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Exterior and symmetric powers of modules for cyclic 2-groups ☆

Frank Himstedt [a], Peter Symonds [b,*]

[a] *Technische Universität München, Zentrum Mathematik – M11, Boltzmannstr. 3, 85748 Garching, Germany*
[b] *School of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom*

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We prove a recursive formula for the exterior and symmetric powers of modules for a cyclic 2-group. This makes computation straightforward. Previously, a complete description was only known for cyclic groups of prime order.

1. Introduction

The aim of this paper is to provide a recursive procedure for calculating the exterior and symmetric powers of a modular representation of a cyclic 2-group. Let $G \cong C_{2^n}$ be a cyclic group of order 2^n and k a field of characteristic 2. Recall that there are 2^n indecomposable kG-modules $V_1, V_2, \dots, V_{2^n}$ for which $\dim V_r = r$.

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* Corresponding author.
E-mail addresses: himstedt@ma.tum.de (F. Himstedt), Peter.Symonds@manchester.ac.uk (P. Symonds).

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Theorem 1.1. *For all $n \geqslant 1$, $r \geqslant 0$ and $0 \leqslant s \leqslant 2^{n-1}$ we have*

$$\Lambda^r(V_{2^{n-1}+s}) \cong \bigoplus_{\substack{i,j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i+j}\big(\Lambda^i(V_s)\otimes_k \Lambda^j(V_{2^{n-1}-s})\big) \oplus tV_{2^n},$$

where t is a non-negative integer chosen so that both sides have the same dimension.

Here Ω_{2^n} is the syzygy or Heller operator over C_{2^n}, so $\Omega_{2^n} V_s = V_{2^n-s}$ for $1 \leqslant s \leqslant 2^n$. The group action on $V_1, \dots, V_{2^{n-1}}$ factors through $C_{2^{n-1}}$ so that exterior powers of these modules can be computed by applying the formula for this smaller group. In particular, one can determine the exterior powers on the right hand side of the formula in this way. We also show that there is a simple recursive procedure for calculating tensor products. Since $\Lambda(A \oplus B) \cong \Lambda(A) \otimes \Lambda(B)$, we obtain a complete recursive procedure for calculating exterior powers of all possible modules. It is sufficiently efficient that it is easy to calculate even by hand far beyond the range that was previously attainable by machine computation.

For symmetric powers we use the following result from [21].

Theorem 1.2. *For all $n \geqslant 1$, $r \geqslant 0$ and $0 \leqslant s \leqslant 2^{n-1}$ we have*

$$S^r(V_{2^{n-1}+s}) \cong_{\mathrm{ind}} \Omega_{2^n}^{r'} \Lambda^{r'}(V_{2^{n-1}-s}),$$

where $0 \leqslant r' < 2^n$ and $r' \equiv r \pmod{2^n}$. Here the symbol $\cong_{\mathrm{ind}}$ means up to direct summands induced from subgroups $H \lneqq G$.

Thus a knowledge of the exterior powers determines the symmetric powers up to induced summands. In fact it is shown in [21] how such a formula determines the symmetric powers completely, using a recursive procedure.

Formulas for the exterior and symmetric powers of a module for a cyclic group of prime order p were given by Almkvist and Fossum [1] and Renaud [19]. These were extended to cyclic p-groups by Hughes and Kemper [14] provided that the power is at most $p-1$. A formula for Λ^2 in the case of cyclic 2-groups was given by Gow and Laffey [11]. Also Kouwenhoven [15] obtained important results on exterior powers of modules for cyclic p-groups, including recursion formulas for $\Lambda(V_{q\pm 1})$ where q is a power of p. For $p = 2$ these formulas are special cases or direct consequences of Theorem 1.1, so we obtain independent proofs for some of the results in [11,15].

Our strategy is to consider $\Lambda(V_{2^{n-1}+s})$ as the quotient of $S(V_{2^{n-1}+s})$ by the ideal generated by the squares of elements of $V_{2^{n-1}+s}$. It turns out that we need to consider an intermediate ring $\widetilde{S}(V_{2^{n-1}+s})$, in which we only quotient out the squares of the elements of $V_s \subseteq V_{2^{n-1}+s}$. We show that $\widetilde{S}^r(V_{2^{n-1}+s}) \cong_{\mathrm{ind}} \Lambda^r(V_{2^{n-1}+s})$ for $r < 2^n$. But $\widetilde{S}(V_{2^{n-1}+s})$ can be resolved by the Koszul complex over $S(V_{2^{n-1}+s})$ on the squares of the elements of a basis for V_s. We show that this Koszul complex is separated in the sense of [21], that is that the image of a boundary map is contained in a projective submodule. This leads to the formula

$$\widetilde{S}^r(V_{2^{n-1}+s}) \cong_{\text{proj}} \bigoplus_{2i+j=r} \Omega^i_{2^n}\big(\Lambda^i(V_s) \otimes_k S^j(V_{2^{n-1}+s})\big),$$

where the symbol $\cong_{\text{proj}}$ means up to projective summands. Using Theorem 1.2, the right hand side is easily seen to be equal to the right hand side of the formula in Theorem 1.1 modulo induced summands. This yields the formula of Theorem 1.1 modulo induced summands. The strengthening to an equality modulo just projective summands is a formal inductive argument.

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2. Koszul complexes

Let G be a finite group, H a subgroup of G and k a field of characteristic $p > 0$. All tensor products will be over k if not otherwise specified. We recall some general facts about chain complexes of kG-modules from [21, Section 3].

Definition 2.1. (See [21, Definition 3.2].) A chain complex C_* of kG-modules is called:

(a) *acyclic* if it is 0 in negative degrees and it only has homology in degree 0;
(b) *weakly induced from* H if each module is induced from H, and *weakly induced from* H *except in degrees* I if each C_i, $i \notin I$, is induced from H;
(c) *separated at* C_i if $\operatorname{Im}(d_{i+1}) \to C_i$ factors through a projective kG-module;
(d) *separated* if it is separated at each C_i.

Write B_i for $\operatorname{Im}(d_{i+1}) \subseteq C_i$. If the inclusion $B_i \hookrightarrow C_i$ factors through a projective then it factors through the injective hull of B_i, call it P_i (injective is equivalent to projective for modular representations), and $P_i \hookrightarrow C_i$ is injective since it is so on the socle. Thus we can write $C_i = P_i \oplus C'_i$ and $B_i \subseteq P_i$.

Lemma 2.2. *(See [21, Lemma 3.9].) If the chain complexes C_*, C'_* of kG-modules are separated then so is the (total) tensor product $C_* \otimes C'_*$. Similarly for a product of finitely many chain complexes.*

Proof. Let P_i be a projective module such that $\operatorname{Im}(d_{i+1}) \subseteq P_i \subseteq C_i$ and similarly for P'_i. Then, summing over all degrees, $\operatorname{Im}(d \otimes d') \subseteq P \otimes C' + C \otimes P'$ where $P := \bigoplus_i P_i$ and $P' := \bigoplus_i P'_i$. There is a short exact sequence $0 \to P \otimes P' \to P \otimes C' \oplus C \otimes P' \to P \otimes C + C \otimes P \to 0$. The first two terms are projective, hence so is the third. $\square$

We need to consider tensor-induced complexes. For details of the construction see [4, II 4.1].

Lemma 2.3. *Suppose that every elementary abelian p-subgroup of G is conjugate to a subgroup of H, and let C_* be a complex of kH-modules that is separated. Then the tensor-induced complex $C_*{\uparrow}_H^{\otimes G}$ is also separated.*

Proof. By the proof of Lemma 2.2 above, the image $\mathrm{Im}(d{\uparrow}_H^{\otimes G})$ is contained in

$$P \otimes C \otimes \cdots \otimes C + C \otimes P \otimes \cdots \otimes C + \cdots + C \otimes C \otimes \cdots \otimes P,$$

which is a kG-submodule of $C_*{\uparrow}_H^{\otimes G}$. But the same proof shows that this module is projective on restriction to H, so it is projective, by Chouinard's Theorem [7, Corollary 1.1]. □

The next two results comprise a variation on [21, Proposition 3.3] and have the same proof.

Proposition 2.4. *Let H be an arbitrary subgroup of G. Suppose that the complex $K_* : K_w \to \cdots \to K_0$ of kG-modules is:*

(a) *acyclic,*
(b) *weakly induced from H except in at most one degree and*
(c) *K_* is separated on restriction to H.*

Then K_* is separated.

Recall that the Heller translate ΩV of a kG-module V is defined to be the kernel of the projective cover $P(V) \to V$ and $\Omega^i V$ for $i \geqslant 1$ denotes Ω iterated i times. Similarly $\Omega^{-1}V$ is the cokernel of the injective hull $V \to I(V)$ and $\Omega^{-i}V$ for $i \geqslant 1$ is its iteration. We let $\Omega^0 V$ denote V with any projective summands removed. These have the properties that $\Omega^i\Omega^j V \cong \Omega^{i+j}V$ and that if V is induced so is $\Omega^i V$.

Lemma 2.5. *Suppose that the complex $K_* : K_w \to \cdots \to K_0$ of kG-modules is:*

(a) *acyclic with $H_0(K_*) = L$, and*
(b) *separated.*

Then $L \cong_{\mathrm{proj}} K_0 \oplus \Omega^{-1}K_1 \oplus \Omega^{-2}K_2 \oplus \cdots \oplus \Omega^{-w}K_w$.

Let V be a kG-module, finite-dimensional as a k-vector space, and W a submodule of V. We write $S = S(V) = \bigoplus_{r=0}^{\infty} S^r(V)$ for the symmetric algebra on V and $\Lambda(W) = \bigoplus_{r=0}^{\infty} \Lambda^r(W)$ for the exterior algebra on W. For $r < 0$ let $S^r(V)$ denote the 0 module.

Definition 2.6. Let W be a submodule of a kG-module V and let W^p denote the kG-submodule of $S^p(V)$ spanned by the p-th powers of elements of W. Let $K(V, W^p)$ denote the Koszul complex of graded kG-modules:

$$\cdots \xrightarrow{d} S(V) \otimes \Lambda^3(W^p) \xrightarrow{d} S(V) \otimes \Lambda^2(W^p) \xrightarrow{d} S(V) \otimes W^p \xrightarrow{d} S(V),$$

where $d(f \otimes w_1^p \wedge w_2^p \wedge \cdots \wedge w_i^p) = \sum_{j=1}^{i} (-1)^{j+1} f w_j^p \otimes w_1^p \wedge \cdots \wedge \widehat{w_j^p} \wedge \cdots \wedge w_i^p$ for $w_j \in W$ and $f \in S(V)$. We write $K^r(V, W^p)$ when we consider the complex $K(V, W^p)$ in grading-degree r.

If $k = \mathbb{F}_2$ then the squaring map gives an isomorphism between W and W^2, so we can regard W^2 as a copy of W in degree 2 equipped with a squaring map into $S^2(V)$. From this point of view, the boundary map is given by $d(f \otimes w_1 \wedge w_2 \wedge \cdots \wedge w_i) = \sum_{j=1}^{i} f w_j^2 \otimes w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_i$ for $w_j \in W$ and $f \in S(V)$.

We will normally take the second point of view, so we will assume that $k = \mathbb{F}_2$ in a large part of this paper. Since any kC_{2^n}-module can be written in $\mathbb{F}_2$, this is not a significant restriction.

Lemma 2.7. *In the context of Definition 2.6, the complex $K(V, W^p)$ is acyclic and its homology in degree 0 is $S(V)/(W^p)$, where (W^p) is the ideal generated by all elements w^p, $w \in W$.*

Proof. If $\{w_1, \ldots, w_r\}$ is a basis for W then $\{w_1^p, \ldots, w_r^p\}$ is a regular sequence in $S(V)$ and it spans W^p. This is now a standard result about Koszul complexes. $\square$

Lemma 2.8. *Let V, V' be kG-modules, finite-dimensional as k-vector spaces and let W, W' be submodules of V and V', respectively. The complex $K(V \oplus V', (W \oplus W')^p)$ is isomorphic to the (total) tensor product $K(V, W^p) \otimes K(V', W'^p)$ as a complex of graded kG-modules.*

Proof. This is analogous to [21, Lemma 3.8]. $\square$

We also need to deal with tensor induction of graded modules and complexes.

Lemma 2.9. *Let H be a subgroup of G and let V, W be kH-modules. Then $S(V{\uparrow}_H^G) \cong S(V){\uparrow}_H^{\otimes G}$, $\Lambda(V{\uparrow}_H^G) \cong \Lambda(V){\uparrow}_H^{\otimes G}$ as graded kG-modules, and if the characteristic of k is 2 then $K(V{\uparrow}_H^G, (W{\uparrow}_H^G)^2) \cong K(V, W^2){\uparrow}_H^{\otimes G}$ as complexes of graded kG-modules.*

Without the restriction on the characteristic of k we would have to deal with the sign convention that appears in the definition of the action of G on the tensor-induced complex.

Proof. Let $\{g_i\}$ be a set of coset representatives for G/H and write $V{\uparrow}_H^G = \bigoplus g_i \otimes V$. The formulas now follow from the usual formulas for S and Λ of a sum and the definition of the group action on a tensor induced module. □

3. Modules for cyclic 2-groups

From now on, let $G = \langle g \rangle \cong C_{2^n}$ be a cyclic group of order 2^n, $n \geqslant 1$, and k a field of characteristic 2. We write $a(G)$ for the Green ring of kG-modules. Up to isomorphism, there are 2^n indecomposable kG-modules $V_1, V_2, \dots, V_{2^n}$ and we choose the notation so that $\dim_k(V_i) = i$. For convenience we write V_0 for the 0 module. The generator $g \in G$ acts on V_i with matrix a Jordan block with ones on the diagonal. Choose a k-basis $\{x_1, x_2, \dots, x_n\}$ of V_n such that $gx_i = x_i + x_{i-1}$ for all $2 \leqslant i \leqslant n$ and $gx_1 = x_1$. Each element of $S(V_i)$ can be written uniquely as a polynomial in $x_1, \dots, x_i$, and for $j \leqslant i$, we can identify V_j with the kG-submodule of V_i spanned by $x_1, x_2, \dots, x_j$. Each V_i is uniserial with composition series $0 < V_1 < V_2 < \cdots < V_{i-1} < V_i$. Note that for $i \leqslant 2^{n-1}$ the kernel of V_i is nontrivial and so V_i can be identified with the i-dimensional indecomposable module for the quotient group $C_{2^{n-1}}$.

Decompositions of tensor products into indecomposables have been studied by several authors, see for example [2,3,9,13,16–18,20]. In our case, this decomposition can easily be computed using the Heller translate Ω. We write Ω_{2^n} instead of Ω when we want to emphasize that we are working with modules for the group C_{2^n}. It is easy to check that $\Omega_{2^n} V_i \cong_{\mathrm{proj}} V_{2^n - i}$ for $0 \leqslant i \leqslant 2^n$, where proj means modulo projective modules for C_{2^n}.

Recall that $\Omega_{2^n}(V \otimes V') \cong_{\mathrm{proj}} (\Omega_{2^n} V) \otimes V'$, where the projective part can be determined by comparing dimensions. For cyclic groups, $\Omega_{2^n}^2 \cong_{\mathrm{proj}} \mathrm{Id}$ and $\Omega_{2^n} V \otimes \Omega_{2^n} V' \cong_{\mathrm{proj}} V \otimes V'$. This provides an easy recursive method for calculating the decomposition of tensor products in the case of cyclic 2-groups.

In order to calculate $V_a \otimes V_b$, we may assume $a \geqslant b$ and write $a = 2^r - a'$ for the smallest possible r such that $a' \geqslant 0$. Then $V_a \otimes V_b \cong (\Omega_{2^r} V_{a'}) \otimes V_b \cong \Omega_{2^r}(V_{a'} \otimes V_b)$ modulo copies of V_{2^r}. If $b \geqslant 2^{r-1}$ then it is more efficient to write $b = 2^r - b'$ too.

Example. For $V_9, V_{13} \in a(C_{16})$ we have: $V_9 \otimes V_{13} \cong (\Omega_{16} V_7) \otimes (\Omega_{16} V_3) \cong (V_7 \otimes V_3) \oplus$ copies of V_{16}. By comparing dimensions we get $V_9 \otimes V_{13} \cong (V_7 \otimes V_3) \oplus 6V_{16}$. Now we consider the non-faithful module $V_7 \otimes V_3$ as a module for the factor group $\cong C_8$ and get $V_7 \otimes V_3 \cong \Omega_8(V_1 \otimes V_3) \oplus$ copies of V_8. Again by comparing dimensions we obtain $V_7 \otimes V_3 \cong \Omega_8(V_1 \otimes V_3) \oplus 2V_8 \cong V_5 \oplus 2V_8$ and hence $V_9 \otimes V_{13} \cong V_5 \oplus 2V_8 \oplus 6V_{16}$.

Let $H = \langle g^2 \rangle$ be the unique maximal subgroup of G. For $1 \leqslant j \leqslant 2^{n-1}$ we also denote the indecomposable kH-module of dimension j by V_j. Of course, this is an abuse of notation, but we will always make it clear whether we consider V_j as a kG-module or as a kH-module. An elementary calculation with Jordan canonical forms shows that the restriction operator ${\downarrow}_H^G \colon a(G) \to a(H)$ is given by $V_i{\downarrow}_H^G = V_{i'} \oplus V_{i''}$ where $V_{i'}$ is the kH-submodule generated by $\{x_i, x_{i-2}, x_{i-4}, \dots\}$ and $V_{i''}$ is the kH-submodule generated by $\{x_{i-1}, x_{i-3}, x_{i-5}, \ldots\}$. In particular, we have $(i', i'') = (\frac{i+1}{2}, \frac{i-1}{2})$ if i is odd, and $(i', i'') = (\frac{i}{2}, \frac{i}{2})$ if i is even. The induction operator $\uparrow_H^G\colon a(H) \to a(G)$ is given by $V_j\uparrow_H^G = V_{2j}$ for $1 \leqslant j \leqslant 2^{n-1}$.

We say that a kG-module is induced if it is induced from proper subgroups. Let $a_P(G)$ be the submodule of $a(G)$ generated by the projective modules and $a_I(G)$ the submodule generated by the induced modules. Notice that $a_P(G)$ and $a_I(G)$ are ideals of $a(G)$ and that induction maps $a_P(H)$ into $a_P(G)$ and $a_I(H)$ into $a_I(G)$, but restriction only maps $a_P(G)$ into $a_P(H)$.

The following lemmas deduce information on kG-modules and short exact sequences of kG-modules from their restriction to H.

Lemma 3.1. *Let A be a kG-module such that $A\downarrow_H^G$ is induced from a proper subgroup of H. Then A is induced from H.*

Proof. We can assume that A is indecomposable. Since $A\downarrow_H^G$ is induced, each indecomposable direct summand of $A\downarrow_H^G$ has even dimension. Thus $\dim_k(A)$ is even and so A is induced. $\square$

As in the introduction we write $\cong_{\mathrm{ind}}$ and $\cong_{\mathrm{proj}}$ for isomorphisms modulo induced and modulo projective summands, respectively.

Lemma 3.2. *Let A, B be induced kG-modules such that $A\downarrow_H^G \cong_{\mathrm{proj}} B\downarrow_H^G$. Then $A \cong_{\mathrm{proj}} B$.*

Proof. Since A is induced, $A\downarrow_H^G\uparrow_H^G \cong 2A$; the same is true for B. Inducing a projective yields a projective, so we obtain $2A \cong_{\mathrm{proj}} 2B$ and the claim follows. $\square$

Lemma 3.3. *Let A and B be kG-modules such that $A \cong_{\mathrm{ind}} B$ and $A\downarrow_H^G \cong_{\mathrm{proj}} B\downarrow_H^G$. Then $A \cong_{\mathrm{proj}} B$.*

Proof. We have $A \oplus X \cong B \oplus Y$ for some induced modules X and Y. On restriction, we obtain $X\downarrow_H^G \cong_{\mathrm{proj}} Y\downarrow_H^G$ and so $X \cong_{\mathrm{proj}} Y$ by Lemma 3.2. Now cancel the non-projective summands of X and Y in the original formula. $\square$

Lemma 3.4. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of kG-modules that is separated at B on restriction to H and such that $C \cong_{\mathrm{ind}} B \oplus \Omega_{2^n}^{-1} A$ as kG-modules. Then the sequence is separated at B (as a sequence of kG-modules).*

Proof. The hypotheses imply that there are induced modules X and Y such that $C \oplus X \cong B \oplus \Omega_{2^n}^{-1} A \oplus Y$ and also $C\downarrow_H^G \cong_{\mathrm{proj}} B\downarrow_H^G \oplus \Omega_{2^{n-1}}^{-1} A\downarrow_H^G \cong_{\mathrm{proj}} (B \oplus \Omega_{2^n}^{-1} A)\downarrow_H^G$ by Lemma 2.5 applied to $0 \to A\downarrow_H^G \to B\downarrow_H^G \to 0$. It follows that $X\downarrow_H^G \cong_{\mathrm{proj}} Y\downarrow_H^G$, hence, by Lemma 3.2, $X \cong_{\mathrm{proj}} Y$ and then $C \cong_{\mathrm{proj}} B \oplus \Omega_{2^n}^{-1} A$.

Thus our short exact sequence is $0 \to A \xrightarrow{d} B \xrightarrow{e} B \oplus \Omega_{2^n}^{-1} A \to 0$, up to projective summands. Consider the long exact sequence for Tate Ext:

$$\cdots \to \overline{\mathrm{Hom}}_{kG}(A, A) \xrightarrow{d_*} \overline{\mathrm{Hom}}_{kG}(A, B) \xrightarrow{e_*} \overline{\mathrm{Hom}}_{kG}\bigl(A, B \oplus \Omega_{2^n}^{-1} A\bigr) \\ \to \mathrm{Ext}^1(A, A) \to \cdots, \tag{1}$$

where $\overline{\mathrm{Hom}}_{kG}$ denotes homomorphisms modulo those that factorize through a projective.

Since $\mathrm{Ext}^1(A, A) \cong \overline{\mathrm{Hom}}_{kG}(A, \Omega_{2^n}^{-1} A)$ we have

$$\begin{aligned} \dim \mathrm{Im}(e_*) + \dim \mathrm{Ext}^1(A, A) &\geqslant \dim \overline{\mathrm{Hom}}_{kG}\bigl(A, B \oplus \Omega_{2^n}^{-1} A\bigr) \\ &= \dim \overline{\mathrm{Hom}}_{kG}(A, B) + \dim \overline{\mathrm{Hom}}_{kG}\bigl(A, \Omega_{2^n}^{-1} A\bigr) \\ &= \dim \overline{\mathrm{Hom}}_{kG}(A, B) + \dim \mathrm{Ext}^1(A, A) \end{aligned}$$

and therefore $\dim \mathrm{Im}(e_*) \geqslant \dim \overline{\mathrm{Hom}}_{kG}(A, B)$. Hence e_* is injective and so $d_* = 0$. But $d = d_*(\mathrm{Id}_A)$, so d factors through a projective, as required. □

The next lemma describes tensor induction from H to G modulo induced modules and gives information on the structure of the exterior algebra $\Lambda(V_{2j})$ as a kG-module in terms of the kH-module $\Lambda(V_j)$.

Lemma 3.5. *Let r, j be integers such that $r \geqslant 0$ and $1 \leqslant j \leqslant 2^{n-1}$. We consider V_j as a kH-module and $V_{2j} = V_j{\uparrow}_H^G$ as a kG-module.*

(a) *Let A and B be kH-modules. Then $(A \oplus B){\uparrow}_H^{\otimes G} \cong A{\uparrow}_H^{\otimes G} \oplus B{\uparrow}_H^{\otimes G} \oplus X$ for some induced kG-module X.*
(b) *There is an induced kG-module X such that $\Lambda^{2r}(V_{2j}) \cong \Lambda^r(V_j){\uparrow}_H^{\otimes G} \oplus X$.*
(c) *If r is odd, then the kG-module $\Lambda^r(V_{2j})$ is induced from H.*
(d) *If j is even, then the kG-module $V_j{\uparrow}_H^{\otimes G}$ is induced from H.*
(e) *If j is odd, then $V_j{\uparrow}_H^{\otimes G} \cong_{\mathrm{ind}} V_1$.*

Proof. (a) follows from [4, I 3.15.2 (iii)].

(b) By the construction of induced modules, we have $V_{2j} = V_j \oplus gV_j$ as vector spaces and the action of the generator g of G on V_{2j} is given by $g(v + gv') = g^2 v' + gv$. So there is a natural isomorphism

$$\Lambda^{2r}(V_{2j}) = \Lambda^{2r}(V_j \oplus gV_j) \cong \bigoplus_{\substack{r', r'' \geqslant 0 \\ r' + r'' = 2r}} \bigl(\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j)\bigr)$$

of vector spaces, and thus

$$\Lambda^{2r}(V_{2j}) \cong \bigl(\Lambda^r(V_j) \otimes g\Lambda^r(V_j)\bigr) \oplus \bigoplus_{\substack{0 \leqslant r' < r'' \\ r' + r'' = 2r}} \bigl(\bigl(\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j)\bigr) \\ \oplus g\bigl(\Lambda^{r'}(V_j) \otimes \Lambda^{r''}(gV_j)\bigr)\bigr).$$

Via this isomorphism, the right hand side becomes a kG-module and from the action of g, we see that $(\Lambda^{r'}(V_j)\otimes\Lambda^{r''}(gV_j))\oplus g(\Lambda^{r'}(V_j)\otimes\Lambda^{r''}(gV_j))$ is a kG-submodule isomorphic to $(\Lambda^{r'}(V_j)\otimes\Lambda^{r''}(gV_j))\uparrow_H^G$ and $\Lambda^r(V_j)\otimes g\Lambda^r(V_j)$ is a submodule isomorphic to $\Lambda^r(V_j)\uparrow_H^{\otimes G}$.

(c) The proof is similar to that of (b). Note that, if r is odd, the summand corresponding to $r'=r''$ which leads to the tensor induced submodule in (b) does not occur.

(d), (e) We say that a kG-module is *induced except for possibly one trivial summand* if it is isomorphic to $A\uparrow_H^G$ or $A\uparrow_H^G\oplus V_1$ for some kH-module A. We prove (d) and (e) simultaneously by showing that for all $1\leqslant j\leqslant 2^{n-1}$ the kG-module $V_j\uparrow_H^{\otimes G}$ is induced except for possibly one trivial summand. The claim then follows from the fact that $\dim_k(V_j\uparrow_H^{\otimes G})$ is even if and only if j is even.

The proof is by induction on j. Because $V_1\uparrow_H^{\otimes G}\cong V_1$ we can assume $j>1$. If j is even, then the kH-module V_j is induced from a proper subgroup of H. So [4, I 3.15.2 (iv)] implies that $V_j\uparrow_H^{\otimes G}$ is a direct sum of modules induced from H (even from proper subgroups of H). Assume that j is odd. So we can write $j=2^m+j'$ with $1\leqslant m<n-1$ and $1\leqslant j'<2^m$. First, we treat the case $j'=1$. By the Mackey formula for tensor induction [4, I 3.15.2 (v)] we have $V_j\uparrow_H^{\otimes G}\downarrow_H^G\cong V_{2^m+1}\otimes V_{2^m+1}\cong V_1\oplus(2^m-2)V_{2^m}\oplus 2V_{2^{m+1}}$, and so $V_j\uparrow_H^{\otimes G}\cong V_1\oplus(2^{m-1}-1)V_{2^{m+1}}\oplus V_{2^{m+2}}$, which is induced up to one trivial summand. Now assume $j'>1$. Then $V_{j'}\otimes V_{2^m+1}\cong V_j\oplus(j'-1)V_{2^m}$ as kH-modules. By [4, I 3.15.2 (i)] and (a) we get

$$\big(V_{j'}\uparrow_H^{\otimes G}\big)\otimes\big(V_{2^m+1}\uparrow_H^{\otimes G}\big)\cong_{\mathrm{ind}}\big(V_j\uparrow_H^{\otimes G}\big)\oplus\big(j'-1\big)\big(V_{2^m}\uparrow_H^{\otimes G}\big). \tag{2}$$

By induction and the case $j'=1$, we know that the left hand side of (2) is induced except for possibly one trivial summand. Hence, $V_j\uparrow_H^{\otimes G}$ is induced except for possibly one trivial summand. □

We can now see that the symmetric and exterior powers of even dimensional indecomposable modules have a particularly restricted form.

Corollary 3.6. *Suppose that we have non-negative integers j,s,t,u with u,j odd and $s\geqslant 1$. Furthermore, assume that $2^tu<2^n$ and $2^sj\leqslant 2^n$. Then $\Lambda^{2^tu}(V_{2^sj})$ and $S^{2^tu}(V_{2^sj})$ are both induced unless $t\geqslant s$. If $t\geqslant s$ then $\Lambda^{2^tu}(V_{2^sj})\cong mV_1\oplus X$ and $S^{2^tu}(V_{2^sj})\cong m'V_1\oplus Y$, where X,Y are induced modules and m and m' are the numbers of non-induced indecomposable summands in $\Lambda^{2^{t-s}u}(V_j)$ and $\Lambda^{2^{t-s}u}(V_{2^{n-s}-j})$, respectively.*

Proof. Using Lemma 3.5 (a), (b), (d) we see that, up to induced direct summands, $\Lambda^{2^tu}(V_{2^sj})$ is tensor-induced from a subgroup of index $2^{\min\{s,t\}}$. If $t<s$ then, up to induced direct summands, it is tensor-induced from $\Lambda^u(V_{2^{s-t}j})$ and thus is induced, by part (c) of the same lemma. If $t\geqslant s$ then, again up to induced direct summands, it is tensor-induced from $\Lambda^{2^{t-s}u}(V_j)$; the description given is then seen to be valid using parts (a), (d) and (e). The case of $S^{2^tu}(V_{2^sj})$ reduces to that of $\Lambda^{2^tu}(V_{2^n-2^sj})$, by Theorem 1.2. □

Corollary 3.7. *If X is a kG-module such that every direct summand has dimension divisible by 4 then $S^2(X)$ is induced.*

Proof. By the identity $S^2(A \oplus B) \cong S^2(A) \oplus S^2(B) \oplus A \otimes B$, we may assume that X is indecomposable, say $X = V_{4u}$. The claim now follows from Corollary 3.6. □

In the proof of our main result we will often have information only modulo induced direct summands. The following definition and lemmas deal with the splitting of maps in such situations.

Recall that a map $f : A \to B$ of kG-modules is *split injective*, if there is a map $g : B \to A$ of kG-modules such that $g \circ f = \mathrm{Id}_A$. For maps $f : A \to B$, $f' : A \to B'$ of kG-modules we write $(f, g) : A \to B \oplus B'$, $a \mapsto (f(a), f'(a))$.

Definition 3.8. Let $f : A \to B$ be a map of kG-modules. We say that f is *split injective modulo induced summands* if there exists an induced kG-module X and a map $f' : A \to X$ of kG-modules such that $(f, f') : A \to B \oplus X$ is split injective.

Split injective modulo induced summands behaves in much the same way as split injective.

Lemma 3.9. *Given maps $f : A \to B$, $g : B \to C$ and $h : D \to E$ of kG-modules:*

(a) *if f and g are split injective modulo induced summands then so is $g \circ f$,*
(b) *if $g \circ f$ is split injective modulo induced summands then so is f,*
(c) *if f, h are split injective modulo induced summands then so is $f \otimes h : A \otimes D \to B \otimes E$.*

Proof. (a) By assumption, we have induced modules X, Y and maps $f' : A \to X$, $u : B \to A$, $u' : X \to A$, $g' : B \to Y$, $v : C \to B$, $v' : Y \to B$ such that $u \circ f + u' \circ f' = \mathrm{Id}_A$ and $v \circ g + v' \circ g' = \mathrm{Id}_B$. We define $(g \circ f)' : A \to X \oplus Y, a \mapsto (f'(a), g' \circ f(a))$, $w : C \to A, c \mapsto u \circ v(c)$ and $w' : X \oplus Y \to A, (x, y) \mapsto u'(x) + u \circ v'(y)$. Then $w \circ (g \circ f) + w' \circ (g \circ f)' = \mathrm{Id}_A$.

Parts (b) and (c) are proved in a similar way; the proofs are left to the reader. □

Lemma 3.10. *Let $f : A \to B$ be a map of kG-modules and write $A = A' \oplus A''$, where A' has only non-induced summands and A'' has only induced summands. Let i denote the inclusion of A' in A. Then f is split injective modulo induced summands if and only if $f \circ i$ is split injective.*

Proof. Suppose that f is split injective modulo induced summands; we want to show that $f \circ i$ is split injective. By Lemma 3.9 (a), the map $f \circ i$ is split injective modulo induced summands, so we can assume that $A = A'$ and we have to show that f is split injective.

Since f is split injective modulo induced summands we have an induced module X and maps $f' : A \to X$, $u : B \to A$, $u' : X \to A$ such that $u \circ f + u' \circ f' = \mathrm{Id}_A$. Since X and A have no summands in common, we know that $u' \circ f'$ lies in the radical of $\mathrm{End}_{kG}(A)$ (note that if $A = \bigoplus A_i$ with A_i indecomposable and we write elements of $\mathrm{End}_{kG}(A)$ as matrices with entries in $\mathrm{Hom}_{kG}(A_i, A_j)$ then the radical consists of the morphisms for which no component is an isomorphism). Thus $u \circ f$ is surjective, hence an automorphism of A, and f is split injective.

Conversely, suppose that $f \circ i : A' \to B$ is split injective, so there is a map $g : B \to A'$ such that $g \circ (f \circ i) = \mathrm{Id}_{A'}$. Let j denote the inclusion of $X := A''$ in A and f' the projection of A onto A''. We define $v := i \circ g : B \to A$, and $v' : X \to A$, $x \mapsto -(i \circ g \circ f \circ j)(x) + j(x)$. Then $v \circ f + v' \circ f' = \mathrm{Id}_A$, so f is split injective modulo induced summands. □

Remark. The proof above shows that the induced module X in Definition 3.8 can always be chosen in such a way that X only contains indecomposable direct summands that also occur in A.

Remark. Definition 3.8 makes sense for any finite group and any class of indecomposable modules and Lemmas 3.9 (a), (b) and 3.10 remain true.

It will turn out that certain symmetric and exterior powers of modules for cyclic 2-groups are contained in the $\mathbb{Z}$-submodule $c(G)$ of the Green ring $a(G)$ spanned by the indecomposable modules V_r for r satisfying $r \not\equiv 2 \pmod 4$. We describe some properties of $c(G)$.

Lemma 3.11. *The submodule $c(G)$ is*

(a) *a subring of $a(G)$ and*
(b) *closed under Ω_{2^n}.*

Proof. Part (b) is clear from the definitions.

For part (a) we need to show that $V_i \otimes V_j \in c(G)$ for all $0 \leqslant i, j \leqslant 2^n$, $i, j \not\equiv 2 \pmod 4$. For $n = 1$ we only have $V_1 \otimes V_1 = V_1 \in c(G)$. Suppose that $n > 1$. By the remarks on the computation of tensor products at the beginning of this section, we have $V_i \otimes V_j = \Omega^m_{2^n}(V_{i'} \otimes V_{j'}) \oplus m' V_{2^n}$ for some integers m, m', where $0 \leqslant i', j' \leqslant 2^{n-1}$ and $i' \equiv \pm i \pmod 4$ and $j' \equiv \pm j \pmod 4$. We can consider $V_{i'}$ and $V_{j'}$ as modules for $H \cong C_{2^{n-1}}$ and the claim follows from induction and part (b). □

4. Main theorem

From now on we assume that $k = \mathbb{F}_2$ is a field with 2 elements and G is a cyclic group of order 2^n. For $0 \leqslant s \leqslant 2^{n-1}$, we know from Lemma 2.7 that $K(V_{2^{n-1}+s}, V_s^2)$ is acyclic and that its homology in degree 0 is $S(V_{2^{n-1}+s})/(V_s^2)$. It will turn out that $S(V_{2^{n-1}+s})/(V_s^2)$

is closely related to the exterior algebra $\Lambda(V_{2^{n-1}+s})$, so it is natural to study the structure of the graded ring $\widetilde{S}(V_{2^{n-1}+s}) = \bigoplus_{r\geqslant 0} \widetilde{S}^r(V_{2^{n-1}+s}) := S(V_{2^{n-1}+s})/(V_s^2)$ as a kG-module. For a non-negative integer m write $\widetilde{S}^{<m}(V_{2^{n-1}+s}) = \bigoplus_{r=0}^{m-1} \widetilde{S}^r(V_{2^{n-1}+s})$ and use a similar notation for other graded modules. Let $\widetilde{N}(V_{2^{n-1}+s})$ denote the kernel of the natural epimorphism $\widetilde{S}(V_{2^{n-1}+s}) \to \Lambda(V_{2^{n-1}+s})$.

Choose a k-basis $\{x_1, x_2, \dots, x_{2^{n-1}+s}\}$ of $V_{2^{n-1}+s}$ as in Section 3. For simplicity, write $x_{top} := x_{2^{n-1}+s}$, $x_{top-1} := x_{2^{n-1}+s-1}$ and so on. Each element of $S(V_{2^{n-1}+s})$ can be written uniquely as a polynomial in $x_1, x_2, \dots, x_{top}$. Set $a := \prod_{i=1}^{2^n}(g^i x_{top}) \in S(V_{2^{n-1}+s})$. If $s < 2^{n-1}$ let $\widetilde{a}$ be the image of a in $\widetilde{S}(V_{2^{n-1}+s})$, and if $s = 2^{n-1}$ let $\widetilde{a}$ be the image of the element $\prod_{t\in G/C_2}(tx_{top}^2)$ in $\widetilde{S}(V_{2^n})$. In the latter case $\widetilde{a}$ is still invariant, because $g^{2^{n-1}} x_{2^n} = x_{2^n} + x_{2^{n-1}}$ and so $x_{top}^2 \in \widetilde{S}(V_{2^n})$ is invariant under C_2. In all cases, a is homogeneous of degree 2^n, has degree 2^n when considered as a polynomial in x_{top}, the elements a and $\widetilde{a}$ are invariant under the action of G, and the image of $\widetilde{a}$ in $\Lambda(V_{2^{n-1}+s})$ is 0. If $s = 2^{n-1}$ then we also write $\widetilde{b}$ for the image of the element $\prod_{i=1}^{2^n}(g^i x_{top})$ in $\widetilde{S}(V_{2^n})$.

The next theorem is our main result. Since any representation of $G \cong C_{2^n}$ over a field of characteristic 2 can be written in $\mathbb{F}_2$, part (d) implies Theorem 1.1, but we record the other parts since they are also of interest and they form an integral part of the proof.

Theorem 4.1. *Let n and s be integers such that $n \geqslant 1$ and $0 \leqslant s \leqslant 2^{n-1}$.*

(a) *(Separation) The complex $K(V_{2^{n-1}+s}, V_s^2)$ of kG-modules is separated.*
(b) *(Periodicity) For $s < 2^{n-1}$ we have $\widetilde{S}(V_{2^{n-1}+s}) \cong_{\mathrm{ind}} k[\widetilde{a}] \otimes \widetilde{S}^{<2^n}(V_{2^{n-1}+s})$ as graded kG-modules. For $s = 2^{n-1}$ we have $\widetilde{S}(V_{2^n}) \cong_{\mathrm{ind}} k[\widetilde{a}] \otimes (\widetilde{S}^{<2^n}(V_{2^n}) \oplus k\widetilde{b})$. In both cases the isomorphism from right to left is induced by the product in $\widetilde{S}(V_{2^{n-1}+s})$.*
(c) *(Splitting) The short exact sequence of graded kG-modules*

$$0 \to \widetilde{N}(V_{2^{n-1}+s}) \to \widetilde{S}(V_{2^{n-1}+s}) \to \Lambda(V_{2^{n-1}+s}) \to 0$$

is split and $\widetilde{N}(V_{2^{n-1}+s}) = \widetilde{a}\widetilde{S}(V_{2^{n-1}+s}) \oplus \widetilde{I}$, where $\widetilde{I}$ is a kG-module induced from H.
(d) *(Exterior powers) For each $r \geqslant 0$ we have the following isomorphism of kG-modules*

$$\Lambda^r(V_{2^{n-1}+s}) \cong_{\mathrm{proj}} \bigoplus_{\substack{i,j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i+j}\big(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^{n-1}-s})\big).$$

The case $s = 0$ is a little unnatural, but we need it for the induction, because the restriction of $V_{2^{n-1}+1}$ is $V_{2^{n-2}+1} \oplus V_{2^{n-2}}$.

It is sometimes more succinct to consider Hilbert series with coefficients in the Green ring (possibly modulo projectives or induced modules). For more details see [12]. In particular, we consider the following series associated to a kG-module V:

$$\lambda_t(V)=\sum_{r=0}^{\infty}\Lambda^r(V)t^r, \qquad \sigma_t(V)=\sum_{r=0}^{\infty}S^r(V)t^r,$$
$$\widetilde{\sigma}_t(V)=\sum_{r=0}^{\infty}\widetilde{S}^r(V)t^r, \qquad \lambda_t^{\Omega}(V)=\sum_{r=0}^{\infty}\Omega^r\Lambda^r(V)t^r.$$

The last of these requires G to be specified in order for the Ω to be determined; it is naturally considered modulo projectives. They all commute with restriction and turn direct sums of modules into products of series. This is all an easy consequence of the corresponding properties of the corresponding functors on modules, except perhaps for $\lambda_t^{\Omega}(V\oplus W)$, where we need the formula $\Omega^rV\otimes\Omega^sW\cong_{\mathrm{proj}}\Omega^{r+s}(V\otimes W)$.

Many of our statements about modules imply Hilbert series versions.

$$\begin{aligned}
\sigma_t(V_{2^{n-1}+s}) &=_{\mathrm{ind}} \lambda_t^{\Omega}(V_{2^{n-1}-s})\bigl(1-t^{2^n}\bigr)^{-1} && \text{Theorem 1.2}\\
\widetilde{\sigma}_t(V_{2^{n-1}+s}) &=_{\mathrm{ind}} \lambda_{t^2}^{\Omega}(V_s)\sigma_t(V_{2^{n-1}+s}) && \text{Separation 4.1 (a)}\\
\widetilde{\sigma}_t(V_{2^{n-1}+s}) &=_{\mathrm{ind}} \lambda_t(V_{2^{n-1}+s})\bigl(1-t^{2^n}\bigr)^{-1} && \text{Splitting and periodicity 4.1 (b), (c)}\\
\lambda_t(V_{2^{n-1}+s}) &=_{\mathrm{proj}} \lambda_{t^2}^{\Omega}(V_s)\lambda_t^{\Omega}(V_{2^{n-1}-s}) && \text{Exterior powers 4.1 (d),}
\end{aligned} \tag{3}$$

where the symbols $=_{\mathrm{ind}}$ and $=_{\mathrm{proj}}$ mean that we consider the coefficients only modulo induced or projective direct summands, respectively. The first and last of the above identities are, in fact, equivalent to the original versions. The second identity follows from Theorem 4.1 (a), Lemma 2.5 and Lemma 2.7 (once the theorem is proved).

Remark. An easy calculation shows that, for fixed n and s, the last of the formulas in (3) follows formally from the first three if we are satisfied with only $=_{\mathrm{ind}}$.

Remark. The proof of Theorem 1.2 given in [21] actually gives a more precise formula than the first one in (3). It works by showing that the complex $K(V_{2^n},V_{2^{n-1}-s})$ defined in [21] is separated and then applying Lemma 2.5; note that the definition of $K(V_{2^n},V_{2^{n-1}-s})$ in [21] is different from our Definition 2.6. The result is that $\sigma_t(V_{2^{n-1}+s})=_{\mathrm{proj}}\sigma_t(V_{2^n})\lambda_t^{\Omega}(V_{2^{n-1}-s})$. Since V_{2^n} can be given a basis that is permuted by G, each $S^r(V_{2^n})$ has a monomial basis that is permuted. For small n, the decomposition of $\sigma_t(V_{2^n})$ can be calculated by hand; in general the calculation can be organized using [21, Proposition 2.2]. Alternatively, [21, Proposition 2.2] can be applied directly to $\sigma_t(V_{2^{n-1}+s})$.

The next six sections are devoted to the proof of Theorem 4.1 by induction on n.

5. The case $n=1$

In this section we start the inductive proof of Theorem 4.1. Suppose that $n=1$, so we have to prove the statements in Theorem 4.1 for $s\in\{0,1\}$. With these assumptions on n

and s, parts (b)–(d) of Theorem 4.1 can easily be verified by a direct calculation. In fact, in (b) one obtains isomorphisms of kG-modules (not only modulo induced summands), and in (c) one gets $\widetilde{I} = 0$. Separation for $s = 0$ is trivial.

Let us consider part (a) for $n = s = 1$. We have to show that for each $r > 0$ the short exact sequence $0 \to S^{r-2}(V_2) \to S^r(V_2) \to \widetilde{S}^r(V_2) \to 0$ of kG-modules is separated at $S^r(V_2)$. If r is odd, then $S^r(V_2)$ is induced by Theorem 1.2, hence projective, and so separation is obviously true. Separation is trivial for $r = 0$. For even $r > 0$, a direct calculation and Theorem 1.2 show that $\widetilde{S}^r(V_2) \cong V_1 \oplus V_1 \cong_{\mathrm{ind}} S^r(V_2) \oplus \Omega_2^{-1} S^{r-2}(V_2)$, and so separation follows from Lemma 3.4.

Sections 6–10 comprise the inductive step in the proof of Theorem 4.1. In these sections we always assume that $n > 1$ is an integer and that Theorem 4.1 holds for all smaller values of n. Throughout these sections the notation remains the same as in Sections 3 and 4; thus $G = \langle g \rangle \cong C_{2^n}$ is a cyclic group of order 2^n, $k = \mathbb{F}_2$ is a field with two elements and s is an integer such that $0 \leqslant s \leqslant 2^{n-1}$.

6. Periodicity

In this section we prove part (b) of Theorem 4.1, assuming that parts (a)–(d) of the theorem hold for all smaller values of n.

Let H be the unique maximal subgroup of G and let $\{x_1, x_2, \dots, x_{top}\}$ be a k-basis of $V_{2^{n-1}+s}$ as in Section 4. We choose G-invariant elements $a \in S^{2^n}(V_{2^{n-1}+s})$ and $\widetilde{a} \in \widetilde{S}^{2^n}(V_{2^{n-1}+s})$ as in Section 4. Let $T(V_{2^{n-1}+s})$ be the kG-submodule of $S(V_{2^{n-1}+s})$ spanned by the monomials in $x_1, \dots, x_{top}$ that are not divisible by $x_{top}^{2^n}$. We have $S(V_{2^{n-1}+s}) \cong k[a] \otimes T(V_{2^{n-1}+s})$ as kG-modules; see [21, Lemma 1.1]. So $T^{<2^n}(V_{2^{n-1}+s}) = S^{<2^n}(V_{2^{n-1}+s})$. Notice that the periodicity of $S(V_{2^{n-1}+s})$ in [21, Theorem 1.2] is equivalent to $T^{\geqslant 2^n}(V_{2^{n-1}+s})$ being induced. In fact, we know something stronger from [21, Corollary 3.11], namely that $T^{>2^{n-1}-s}(V_{2^{n-1}+s})$ is induced.

We can make the same construction for $\widetilde{S}(V_{2^{n-1}+s})$, obtaining $\widetilde{S}(V_{2^{n-1}+s}) \cong k[\widetilde{a}] \otimes \widetilde{T}(V_{2^{n-1}+s})$ as kG-modules.

Define $L(V_{2^{n-1}+s}, V_s^2)$ to be the subcomplex of $K(V_{2^{n-1}+s}, V_s^2)$ defined using $T(V_{2^{n-1}+s})$ instead of $S(V_{2^{n-1}+s})$, that is

$$\cdots \xrightarrow{d} T(V_{2^{n-1}+s}) \otimes \Lambda^2(V_s) \xrightarrow{d} T(V_{2^{n-1}+s}) \otimes V_s \xrightarrow{d} T(V_{2^{n-1}+s}),$$

where the boundary morphisms are as in Definition 2.6 (this can be done since the x_{top} used in the definition of $T(V_{2^{n-1}+s})$ is not contained in V_s). Thus $L(V_{2^{n-1}+s}, V_s^2)$ is a complex of graded kG-modules; it is exact except in degree 0, where the homology is $H_0(L(V_{2^{n-1}+s}, V_s^2))$, which is isomorphic to $\widetilde{T}(V_{2^{n-1}+s})$ as a kG-module. Notice that, by construction, the complexes $K(V_{2^{n-1}+s}, V_s^2)$ and $k[a] \otimes L(V_{2^{n-1}+s}, V_s^2)$ of kG-modules are isomorphic. In particular, note for later use that one of them is separated (over G or over H) if and only if the other is so too.

From now on we fix s and abbreviate the notation to just S, T, K, L, etc.

Suppose that $s < 2^{n-1}$. We claim that $L_i^r = T^{r-2i} \otimes \Lambda^i(V_s)$ is induced for all $i \geqslant 0$, $r \geqslant 2^n$. We may assume that $i \leqslant s$. Then $r - 2i \geqslant 2^n - 2s > 2^{n-1} - s$ and so T^{r-2i} is induced. Thus L^r is a complex of induced kG-modules for each $r \geqslant 2^n$.

Consider the restriction of the complex K to the subgroup H. It decomposes as a tensor product of two complexes, by Lemma 2.8. Each of these is separated, by induction and Theorem 4.1 (a), hence so is their product, by Lemma 2.2. It follows that for each $r \geqslant 0$, the complex L^r is separated on restriction to H. We have just seen that L^r is a complex of induced modules for all $r \geqslant 2^n$. Thus, for each $r \geqslant 2^n$, the complex L^r is separated, by Proposition 2.4. Now Lemma 2.5 shows that $H_0(L^{\geqslant 2^n})$ is induced. But this is exactly $\widetilde{T}^{\geqslant 2^n}$, so $\widetilde{S}(V_{2^{n-1}+s}) \cong k[\widetilde{a}] \otimes \widetilde{T} \cong_{\text{ind}} k[\widetilde{a}] \otimes \widetilde{T}^{<2^n} = k[\widetilde{a}] \otimes \widetilde{S}^{<2^n}$ is periodic if $s < 2^{n-1}$.

Now suppose that $s = 2^{n-1}$. By the same argument as for $s < 2^{n-1}$, we see that $\widetilde{T}^{>2^n}$ is induced. To complete the proof of Theorem 4.1 (b) we have to show that $\widetilde{S}^{2^n} \cong_{\text{ind}} k\widetilde{a} \oplus k\widetilde{b}$. Set $y_i := g^{2^n-i}x_{2^n}$ for $i = 1, 2, \ldots, 2^n$, so $\{y_1, \ldots, y_{2^n}\}$ is a k-basis of V_{2^n} which is permuted by G. A basis for $V_{2^{n-1}} < V_{2^n}$ is given by $g^{2^{n-1}}y_i - y_i = y_{i+2^{n-1}} - y_i$ for $i = 1, 2, \ldots, 2^{n-1}$. Write $\widetilde{y}_i$ for the image of y_i in $\widetilde{S}(V_{2^n})$, so $\widetilde{y}_i^2 = \widetilde{y}_{i+2^{n-1}}^2$ for $i = 1, \ldots, 2^{n-1}$. The set consisting of all monomials of degree 2^n in all the $\widetilde{y}_i$ such that $\widetilde{y}_1, \ldots, \widetilde{y}_{2^{n-1}}$ only occur to the power at most 1 forms a k-basis for $\widetilde{S}(V_{2^n})$. The group G permutes these monomials and it is straightforward to check that there are two invariant monomials, namely $\widetilde{y}_1\widetilde{y}_2 \cdots \widetilde{y}_{2^n} = \widetilde{b}$ and $\widetilde{y}_{2^{n-1}+1}^2\widetilde{y}_{2^{n-1}+2}^2 \cdots \widetilde{y}_{2^n}^2 = \widetilde{a}$; the rest span induced submodules. This completes the proof of periodicity.

7. Splitting

In this section we prove part (c) of Theorem 4.1, assuming the whole of the theorem for smaller n.

Let H be the unique maximal subgroup of G and $\{x_1, x_2, \ldots, x_{top}\}$ a k-basis of $V_{2^{n-1}+s}$ as in Section 4. As in Theorem 4.1 we write $\widetilde{N}(V_{2^{n-1}+s})$ for the kernel of the natural surjection $\widetilde{S}(V_{2^{n-1}+s}) \xrightarrow{f} \Lambda(V_{2^{n-1}+s})$. The following proposition deals with the structure of $\widetilde{S}(V_{2^{n-1}+s})$ in degrees less than 2^n.

Proposition 7.1. *For any integer s such that $0 \leqslant s \leqslant 2^{n-1}$, the short exact sequence*

$$0 \to \widetilde{N}^{<2^n}(V_{2^{n-1}+s}) \to \widetilde{S}^{<2^n}(V_{2^{n-1}+s}) \xrightarrow{f} \Lambda^{<2^n}(V_{2^{n-1}+s}) \to 0 \tag{4}$$

of graded kG-modules is split, and $\widetilde{N}^{<2^n}(V_{2^{n-1}+s})$ is induced from H.

Before starting with the proof of Proposition 7.1 we introduce some further notation. As described at the beginning of Section 3, we have $V_{2^{n-1}+s}\downarrow_H^G = V_{2^{n-2}+s'} \oplus V_{2^{n-2}+s''}$ where $0 \leqslant s', s'' \leqslant 2^{n-2}$ and $s' = s''$ or $s' = s'' + 1$. The kH-submodule $V_{2^{n-2}+s'}$ of $V_{2^{n-1}+s}$ has the k-basis $\{x_{top}, x_{top-2}, x_{top-4}, \ldots\}$ and the kH-submodule $V_{2^{n-2}+s''}$ has the k-basis $\{x_{top-1}, x_{top-3}, x_{top-5}, \ldots\}$. We write $\widetilde{S}'(V_{2^{n-2}+s'})$ and $\widetilde{S}''(V_{2^{n-2}+s''})$ for

$S(V_{2^{n-2}+s'})/(V_{s'}^2)$ and $S(V_{2^{n-2}+s''})/(V_{s''}^2)$, respectively. So the x_i with odd i and the x_i with even i provide natural embeddings $\widetilde{S}'(V_{2^{n-2}+s'}) \to \widetilde{S}(V_{2^{n-1}+s})$ and $\widetilde{S}''(V_{2^{n-2}+s''}) \to \widetilde{S}(V_{2^{n-1}+s})$ of kH-modules, and we have

$$\widetilde{S}(V_{2^{n-1}+s}) \cong \widetilde{S}'(V_{2^{n-2}+s'}) \otimes \widetilde{S}''(V_{2^{n-2}+s''})$$

as kH-modules, where the isomorphism is given by $f_1 \otimes f_2 \mapsto f_1 \cdot f_2$.

Choose $a' \in S(V_{2^{n-2}+s'})$, $\widetilde{a}' \in \widetilde{S}(V_{2^{n-2}+s'})$ according to the description preceding Theorem 4.1, but working over H. Thus a' is homogeneous of degree 2^{n-1}, has degree 2^{n-1} when considered as a polynomial in x_{top}. Furthermore, a' and $\widetilde{a}'$ are invariant under the action of H and $\widetilde{a}'$ has image 0 in $\Lambda(V_{2^{n-2}+s'})$. Similarly, choose $a'' \in S(V_{2^{n-2}+s''})$ and $\widetilde{a}'' \in \widetilde{S}''(V_{2^{n-2}+s''})$. So a'' is homogeneous of degree 2^{n-1}, has degree 2^{n-1} when considered as a polynomial in x_{top-1} and a'' and $\widetilde{a}''$ are invariant under the action of H and $\widetilde{a}''$ has image 0 in $\Lambda(V_{2^{n-2}+s''})$.

By induction, $\widetilde{a}'$ and $\widetilde{a}''$ are periodicity generators of $\widetilde{S}'(V_{2^{n-2}+s'})$ and $\widetilde{S}''(V_{2^{n-2}+s''})$, respectively. That is, we have

$$\widetilde{S}'(V_{2^{n-2}+s'}) \cong_{\text{ind}} k[\widetilde{a}'] \otimes \widetilde{S}'^{\,<2^{n-1}}(V_{2^{n-2}+s''}) \quad \text{and}$$
$$\widetilde{S}''(V_{2^{n-2}+s''}) \cong_{\text{ind}} k[\widetilde{a}''] \otimes \widetilde{S}''^{\,<2^{n-1}}(V_{2^{n-2}+s''})$$

or the variant with $\widetilde{b}'$ or $\widetilde{b}''$ if $s' = 2^{n-2}$ or $s'' = 2^{n-2}$.

Lemma 7.2. *Let s be an integer such that $0 \leqslant s \leqslant 2^{n-1}$ and let $\widetilde{a}'$ be a periodicity generator for $\widetilde{S}'(V_{2^{n-2}+s'})$ as above. Then*

$$\widetilde{S}^{<2^n}(V_{2^{n-1}+s})\downarrow_H^G = \widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s}) \oplus \widetilde{a}'\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s}) \oplus (g\widetilde{a}')\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s}) \oplus \widetilde{X}$$

as kH-modules, where the kH-submodule $\widetilde{X}$ is generated as a k-vector space by the images of all monomials $x \in \bigoplus_{r=2^{n-1}}^{2^n-1} S^r(V_{2^{n-1}+s})$ such that x has degree strictly less than 2^{n-1} when considered as a polynomial in x_{top} and x has degree strictly less than 2^{n-1} when considered as a polynomial in x_{top-1}.

Proof. We give all monomials in $S(V_{2^{n-1}+s})$ the lexicographic order with $x_{top-1} > x_{top} > x_{top-2} > \cdots > x_1$. Let $h \in \bigoplus_{r=2^{n-1}}^{2^n-1} S^r(V_{2^{n-1}+s})$. Since $g\widetilde{a}'$ has leading term $x_{top-1}^{2^{n-1}}$ we can write h as $h = h_1 \cdot g\widetilde{a}' + h_2$ where $h_1 \in \widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})$ and $h_2 \in \widetilde{S}^{<2^n}(V_{2^{n-1}+s})$ such that h_2 has degree $< 2^{n-1}$ when considered as a polynomial in x_{top-1}. Then, because $\widetilde{a}'$ has leading term $x_{top}^{2^{n-1}}$ and only involves monomials in $x_{top}, x_{top-2}, x_{top-4}, \ldots$, we can find $h_3 \in \widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})$ and $h_4 \in \widetilde{X}$ such that $h_2 = h_3 \cdot \widetilde{a}' + h_4$. Thus, $\bigoplus_{r=2^{n-1}}^{2^n-1} S^r(V_{2^{n-1}+s})$ is the sum of $\widetilde{a}'\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})$, $(g\widetilde{a}')\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})$ and $\widetilde{X}$. Comparing dimensions, we see that this sum has to be direct and Lemma 7.2 follows. □

We are now ready to prove Proposition 7.1.

Proof of Proposition 7.1. We study the restriction of the sequence (4) to the maximal subgroup H. By Lemma 7.2, the middle term is

$$\widetilde{S}^{<2^n}(V_{2^{n-1}+s})\downarrow_H^G = \widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s}) \oplus \widetilde{J} \oplus \widetilde{X},$$

where $\widetilde{J} := \widetilde{a}'\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s}) \oplus (g\widetilde{a}')\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})$. Owing to the choice of $\widetilde{a}'$, we have $\widetilde{a}' \in \widetilde{N}^{<2^n}(V_{2^{n-1}+s})$ and therefore $\widetilde{J} \subseteq \widetilde{N}^{<2^n}(V_{2^{n-1}+s})$. In fact, by construction, $\widetilde{J}$ is a kG-submodule of $\bigoplus_{r=2^{n-1}}^{2^n-1} \widetilde{N}^r(V_{2^{n-1}+s})$ and is induced from H. We consider the exact sequence of kG-modules

$$0 \to \widetilde{J} \to \bigoplus_{r=2^{n-1}}^{2^n-1} \widetilde{S}^r(V_{2^{n-1}+s}) \to \overline{X} \to 0, \tag{5}$$

where $\overline{X} := \bigoplus_{r=2^{n-1}}^{2^n-1} \widetilde{S}^r(V_{2^{n-1}+s})/\widetilde{J}$. We know from Lemma 7.2 that the sequence (5) is split when restricted to H and $\overline{X}\downarrow_H^G \cong \widetilde{X}$ as kH-modules. Since $\widetilde{J}$ is induced from H it is relatively H-injective, and so the sequence (5) splits over kG (see [8, Theorem 19.2]). Thus $\widetilde{J}$ is a direct summand of $\widetilde{S}^{<2^n}(V_{2^{n-1}+s})$ over kG, so there is a kG-submodule $\widetilde{J}''$ of $\widetilde{S}^{<2^n}(V_{2^{n-1}+s})$ such that $\widetilde{S}^{<2^n}(V_{2^{n-1}+s}) = \widetilde{J} \oplus \widetilde{J}''$. Since $\widetilde{J} \subseteq \widetilde{N}^{<2^n}(V_{2^{n-1}+s})$, it follows that $\widetilde{N}^{<2^n}(V_{2^{n-1}+s}) = \widetilde{J} \oplus \widetilde{J}'$, where $\widetilde{J}' := \widetilde{J}'' \cap \widetilde{N}^{<2^n}(V_{2^{n-1}+s})$. We have

$$\widetilde{J}\downarrow_H^G = \widetilde{a}'\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})\downarrow_H^G \oplus \big(g\widetilde{a}'\big)\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})\downarrow_H^G.$$

Because $\widetilde{S}^{<2^{n-1}}(V_{2^{n-1}+s})\downarrow_H^G \cong (\widetilde{S}'(V_{2^{n-2}+s'}) \otimes \widetilde{S}''(V_{2^{n-2}+s''}))^{<2^{n-1}}$, we have, by induction and ignoring the grading,

$$\widetilde{J}\downarrow_H^G \cong_{\mathrm{ind}} \big(\Lambda' \otimes \Lambda''\big)^{<2^{n-1}} \oplus \big(\Lambda' \otimes \Lambda''\big)^{<2^{n-1}}. \tag{6}$$

Here we write $\Lambda' \otimes \Lambda''$ for the graded kH-module $\Lambda(V_{2^{n-2}+s'}) \otimes \Lambda(V_{2^{n-2}+s''})$. Restricting the sequence (4) to H, we obtain the sequence

$$0 \to \widetilde{N}^{<2^n}(V_{2^{n-1}+s})\downarrow_H^G \to \big(\widetilde{S}'(V_{2^{n-2}+s'}) \otimes \widetilde{S}''(V_{2^{n-2}+s''})\big)^{<2^n} \to \big(\Lambda' \otimes \Lambda''\big)^{<2^n} \to 0$$

of kH-modules, which is split by induction. Thus, by induction again, we obtain

$$\widetilde{N}^{<2^n}(V_{2^{n-1}+s})\downarrow_H^G \cong_{\mathrm{ind}} \widetilde{a}'\big(\Lambda' \otimes \Lambda''\big)^{<2^{n-1}} \oplus \widetilde{a}''\big(\Lambda' \otimes \Lambda''\big)^{<2^{n-1}}. \tag{7}$$

Eqs. (6) and (7) imply that $\widetilde{J}\downarrow_H^G \oplus \widetilde{J}'\downarrow_H^G \cong (\widetilde{J} \oplus \widetilde{J}')\downarrow_H^G \cong \widetilde{N}^{<2^n}(V_{2^{n-1}+s})\downarrow_H^G \cong_{\mathrm{ind}} \widetilde{J}\downarrow_H^G$. It follows that $\widetilde{J}'\downarrow_H^G$ is induced from proper subgroups of H. By Lemma 3.1 the kG-module $\widetilde{J}'$ is induced from H, and hence $\widetilde{N}^{<2^n}(V_{2^{n-1}+s}) = \widetilde{J} \oplus \widetilde{J}'$ is induced from H. We have just seen that sequence (4) is split on restriction to H; since $\widetilde{N}^{<2^n}(V_{2^{n-1}+s})$ is relatively H-injective the sequence must split over kG. This completes the proof of Proposition 7.1. □

The following corollary provides a connection between $\widetilde{S}(V_{2^{n-1}+s})$ and the exterior powers of $V_{2^{n-1}+s}$ in degrees less than 2^n.

Corollary 7.3. *For r and s integers such that $0 \leqslant s \leqslant 2^{n-1}$ and $0 \leqslant r < 2^n$, the map f induces an isomorphism of kG-modules modulo induced summands*

$$\widetilde{S}^r(V_{2^{n-1}+s}) \cong_{\mathrm{ind}} \Lambda^r(V_{2^{n-1}+s}).$$

Proof. This is clear from Proposition 7.1 (for $n > 1$) and Section 5 (for $n = 1$). □

We can now prove Theorem 4.1 (c). For $s < 2^{n-1}$ we have $\widetilde{S}^{<2^n}(V_{2^{n-1}+s}) \cong \Lambda(V_{2^{n-1}+s}) \oplus X$, where X is induced, so part (c) of Theorem 4.1 follows from part (b). For $s = 2^{n-1}$ we have $\widetilde{S}^{<2^n}(V_{2^{n-1}+s}) \oplus k\widetilde{b} \cong \Lambda(V_{2^{n-1}+s}) \oplus X'$, where X' is induced. Note that $\widetilde{b}$ maps to a generator of $\Lambda^{2^n}(V_{2^{n-1}+s})$. Again, part (c) of Theorem 4.1 is a consequence of (b).

8. Preparation for separation

In this section we prepare for the proof of part (a) of Theorem 4.1, assuming the whole of the theorem for smaller n.

Let H be the unique maximal subgroup of G and let $\{x_1, x_2, \ldots, x_{top}\}$ be a k-basis of $V_{2^{n-1}+s}$ as in Section 4. The main goal of this section is to develop useful criteria for the complex $K(V_{2^{n-1}+s}, V_s^2)$ to be separated.

Lemma 8.1. *Let r, s be non-negative integers such that $0 \leqslant s \leqslant 2^{n-1}$. Suppose that for each $0 \leqslant r' < r$ with $r' \equiv r \bmod 2$, the complex $K^{r'}(V_{2^{n-1}+s}, V_s^2)$ is separated at $K_0^{r'}(V_{2^{n-1}+s}, V_s^2) = S^{r'}(V_{2^{n-1}+s})$. Then $K^r(V_{2^{n-1}+s}, V_s^2)$ is separated at $K_i^r(V_{2^{n-1}+s}, V_s^2) = S^{r-2i}(V_{2^{n-1}+s}) \otimes \Lambda^i(V_s)$ for all $i \geqslant 1$. The same is true when K and S are replaced by L and T from Section 6.*

Proof. We only demonstrate the proof for K and S; the proof for L and T is analogous.

We write $V := V_{2^{n-1}+s}$ and $W := V_s$ for short. Fix $i \geqslant 1$ and consider the boundary morphism $d_{i+1} : S^{r-2i-2}(V) \otimes \Lambda^{i+1}(W) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ in $K^r(V, W^2)$. We have to show that $\mathrm{Im}(d_{i+1}) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ factors through a projective kG-module. Since $K^{r-2i}(V, W^2)$ is separated at $K_0^{r-2i}(V, W^2)$ the inclusion $(W^2)^{r-2i} \to S^{r-2i}(V)$ factors through a projective kG-module P^{r-2i}. We can write the inclusion $\mathrm{Im}(d_{i+1}) \to S^{r-2i}(V) \otimes \Lambda^i(W)$ as a composition of inclusions

$$\mathrm{Im}(d_{i+1}) \to \left(W^2\right)^{r-2i} \otimes \Lambda^i(W) \to S^{r-2i}(V) \otimes \Lambda^i(W),$$

where the last map factors through the projective kG-module $P^{r-2i} \otimes \Lambda^i(W)$. □

Lemma 8.2. *Let s and r be integers such that $0 \leqslant s \leqslant 2^{n-1}$ and $0 < r < 2^n$, and suppose that the complex $K^i(V_{2^{n-1}+s}, V_s^2)$ is separated for all $0 \leqslant i < r$. Then the following statements are equivalent:*

(a) *$K^r(V_{2^{n-1}+s}, V_s^2)$ is separated,*
(b) *the natural map $S^r(V_{2^{n-1}+s}) \xrightarrow{g} \Lambda^r(V_{2^{n-1}+s})$ is split injective modulo induced summands,*
(c) *$\Lambda^r(V_{2^{n-1}+s}) \cong_{\mathrm{ind}} \bigoplus_{\substack{i,j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i+j}(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^{n-1}-s}))$.*

Proof. We write $S^r := S^r(V_{2^{n-1}+s})$, $\widetilde{S}^r := \widetilde{S}^r(V_{2^{n-1}+s})$ and $K^i := K^i(V_{2^{n-1}+s}, V_s^2)$. The conditions on K^i and Lemma 8.1 show that K^r is separated except, perhaps, at $K_0^r = S^r$. The restriction of the complex K^r to H decomposes as a tensor product of two complexes, by Lemma 2.8. Each of these is separated by our continuing induction hypothesis, hence so is their product, by Lemma 2.2, and so K^r is separated on restriction to H. Thus the short exact sequence

$$0 \to \mathrm{Im}(d_1) \xrightarrow{i} S^r \xrightarrow{j} \widetilde{S}^r \to 0 \tag{8}$$

from K^r is separated at S^r on restriction to H (the maps i, j should not be confused with the indices in part (c) of the lemma). The separation of K^r in positive (complex-) degrees and Lemma 2.5 yield the formula $\mathrm{Im}(d_1) \cong_{\mathrm{proj}} \bigoplus_{\substack{i\geqslant 1,\, j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i-1}(\Lambda^i(V_s)) \otimes S^j(V_{2^{n-1}+s})$. Theorem 1.2 now shows that

$$\mathrm{Im}(d_1) \cong_{\mathrm{ind}} \bigoplus_{\substack{i\geqslant 1,\, j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i+j-1}\bigl(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^{n-1}-s})\bigr). \tag{9}$$

(a) ⇒ (b) Let $f : \widetilde{S}^r \to \Lambda^r$ be the natural surjection, so $g = f \circ j$. By Proposition 7.1, the map f is split injective modulo induced summands, and, by Lemma 3.9, it is enough to show that j is split injective modulo induced summands. By assumption, $S^r = X \oplus M$ for some submodules X and M of S^r such that X is projective and $\ker(j) = \mathrm{Im}(d_1) \subseteq X$. Let $j' : S^r \to X$ be the projection onto X and $u' : X \to S^r$ the natural embedding. Define $u : \widetilde{S}^r = j(X) \oplus j(M) \to S^r, j(x) + j(m) \mapsto m$ (note that the restriction of j to M is injective). Then $u \circ j + u' \circ j' = \mathrm{Id}_{S^r}$ and so j is split injective modulo induced summands.

(b) ⇒ (c) Assume (b). The factorization $g = f \circ j$ and Lemma 3.9 (b) imply that j is also split injective modulo induced summands. Write $S^r = A' \oplus A''$, where A' has only non-induced summands and A'' is induced. By Lemma 3.10, the restriction of j to A' is split injective, so j maps A' injectively into $\widetilde{S}^r$ and $j(A')$ is a direct summand of $\widetilde{S}^r$. Factoring out A' and $j(A')$ in (8) we obtain the short exact sequence $0 \to \mathrm{Im}(d_1) \xrightarrow{\bar{i}} S^r/A' \xrightarrow{\bar{j}} \widetilde{S}^r/j(A') \to 0$.

As we have seen at the beginning of the proof, i factors through a projective on restriction to H, and so the same is true for $\bar{i}$. Thus the complex $\mathrm{Im}(d_1) \to S^r/A'$ is separated on restriction to H. Because $S^r/A' \cong A''$ is induced, the complex is separated, by Proposition 2.4. Lemma 2.5 yields $\widetilde{S}^r/j(A') \cong_{\mathrm{proj}} S^r/A' \oplus \Omega_{2^n} \mathrm{Im}(d_1) \cong_{\mathrm{ind}} \Omega_{2^n} \mathrm{Im}(d_1)$. Using (9) we obtain

$$\widetilde{S}^r/j(A') \cong_{\mathrm{ind}} \bigoplus_{\substack{i\geqslant 1,\, j\geqslant 0\\ 2i+j=r}} \Omega_{2^n}^{i+j}\big(\Lambda^i(V_s) \otimes \Lambda^j(V_{2^{n-1}-s})\big). \tag{10}$$

Theorem 1.2 implies $j(A') \cong A' \cong_{\mathrm{ind}} S^r \cong_{\mathrm{ind}} \Omega_{2^n}^r \Lambda^r(V_{2^{n-1}-s})$. Adding the summand $j(A')$ to both sides of (10) and using Corollary 7.3 gives us the formula in (c).

(c) ⇒ (a) Assume that (c) holds. From Corollary 7.3, Theorem 1.2 and (9) we get $\widetilde{S}^r \cong_{\mathrm{ind}} S^r \oplus \Omega_{2^n}^{-1} \mathrm{Im}(d_1)$. Separation of K^r now follows from applying Lemma 3.4 to the short exact sequence (8). □

Separation of $K^r(V_{2^{n-1}+s}, V_s^2)$ for $r = 0, 1$ is trivial. We will now prove it for $r = 2$. Notice that if a non-zero map $V_a \to V_b$ of kG-modules is to factor through a projective over C_{2^n}, then we must have $a + b > 2^n$. This is because the map must factor through the projective cover $V_{2^n} \twoheadrightarrow V_b$, which has kernel V_{2^n-b}, into which V_a will certainly be mapped if $a \leqslant 2^n - b$.

Lemma 8.3. *For any integer s such that $0 \leqslant s \leqslant 2^{n-1}$ the complex $K^2(V_{2^{n-1}+s}, V_s^2)$ is separated.*

Proof. The complex in question is $V_s \hookrightarrow S^2(V_{2^{n-1}+s})$. By induction, the map factors through a projective on restriction to H. Write $S^2(V_{2^{n-1}+s}) = A' \oplus A''$, where A' has only non-induced summands and A'' has only induced summands. The component $V_s \to A''$ factors through a projective, by Proposition 2.4.

We claim that the component $V_s \to A'$ must be 0. From Theorem 1.2, we know that $S^2(V_{2^{n-1}+s}) =_{\mathrm{ind}} \Lambda^2(V_{2^{n-1}-s})$; but $V_{2^{n-1}-s}$ is a module for $C_{2^{n-1}}$, and it follows that A' contains only summands of dimension $\leqslant 2^{n-1}$. Let V_t be such a summand, so $t \leqslant 2^{n-1}$ and suppose that there is a non-zero component $V_s \to V_t$. It must factor through a projective on restriction, where it is a map $V_{s'} \oplus V_{s''} \to V_{t'} \oplus V_{t''}$, with $s', s'', t', t'' \leqslant 2^{n-2}$. By the discussion above, none of the components can factor through a projective module over $C_{2^{n-1}}$ unless they are 0. □

We can readily prove separation when s is even.

Lemma 8.4. *For any even integer s such that $0 \leqslant s \leqslant 2^{n-1}$, the complex $K(V_{2^{n-1}+s}, V_s^2)$ of kG-modules is separated.*

Proof. Write $s = 2s'$. From Lemma 2.9 we know that $K(V_{2^{n-1}+s}, V_s^2) \cong K(V_{2^{n-2}+s'}, V_{s'}^2)\uparrow_H^{\otimes G}$. The right hand side is separated by Lemma 2.3 and our induction hypothesis. $\square$

In view of this lemma, we assume now that s is odd.

Lemma 8.5. *Let s be an odd integer such that $0 < s < 2^{n-1}$. Then (given our induction hypothesis):*

(a) $\Lambda^r(V_{2^{n-1}-s}) \in c(G)$ *for all* $r \geqslant 0$ *and*
(b) $S^r(V_{2^{n-1}+s}) \in c(G)$ *for all* $0 \leqslant r < 2^{n-1}$.

Here $c(G)$ is the subgroup of the Green ring in Lemma 3.11.

Proof. For part (a), the dimension $d = 2^{n-1} - s$ of the module is in the range where we know that our formula for exterior powers (see Theorem 4.1 (d)) is valid by our continuing induction hypothesis. The statement is clearly true for $d = 1$ and we can employ induction on d, using the formula and the properties of $c(G)$ in Lemma 3.11.

For part (b) we use the formula $\sigma_t(V_{2^{n-1}+s}) =_{\text{proj}} \sigma_t(V_{2^n})\lambda_t^{\Omega}(V_{2^{n-1}-s})$ from the remark at the end of Section 4 and part (a). The summands of $S(V_{2^n})$ are permutation modules on a monomial basis, so are in $c(G)$ unless the stabilizer of a monomial is of index 2. But this first happens in degree 2^{n-1}, because if a monomial fixed by a subgroup of order 2^{n-1} contains y_i, it must also contain all 2^{n-1} elements of the orbit of y_i. $\square$

9. Separation

First we make some general constructions related to symmetric and exterior powers of vector spaces. It is convenient to do this integrally first and then reduce modulo 2. Let U be a free module over the integers localized at 2, $\mathbb{Z}_{(2)}$. For $r \geqslant 0$ set

$$T^r(U) = U \otimes_{\mathbb{Z}_{(2)}} \cdots \otimes_{\mathbb{Z}_{(2)}} U \quad (r \text{ times}).$$

Let the symmetric group Σ_r act on $T^r(U)$ by permuting the factors. Factoring out the action of Σ_r we get $S^r(U) = T^r(U)/\Sigma_r = T^r(U) \otimes_{\mathbb{Z}_{(2)}\Sigma_r} \mathbb{Z}_{(2)}$. We can also let Σ_r act on $T^r(U)$ by permuting the factors and multiplying by the signature of the permutation, in which case we write $T^r(U)_\sigma$. Similarly, on factoring out the action of Σ_r we obtain $\Lambda^r(U) = T^r(U)_\sigma/\Sigma_r$.

For any subset $I \subseteq \{1, \ldots, r\}$ we set $\Sigma_I := \{\pi \in \Sigma_r \mid \pi(i) = i \text{ for all } i \notin I\}$ (so Σ_I is a subgroup of Σ_r isomorphic to $\Sigma_{|I|}$). For $r \geqslant 2$, write $r = 2^p + t$ with $1 \leqslant t \leqslant 2^p$. Consider the subgroup

$$Q_r := \begin{cases} \Sigma_{\{1,\ldots,2^p\}} \times \Sigma_{\{2^p+1,\ldots,r\}} & \text{if } t < 2^p \\ (\Sigma_{\{1,\ldots,2^p\}} \times \Sigma_{\{2^p+1,\ldots,2^{p+1}\}}) \rtimes \langle\tau\rangle & \text{if } t = 2^p \end{cases}$$

of Σ_r, where $\tau \in \Sigma_r$ is the involution mapping i to $2^p + i$ for $i = 1, 2, \ldots, 2^p$. The importance of Q_r lies the fact that the index $|\Sigma_r : Q_r|$ is odd. This can been seen as follows: $|\Sigma_{2^p+t}|/(|\Sigma_{2^p}| \cdot |\Sigma_t|) = \binom{2^p+t}{t}$, which is equal to the coefficient of x^t in $(1+x)^{2^p+t} \equiv (1+x^{2^p})(1+x)^t \pmod 2$; also $(1+x)^{2^{p+1}} = (1+x^{2^p}+2X)^2 \equiv 1+2x^{2^p}+x^{2^{p+1}}$ (mod 4).

Define $L_S^r(U) := T^r(U)/Q_r$ and $L_\Lambda^r(U) := T^r(U)_\sigma/Q_r$. There are natural quotient maps $q_S : L_S^r(U) \to S^r(U)$ and $q_\Lambda : L_\Lambda^r(U) \to \Lambda^r(U)$, which have sections $\operatorname{tr}_S : S^r(U) \to L_S^r(U)$ and $\operatorname{tr}_\Lambda : \Lambda^r(U) \to L_\Lambda^r(U)$ given by $\operatorname{tr} x := \frac{1}{|\Sigma_r : Q_r|} \sum_{\pi \in \Sigma_r/Q_r} \pi x$. These have the property that $q_S \circ \operatorname{tr}_S = \operatorname{Id}_{S^r(U)}$ and $q_\Lambda \circ \operatorname{tr}_\Lambda = \operatorname{Id}_{\Lambda^r(U)}$. These maps are all natural transformations of functors on free $\mathbb{Z}_{(2)}$-modules.

Writing $r = 2^p + t$ as before, we see from the description of Q_r that

$$L_S^r(U) \cong \begin{cases} S^{2^p}(U) \otimes S^t(U) & \text{if } t < 2^p \\ (S^{2^p}(U) \otimes S^{2^p}(U))/C_2 \cong S^2(S^{2^p}(U)) & \text{if } t = 2^p. \end{cases}$$

Similarly, if $r \geqslant 3$ we have

$$L_\Lambda^r(U) \cong \begin{cases} \Lambda^{2^p}(U) \otimes \Lambda^t(U) & \text{if } t < 2^p \\ (\Lambda^{2^p}(U) \otimes \Lambda^{2^p}(U))/C_2 \cong S^2(\Lambda^{2^p}(U)) & \text{if } t = 2^p, \end{cases}$$

because the involution τ has signature 1, provided that $p \geqslant 1$.

Now let V be an $\mathbb{F}_2$-vector space and let U be a free $\mathbb{Z}_{(2)}$-module such that $V \cong \mathbb{F}_2 \otimes_{\mathbb{Z}_{(2)}} U$. Let L^r denote one of the functors $S^r, \Lambda^r, L_S^r, L_\Lambda^r$ above and use it to define a functor with the same name on $\mathbb{F}_2$-vector spaces by $L^r(V) = \mathbb{F}_2 \otimes_{\mathbb{Z}_{(2)}} L^r(U)$. This gives the expected result for $S^r(V)$ and $\Lambda^r(V)$.

In order to verify that L^r is really a functor on vector spaces, notice that if U and U' are two free $\mathbb{Z}_{(2)}$-modules then the natural map $\operatorname{Hom}_{\mathbb{Z}_{(2)}}(U, U') \to \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathbb{Z}_{(2)}} U, \mathbb{F}_2 \otimes_{\mathbb{Z}_{(2)}} U')$ is surjective, so all maps of vector spaces lift. Furthermore, a map in the kernel has image in $2U'$, so factors through multiplication by 2 on U'. But multiplication by 2 on U' induces multiplication by 2^r on $T^r(U')_\sigma$, thus it induces 0 on $\mathbb{F}_2 \otimes_{\mathbb{Z}_{(2)}} L^r(U')$.

It follows that the formulas above are also valid for $\mathbb{F}_2$-vector spaces. A difference is that we now have natural transformations $e^r : S^r \to \Lambda^r$ and $L_e^r : L_S^r \to L_\Lambda^r$ induced by reducing modulo squares.

The above functors induce functors on modules for a group in the obvious way.

Remark. Any representation of G over a field of characteristic 2 can be written in $\mathbb{F}_2$, so this is sufficient for our purposes. If we really needed functors on vector spaces over a bigger field, this could be achieved by starting with a larger ring than $\mathbb{Z}_{(2)}$.

In the rest of this section we prove part (a) of Theorem 4.1, assuming the whole of the theorem for smaller n. We use the same notation as before.

Lemma 9.1. *Suppose that V is a kG-module, $r \geqslant 2$ and $L_e^r : L_S^r(V) \to L_\Lambda^r(V)$ is split injective modulo induced summands. Then $e^r : S^r(V) \to \Lambda^r(V)$ is split injective modulo induced summands.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} S^r(V) & \xrightarrow{e^r} & \Lambda^r(V) \\ {\scriptstyle \mathrm{tr}_S}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}_\Lambda} \\ L_S^r(V) & \xrightarrow{L_e^r} & L_\Lambda^r(V). \end{array}$$

The map tr_S is split injective, so if L_e^r is split injective modulo induced summands then so is $L_e^r \circ \mathrm{tr}_S$, by Lemma 3.9 (a). But this is equal to $\mathrm{tr}_\Lambda \circ e^r$ and Lemma 3.9 (b) shows that e^r is split injective modulo induced summands. $\square$

Lemma 9.2. *If s is an odd integer such that $0 < s < 2^{n-1}$ and r is an integer such that $0 \leqslant r < 2^n$, then $e^r : S^r(V_{2^{n-1}+s}) \to \Lambda^r(V_{2^{n-1}+s})$ is split injective modulo induced summands.*

Proof. We use induction on r. The cases $r = 0, 1$ are trivial and $r = 2$ is covered by Lemma 8.3 combined with Lemma 8.2. Let $r \geqslant 3$ and write $r = 2^p + t$ with $1 \leqslant t \leqslant 2^p$. Abbreviate $V_{2^{n-1}+s}$ to V. By Lemma 9.1, it is sufficient to check that $L_e^r : L_S^r(V) \to L_\Lambda^r(V)$ is split injective modulo induced summands.

If $t < 2^p$ then $L_e^r = e^{2^p} \otimes e^t : S^{2^p}(V) \otimes S^t(V) \to \Lambda^{2^p}(V) \otimes \Lambda^t(V)$. This is split injective modulo induced summands by induction and Lemma 3.9 (c).

If $t = 2^p$ then $L_e^r = S^2(e^{2^p}) : S^2(S^{2^p}(V)) \to S^2(\Lambda^{2^p}(V))$. By induction, e^{2^p} is split injective modulo induced summands, so it extends to a split injective map $M : S^{2^p}(V) \to \Lambda^{2^p}(V) \oplus X$ with left inverse N, where X is induced. By the remark after Lemma 3.10, we may assume that X only contains summands that are also summands of $S^{2^p}(V)$; by Lemma 8.5 and the assumption that $r < 2^n$ (and hence $2^p < 2^{n-1}$, since $t = 2^p$), these are of dimension divisible by 4. Applying S^2, we see that $S^2(e^{2^p})$ extends to

$$S^2(M) : S^2\big(S^{2^p}(V)\big) \to S^2\big(\Lambda^{2^p}(V)\big) \oplus S^2(X) \oplus \big(\Lambda^{2^p}(V) \otimes X\big),$$

with left inverse $S^2(N)$. Certainly $\Lambda^{2^p}(V) \otimes X$ is induced, and $S^2(X)$ is induced, by Corollary 3.7. Thus L_e^r and thus e^r are split injective modulo induced summands. $\square$

Again, let s be an odd integer such that $0 < s < 2^{n-1}$. It follows from Lemmas 8.1, 8.2 and 9.2 that the complex $K^r(V_{2^{n-1}+s}, V_s^2)$ is separated for all $0 \leqslant r < 2^n$.

Recall that $K(V_{2^{n-1}+s}, V_s^2)$ is separated if and only if the complex $L(V_{2^{n-1}+s}, V_s^2)$ from Section 6 is separated. For the rest of this section we will write just K, L, etc. Now L^r is separated for all $0 \leqslant r < 2^n$, because it coincides with K^r in this range. We will show that L^r is separated for $r \geqslant 2^n$ by induction on r, so let $r \geqslant 2^n$ and assume that the complex is separated in all lower degrees.

By Lemma 8.1, we can also assume that L^r is separated in positive (complex-)degrees, so it is enough to prove that the short exact sequence

$$0 \to \operatorname{Im}\left(d_1^r\right) \to T^r \to \widetilde{T}^r \to 0 \tag{11}$$

is separated at T^r. By Lemma 2.8, the restriction of K to the maximal subgroup H of G decomposes as a tensor product of two complexes, and each of these is separated, by our continuing induction hypothesis and Theorem 4.1 (a). Their product is also separated, by Lemma 2.2, hence so is L^r. It follows that the sequence (11) is separated at T^r on restriction to H.

But T^r is induced for this range of r. Separation of (11) follows immediately from Proposition 2.4 applied to $\operatorname{Im}(d_1^r) \to T^r$.

This proves that the complex $K^r(V_{2^{n-1}+s}, V_s^2)$ is separated for all $r \geqslant 0$, and part (a) of Theorem 4.1 follows.

10. Exterior powers

In this section we prove part (d) of Theorem 4.1, assuming the whole of the theorem for smaller n.

Because we have already proved separation, periodicity and splitting we know that

$$\lambda_t(V_{2^{n-1}+s}) =_{\text{ind}} \lambda_{t^2}^{\Omega}(V_s)\lambda_t^{\Omega}(V_{2^{n-1}-s});$$

see the first remark at the end of Section 4. In order to obtain the formula with $=_{\text{proj}}$, we first consider the restriction to the subgroup H of index 2. Writing $V_{2^{n-1}+s}\downarrow_H^G = V_{2^{n-2}+s'} \oplus V_{2^{n-2}+s''}$, the two sides of the formula become

$$\lambda_t(V_{2^{n-2}+s'})\lambda_t(V_{2^{n-2}+s''}) \quad \text{and} \quad \lambda_{t^2}^{\Omega}(V_{s'})\lambda_{t^2}^{\Omega}(V_{s''})\lambda_t^{\Omega}(V_{2^{n-2}-s'})\lambda_t^{\Omega}(V_{2^{n-2}-s''}).$$

But we know, by induction, that $\lambda_t(V_{2^{n-2}+s'}) =_{\text{proj}} \lambda_{t^2}^{\Omega}(V_{s'})\lambda_t^{\Omega}(V_{2^{n-2}-s'})$ and similarly for s''. Thus, on restriction, the two sides are equal modulo projectives. Now use Lemma 3.3 in order to see that the two sides are equal modulo projectives even before restriction. This finally completes the proof of Theorem 4.1.

11. A bound on the number of non-induced summands

The description of the tensor product given in Section 3 shows that the decomposition of $V_r \otimes V_s$ into indecomposable summands involves a summand of odd dimension if and only if both r and s are odd, in which case it contains precisely one odd-dimensional summand.

Let us write $\operatorname{summ}(V)$ for the number of indecomposable summands in the module V.

Proposition 11.1. *The number of non-induced summands in $\Lambda(V)$ is at most*

$$2^{\frac{1}{2}(\dim(V)+\operatorname{summ}(V))} = \sqrt{2^{\operatorname{summ}(V)}\dim\bigl(\Lambda(V)\bigr)}.$$

Proof. Let $f(V)$ denote the number of non-induced summands in $\Lambda(V)$. The comment about the tensor product above shows that $f(V \oplus W) = f(V)f(W)$. The proposed bound also turns sums into products, so it suffices to consider the case when $V = V_r$ is indecomposable and show that $f(V_r) \leqslant 2^{\frac{1}{2}(r+1)}$.

We use induction on r. Since the cases $r = 0, 1$ are trivial we can assume that $r \geqslant 2$, and we can write $r = 2^{n-1} + s$, where $1 \leqslant s \leqslant 2^{n-1}$. Setting $t = 1$ in the formula $\lambda_t(V_{2^{n-1}+s}) =_{\text{proj}} \lambda^{\Omega}_{t^2}(V_s)\lambda^{\Omega}_t(V_{2^{n-1}-s})$ and using induction we obtain

$$\begin{aligned} f\bigl(\Lambda(V_{2^{n-1}+s})\bigr) &= f\bigl(\lambda_1(V_{2^{n-1}+s})\bigr) \\ &= f\bigl(\lambda^{\Omega}_1(V_s)\bigr)f\bigl(\lambda^{\Omega}_1(V_{2^{n-1}-s})\bigr) = f\bigl(\lambda_1(V_s)\bigr)f\bigl(\lambda_1(V_{2^{n-1}-s})\bigr) \\ &\leqslant 2^{\frac{1}{2}(2^{n-1}-s+1)}2^{\frac{1}{2}(s+1)} = 2^{\frac{1}{2}(2^{n-1}+2)} \leqslant 2^{\frac{1}{2}(2^{n-1}+s+1)}. \qquad \square \end{aligned}$$

For an indecomposable kG-module V_r, if we assume that the group acts faithfully then $r > \frac{1}{2}|G|$, and the dimension of any direct summand of $\Lambda(V_r)$ is at most $|G|$. It follows that the dimension of the non-induced part of $\Lambda(V_r)$ divided by the dimension of the whole of $\Lambda(V_r)$ is at most $2^{\frac{1}{2}(3-r)}r$.

12. Remarks

(a) As already mentioned in the introduction, the formula in Theorem 1.1 reduces the computation of $\Lambda^r(V_{2^{n-1}+s})$ to the computation of tensor products of exterior powers of modules of smaller dimension. Since tensor products can easily be determined recursively (see Section 3), this gives an efficient recursive method for calculating the decomposition of exterior powers of modules for cyclic 2-groups into indecomposables. A program based on this recurrence relation was implemented in GAP [10] by the first author.

A restriction on the use of Theorem 1.1 is the growth of the multiplicities of direct summands of the form V_{2^m}. For example, the multiplicity of V_{128} as a direct summand of $\Lambda^{57}(V_{147})$ is 8 197 519 886 357 582 844 587 268 803 532 720. If one is only interested in the non-induced part of $\Lambda^r(V_{2^{n-1}+s})$ the recurrence relation can be applied modulo induced summands to keep the multiplicities relatively small.

Together with the results in [21], the recurrence relation in Theorem 1.1 also provides an algorithm for computing the decomposition of the symmetric powers $S^r(V_m)$ into indecomposables for arbitrary m and r.

Example. We determine the decomposition of $\Lambda^6(V_{13})$ into indecomposables:

$$\begin{aligned}\Lambda^6(V_{13}) &\cong_{\mathrm{proj}} \Omega_{16}^{0+6}\big(\Lambda^0(V_5)\otimes\Lambda^6(V_3)\big)\oplus\Omega_{16}^{1+4}\big(\Lambda^1(V_5)\otimes\Lambda^4(V_3)\big)\\ &\qquad\oplus\Omega_{16}^{2+2}\big(\Lambda^2(V_5)\otimes\Lambda^2(V_3)\big)\oplus\Omega_{16}^{3+0}\big(\Lambda^3(V_5)\otimes\Lambda^0(V_3)\big)\\ &\cong_{\mathrm{proj}} \big(\Lambda^2(V_5)\otimes V_3\big)\oplus\Omega_{16}\big(\Lambda^3(V_5)\big).\end{aligned}$$

Furthermore $\Lambda^3(V_5)\cong\Lambda^2(V_5)$, by duality, and

$$\begin{aligned}\Lambda^2(V_5) &\cong \Omega_8^{0+2}\big(\Lambda^0(V_1)\otimes\Lambda^2(V_3)\big)\oplus\Omega_8^{1+0}\big(\Lambda^1(V_1)\otimes\Lambda^0(V_3)\big)\\ &\cong V_3\oplus\Omega_8(V_1)\cong V_3\oplus V_7.\end{aligned}$$

Thus

$$\begin{aligned}\Lambda^6(V_{13}) &\cong_{\mathrm{proj}} (V_3\oplus V_7)\otimes V_3\oplus\Omega_{16}(V_3\oplus V_7)\\ &\cong_{\mathrm{proj}} (V_3\otimes V_3)\oplus(V_7\otimes V_3)\oplus\Omega_{16}(V_3\oplus V_7)\\ &\cong_{\mathrm{proj}} (V_1\oplus 2V_4)\oplus(V_5\oplus 2V_8)\oplus(V_{13}\oplus V_9)\\ &\cong_{\mathrm{proj}} V_1\oplus 2V_4\oplus V_5\oplus 2V_8\oplus V_9\oplus V_{13}.\end{aligned}$$

Comparing dimensions, we obtain $\Lambda^6(V_{13})\cong V_1\oplus 2V_4\oplus V_5\oplus 2V_8\oplus V_9\oplus V_{13}\oplus 104V_{16}$.

(b) Obviously, Gow and Laffey's formula for exterior squares [11, Theorem 2] is the special case $r=2$ of Theorem 1.1. Furthermore, setting $s=2^{n-1}-1$ in Theorem 1.1 gives Kouwenhoven's formula [15, Theorem 3.4] for $\Lambda^r(V_{q-1})$ when q is a power of 2 (for all r).

In [15, Theorem 3.5] Kouwenhoven proved the formula

$$\lambda_t(V_{q+1}-V_{q-1})=1+(V_{q+1}-V_{q-1})t+t^2 \tag{12}$$

in $a(C_{q\cdot p})[[t]]$, where q is a power of a prime p; see Section 4 for a definition of λ_t. We will show how this can be derived from Theorem 1.1 in the case that $p=2$. Note that, since the dimension series of the two sides match, it is sufficient to prove this modulo projectives. The theorem gives us:

$$\lambda_t(V_{2^{n-1}+1})=\big(1+V_{2^n-1}t^2\big)\lambda_t^{\Omega_{2^n}}(V_{2^{n-1}-1})$$

modulo V_{2^n} and

$$\lambda_t(V_{2^{n-1}-1})=(1+V_{2^{n-1}-1}t)\lambda_{t^2}^{\Omega_{2^{n-1}}}(V_{2^{n-2}-1})$$

modulo $V_{2^{n-1}}$. The latter can be written as

$$\lambda_t(V_{2^{n-1}-1}) = (1 + V_{2^{n-1}-1}t)\big(\lambda_{t^2}^{\Omega_{2^{n-1}}}(V_{2^{n-2}-1}) + V_{2^{n-1}}f(t)\big)$$

exactly (the last term can be written inside the parentheses, since $(1 + V_{2^{n-1}-1}t)$ is invertible). Applying Ω_{2^n} in odd degrees we obtain

$$\lambda_t^{\Omega_{2^n}}(V_{2^{n-1}-1}) = (1 + V_{2^{n-1}+1}t)\big(\lambda_{t^2}^{\Omega_{2^{n-1}}}(V_{2^{n-2}-1}) + V_{2^{n-1}}f(t)\big).$$

Substituting into the left hand side of (12) yields

$$\big(1 + V_{2^n-1}t^2\big)(1 + V_{2^{n-1}+1}t)(1 + V_{2^{n-1}-1}t)^{-1}$$

modulo V_{2^n}, and it is easy to verify that

$$\big(1 + V_{2^n-1}t^2\big)(1 + V_{2^{n-1}+1}t) = (1 + V_{2^{n-1}-1}t)\big(1 + (V_{2^{n-1}+1} - V_{2^{n-1}-1})t + t^2\big)$$

modulo V_{2^n}.

(c) Theorem 1.1 can also be used to calculate the Adams operations on the Green ring $a(C_{2^n})$, as was shown to us by Roger Bryant and Marianne Johnson. For each $r > 0$ and $j \in \{1, \dots, 2^n\}$, define an element $\psi_\Lambda^r(V_j) \in a(C_{2^n})$ by

$$\psi_\Lambda^1(V_j) - \psi_\Lambda^2(V_j)t + \psi_\Lambda^3(V_j)t^2 - \cdots = \psi_{\Lambda,t}(V_j) = \tfrac{d}{dt}\log\lambda_t(V_j).$$

By $\mathbb{Z}$-linear extension there is a map $\psi_\Lambda^r : a(C_{2^n}) \to a(C_{2^n})$, called the rth Adams operation defined by the exterior powers. It can be shown that if r is odd, then ψ_Λ^r is the identity map on $a(C_{2^n})$ and that $\psi_\Lambda^{2^i r} = \psi_\Lambda^{2^i}$ for all $i \geqslant 1$, so all that remains is to describe $\psi_\Lambda^{2^i}$ for $i \geqslant 1$ (see [5] and [6] for details). For $j \geqslant 2$, write $j = 2^m + s$ with $m \geqslant 0$ and $1 \leqslant s \leqslant 2^m$; then

$$\psi_\Lambda^{2^i}(V_{2^m+s}) = 2\psi_\Lambda^{2^{i-1}}(V_s) + \psi_\Lambda^{2^i}(V_{2^m-s})$$

for all $i \geqslant 2$ and

$$\psi_\Lambda^2(V_{2^m+s}) = 2V_{2^{m+1}} - 2V_{2^{m+1}-s} + \psi_\Lambda^2(V_{2^m-s}).$$

This can be seen by applying the definition of the Adams operations to the Hilbert series form of Theorem 1.1, obtaining (in the obvious notation)

$$\psi_{\Lambda,t}(V_{2^m+s}) =_{\mathrm{proj}} 2t\psi_{\Lambda,t^2}^{\Omega}(V_s) + \psi_{\Lambda,t}^{\Omega}(V_{2^m-s}).$$

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