 5.1–5.5.

Proof. Let $T = \mathrm{soc}(G)$ be a classical simple group and M a solvable maximal subgroup of G. Then there exists a minimal normal subgroup $G_0 \lhd G$ such that $M_0 = M \cap G_0$ is maximal in G_0. All such pairs (G_0, M_0) are listed in [25, Tables 16–19]). Except for those listed in Lemmas 5.1–5.5, one can exclude all other entries of [25, Tables 16–19]) by applying lemmas in Section 2 and combining with computation. We take three cases as examples. Recall that $K = K(\alpha)$ is the kernel of M acting on $\Delta(\alpha)$ and E is the kernel of K acting on $\Delta'(\beta)$ for $\beta \in \Delta(\alpha)$ (cf. Lemma 2.1).

Case 1. $T = PSp(4,q)$ where $q = 2^f$ for some $f \geqslant 2$. G contains a graph automorphism and $M \cap T = [q^4]:Z_{q-1}^2$.

In this case $M = N_G(X)$ for $X \in Syl_2(T)$. Choose a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ of V satisfying

$$V = \langle \varepsilon_1, \varepsilon_4 \rangle \perp \langle \varepsilon_2, \varepsilon_3 \rangle,$$

where $\langle \varepsilon_1, \varepsilon_4 \rangle$ and $\langle \varepsilon_2, \varepsilon_3 \rangle$ are both hyperbolic planes. Then X can be written as

$$X = \left\{ \begin{pmatrix} 1 & t & u & v \\ 0 & 1 & w & x \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} t,u,v,w,x \in GF(q), \\ tw+u+x=0 \end{matrix} \right\}.$$

It is not hard to verify that $N_T(X) = X{:}H$, where

$$H=\left\{\begin{pmatrix} y & & & \\ & z & & \\ & & z^{-1} & \\ & & & y^{-1}\end{pmatrix} \;\middle|\; y,z\in GF(q)^{\times}\right\}.$$

Hence $|M| \mid q^4(q-1)^2\cdot 2f$.

If $XK/K\neq 1$ then by Lemma 2.8, $d=2^a$ and $XK/K\cong 2^a$. Suppose that E is an r-group. Then by Lemma 2.1, $r\mid 2^a-1$ and hence $r\neq 2$. Write $D=X\cap K$. Then $D\neq 1$ because X is non-abelian. Thus D is the normal Sylow 2-subgroup of K. It follows that

$$1\neq DE/E\operatorname{char} K/E\trianglelefteq G_{\alpha\beta}^{\Delta'(\beta)}\cong G_{\alpha\beta}^{\Delta(\alpha)}.$$

However, by [14, II 3.2], $G_{\alpha\beta}^{\Delta(\alpha)}$ cannot have normal 2-subgroup. This contradiction shows that $X\leqslant K$.

Now if $E\neq 1$ is an r-subgroup and $r\neq 2$, then by Lemma 2.1, G_α is strongly r-constrained and hence $O_2(G_\alpha)=1$, a contradiction. Therefore, either $E=1$ or E is a 2-group. Furthermore, if $X{:}H\leqslant K$ then $G_\alpha^{\Delta(\alpha)}\leqslant Z_f.Z_2$, which is impossible. Hence by Lemmas 2.8 and 2.9 we have $(X{:}H)K/K\cong p^a$, $a=2$ and $p\mid q-1$. It follows that

$$\frac{(q-1)^2}{p^2}\;\Big|\;|K/E|\;\Big|\;\big|G_{\alpha\beta}^{\Delta(\alpha)}\big|\;\Big|\;2f$$

and $(p^2-1)\leqslant f$ for $G_\alpha^{\Delta(\alpha)}$ is non-sharply 2-transitive. Hence we get $(2^f-1)^2\leqslant 2f(f+1)$, which forces $f=2$, excluded by [7].

Case 2. $T=PSU(4,q^2)$ and $M\cap T=Q.S_4$ where

$$Q\cong Z_{q+1}^2\times Z_{\frac{q+1}{(q+1,4)}}$$

is abelian.

Let $P\lhd G_\alpha^{\Delta(\alpha)}$ be the unique normal subgroup of order p^a and $\bar S=S_4K/K$. If $\bar S\neq 1$, then $P\leqslant\bar S$, which is impossible because $p^a\geqslant 8$. It follows that $S_4\leqslant K$. If $Q\leqslant K$, then $Q.S_4\leqslant K$. It follows that $G_\alpha^{\Delta(\alpha)}$ is a homomorphism image of $Z_{(q+1,4)}.Z_f.Z_2$, which cannot be a 2-transitive group of degree $d\geqslant 8$. This implies that $Q\nleqslant K$ and hence $QK/K\cong p^a$, which leads to

$$(q+1)^3\leqslant 4f^2(4,q+1)^3\cdot(4!)^2,$$

where $q=t^f$ for some prime t. It follows that t^f is a divisor of 2^5, 3^3, 5^2, $7,\dots,47$. Elementary calculation shows that the only possible values for $q=t^f$ are 5, 11, 23 and 47. If $q=5$ and $T=PSU(4,5^2)$, then

$$\big(Z_6^2\times Z_3\big).S_4\trianglelefteq M\leqslant\big(Z_6^2\times Z_3\big).S_4.[4]$$

because $|\mathrm{Out}(T)|=4$. It follows from Lemma 2.2 and [13] that $d=9$ and $G_{\alpha\beta}^{\Delta(\alpha)}=Q_8.3$ or $Q_8.S_3$. If $E\neq 1$ then by Lemma 2.1 it is a 2-group and M is strongly 2-constrained, contradicting the fact that $O_3(M)>1$. Thus $E=1$ and K is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Delta(\alpha)}$. If $G=T$, then $G_{\alpha\beta}^{\Delta(\alpha)}=Q_8.3$ and $|K|=12$. However, $Q_8.3$ has no normal subgroup of order 12. If $G=T.2$ and $G_{\alpha\beta}^{\Delta(\alpha)}=Q_8.3$, then $|K|=24$, which leads to a contradiction that $(2^2\times 3).2=K\cong Q_8.3$. If $G=T.2$ and $G_{\alpha\beta}^{\Delta(\alpha)}=Q_8.S_3$ then $|K|=12$. But now $G_{\alpha\beta}^{\Delta(\alpha)}$ contains no normal subgroup of order 12. Similarly one can prove that $G=T.4$ does not satisfy hypothesis $(*)$. For $q=11$, 23 or 47, one can prove that

$\pi(M) = \{2, 3\}$ and $O_{2'}(M) \neq 1$. Hence $d = 9$ and $E = 1$. However, it follows that $|K| = |K/E| > |G_{\alpha\beta}^{\Delta(\alpha)}|$, contradicting Lemma 2.1.

Case 3. $T = P\Omega^+(8, q)$, M has a section isomorphic to $P\Omega^\varepsilon(2, q) \wr S_4$ ($\varepsilon = \pm 1$) and G contains no graph automorphism of order 3.

In this case M is the stabilizer of $\{V_1, \ldots, V_4\}$ for an orthogonal decomposition $V = V_1 \perp \cdots \perp V_4$, where V_i are isomorphic non-degenerate subspaces of dimension 2.

First assume that $q = 2^f$. By [16] and the assumption that G contains no graph automorphism of order 3, we have

$$Q.2^3.S_4 \leqslant M \leqslant Q.2^3.S_4.Z_f.Z_2,$$

where $Q = Z_{(q-\varepsilon)}^4$. If $QK/K \neq 1$, then by Lemma 2.8, $QK/K \cong p^a$. If $E \neq 1$ is an r_1-subgroup such that r_1 divides $|Q|$, then $Q = Q_{r_1} \times Q_{r_1'}$ where Q_{r_1} is the Sylow r_1-subgroup of Q and $Q_{r_1'}$ the Hall-r_1'-subgroup of Q. Thus we have $Q_{r_1'} \operatorname{char} Q \lhd G_\alpha$. It follows that $O_{r_1'}(G_\alpha) \neq 1$, contradicting Lemma 2.1. Therefore either $E = 1$ or $(|E|, q - \varepsilon) = 1$. It follows that

$$\frac{(q-\varepsilon)^4}{p^a} \mid |K/E| \mid (p^a - 1)l \mid 8 \cdot 24 \cdot 2f$$

for some $l \geqslant 2$. Thus

$$(q-\varepsilon)^4 = \left(2^f - \varepsilon\right)^4 \leqslant p^a\left(p^a - 1\right)l \leqslant \left(p^a - 1\right)^2 \cdot l^2 \leqslant 147456 f^2.$$

Hence we have $f \leqslant 5$. Furthermore, by [20, Table 3.5.E], $\varepsilon = -1$ when $f \leqslant 2$. It is not hard to show that none of them satisfies hypothesis $(*)$. Therefore we assume that $Q \leqslant K$. If $Q.2^3 \leqslant K$, then $G_\alpha^{\Delta(\alpha)}$ is a section of $S_4.Z_f.Z_2$, which cannot be a 2-transitive group of degree $d \geqslant 8$. So by Lemma 2.8, $(Q.2^3)K/K \cong 2^3$ and $|G_{\alpha\beta}^{\Delta(\alpha)}| = 21$. Thus we have $8 \mid |K|$. On the other hand, by Lemma 2.1, either $E = 1$ or it is a 7-group, which leads to a contradiction that 8 is divides $|G_{\alpha\beta}^{\Delta(\alpha)}|$.

Next assume that $q = t^f$ is odd. It follows from [16] that $M \cap T = Q.[2^6].S_4$, where Q is an abelian subgroup of order $(q-\varepsilon)^4/2^5$ with $\varepsilon = \pm 1$. If $QK/K \neq 1$ then by Lemmas 2.8 and 2.9 we get $a = 4$, $p \mid q - \varepsilon$ and $p > 2$. Similarly as in the above paragraph, we can prove that either $E = 1$ or $(|E|, |Q|) = 1$. It follows that

$$\frac{(q-\varepsilon)^4}{32p^4} \mid |K/E| \mid \left|G_{\alpha\beta}^{\Delta(\alpha)}\right| \mid 2^6 \cdot 24 \cdot 8f,$$

as G contains no graph automorphism of order 3. Elementary calculation shows that $q = t^f$ is a divisor of 3^5, 5^3, 7^3, $11^2, \ldots, 19^2$, $23, \ldots, 173$. If $q = 3^5$, then $p = 61$ and $31 \mid p^4 - 1 = d - 1$ should be a divisor of $|M|$, a contradiction. For the other values of q, $\pi(G_{\alpha\beta}^{\Delta(\alpha)}) = \{2, 3\}$. It follows from Lemma 2.4 that $a = 2$, a contradiction. So next we assume that $Q \leqslant K$. If $Q.[2^6] \leqslant K$, then $G_\alpha^{\Delta(\alpha)}$ is a section of $S_4.Z_4.Z_f.Z_2$. It is not difficult to show that $G_\alpha^{\Delta(\alpha)}$ cannot be a 2-transitive group of degree $d \geqslant 8$. Thus by Lemma 2.8 we have $(Q.[2^6])K/K \cong p^a$, which implies that $p = 2$ and $3 \leqslant a \leqslant 6$. By [13], if $G_\alpha^{\Delta(\alpha)}$ is a non-sharply 2-transitive group of degree 2^a, then $8 \nmid |G_{\alpha\beta}^{\Delta(\alpha)}|$. However, it is clear that E is not a 2-group and hence 8 divides $|K/E|$ which is a divisor of $|G_{\alpha\beta}^{\Delta(\alpha)}|$.

Similarly, one can prove that all other entries of [25, Tables 16–19]) do not satisfy hypothesis $(*)$. $\square$

6. The exceptional groups

In this section we treat the case where $T = \mathrm{soc}(G)$ is an exceptional simple group of Lie type over $GF(q)$, where $q = t^f$ for some prime t. Suppose that G has a solvable maximal subgroup M such that (G, M) satisfies hypothesis $(*)$. If $T = {}^2B_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, $G_2(q)$ and ${}^2G_2(q)$, all maximal subgroups of G are determined (see [8,17,18,29,40]).

For the cases that $T = F_4(q)$, $E_i(q)$ $(i = 6, 7, 8)$ or ${}^2E_6(q)$, where $q = t^f$ for some prime t, write A as a minimal normal subgroup of M and $\mathrm{Inndiag}(T)$ the group generated by all inner and diagonal automorphisms of T. Then A is elementary abelian and $M = N_G(A)$. Since M is solvable, it follows from [5] that there are three cases to be considered:

(1) $t \mid |A|$, i.e. M is a maximal parabolic subgroup of G;
(2) $A < \mathrm{Inndiag}(T)$ and M is of maximal rank (see [5,27] for details);
(3) $A < \mathrm{Inndiag}(T)$ but M is not of maximal rank.

In addition, denote $G_0 \lhd G$ as the minimal normal subgroup of G such that $M_0 = M \cap G_0$ is maximal in G_0. Then all such pairs (G_0, M_0) are listed in [25, Table 20]).

In what follows, we first give an example of (G, M) that satisfies hypothesis $(*)$. Then we prove there exist no other entries of [25, Table 20]) satisfying $(*)$, which concludes the proof of Theorem 1.1. As in Section 5, we always assume that $K = K(\alpha)$ is the kernel of G_α acting on $\Delta(\alpha)$ and $E \trianglelefteq K$ is the kernel of K acting on $\Delta'(\beta)$ for a fixed $\beta \in \Delta(\alpha)$ (cf. Lemma 2.1).

Lemma 6.1. *Suppose $T = F_4(q)$ $(q = 2^f)$ and G contains a graph automorphism. Then G satisfies hypothesis $(*)$ if and only if $G = F_4(2).2$ and $M = 7^2{:}(3 \times 2S_4)$ with $d = 49$.*

Proof. All the possible pairs (G_0, M_0) are listed in [25, Table 20]. First assume that $G = F_4(2).2$ and $M = [2^{22}].(S_3 \times S_3){:}2$. It follows from Lemma 2.2 that $d = 9$ and $[2^{22}] \leqslant K$. Thus we have $|G_\alpha^{\Delta(\alpha)}| \leqslant 72$, which cannot be a non-sharply 2-transitive group of degree 9.

Next consider the case that $G = F_4(q).2$ for some $q = 2^f$ and $M \cap T = (q \pm 1)^4.W(F_4)$. Notice that now the Weyl group $W(F_4) \cong 2^3{:}S_4{:}S_3$ and $|\mathrm{Out}(T)| = 2f$. It is not hard to prove that $Z_{q\pm1}^4 K/K \cong p^a$. Therefore $(2^f \pm 1)^2 \leqslant 2^8 \cdot 3^2 f$. Calculation shows that $f \leqslant 8$ and no case satisfies hypothesis $(*)$.

Next consider the case that $M \cap T = (q^2 \pm q + 1)^2.(3 \times SL(2,3))$. Similarly one can get $4^f \pm 2^f + 1 \leqslant 144f$. Therefore either $G = F_4(2).2$ with $M = 7^2{:}(3 \times 2S_4)$, which is listed in Table 2, or $d = 49$, $G = F_4(4).2$ and $M = Z_{21}^2.(3 \times 2A_4).2$. In the latter case, if E is a 3-group, then by Lemma 2.1 $M = G_\alpha$ is strongly 3-constrained. This contradicts the fact that $O_7(M) \neq 1$. Thus 3 is not a divisor of $|E|$. It follows that, if $|G_{\alpha\beta}^{\Delta(\alpha)}| = 48 \cdot 2$, then 3^3 should be a divisor of $|G_{\alpha\beta}^{\Delta(\alpha)}|$. This contradiction forces $|G_{\alpha\beta}^{\Delta(\alpha)}| = 144$ and $G_{\alpha\beta}^{\Delta(\alpha)} \cong (Z_3 \times Q_8){:}S_3$. Hence Z_3^2 is a characteristic subgroup of K, which implies that $G_{\alpha\beta}^{\Delta(\alpha)}$ contains a normal subgroup isomorphic to Z_3^2, a contradiction.

In the former case, $F_4(2) = T \lhd G = F_4(2).2$ and $M = 7^2{:}(3 \times 2S_4)$. Let $M_1 = M \cap T = 7^2{:}(3 \times 2A_4)$, $L = Z_3 \times 2S_4$ and $L_1 = L \cap T = Z_3 \times 2A_4$. Computation by using the permutation representation of T of degree 69 888 shows that L_1 is maximal in M_1, $|N_T(L_1)| = 144$ and $|N_{M_1}(L_1)| = |L_1| = 72$. It follows from Corollary 2.7 that G has indeed a suborbit $\Delta(\alpha)$ of length 49 with $L = G_{\alpha\beta}$. It is not hard to show that $M = G_\alpha$ is 2-transitive on $\Delta(\alpha)$.

Similar arguments can be applied to the remained cases to prove that no other entries of [25, Table 20] satisfy hypothesis $(*)$. $\square$

Proposition 6.2. *There are no almost simple groups of exceptional Lie type satisfying hypothesis $(*)$ except for $G = F_4(2).2$, $M = 7^2{:}(3 \times 2S_4)$ with $d = 49$.*

Proof. It is sufficient to prove that all other pairs (G_0, M_0) in [25, Table 20] do not satisfy hypothesis $(*)$. We take the case that $T = {}^3D_4(q)$ as example. Now three classes of maximal subgroups have to be considered.

(1) The parabolic subgroup $M \geqslant [q^{11}]{:}(Z_{q^3-1} \circ SL(2,q)).Z_{(2,q-1)}$. It follows that $q=2$ or 3. If $q=2$ then by [7, p. 89], $M \leqslant 2^2.[2^9]{:}(7 \times S_3).3$. Let $Q = 2^2.[2^9]$. Then it is easy to show that $Q \nleqslant K$. It follows from Lemma 2.1 that $p=2$ and $3 \leqslant a \leqslant 10$ as Q is non-abelian. It is clear that $p^a - 1$ must be a divisor of $14 \cdot 9$. This implies that $a=3$ or 6, excluded by elementary calculation. The case $q=3$ can be excluded similarly.

(2) $M \cap T = (Z_{q^2-q+1}) \circ SU(3,q^2).Z_{(q^2-q+1,3)}.Z_2$, which leads to $q=2$. Hence we have $G = {}^3D_4(2)$ with $M = 3_+^{1+2}.2S_4$. It follows form Lemmas 2.1, 2.2 and [13] that $d=9$ and $G_{\alpha\beta}^{\Delta(\alpha)} = Q_8{:}3$ or $Q_8{:}S_3$. However, it implies that $|K|=3$ or 6 and

$$Z_3 \operatorname{char} K/E \trianglelefteq G_{\alpha\beta}^{\Delta'(\beta)} \cong G_{\alpha\beta}^{\Delta(\alpha)},$$

a contradiction.

(3) $M \cap T = Z^2_{q^2 \pm q+1}.SL(2,3)$. Therefore $p \mid q^2 \pm q + 1$ and $a=2$ by Lemma 2.9. Thus we have

$$\frac{(q^2 \pm q + 1)^2}{p^2} \,\Big|\, |K/E| \,\Big|\, 24 \cdot 3f,$$

where $q = t^f$. Calculation shows that $q=2$ or 4. If $q=2$ then $G = {}^3D_4(2)$ or ${}^3D_4(2).3$ with $d=9$. In the former case, $G_\alpha = 3^2{:}2A_4$. By using a permutation representation of G with degree 819, computation shows that G_α has 4604 suborbits with lengths 1, 8, 12, 24, 27, 36, 54, 72, 108 and 216 but has no 2-transitive action on these suborbits. In the latter case, $G_\alpha = 3^2{:}2A_4 \times 3 = E{:}L$. Take the 24 dimension representation of G over $GF(2)$. Computation shows that $N_G(L) = L$ and there is only one conjugacy class of subgroups of order 72 in G_α. It follows from Theorem 2.6 that G has no 2-transitive subconstituent of degree 9. If $q=4$ then $G = {}^3D_4(4).[6]$, $d=49$ and $G_\alpha^{\Delta(\alpha)} = 7^2{:}(3 \times Q_8).S_3$ since it is non-sharply 2-transitive on $\Delta(\alpha)$. Now $M = G_\alpha = (Z_{21} \times Z_{21}).2A_4.[6]$. Thus $K \cong Z_3^2$. If $E \neq 1$ is a 3-group, then by Lemma 2.1, G_α is strongly 3-constrained, which yields a contradiction that $Z_7^2 \leqslant O_{3'}(G_\alpha) = 1$. Therefore, $E=1$ and $Z_3^2 \cong K/E$ is a normal subgroup of $G_{\alpha\beta}^{\Delta(\alpha)} = (3 \times Q_8).S_3$. However, $G_{\alpha\beta}^{\Delta(\alpha)}$ has no normal subgroup of order 9, a contradiction.

Other cases can be excluded similarly. This completes the proof of the proposition and the proof of Theorem 1.1 as well. □

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