Journal of Algebra 404 (2014) 124–151

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The quasi-partition algebra

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ARTICLE INFO

Article history:
Received 8 April 2013
Available online 19 February 2014
Communicated by Volodymyr Mazorchuk

Keywords:
Representation theory
Partition algebras
Diagram algebras
Centralizer algebras
Schur–Weyl duality

ABSTRACT

We introduce the quasi-partition algebra $QP_k(n)$ as a centralizer algebra of the symmetric group. This algebra is a subalgebra of the partition algebra and inherits many similar combinatorial properties. We construct a basis for $QP_k(n)$, give a formula for its dimension in terms of the Bell numbers, and describe a set of generators for $QP_k(n)$ as a complex algebra. In addition, we give the dimensions and indexing set of its irreducible representations. We also provide the Bratteli diagram for the tower of quasi-partition algebras (constructed by letting k range over the positive integers).

Introduction

We introduce the centralizer algebra $QP_k(n)$, the *quasi-partition algebra*. This algebra arises as a subalgebra of the partition algebra, $P_k(n)$, which was introduced independently by Jones [9] and Martin [10] as a generalization of the Temperley–Lieb algebra and the Potts model in statistical mechanics. Jones defined $P_k(n)$ as a centralizer algebra and explicitly described the Schur–Weyl duality between $P_k(n)$ and the symmetric group S_n. Specifically, $P_k(n)$ generically centralizes the action of the symmetric group on the k-fold tensor product of the permutation representation V of S_n, i.e.

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.028

$$P_k(n) \cong \mathrm{End}_{S_n}\left(V^{\otimes k}\right) \quad \text{when } n \geqslant 2k.$$

The partition algebra has a basis indexed by set partitions. These set partitions can be encoded into graphs that make the partition algebra into a diagram algebra with multiplication given by concatenation of diagrams.

The permutation representation V decomposes into a direct sum of the trivial representation $S^{(n)}$ and the irreducible reflection representation $W = S^{(n-1,1)}$. We define $QP_k(n)$ as the centralizer

$$QP_k(n) = \mathrm{End}_{S_n}\left(W^{\otimes k}\right).$$

We describe a basis for $QP_k(n)$, which is indexed by set partitions of $2k$ elements without sets of size one. The dimension is therefore the number of such partitions, given by a formula in terms of the Bell numbers.

From our construction, $QP_k(n)$ is a subalgebra of $P_k(n)$. In addition we show that $QP_k(n)$ is also isomorphic to a subalgebra of $P_k(n-1)$; we exploit this relationship to provide a set of generators for $QP_k(n)$ and find relations satisfied by these generators. We give a formula for the product in $QP_k(n)$ and show that it is dominated by the relations in $P_k(n-1)$. Using the rule for decomposing the tensor product of W with any other irreducible representation S^λ of the symmetric group, we show that for $k \geqslant 2$ the irreducible representations of $QP_k(n)$ are indexed by the set of integer partitions of $0, 1, 2, \dots, k$. We construct the Bratteli diagram, which encodes inclusion and restriction rules between $QP_{k-1}(n)$ and $QP_k(n)$. We also give a formula for the dimensions of the irreducible representations for $QP_k(n)$.

While the study of rook algebras often follows from the study of a pre-existing centralizer algebra, in our case, it is the partition algebra that can be interpreted as the rook algebra of our quasi-partition algebra. If A centralizes B on $V^{\otimes k}$, the *rook version* of A centralizes B on $U^{\otimes k}$, where U is the direct sum of V and the trivial B-module. There has been a lot of work related to rook monoid algebras in recent years. The Motzkin algebra of Benkart and Halverson is the rook version of the Temperley–Lieb algebra [1]; the rook Brauer algebra studied in [6] generalizes the classical Brauer algebra; and the rook partition algebra studied in [7] generalizes the classical partition algebra. However, the partition algebra centralizes the action of the symmetric group on the k-fold tensor product of the permutation representation, which itself decomposes into the direct sum of the trivial module and the irreducible reflection representation. As we will see in the remark following Corollary 2.6, for example, our dimension formula agrees with this point of view.

We also expect some results in the representation theory of $QP_k(n)$ to correspond to results about the Kronecker product. Similar as in [2], this comes from the duality between the symmetric group and the quasi-partition algebra. However, in order to apply similar results to those obtained in [2], we first need to understand the non-semisimple representation theory of $QP_k(n)$ more deeply. We plan to further explore the structure of this new algebra and seek applications to symmetric functions and to the representation theory of the symmetric group.

1. The partition algebra

The structure of the quasi-partition algebra $QP_k(n)$ is understood through the structure of the partition algebra $P_k(n)$. A basis for each algebra is encoded both as set partitions and as diagrams, and actions of both algebras on tensor space are calculated from those diagrams. Combinatorial results about the irreducible representations of $QP_k(n)$ will resemble those for the partition algebra as well. In this section we set the stage by describing the general partition algebra $P_k(x)$ in terms of the partition diagrams and describing its action on tensor space.

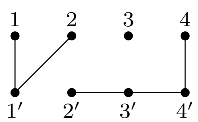
1.1. Set partitions and partition diagrams

A *set partition* of a set S is a set of pairwise disjoint subsets of S, called *blocks*, whose union is S. Fix $k \in \mathbb{Z}_{>0}$, and denote

$$[k] = \{1, \ldots, k\} \quad \text{and} \quad \left[k'\right] = \left\{1', \ldots, k'\right\},$$

so that $[k] \cup [k'] = \{1, \ldots, k, 1', \ldots, k'\}$ is formally a set with $2k$ elements.

For each set partition of $[k] \cup [k']$, we associate a diagram as follows. Consider the set of simple graphs with $2k$ vertices labeled from $[k] \cup [k']$, and draw the graph so that the vertices appear in two rows, $1, \ldots, k$ on the top and $1', \ldots, k'$ on the bottom. Any two vertices in the same block of a set partition are connected by a path. In particular, the connected components of the graph correspond to the blocks in the set partition. Define two graphs to be equivalent if their connected components partition the $2k$ (labeled) vertices in the same way. Then we define a *k-partition diagram* or simply *diagram* as the equivalence class of graphs corresponding to the same set partition of $[k] \cup [k']$. For example,



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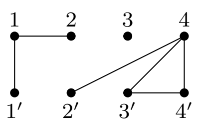
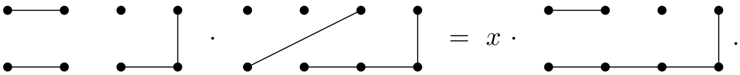


are equivalent, and both represent diagrams for the set partition $\{\{1, 2, 1'\}, \{3\}, \{2', 3', 4', 4\}\}$.

Let $\mathbb{C}(x)$ be the field of rational functions with complex coefficients in an indeterminate x. We define the product $d_1 \cdot d_2$ of two diagrams d_1 and d_2 using the concatenation of d_1 above d_2, where we identify the southern vertices of d_1 with the northern vertices of d_2. If there are c connected components consisting only of middle vertices, then the product is set equal to x^c times the diagram with the middle components removed. Extending this linearly defines a multiplication on $P_k(x)$.

For example,

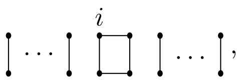


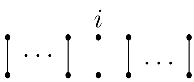
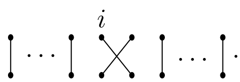
This product is associative and independent of the representative graphs.

The *partition algebra* $P_k(x)$ is the $\mathbb{C}(x)$-span of the k-partition diagrams with this product (with $P_0(x) = \mathbb{C}(x)$). Under this product, $P_k(x)$ is an associative algebra with identity given by the diagram corresponding to $\{\{1, 1'\}, \ldots, \{k, k'\}\}$. The dimension of $P_k(x)$ is the number of set partitions of $2k$ elements, i.e. the *Bell number* $B(2k)$.

1.2. Generators and relations of $P_k(x)$

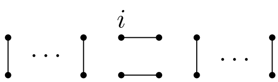
A presentation for $P_k(n)$ has been given in [8] and in [4]. Let

$b_i =$  ,

$p_i =$  and $s_i =$  .

Theorem 1.1. *(See Theorem 1.11 of [8].) Fix $k \in \mathbb{Z}_{>0}$. The partition algebra $P_k(x)$ is the unital associative $\mathbb{C}$-algebra presented by the generators b_i, s_i, and p_j for $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant k$, together with Coxeter, idempotent, commutation, and contraction relations (see (15)).*

It is often useful to additionally distinguish the element

$e_i = b_i p_i p_{i+1} b_i =$ 

for $1 \leqslant i \leqslant k-1$. Using subsets of $\{s_i, e_i \mid 1 \leqslant i \leqslant k-1\}$, one can generate the Temperley–Lieb algebra $TL_k(x)$, the group algebra of the symmetric group $\mathbb{C}(x)S_k$, and the Brauer algebra $B_k(x)$ all as subalgebras of the partition algebra $P_k(x)$.

1.3. $P_k(n)$ as a centralizer algebra of S_n

Let V denote the n-dimensional permutation representation of the symmetric group S_n. That is, $V = \mathbb{C}\text{-span}\{v_i \mid 1 \leqslant i \leqslant n\}$, where

$$\sigma \cdot v_i = v_{\sigma(i)} \quad \text{for } \sigma \in S_n. \tag{1}$$

Let S_n act diagonally on the basis of simple tensors in $V^{\otimes k}$:

$$\sigma \cdot (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)},$$

and extend this action linearly to $V^{\otimes k}$. This defines a S_n-module structure for $V^{\otimes k}$.

As in Section 1.1, number the vertices of a k-partition diagram $1, \dots, k$ from left to right in the top row and $1', \dots, k'$ from left to right on the bottom row. For each k-partition diagram d and each integer sequence $i_1 \dots, i_k, i_{1'}, \dots, i_{k'}$ with $1 \leqslant i_r \leqslant n$, define

$$\delta(d)^{i_1,\dots,i_k}_{i_{1'},\dots,i_{k'}} = \begin{cases} 1 & \text{if } i_t = i_s \text{ whenever vertices } t \text{ and } s \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Define an action of a partition diagram $d \in P_k(n)$ on $V^{\otimes k}$ by

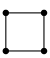
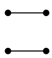
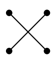
$$d \cdot (v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}}) = \sum_{1 \leqslant i_1,\dots,i_k \leqslant n} \delta(d)^{i_1,\dots,i_k}_{i_{1'},\dots,i_{k'}} v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k},$$

and extending linearly. For example, the action of $P_2(n)$ on $V^{\otimes 2}$ is determined by

$$b \cdot (v_i \otimes v_j) = \delta_{ij} v_i \otimes v_i, \qquad e \cdot (v_i \otimes v_j) = \delta_{ij} \sum_{\ell=1}^{n} v_\ell \otimes v_\ell,$$

$$s \cdot (v_i \otimes v_j) = v_j \otimes v_i, \quad \text{and} \quad p \otimes \mathrm{id} \cdot (v_i \otimes v_j) = \left(\sum_{\ell=1}^{n} v_\ell \right) \otimes v_j, \tag{3}$$

where

$b =$ , $e =$ , $s =$ , and $p =$. (4)

Then s_i, e_i, and b_i, for $i = 1, \dots, k-1$ can be identified with the maps in $\mathrm{End}(V^{\otimes k})$ given by

$$d_i = \mathrm{id}^{\otimes i-1} \otimes d \otimes \mathrm{id}^{\otimes k-i+1}, \quad \text{where } d = s, e, \text{ or } b;$$

the elements p_i for $i = 1, \dots, k$ can be identified with the maps in $\mathrm{End}(V^{\otimes k})$ given by

$$p_i = \mathrm{id}^{\otimes i-1} \otimes p \otimes \mathrm{id}^{\otimes k-i}.$$

Theorem 1.2. *(See [9].) S_n and $P_k(n)$ generate full centralizers of each other in $\mathrm{End}(V^{\otimes k})$. In particular,*

(a) *$P_k(n)$ generates $\mathrm{End}_{S_n}(V^{\otimes k})$, and when $n \geqslant 2k$, $P_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$;*
(b) *S_n generates $\mathrm{End}_{P_k(n)}(V^{\otimes k})$.*

For the remainder of the paper, we identify the elements of $P_k(n)$ (with $n \geqslant 2k$ integers) with endomorphisms of $V^{\otimes k}$.

2. The quasi-partition algebra

In this section we are interested in studying the centralizer algebra

$$QP_k(n) = \mathrm{End}_{S_n}\left(W^{\otimes k}\right), \quad \text{where } W = S^{(n-1,1)}$$

is the irreducible representation of S_n indexed by the partition $(n-1,1)$. With V as in (1), it is known that V decomposes as $V = T \oplus W$, where T is the trivial representation (indexed by the partition (n)).

2.1. Action of S_n on $W = S^{(n-1,1)}$

Using the same basis $\{v_1, \ldots, v_n\}$ for V as in Section 1.3, we fix a basis $\{w_2, \ldots, w_n\}$ for W, where $w_i = v_i - v_1$. The permutation action of S_n on V in (1) induces an action of S_n on W given by

$$\sigma \cdot w_i = w_{\sigma(i)}, \quad \text{for } \sigma \in S_{\{2,\ldots,n\}} \quad \text{and} \quad s_1 \cdot w_i = \begin{cases} w_i - w_2 & \text{for } i \neq 2, \\ -w_2 & \text{for } i = 2. \end{cases}$$

With S_n acting diagonally on $W^{\otimes k}$, we define the *quasi-partition algebra* as the centralizer algebra

$$QP_k(n) = \mathrm{End}_{S_n}(W^{\otimes k}) = \left\{ g : W^{\otimes k} \to W^{\otimes k} \;\middle|\; g\sigma = \sigma g \ \forall \sigma \in S_n \right\}.$$

The partition algebra $P_k(n-1)$ can be recognized as a subalgebra of $\mathrm{End}(W^{\otimes k})$ via the following change of basis. Define

$$f : \{v_1, \ldots, v_{n-1}\} \to \{w_2, \ldots, w_n\} \quad \text{by } f : v_i \mapsto w_{i+1}, \tag{5}$$

and extend linearly. Then if d is a diagram in $P_k(n-1)$, set

$$[d] = f \circ d \circ f^{-1}. \tag{6}$$

Hence, the action of $[d]$ on $W^{\otimes k}$ is the same as the action of d on $V^{\otimes k}$ in one fewer dimension, i.e. n has decreased by 1. Specifically, since the map $d \mapsto [d]$ is a homomorphism, we have the property that if $d_1, d_2 \in P_k(n-1)$, then $[d_1][d_2] = [d_1 d_2]$. We remark that the maps $[d] : W^{\otimes k} \to W^{\otimes k}$ are not necessarily elements in the centralizer algebra $QP_k(n)$. We will, however, use them to construct the elements of $QP_k(n)$.

2.2. Projections

Since $V = W \oplus T$, we have

$$V^{\otimes k} \cong W^{\otimes k} \oplus \left(\bigoplus_{i=1}^{k} \binom{k}{i} \left(W^{\otimes k-i} \otimes T^{\otimes k}\right) \right).$$

In this section, we construct the maps in $QP_k(n)$ by applying maps in $P_k(n)$ on $W^{\otimes k}$ and then projecting the result (which have their images in the whole of $V^{\otimes k}$) back onto $W^{\otimes k}$.

To this end, first define the projection $\varpi : V \to T$ to be a projection of V onto T. Hence,

$$\varpi(v_i) = \frac{1}{n}(v_1 + \cdots + v_n) \quad \text{for } i = 1, \ldots, n.$$

The matrix representation of $n\varpi$ is the matrix of all 1's, i.e. $n\varphi = \sum_{1 \leqslant i,j, \leqslant n} E_{ij}$, were E_{ij} is the i,j-matrix unit. Now define $\varpi_\ell = \mathbf{1}^{\otimes \ell - 1} \otimes \varpi \otimes \mathbf{1}^{\otimes k - \ell}$. The endomorphism ring $\operatorname{End}(V^{\otimes k})$ has basis $E^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$, and as a matrix ϖ_ℓ is given by

$$n\varpi_\ell = \sum_{\substack{1 \leqslant i,j \leqslant n \\ 1 \leqslant a_m \leqslant n \text{ for } m \neq \ell}} E^{a_1,\ldots,a_{\ell-1},i,a_{\ell+1},\ldots,a_k}_{a_1,\ldots,a_{\ell-1},j,a_{\ell+1},\ldots,a_k}.$$

So as an operator on $V^{\otimes k}$, $n\varpi_\ell = p_\ell$, the element of $P_k(n)$ with isolated vertices at ℓ and ℓ', and all other blocks of the form $\{i, i'\}$ for $i \neq \ell$ (see diagram at the beginning of Section 1.2). We will show that a basis of $QP_k(n)$ is given by diagrams that do not have isolated vertices. As we will see in Lemma 2.1, this is a result of the fact that isolated vertices occur when $d = p_\ell d'$ or $d = d' p_\ell$ for some diagram d' and some $1 \leqslant \ell \leqslant k$.

Now we can define the projection $\pi : V \to W$ by

$$\pi = \mathrm{id} - \varpi = \mathrm{id} - \frac{1}{n}p, \quad \text{so that } \pi^{\otimes k} = \pi \otimes \pi \otimes \cdots \otimes \pi$$

projects $V^{\otimes k}$ onto $W^{\otimes k}$. To simplify computations, we transform bases of V from $\{v_1, \ldots, v_n\}$ to $\{v, w_2, \ldots, w_n\}$, where

$$v = \sum_{i=1}^{n} v_i \quad \text{and} \quad w_j = v_j - v_1.$$

So

$$\pi(v) = 0, \qquad \pi(w_i) = w_i, \quad \text{and} \quad \pi(v_i) = w_i - \frac{1}{n}w, \quad \text{where } w_1 = 0 \text{ and } w = \sum_{i=2}^{n} w_i.$$

Lemma 2.1. *For all diagrams $d \in P_k(n)$, the projection $\pi^{\otimes k} \circ d$ is an element of $QP_k(n)$. Furthermore, if d is a diagram with one or more isolated vertices, then $\pi^{\otimes k} \circ d = 0$.*

Proof. Since $\pi = \mathrm{id} - \frac{1}{n} p$ is an operator on V (with p as in (4)), π commutes with the action of S_n. So $\pi^{\otimes k} \circ d \in \mathrm{End}_{S_n}(V^{\otimes k}) = P_k(n)$. Now considering $W^{\otimes k} \subset V^{\otimes k}$, we have

$$\pi^{\otimes k} \circ d : W^{\otimes k} \xrightarrow{d} V^{\otimes k} \xrightarrow{\pi^{\otimes k}} W^{\otimes k}.$$

So $\pi^{\otimes k} \circ d$ is also an element of $\mathrm{End}_{S_n}(W^{\otimes k}) = QP_k(n)$.

Now suppose that d is a diagram with an isolated vertex. If the isolated vertex occurs in the bottom row of the diagram on the ith vertex, then $d = d' p_i$ for some diagram d'. But $p = n\varpi$ as operators on V, so p acts as 0 on W. So $(\pi^{\otimes k} \circ d) \cdot (w_{i_1} \otimes \cdots \otimes w_{i_k}) = 0$.

If instead, the isolated vertex occurs in the top of the diagram on the ith vertex, then $d = p_i d'$. So again since $p = n\varpi$ as operators on V, we have

$$\pi^{\otimes k}(p_i d') = n\big(\pi^{\otimes i-1} \otimes (\pi \circ \varpi) \otimes \pi^{\otimes (k-i)}\big) d' = 0, \quad \text{since } \pi \circ \varpi = 0. \qquad \square$$

2.3. Basis for the quasi-partition algebra

In Lemma 2.1 we found a spanning set for $QP_k(n)$. We now show that the set of projections of the diagrams without singleton vertices will form a basis. That is,

$$QP_k(n) = \mathbb{C}\text{-span}\{\bar{d} \mid d \in \mathcal{D}\}, \quad \text{where } \mathcal{D} = \{\text{diagrams } d \text{ without isolated vertices}\},$$

and $\bar{d} = \pi^{\otimes k} \circ d$. In order to prove this, we will show that if d is a diagram without isolated vertices, then one can write $\pi^{\otimes k} \circ d$ as a linear combination of the operator $[d]$ (as defined in (6)) and operators $[d']$ with isolated vertices.

Recall from (6) that for any diagram d, the operator $[d] = f \circ d \circ f^{-1} \in \mathrm{End}(W^{\otimes k})$ acts on $W^{\otimes k}$ the same way that the diagram d acts on $(\mathbb{C}^{n-1})^{\otimes k}$. Notice that $[d]$ is an element of $\mathrm{End}_{S_{\{2,\ldots,k\}}}(W^{\otimes k})$. However, $[d]$ is in general not an element of $\mathrm{End}_{S_n}(W^{\otimes k})$.

Considering d as a set partition,

$$d \cdot (v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) = \begin{cases} v_{i_1} \otimes \cdots \otimes v_{i_k} & \text{if for each } B \in d, \\ & i_\ell = i_m \text{ for all } \ell, m \in B \subseteq [k] \cup [k'], \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Let B be a block in d, and define

$$B^t := B \cap [k] \quad \text{and} \quad B^b := B \cap \big[k'\big],$$

so that B^t (resp. B^b) is the set of vertices in block B which are on the top (resp. bottom) of the diagram. Then let

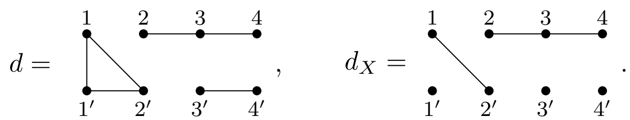
$$d^t = \big\{B \in d \bigm| B^b = \emptyset\big\} \quad \text{and} \quad d^b = \big\{B \in d \bigm| B^t = \emptyset\big\}$$

be the sets of blocks containing vertices only on the top or bottom of the diagram, respectively. See, for example, Example 2.4.

If X is a subset of $[k] \cup [k']$, define the *isolation of d (at X)* as

$$d_X, \quad \text{the diagram constructed from } d \text{ by isolating all vertices in } X.$$

For example, if $k = 4$, $X = \{1', 4'\}$ and $d = \{\{1, 1', 2'\}, \{2, 3, 4\}, \{3', 4'\}\}$, then $d_X = \{\{1, 2'\}, \{1'\}, \{2, 3, 4\}, \{3'\}, \{4'\}\}$. In pictures,



Notice that two different sets X_1 and X_2 can lead to the same isolation of d. In the above example, the set $X_2 = \{1', 3', 4'\}$ and $X_1 = \{1', 4'\}$ give $d_{X_2} = d_{X_1}$.

Lemma 2.2. *The action of $\bar{d} := \pi^{\otimes k} \circ d$ on $W^{\otimes k}$ is equal to the action of a linear combination of $[d]$ and diagrams $[d']$, where d' is an isolation of d. That is,*

$$\bar{d} = [d] + \sum_U c_U [d_U], \quad c_U \in \mathbb{C}(n), \tag{8}$$

where the sum is over non-empty sets U of vertices satisfying

$$\textit{if} \quad U \cap B^b \neq \emptyset \quad \textit{for any block } B \in d, \ \textit{then } B \subseteq U.$$

Proof. The first step is to understand how d acts on an arbitrary element $\mathrm{w} \in W^{\otimes k}$,

$$\mathrm{w} = w_{i_{1'}} \otimes \cdots \otimes w_{i_{k'}} = (v_{i_{1'}} - v_1) \otimes \cdots \otimes (v_{i_{k'}} - v_1).$$

Note that w is the sum of terms v with factors v_{i_ℓ} or $-v_1$. The collection of blocks B in $d \setminus d^t$ (blocks containing bottom vertices) checks for equality of factors in each v corresponding to vertices in those blocks—$d \cdot \mathrm{v}$ is zero if factors corresponding to vertices in the same block are not equal. Since $i_\ell \neq 1$, there are only two types of terms when d does not act by zero: (1) terms v where for each block B, for all $\ell \in B^b$, the i_ℓ's take on the same value and these factors are all equal v_{i_ℓ} and (2) terms where all factors corresponding to vertices in B^b are all $-v_1$.

We now carry out the computation $(\pi^{\otimes k} \circ d) \cdot \mathrm{w}$. Let $S \subseteq d \setminus d^t$, and for $\mathrm{i} = (i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'})$ let $\delta_{S,\mathrm{i}}$ be the characteristic function

$$\delta_{S,\mathrm{i}} = \begin{cases} 1 & \text{if for each } B \notin S, i_\ell = i_m \text{ for all } \ell, m \in B, \text{ and} \\ & \text{for each } B \in S, i_\ell = 1 \text{ for all } \ell \in B^t, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $w_1 = 0$,

$$\begin{aligned}\left(\pi^{\otimes k} \circ d\right) \cdot \mathrm{w} &= \pi^{\otimes k} \cdot \sum_{S \subseteq d \setminus d^t} (-1)^{\sum_{B \in S} |B^b|} \sum_{i_1, \ldots, i_k \in [n]} \delta_{S,\mathrm{i}} (v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &= \sum_{S \subseteq d \setminus d^t} (-1)^{\sum_{B \in S} |B^b|} \sum_{i_1, \ldots, i_k \in [n]} \delta_{S,\mathrm{i}} \left(\left(w_{i_1} - \frac{1}{n} w \right) \otimes \cdots \otimes \left(w_{i_k} - \frac{1}{n} w \right) \right).\end{aligned} \tag{9}$$

If $d^t = \emptyset$, then this implies that as operators on w,

$$\begin{aligned}\pi^{\otimes k} \circ d &= \sum_{\substack{X = \bigcup_{B \in S} B \\ S \subseteq d \setminus d^t}} (-1)^{|X \cap [k']|} \sum_{Y \subseteq [k] \setminus X} \left(-\frac{1}{n} \right)^{|Y \cup (X \cap [k])|} [d_{X \cup Y}] \\ &= \sum_{\substack{X, Y \\ U = X \cup Y}} (-1)^{|U|} \frac{1}{n^{|U \cap [k]|}} [d_U],\end{aligned}$$

where the last sum is a double sum over sets X and Y of vertices satisfying

$$X = \bigcup_{B \in S} B \quad \text{with } S \subseteq d \setminus d^t, \quad \text{and} \quad Y \subseteq [k] \setminus X. \tag{10}$$

However, if $B = \{j_1, \ldots, j_r\} \subseteq [k]$ is a top block, then isolating the factors in positions $j_1, \ldots, j_r$ in $(\pi^{\otimes k} \circ d) \cdot \mathrm{w}$ yields

$$\begin{aligned}\pi^{\otimes k} \cdot \sum_{\ell=1}^{n} v_\ell^{\otimes r} &= \left(\frac{-1}{n} \right)^r w^{\otimes r} + \sum_{\ell=2}^{n} \left(w_\ell - \frac{1}{n} w \right)^{\otimes r} \\ &= n \left(\frac{-1}{n} \right)^r w^{\otimes r} \\ &\quad + \sum_{a=0}^{r-1} \sum_{\sigma \in S_r / (S_a \times S_{r-a})} \sigma \cdot \left(\left(\frac{-1}{n} \right)^a w^{\otimes a} \otimes \left(\sum_{\ell=2}^{n} w_\ell^{\otimes (r-a)} \right) \right),\end{aligned}$$

where $\sigma \in S_r/(S_a \times S_{r-a})$ acts by permuting the r factors, stabilizing the relative positions of the factors of $w^{\otimes a}$ and the factors of $w_\ell^{\otimes (r-a)}$. So the term $w^{\otimes r}$ appears with an extra factor of n.

Thus, in general

$$\pi^{\otimes k} \circ d = \sum_{\substack{X, Y \\ U = X \cup Y}} (-1)^{|U|} \frac{1}{n^{|U \cap [k]| - N(Y)}} [d_U], \tag{11}$$

where

$$N(Y) = \left| \left\{ B \in d^t \;\middle|\; B \subseteq Y \right\} \right|. \qquad \square$$

As noted before Lemma 2.2, a diagram d_U may arise non-uniquely as a function of U, and so Lemma 2.2 does not necessarily combine like-terms in $\bar{d}$. Our next lemma addresses this issue. To that end, define a *viable isolation* $\hat{d}$ of a diagram d as an isolation that may appear in the expansion of $\bar{d}$ with non-zero coefficient, i.e. if any vertex of B^b is isolated in $\hat{d}$, then all vertices of B are isolated in $\hat{d}$.

Lemma 2.3. *Suppose $\hat{d}$ is a viable isolation of d with maximal set of isolated vertices U, and let c be the coefficient of $[\hat{d}]$ after collecting like-terms in (8). Then c falls into one of the following cases*:

1. *Non-unique terms, i.e. c is the sum of multiple terms*:
   (a) *If $B \in d$ has exactly one vertex in $[k']$, and the vertices of B are isolated in $\hat{d}$, then $c = 0$.*
   (b) *Assume there is no block as in* (a) *completely isolated in $\hat{d}$, and that d^t is non-empty.*
   If $\mathcal{B} = \{B_1, \dots, B_\ell\} \subseteq d^t$, and all blocks in $\mathcal{B}$ but no other blocks of d^t are completely isolated in $\hat{d}$, then
   $$c = (-1)^{|U|}\left(\frac{1}{n^{|U\cap[k]|-\ell}}\right)\left(\sum_{\mathcal{B}'\subseteq\{B_1,\dots,B_\ell\}}\left((-1)^{|\mathcal{B}'|}\prod_{B\in\mathcal{B}'}|B|\right)\right)$$
   where $\prod_{B\in\mathcal{B}'}|B| = 1$ when $\mathcal{B}' = \emptyset$.
2. *Unique terms*: *If $\hat{d}$ is not one of the cases in* 1*, then $\hat{d}$ appears uniquely with coefficient $c = (-1)^{|U|}\frac{1}{n^{|U\cap[k]|}}$ where U is the set of isolated vertices of $\hat{d}$. In particular, if the isolated vertices of $\hat{d}$ are all in $[k']$, then $c = (-1)^{\#\{\text{isolated vertices}\}}$.*

If $U \cap [k] \neq \emptyset$, then $\lim_{n\to\infty} c = 0$.

Proof. Use the same notation as in Lemma 2.2, and continue from its proof.

In case 1 note that a diagram $[d_U]$ can appear multiple times, if selecting different sets X and Y breaks up an entire block in different ways. This can happen in two cases.

(a) *If some $B \in d \setminus d^t$ has the property that B^b has a single element, then including B in X or including B^t in Y result in the same d_U.*
In this case, pair each (X, Y) such that $B \subset X$, and (X', Y') given by
$$X' = X \setminus \{B\}, \quad \text{and} \quad Y' = Y \cup B^t$$
yield $[d_{X\cup Y}] = [d_{X'\cup Y'}]$. These terms occur with equal but opposite coefficients, since
$$\left|(X\cup Y)\cap[k]\right| = \left|\left(X'\cup Y'\right)\cap[k]\right| \quad \text{but } \left|X'\cup Y'\right| = |X\cup Y| - 1.$$
Thus the $[d_{X\cup Y}]$ and $[d_{X'\cup Y'}]$ terms cancel.

(b) *If $B \in d^t$, then the diagrams with $B \subseteq Y$ are exactly equal to those with one vertex of B removed from Y ($Y' = Y \setminus \{j\}$ for some $j \in B$).*
Fix a collection of blocks $B_1, \ldots, B_\ell$ in d^t, and let U be some set of the form $U = X \cup Y$ as in (10), such that

$$\bigcup_{1\leqslant i\leqslant \ell} B_i \subseteq U, \quad \text{but } d_U \text{ has no other blocks in } d^t \text{ completely broken up.}$$

Then $[d_U]$ will appear with coefficient

$$c = (-1)^{|U|}\left(\frac{1}{n^{|U\cap[k]|-\ell}}\right) \quad \text{for the set } U,$$
$$\left((-1)^{|\mathcal{B}'|}\prod_{B\in\mathcal{B}'}|B|\right)c \quad \text{for each non-empty } \mathcal{B}' \subseteq \{B_1, \ldots, B_\ell\},$$

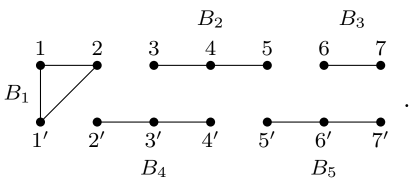
where the second value counts the cases $U = X \cup Y'$, where Y' removes one element from each $B \in \mathcal{B}$ from Y: for each B, there are $|B|$ ways to accomplish this, and each resulting Y' has $N(Y') = N(Y) - |\mathcal{B}'|$, $|U'| = |U| - |\mathcal{B}'|$, and $|U' \cap [k]| = |U \cap [k]| - |\mathcal{B}'|$. So, collecting like-terms, the coefficient on $[d_U]$ is

$$c_U = (-1)^{|U|}\left(\frac{1}{n^{|U\cap[k]|-\ell}}\right)\left(\sum_{\mathcal{B}'\subseteq\{B_1,\ldots,B_\ell\}}\left((-1)^{|\mathcal{B}'|}\prod_{B\in\mathcal{B}'}|B|\right)\right),$$

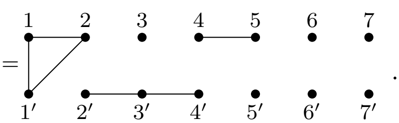
where $\prod_{B\in\mathcal{B}'}|B| = 1$ when $\mathcal{B}' = \emptyset$.

If $U \subseteq [k']$, the term $[d_U]$ appears exactly once; the isolated vertices are unions of blocks in d^b, and so the coefficient is $(-1)^{|U|}$. Otherwise, $[d_U]$ appears exactly once, with coefficient $(-1)^{|U|}\frac{1}{n^{|U\cap[k]|}}$. One can check that in each of these cases, if $U \cap [k] \neq \emptyset$, then $\lim_{n\to\infty} c_U = 0$. $\square$

Example 2.4. Let d be the diagram



So, for example,

$d_{\{3\}\cup B_3\cup B_5} =$ 

Since

$$B_1^t = \{1,2\}, \qquad B_1^b = \{1'\}, \qquad B_2^t = B_2, \qquad B_2^b = \emptyset, \qquad B_3^t = B_3, \qquad B_3^b = \emptyset,$$
$$B_4^t = \emptyset, \qquad B_4^b = B_4, \qquad B_5^t = \emptyset, \quad \text{and} \quad B_5^b = B_5,$$

we have $d^t = \{B_2, B_3\}$ and $d^b = \{B_4, B_5\}$. Then the sum in (9) is over the sets

$$S = \emptyset, \{B_1\}, \{B_4\}, \{B_5\}, \{B_1, B_4\}, \{B_1, B_5\}, \{B_4, B_5\}, \text{ or } \{B_1, B_4, B_5\}.$$

Notice that $d_{\{6\}}$, $d_{\{7\}}$, and d_{B_3} are all the same diagram. In Lemma 2.3, to avoid these redundancies, set $U = \{6, 7\}$ and consolidate $d_u = d_{\{6\}} = d_{\{7\}} = d_{B_3}$ into a single term.

An example of case 1(a) is any isolation of B_1. So the (simplified) coefficients on $[d_{B_1}]$, $[d_{B_1 \cup B_2}]$, $[d_{B_1 \cup \{3\} \cup B_3 \cup B_5}]$, or any other term isolating B_1, will all be 0. For 2(b), consider $[d_{B_2}]$. In this case, $\mathcal{B} = \{B_2\}$ and the coefficient is

$$(-1)^3 \left(\frac{1}{n^{3-1}}\right)\left((-1)^0 * 1 + (-1) * 3\right) = \frac{2}{n^2}.$$

This case also covers, for example, $[d_{\{1,5\} \cup B_3 \cup B_4}]$. Even though one vertex of B_2 is isolated, B_2 is not completely isolated, and so $\mathcal{B} = \{B_3\}$ and the coefficient is

$$(-1)^7 \left(\frac{1}{n^{4-1}}\right)\left((-1)^0 * 1 + (-1)^2 * 2\right) = \frac{3}{n^3}.$$

The terms where $U \subseteq [k']$ are when $S = \emptyset, \{B_4\}, \{B_5\}, \{B_4, B_5\}$, and expand to

$$(-1)^0[d] + (-1)^3[d_{B_4}] + (-1)^3[d_{B_5}] + (-1)^6[d_{B_4 \cup B_5}].$$

For another example, a complete expansion of $\bar{e}_1$ is given in (14).

Theorem 2.5. *A basis for $QP_k(n)$ is given by*

$$\{\bar{d} \mid d \in \mathcal{D}\} \quad \textit{where } \mathcal{D} = \{\textit{diagrams } d \textit{ without isolated vertices}\}$$

as before.

Proof. We have already established $\{\bar{d} \mid d \in \mathcal{D}\}$ as a spanning set. It remains to show that it is also linearly independent. By Lemma 2.2, for each diagram d without singletons, there is exactly one element $\bar{d}$ which has $[d]$ with non-zero coefficient the expansion of its projection (namely, d itself). Since the diagrams form a basis of $P_k(n-1)$ when $n-1 > 2k$, and the elements $[d] \in \mathrm{End}(W^{\otimes k})$ generate an isomorphic subalgebra when $n-1 > 2k$, this implies that $\{\bar{d} \mid d \in \mathcal{D}\}$ is linearly independent. $\square$

We obtain as easy consequences of this theorem both dimension formulas and a result concerning diagram multiplication.

Corollary 2.6. *If $n > 2k+1$, then the dimension of $QP_k(n)$ is the number of set partitions of $2k$ without blocks of size one. This number is given by*

$$\sum_{j=1}^{2k}(-1)^{j-1}B(2k-j)+1,$$

where $B(r)$ is the Bell number and $B(0)=1$.

This can be easily checked by noticing that the number of set partitions without singletons, $a(r)$, satisfies the recurrence $a(r+1)+a(r)=B(r)$ subject to $a(1)=0$ and $a(2)=1$, and then that the formula satisfies this recurrence. These values $a(r)$ can alternately be described with the exponential generating function $\exp(\exp(x)-1-x)$, which follows from the above recurrence relation for $a(r)$ and the exponential generating function for the Bell numbers, $\exp(\exp(x)-1)$. For reference, the values of $a(2k)$ expand to $1, 1, 4, 41, 715, 17722, \ldots$ (A000296)—we are only interested in the even terms of the sequence $a(r)$ since our diagrams have $r = 2k$ vertices.

Remark. We notice that the formula in Corollary 2.6 is the inverse binomial transform of the Bell numbers. Hence, we can write the Bell numbers as a binomial transform of the dimensions of the quasi-partition algebra, i.e., $B(n)=\sum_{i=1}^{n}\binom{n}{i}a(i)$. This is consistent with the interpretation that the partition algebra is the rook version of $QP_k(n)$.

We give the following Corollary as a statement of which terms can appear with non-zero coefficient in a product, not as a calculation of the exact values of those coefficients. To compute the exact values, one begins by using Lemma 2.3, together with the fact that diagrams $[d]$ multiply by concatenation; then Lemma 2.2 allows for a straightforward transition back into diagrams $\bar{d}$, as we explain in the following proof. We do this in certain cases in Section 3.

Corollary 2.7. *Given any two diagrams d_1 and d_2 without singleton vertices, we have*

$$\bar{d}_1\bar{d}_2=\sum_{\substack{d\in\mathcal{D}\\ d\leqslant d_1d_2}}c^{d}_{d_1,d_2}\bar{d},\qquad c^{d}_{d_1,d_2}\in\mathbb{C}(n),$$

where $d \leqslant d'$ if every block of d' is the union of blocks of d, i.e. d is a refinement *of d'.*

Proof. If $d_1, d_2 \in \mathcal{D}$, and d_1', d_2' are isolations of d_1, d_2, then $d_1'd_2' \leqslant d_1d_2$. By Lemma 2.2

$$\bar{d}_1=[d_1]+\sum_{U}a_U\big[(d_1)_U\big]\quad\text{and}\quad\bar{d}_2=[d_2]+\sum_{V}b_V\big[(d_2)_V\big]$$

where sets U and V determine viable isolations of d_1 and d_2, and coefficients a_U, b_V are determined in Lemma 2.3. So since f in (5) is a homomorphism,

$$\bar{d_1}\bar{d_2} = \sum_{d \leqslant d_1 d_2} c^d_{d_1,d_2}[d]. \tag{12}$$

Since $QP_k(n)$ is closed under composition, we can also expand $\bar{d_1}\bar{d_2}$ in the basis $\{\bar{d} \mid d \in \mathcal{D}\}$. Again by Lemma 2.2, for each $d \in \mathcal{D}$, $[d]$ appears with non-zero coefficient in the bracket expansions of the elements of this basis: it appears with coefficient 1 in $\bar{d}$ and with coefficient 0 in $\bar{d'}$ for all $d \neq d' \in \mathcal{D}$. Therefore,

$$\bar{d_1}\bar{d_2} = \sum_{\substack{d \in \mathcal{D} \\ d \leqslant d_1 d_2}} c^d_{d_1,d_2}\bar{d}$$

(where $c^d_{d_1,d_2}$ takes the same value as in (12)). $\square$

2.4. The generic quasi-partition algebra

By Lemma 2.3 and the multiplication rules for $P_k(n-1)$, the coefficients $c^d_{d_1,d_2} = c^d_{d_1,d_2}(n)$ are well-defined rational functions of n (with poles only at $n = 0$). Now fix a non-zero indeterminant x. Using the multiplication determined in Corollary (2.7) (by way of Lemmas 2.2 and 2.3), define the *general quasi-partition algebra* $QP_k(x)$ formally as

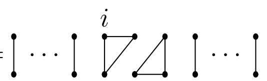
$$QP_k(x) = \mathbb{C}(x)\text{-span } \mathcal{D} \quad \text{with multiplication } d_1 d_2 = \sum_{\substack{d \in \mathcal{D} \\ d \leqslant d_1 d_2}} c^d_{d_1,d_2}(x)d.$$

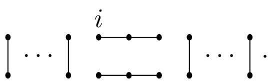
Note that when you specialize x to an integer greater than $2k+1$, $QP_k(x)$ is isomorphic both the centralizer of S_x in $\mathrm{End}(W^{\otimes k})$ and to a subalgebra of $P_k(x-1)$.

3. Generators and relations

In this section we give a set of generators for $QP_k(n)$. We first give a set of generators for the partition diagrams that do not have singleton vertices. We then proceed to show that the image of $\pi^{\otimes k}$ of these generators form a generating set for $QP_k(n)$.

It was shown in [12, Lemma 3.1] and [11, Lemma 5.2] that the generators s_i, e_i, and b_i, for $i = 1, \ldots, k-1$, generate those diagrams where all blocks contain an even number of vertices. The quasi-partition algebra additionally contains diagrams which has blocks with an odd number of vertices (and therefore an even number of odd blocks). To generate these additional diagrams, we will also need the generators

$t_i = b_i p_{i+1} b_{i+1} =$  and

$h_i = b_i e_{i+1} p_i b_i =$  .

In fact, we only need h_1 when $k = 3$, since we have that $h_1 = t_2e_1(s_2s_3t_2s_2s_3)$ for $k \geqslant 4$.

Theorem 3.1. *In $P_k(n)$, all diagrams in $\mathcal{D}$ (i.e. all diagrams without isolated vertices) are generated by*

$$\mathcal{G} = \{s_1, \ldots, s_{k-1}, e_1, b_1, t_1, h_1\}. \tag{13}$$

Proof. Note that by conjugating b_1 or e_1 by words in $\{s_i \mid i = 1, \ldots, k-1\}$, one can generate e_i and b_i for $i = 1, \ldots, k-1$, and therefore, by [12, Lemma 3.1] and [11, Lemma 5.2], one can generate all diagrams with even-sized blocks. For any diagram d with an odd-sized block I (and therefore at least one more, J), we will build d from a diagram with two fewer odd blocks with the distinguished generators, showing d is generated by $\mathcal{G}$ by induction on the number of odd-sized blocks. The generators $t_1, s_1, \ldots, s_{k-1}$ generate all set partitions of the form

$$t_{i_1,i_2,i_3} = \{i_1, i_2, i_1'\}, \{i_3, i_2', i_3'\}, \{1, 1'\}, \{2, 2'\}, \ldots, \{k, k'\}$$

and the generators $h_1, s_1, \ldots, s_{k-1}$ generate all set partitions of the form

$$h_{i_1,i_2,i_3} = \{i_1, i_2, i_3\}, \{i_1', i_2', i_3'\}, \{1, 1'\}, \{2, 2'\}, \ldots, \{k, k'\}.$$

Let d be a diagram in $\mathcal{D}$ with at least one pair of odd-sized blocks. We have two cases to consider.

Case 1: *Let I and J be two blocks of odd size in d and assume $I \subseteq [k]$ and $J \subseteq [k']$.*

Since d has no singletons, each set has at least 3 elements. So let $i_1, i_2, i_3 \in I$ and $j_1', j_2', j_3' \in J$. Let d' be the diagram with the same blocks as d except that I and J in d have been replaced by the following sets in d':

$$I \setminus \{i_1, i_2, i_3\}, \qquad J \setminus \{j_1', j_2', j_3'\}, \qquad \{i_1, j_1'\}, \qquad \{i_2, j_2'\}, \qquad \{i_3, j_3'\}.$$

Then d' has two fewer sets of odd size than d, and hence inductively is the product of elements in $\mathcal{G}$. The diagram

$$h_{i_1,i_2,i_3}d'$$

is the diagram obtained from d by replacing I and J by the sets

$$I \setminus \{i_1, i_2, i_3\}, \qquad J \setminus \{j_1', j_2', j_3'\}, \qquad \{i_1, i_2, i_3\}, \qquad \{j_1', j_2', j_3'\}.$$

If I or J has more than 3 elements (i.e. $I \setminus \{i_1, i_2, i_3\}$ or $J \setminus \{j_1', j_2', j_3'\}$ are non-empty), then for example

$$I \setminus \{i_1, i_2, i_3\} \quad \text{and} \quad \{i_1, i_2, i_3\}$$

can be joined by right-multiplication of the even-block diagram

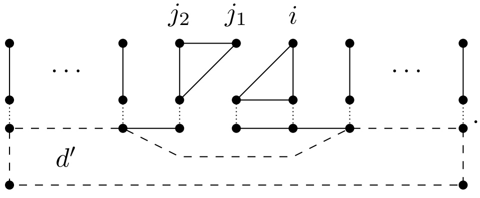
$$\{i_1, i_4, i_1', i_4'\}, \{1, 1'\}, \ldots, \{k, k'\},$$

where $i_4 \in I \setminus \{i_1, i_2, i_3\}$. Hence, d is the product of elements in $\mathcal{G}$.

Case 2: *Both* I *and* J *have at least one element in* $[k]$, *and one of them has at least two elements in* $[k]$. (Otherwise, the same statement is true for $[k']$, and a similar construction can be used.)

Assume that $J \cap [k]$ has at least two elements j_1 and j_2. Let d' be a new diagram with the sets $J \setminus j_1$ and $I \cup \{j_1\}$ in place of I and J in d. Then d' has two fewer sets of odd size than d, and if $i \in I \cap [k]$, then

$$d = t_{j_2, j_1, i} d',$$

$d =$  . □

We have just shown that one can write $d = w$ as diagrams with $w = w_1 w_2 \cdots w_\ell$ and $w_i \in \mathcal{G}$. Further, the algorithm presented inductively on the number of pairs of odd blocks in d provides a process for finding a w satisfying further that each consecutive subword has no isolated vertices.

Corollary 2.7 tells us that if $d_1, d_2 \in \mathcal{D}$ and the diagram $d_1 d_2$ has an isolated vertex, then $\bar{d}_1 \bar{d}_2 = 0$. However, if the diagram $d_1 d_2$ does not have an isolated vertex, Corollary 2.7 does not tell us which terms appear with non-zero coefficient. In particular, it is not immediately obvious that $c_{d_1, d_2}^{d_1 d_2}$ is non-zero. A case-by-case calculation (done in Appendix A) gives the following lemma.

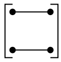
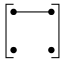
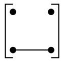
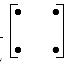
Lemma 3.2. *If* $d_1 \in \mathcal{G}$ *is one of the generators of* $\mathcal{D}$ *and* $d_2 \in \mathcal{D}$, *then either*

1. $d_1 d_2 \in \mathcal{D}$ *and then coefficient* $c_{d_1, d_2}^{d_1 d_2}$ *of* $\overline{d_1 d_2}$ *in* $\bar{d}_1 \bar{d}_2$ *is non-zero, or*
2. $d_1 d_2 \notin \mathcal{D}$, *and then* $\bar{d}_1 \bar{d}_2 = 0$.

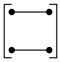
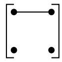
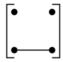
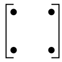
Corollary 3.3. *As before, let* $\bar{d} = \pi^{\otimes k} \circ d$. *Then* $QP_k(n)$ *is generated by* $\{\bar{d} \mid d \in \mathcal{G}\}$, *where* $\mathcal{G}$ *is as in* (13).

Proof. Let $d \in \mathcal{D}$. By Theorem 3.1, we can generate d as a diagram from $\mathcal{G}$; let $w = w_1 w_2 \cdots w_\ell$ be a word in the elements of $\mathcal{G}$ so that $d = w$ as diagrams, satisfying for all $i = 1, \ldots, \ell - 1$, the diagram $w_i w_{i+1} \cdots w_\ell \in \mathcal{D}$ (as explained immediately after the proof of Theorem 3.1). Using Lemma 3.2 inductively, $\bar{w}$ appears with non-zero coefficient in $\bar{w}_1 \cdots \bar{w}_\ell$. Using the triangularity of Corollary 2.7, we can then generate all of $\{\bar{d} \mid d \in \mathcal{D}\}$. $\square$

We can use Lemmas 2.2 and 2.3 to calculate the expansion of these elements of $QP_k(n)$ in terms of maps $[d]$, and thus use relations in $P_k(n-1)$ to determine relations in $QP_k(n)$. For example, Lemma 2.3 says that

$\bar{e}_1 =$  $+$  $- \frac{1}{n}$  $- \frac{1}{n}$  . (14)

So since bracketed terms multiply by diagram concatenation,

$\bar{e}_1^2 = (n-1)$  $+ (n-1)$  $- \frac{n-1}{n}$  $- \frac{n-1}{n}$  $= (n-1)\bar{e}_1.$

Similarly, one can easily check that the generators satisfy the following relations in $QP_k(n)$:

$$\bar{s}_i^2 = 1, \qquad \bar{s}_i\bar{s}_{i+1}\bar{s}_i = \bar{s}_{i+1}\bar{s}_i\bar{s}_{i+1}, \qquad \bar{s}_i\bar{s}_j = \bar{s}_j\bar{s}_i \quad \text{if } |i-j|>1,$$
$$\bar{e}_i^2 = (n-1)\bar{e}_i, \quad \bar{e}_i\bar{e}_{i\pm1}\bar{e}_i = \bar{e}_i, \qquad \bar{b}_i^2 = \frac{n-2}{n}\bar{b}_i + \frac{1}{n^2}\bar{e}_i$$
$$\bar{s}_i\bar{b}_i = \bar{b}_i\bar{s}_i = \bar{b}_i \quad \text{if } 1 \leqslant i \leqslant n-1, \qquad \bar{s}_i\bar{t}_i = \bar{t}_i\bar{s}_{i+1} = \bar{t}_i \quad \text{if } 1 \leqslant i \leqslant n-2, \quad \text{and}$$
$$\bar{e}_i\bar{t}_i = \bar{t}_i\bar{e}_{i+1} = 0.$$

Comparing these relations to those of the partition algebra $P_k(n-1)$,

$$s_i^2 = 1, \qquad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \qquad s_is_j = s_js_i \quad \text{if } |i-j|>1,$$
$$e_i^2 = (n-1)e_i, \qquad e_ie_{i\pm1}e_i = e_i, \qquad b_i^2 = b_i,$$
$$s_ib_i = b_is_i = b_i \quad \text{if } 1 \leqslant i \leqslant n-1, \qquad s_it_i = t_is_{i+1} = t_i \quad \text{if } 1 \leqslant i \leqslant n-2, \quad \text{and}$$
$$e_it_i = t_ie_{i+1} = 0, \tag{15}$$

we observe that they are very similar but with additional lower terms in some cases. For example, b_i^2 has and additional e_i term; but $e_i \leqslant b_i$, and in the limit $n \to \infty$, we have $\bar{b}_i^2 \to \bar{b}_i$.

4. Representation theory of the quasi-partition algebra

In this section we describe the representation theory of $QP_k(n)$. Both the representation theory of $QP_k(n)$ and of S_n can be described using integer partitions, which we review here. Fix $n \in \mathbb{Z}_{\geqslant 0}$. An (integer) *partition* λ of n, denoted $\lambda \vdash n$, is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$. If $n = 0$, the unique partition is the empty partition, denoted by $\emptyset$. Given two partitions λ and μ we say that $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all i.

The irreducible representations S^λ of S_n are indexed by partitions λ of n. As usual we identify a partition with its Young diagram, depicted as $|\alpha|$ boxes up and left justified, where the ith row has α_i boxes. In our setting, the combinatorics of the representation theory of $QP_k(n)$ can be simplified by replacing partitions $\lambda \vdash n$ with partitions $(\lambda_2, \ldots, \lambda_\ell)$ of $\lambda_2 + \cdots + \lambda_\ell$. Thus the partitions of n are in bijection with the partitions α of $m < n$ for which $\alpha_1 \leqslant n/2$. For any such partition, let

$$\bar{\alpha} = (n - |\alpha|, \alpha_1, \ldots, \alpha_\ell), \quad \text{for example,} \quad (n-7,3,3,1) = \overline{(3,3,1)} = \overline{\ydiagram{3,3,1}}\,.$$

We will need the following theorem.

Theorem 4.1 *(The centralizer theorem). Let A be a finite dimensional algebra over $\mathbb{C}$. Let M be a semisimple A-module and let $C = \mathrm{End}_A(M)$. Suppose that $M \cong \bigoplus_\lambda (A^\lambda)^{\oplus m_\lambda}$, where A^λ are irreducible A-modules and $m_\lambda \in \mathbb{Z}_{\geqslant 0}$ are multiplicities of A^λ in the decomposition of M. Then*

(a) *$C \cong \bigoplus_\lambda M_{m_\lambda}(\mathbb{C})$, and*
(b) *as an (A, C)-bimodule, $M \cong \bigoplus_\lambda A^\lambda \otimes C^\lambda$, where C^λ are simple C-modules.*

By the centralizer theorem, in order to understand the representation theory of $QP_k(n)$, it suffices to know how to decompose the S_n module $W^{\otimes k}$. The tensor product (or *Kronecker product*) of two irreducible S_n-representations is usually not itself irreducible, and a general rule for decomposing this tensor product is not known. However, there are many results concerning stability of the product. For example, in [3], it is shown that if $n \geqslant |\alpha| + |\beta| + \alpha_1 + \beta_1$ then the Kronecker product

$$S^{\bar{\alpha}} \otimes S^{\bar{\beta}} = \bigoplus_\gamma g_{\alpha\beta}^\gamma S^{\bar{\gamma}}$$

is stable, meaning that for large n the product does not depend on n. In the special case when one of the representations is $S^{(n-1,1)} = S^{\overline{(1)}} = S^{\bar{\square}}$, we have the very well-known result

$$S^{\bar{\alpha}} \otimes S^{\bar{\square}} = c(\alpha)(S^{\bar{\alpha}}) \oplus \bigoplus_{\beta \in \alpha^\pm} S^{\bar{\beta}}, \tag{16}$$

where $c(\alpha)$ is the number of corner boxes of α and $\alpha^\pm$ is the set of partitions β with $\beta_1 \leqslant n/2$ gotten from α by (1) adding a box, (2) removing a box, or (3) moving a corner box of alpha to another corner. That is, β differs from α from the position of a corner. For example, if $n \gg 0$,

$$S^{\overline{(2,1)}} \otimes S^{\overline{(1)}} = 2\, S^{\overline{(2,1)}} \oplus S^{\overline{(2)}} \oplus S^{\overline{(1,1)}} \oplus S^{\overline{(3)}} \oplus S^{\overline{(1,1,1)}} \oplus S^{\overline{(3,1)}} \oplus S^{\overline{(2,2)}} \oplus S^{\overline{(2,1,1)}}.$$

Theorem 4.2. *Let $k \geqslant 2$ and assume that $n > 2k + 1$. If $L_k(\lambda)$ denotes the irreducible representation of $QP_k(n)$ indexed by λ, we have the decomposition of $W^{\otimes k}$ as an $(S_n, QP_k(n))$-bimodule*

$$W^{\otimes k} = \bigoplus S^{\bar{\lambda}} \otimes L_k(\lambda)$$

where the sum is over all partitions $\bar{\lambda}$ of n such that $|\lambda| \leqslant k$.

Proof. Since $n > 2k + 1$, we have that $QP_k(n)$ is isomorphic to the centralizer algebra. Then by the rule for tensoring $S^{(n-1,1)} \otimes S^{\lambda}$ and the centralizer theorem we get the decomposition. $\square$

We get as an immediate consequence a labeling set for the irreducible representations of $QP_k(n)$.

Corollary 4.3. *For $(n - 1)/2 > k \geqslant 2$ the irreducible representations of $QP_k(n)$ are indexed by partitions μ of $0, 1, \ldots, k$.*

4.1. Bratteli diagram

We set $QP_0(n) = \mathbb{C}$. For $k \geqslant 1$ we have that

$$QP_{k-1}(n) \subseteq QP_k(n).$$

We identify the elements in $QP_{k-1}(n)$ with the elements of $QP_k(n)$ that contain the block $\{k, k'\}$. Hence, we have a tower of algebras

$$QP_0(n) \subseteq QP_1(n) \subseteq QP_2(n) \subseteq QP_3(n) \subseteq \cdots. \tag{17}$$

For the remainder of the paper we assume that $n \gg 0$.

We can represent the inclusion of $A \subset B$ of multimatrix algebras (with the same unit) by a bipartite graph. The vertices in the graph are labeled by the simple summands of A and B. The number of edges joining a vertex v for A to a vertex w for B is the number of times the representation v occurs in the restriction of w to A.

In the case that we have a sequence of inclusions $A_0 \subset A_1 \subset A_2 \subset \cdots$ of multimatrix algebras, one may connect the bipartite graphs describing the inclusions $A_i \subset A_{i+1}$, to obtain the *Bratteli diagram*.

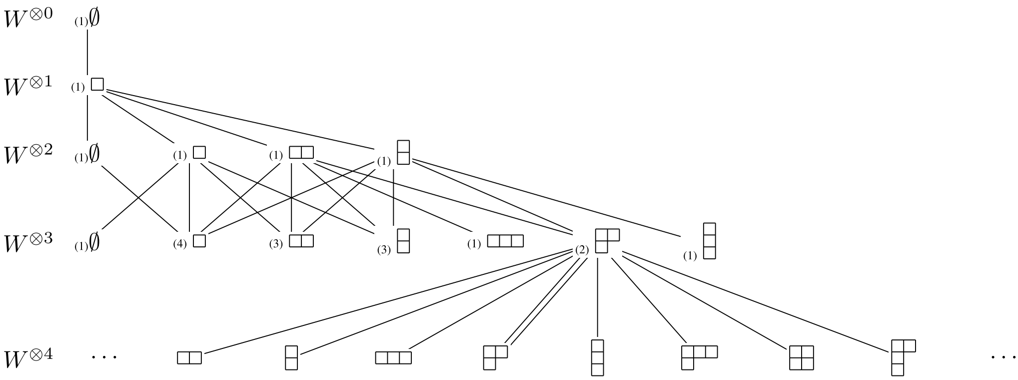


Fig. 1. Bratteli diagram for $QP_0(n) \subseteq QP_1(n) \subseteq QP_2(n) \subseteq \cdots$, levels 0–3, and part of level 4. On levels 0–3, the parenthetical number given next to each partition λ is the number of downward paths from the top to λ.

We build the graph $\hat{P}$ as follows:

Vertices: On level $k = 0$ the vertex set is $\mathcal{P}_0 = \{\emptyset\}$;
on level $k = 1$ the vertex set is $\mathcal{P}_1 = \{(1)\}$;
and on level $k \geqslant 2$, the vertex set is $\mathcal{P}_k = \{\mu \mid |\mu| \leqslant k\}$.
Edges: Connect $\lambda \in \mathcal{P}_{k-1}$ to $\lambda \in \mathcal{P}_k$ with $c(\lambda)$ edges;
otherwise, connect $\lambda \in \mathcal{P}_{k-1}$ to $\mu \in \mathcal{P}_k$ with one edge if $\mu \in \lambda^{\pm}$.

See Fig. 1.

Proposition 4.4. *As $n \to \infty$, the Bratteli diagram of the chain*

$$QP_0(n) \subset QP_1(n) \subset QP_2(n) \subset QP_3(n) \subset \cdots$$

is the graph $\hat{P}$.

Proof. Since $QP_k(n)$ is a centralizer algebra, we know by double centralizer theory that the decomposition of $W^{\otimes k}$ as an S_n-module yields the decomposition as a $QP_k(n)$-module. So we can use the decomposition rule in (16) to construct the Bratteli diagram for the chain in (17), a leveled graph completely described by $\hat{P}$.

The vertices at the kth level of the Bratteli diagram are indexed by the irreducible components of $W^{\otimes k}$ as an S_n-module; the number of edges joining a vertex indexed by α in the $(k-1)$st level to a vertex β in the kth is given by the multiplicity of β in $V_\alpha \otimes W$ as an S_n-module. In other words, the vertices at the ith level index the irreducible components of $W^{\otimes k}$ as an S_n-module, and the number of edges joining α in the ith level to a vertex β in the $(i+1)$th level is the multiplicity of $S^{\overline{\beta}}$ in $S^{\overline{\alpha}} \otimes W$ as an S_n-module. $\square$

Remark. Notice that in general, the Bratteli diagram is not multiplicity free; if λ has more than one corner in its diagram, there are that number of edges from λ in level i to λ in level $i+1$. For example, in Fig. 1, the number of edges from $\yng(2,1)$ to itself is 2.

As usual, we get that the number of paths from $\emptyset$ to λ is equal to the dimension of the irreducible representation of $QP_k(n)$ indexed by λ. These paths can be encoded by the set of tableaux in the next definition as in [5].

Definition. Given a positive integer k and partition λ, a sequence $T = (\mu^0, \ldots, \mu^k)$ is called a *Kronecker tableau* of shape λ if it is a sequence of k Young diagrams such that $\mu^0 = \emptyset$ and $\mu^k = \lambda$ and, for every pair μ^i and μ^{i+1} of consecutive diagrams, either μ^{i+1} is obtained from μ^i by the addition or removal of one corner, or μ^{i+1} differs from μ^i by the position of one corner, or $\mu^{i+1} = \mu^i$ and μ^{i+1} has one distinguished corner.

For example, if $k = 9$ a possible tableaux of shape $\lambda = \yng(2)$ is

$$T = \left(\emptyset, \yng(1), \young(\times), \yng(2), \yng(1,1), \yng(2,1), \young(\ \ ,\times), \yng(1,1,1), \yng(2,1), \yng(2)\right)$$

Here we have indicated the distinguished corner by an $\times$.

The following is a direct consequence of the centralizer theorem as the number of paths from $\emptyset$ to λ is equal to the dimension of $L(\lambda)$.

Lemma 4.5. *Let λ be a partition indexing an irreducible representation $QP_k(n)$. Then the number of Kronecker tableaux of shape λ of length k is equal to the dimension of the $L(\lambda)$.*

In the following theorem we give an exact formula for these dimensions.

Theorem 4.6. *Let k and n be two positive integers and $\bar{\lambda}$ a partition of n such that $n > k + \lambda_2$ with $\lambda = (\lambda_2, \lambda_3, \ldots)$. Then the dimension of the irreducible representation indexed by λ in $QP_k(n)$ is*

$$f^\lambda \sum_{m_1=0}^{|\lambda|} \left(\binom{k}{m_1} \sum_{m_2=|\lambda|-m_1}^{\lfloor \frac{k-m_1}{2} \rfloor} \binom{m_2}{|\lambda| - m_1} sp_2(k - m_1, m_2) \right),$$

where f^α is the number of Kronecker tableaux of shape α and $sp_2(a, b)$ is the number of set partitions of a set with a elements into b parts of size at least 2.

Proof. In [5, Proposition 2] Chauve an Goupil have counted the number of Kronecker tableaux, hence by Lemma 4.5 the result follows. $\square$

The number of paths from $\emptyset$ to λ and back to $\emptyset$ is equal to the square of the dimension of λ. If we sum these values we obtain the dimension of $QP_k(n)$. In our case, we get for the sum of squares of the dimensions of the irreducible representations is $1, 1, 4, 41, 715, 17722, \ldots$ (A000296) is the number of partitions of a $2k$-set into blocks of size greater than 1, as expected.

Acknowledgments

We thank the referee for very thoughtful and detailed comments; in particular, for pointing out a connection to rook algebras. Author Z. Daugherty is partially supported by NSF Grant DMS-1162010.

Appendix A. Finding the non-zero "top terms"

In this appendix we prove Lemma 3.2. To do this we would like to track the coefficient of $[d_1 d_2]$ (and therefore the coefficient of $\overline{d_1 d_2}$) in the expansion of $\bar{d}_1 \bar{d}_2$ according to Lemmas 2.2 and 2.3. If $[(d_1)_X][(d_2)_Y] = [d_1][d_2]$ as diagrams (neglecting coefficients), then necessarily, $X \subseteq [k']$ and $Y \subseteq [k]$. Therefore, we will restrict to focusing on the products of those terms

1. in the expansion of $\bar{d}_1$ which isolate only bottom vertices, and
2. in the expansion of $\bar{d}_2$ which isolate only top vertices.

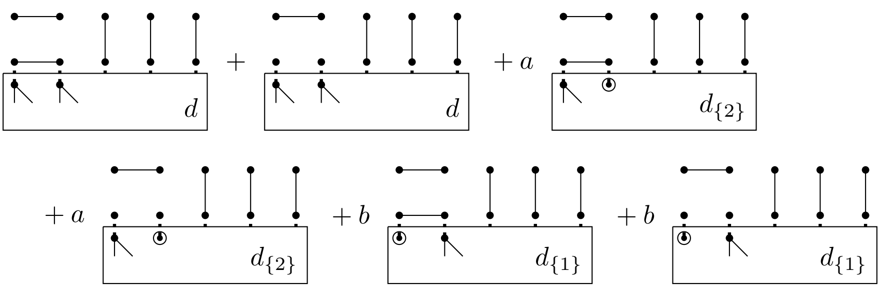
Further, if d_1 has a vertical size-2 block containing i', then Y cannot contain i (and vice-versa). Note that for $i = 1, \ldots, k-1$, since $\bar{s}_i = [s_i]$, we have $\bar{s}_i \bar{d} = \overline{s_i d}$. We continue with calculations for $d_1 = e_1, b_1, t_1$, and h_1, and let $d_2 = d$ be any diagram in $\mathcal{D}$.

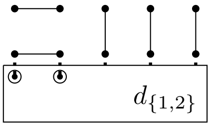
A.1. The coefficient of $[e_1 d]$ in $\bar{e}_1 \bar{d}$

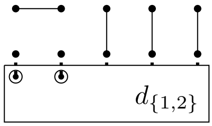
The expansion of $\bar{e}_1$ is

$$\bar{e}_1 = [e_1] + \left[(e_1)_{\{1',2'\}}\right] - \frac{1}{n}\left[(e_1)_{\{1,2\}}\right] - \frac{1}{n}\left[(e_1)_{\{1,2,1',2'\}}\right].$$

The terms $[(e_1)_X]$ for which $X \subseteq [k']$ are $[e_1] + [(e_1)_{\{1',2'\}}]$. The only vertices which can then be isolated in d are 1 and 2. Thus, possible contributions to $[e_1 d]$ are



$+\,c$ 

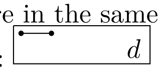
$+\,c$ 

.

In the diagrams we emphasize that a vertex is isolated by circling it.

Case 1: If the vertices 1 and 2 are in separate blocks, then $[e_1 d]$ only appears in one place, since right multiplication of d by e_1 results in 1 and 2 being in the same block. The coefficient of $[e_1 d]$ is 1.

Case 2: The vertices 1 and 2 are in the same block.

(a) If $\{1,2\}$ is a block: 

.
Then $d_{\{1\}} = d_{\{2\}} = d_{\{1,2\}}$, so $a = b = -1/n$ and $c = 1/n$. Then the coefficient on $[e_1 d]$ is

$$(n-1) + (n-1) - (1/n)\big((n-1) + (n-1)^2\big) = \boxed{\;n-1\;}.$$

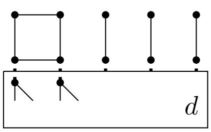
(b) If $\{1,2,m'\}$ or $\{1,2,m\}$ is a block, then $e_1 d$ has an isolated vertex, so $\bar{e}_1 \bar{d} = 0$.

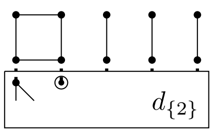
(c) The vertices 1 and 2 are in the same block, and are connected to at least two other vertices.
Then $a = b = -1/n$, $c = 1/n^2$, and the coefficient on $[e_1 d]$ is

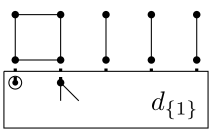
$$1 + 1 - (1/n)\big(1 + (n-1) + 1 + (n-1)\big) + \big(1/n^2\big)\big((n-1) + (n-1)^2\big)$$
$$= \boxed{\frac{n-1}{n}}.$$

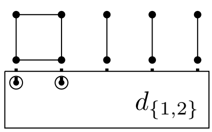
A.2. The coefficient of $[b_1 d]$ in $\bar{b}_1 \bar{d}$

The expansion of $\bar{b}_1$ only has one term where the isolation avoids top vertices, namely $[b_1]$. Again, this restricts us to terms in $\bar{d}$ where the isolation is restricted to the first two vertices. Thus the possible contributions to $[b_1 d]$ are



$+\,a$ 

$+\,b$ 

$+\,c$ 

.

Case 1: The vertices 1 and 2 are in separate blocks.
In this case, $[b_1 d]$ only appears in one place, since b_1 joins their blocks together. Then the coefficient of $[b_1 d]$ is 1.

Case 2: The vertices 1 and 2 are in the same block.

(a) If $\{1,2\}$ is a block, then $d_{\{1\}} = d_{\{2\}} = d_{\{1,2\}}$. So $a = b = -1/n$ and $c = 1/n$, and the coefficient on $[b_1 d]$ is $1 - 1/n - 1/n + 1/n = \boxed{\frac{n-1}{n}}$.

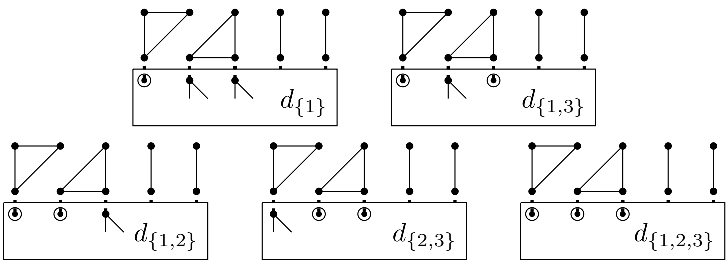
(b) If $\{1,2,m'\}$ or $\{1,2,m\}$ is a block:
Then $d_{\{1,2\}}$ doesn't contribute, and $a = b = -1/n$. Since b_1 joins 1 and 2 back together, $[b_1 d]$ has coefficient $1 - 1/n - 1/n = \boxed{\frac{n-2}{n}}$.

(c) The vertices 1 and 2 are in the same block, and are connected to at least two other vertices.
Then $a = b = -1/n$, $c = 1/n^2$, and the coefficient on $[b_1 d]$ is

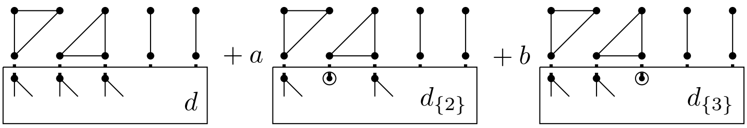
$$1 - 1/n - 1/n + 1/n^2 = \boxed{\frac{(n-1)^2}{n^2}}.$$

A.3. The coefficient of $[t_1 d]$ *in* $\bar{t}_1 \bar{d}$

The expansion of $\bar{t}_1$ only has one term where the isolation avoids top vertices, namely $[t_1]$. This restricts us to terms in $\bar{d}$ where the isolation is in the first three vertices. However, we can also note that the terms



will not contribute to $[t_1 d]$ in any case. Therefore the possible contributors are



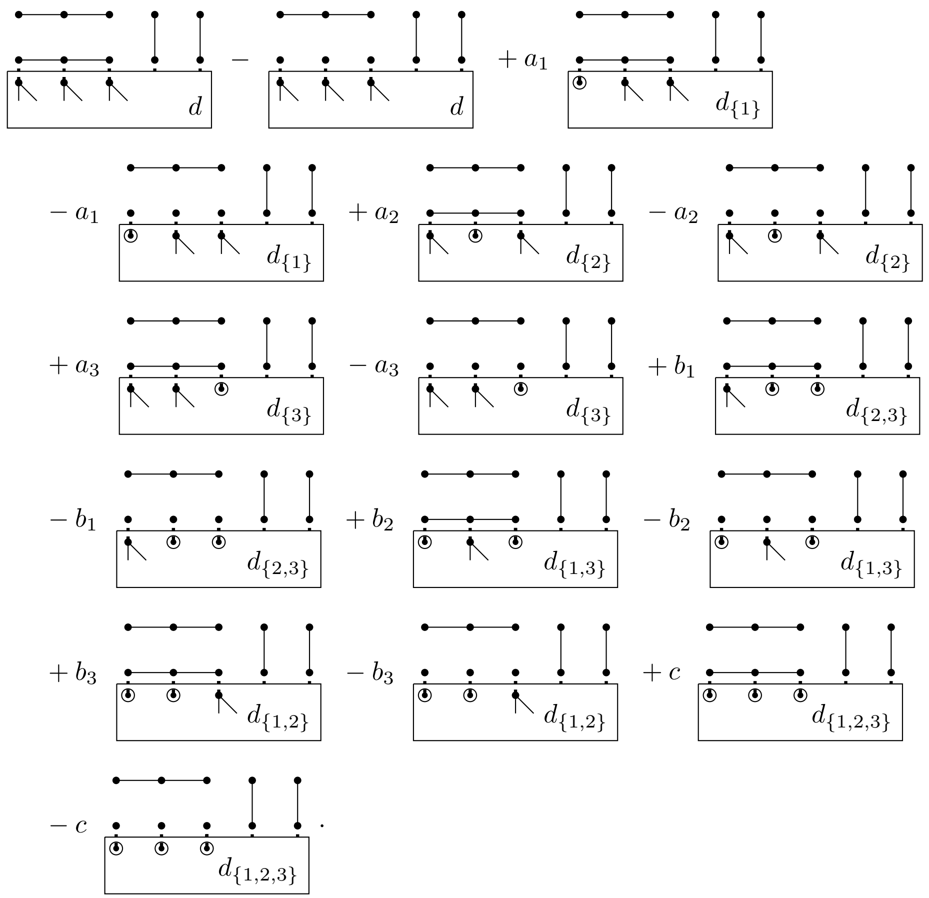
Case 1: If 2 and 3 are in separate blocks, then $[t_1 d]$ only appears in one place, since t_1 joins their blocks together. In this case its coefficient is 1.

Case 2: If $\{2,3\}$ is a block, then $\bar{t}_1 \bar{d} = 0$.

Case 3: Otherwise, $a = b = -1/n$ and the coefficient on $[t_1 d]$ is $1 - 2/n = \boxed{\frac{n-2}{n}}$.

A.4. The coefficient of $[h_1 d]$ in $\bar{h}_1\bar{d}$

The expansion of $\bar{h}_1$ has two terms which have no top vertices isolated, namely $[h_1] - [(h_1)_{\{1',2',3'\}}]$. This restricts us to terms in $\bar{d}$ where the isolation is in the first three vertices, and the possible contributors to $[h_1 d]$ are



Case 1: If 1, 2, and 3 are all in separate blocks, then there is no repetition, and the top term appears with coefficient 1.

Case 2: The vertices 1 and 2 appear in the same block, but separate from 3 (similarly for 1 and 3 or 2 and 3).

(a) If $\{1,2\}$ is a block:
If 3 is also in a block of size 2, then $h_1 d$ has an isolated vertex, and $\bar{h}_1\bar{d} = 0$.
If 3 is in a larger block, then since $a_1 = a_2 = a_3 = -b_3 = -1/n$, $b_1 = b_2 = -c = 1/n^2$, the coefficient is

$$1 - (n-1) - (2/n)\big(1 - (n-1)^2\big) - (1/n)\big((n-1) - (n-1)^2\big)$$

$$+ (2/n^2)\big((n-1)-(n-1)^3\big) + (1/n)\big(1-(n-1)^2\big)$$
$$- (1/n^2)\big((n-1)-(n-1)^3\big) = \boxed{(n-1)(n-2)}.$$

(b) Otherwise, none of the terms with $[(h_1)_{\{1',2',3'\}}]$ will contribute. So the only possible contributors are

$$[h_1][d], \qquad [h_1][d_{\{1\}}], \quad \text{and} \quad [h_1][d_{\{2\}}]$$

(because we need h_1 to bond 3 to 1 and 2, and we need one of 1 or 2 to bond to the rest of their block). Then since $a_1 = a_2 = -1/n$, the top term has a coefficient of $1 - 2/n = \boxed{\frac{n-2}{n}}$.

Case 3: The vertices 1, 2, and 3 are all in the same block.

(a) If $\{1,2,3\}$ is a block:
Then $d_{\{1,2\}} = d_{\{1,3\}} = d_{\{2,3\}} = d_{\{1,2,3\}}$, $b_1 = b_2 = b_3 = -c = 1/n^2$, $a_1 = a_2 = a_3 = -1/n$. The coefficient on the top terms is

$$(n-1)\big(1-1-(3/n)\big(1-(n-1)\big) + \big(2/n^2\big)\big(1-(n-1)^2\big)\big)$$
$$= \boxed{\frac{(n-2)(n-1)}{n}}.$$

(b) If $\{1,2,3,m'\}$ or $\{1,2,3,m\}$ is a block, then $\bar{h}_1\bar{d} = 0$.

(c) Otherwise, the vertices 1, 2, and 3 are in a block of size 5 or more.
Then $a_1 = a_2 = a_3 = -1/n$, $b_1 = b_2 = b_3 = 1/n^2$, and $c = -1/n^3$ and the coefficient on the top term is

$$1 - 1 - (3/n)\big(1-(n-1)\big) + \big(3/n^2\big)\big(1-(n-1)^2\big)$$
$$- \big(1/n^3\big)\big((n-1)-(n-1)^3\big) = \boxed{\frac{(n-1)(n-2)}{n^2}}.$$

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