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The global extension problem, crossed products and co-flag non-commutative Poisson algebras [☆]

A.L. Agore ^{a,1}, G. Militaru ^{b,*}

^a Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

^b Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

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ABSTRACT

Let P be a Poisson algebra, E a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces with $V = \text{Ker}(\pi)$. The global extension problem asks for the classification of all Poisson algebra structures that can be defined on E such that $\pi : E \rightarrow P$ becomes a morphism of Poisson algebras. From a geometrical point of view it means to decompose this groupoid into connected components and to indicate a point in each such component. All such Poisson algebra structures on E are classified by an explicitly constructed classifying set $\mathcal{GPH}^2(P, V)$ which is the coproduct of all non-abelian cohomological objects $\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V))$ which are the classifying sets for all extensions of P by $(V, \cdot_V, [-, -]_V)$. The second classical Poisson cohomology group $H^2(P, V)$ appears as the most elementary piece among all components of $\mathcal{GPH}^2(P, V)$. Several examples are provided in the case of metabelian Poisson algebras or co-flag Poisson algebras over P : the latter being Poisson algebras Q which admit

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* Corresponding author.

E-mail addresses: ana.agore@vub.ac.be, ana.agore@gmail.com (A.L. Agore), gigel.militaru@fmi.unibuc.ro, gigel.militaru@gmail.com (G. Militaru).

¹ Permanent address: Department of Applied Mathematics, Bucharest University of Economic Studies, Piata Romana 6, RO-010374 Bucharest 1, Romania.

a finite chain of epimorphisms of Poisson algebras $P_n := Q \xrightarrow{\pi_n} P_{n-1} \cdots P_1 \xrightarrow{\pi_1} P_0 := P$ such that $\dim(\text{Ker}(\pi_i)) = 1$, for all $i = 1, \dots, n$.

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Introduction

A Poisson algebra is both a Lie algebra and an associative algebra living on the same vector space P such that any bracket $[-, p] : P \rightarrow P$ is a derivation of the associative algebra P . The concept is the abstract algebra counterpart of a Poisson manifold: for a given smooth manifold M , there is a one-to-one correspondence between Poisson brackets on the commutative algebra $C^\infty(M)$ of smooth functions on M and all Poisson structures on M (see for instance [18, Remark 1.2]) – we recall that Poisson structures on M are bivector fields Q such that $[Q, Q] = 0$, where $[-, -]$ is the Schouten bracket of multivector fields. In categorical language the correspondence $M \mapsto C^\infty(M)$ gives a contravariant functor from the category of Poisson manifolds to the category of Poisson algebras. The functor $C^\infty(-)$ is the tool used for translating purely geometrical concepts or problems of study into the algebraic setting of Poisson algebras using a well established dictionary between the two categories: for details we refer to [19,24] and the references therein. The first example of a Poisson algebra structure was given by S.D. Poisson in 1809 related to the study of the three-body problem in celestial mechanics. Since then, Poisson algebras have become a very active subject of research in several areas of mathematics and mathematical physics such as: Hamiltonian mechanics [6,26,13], differential geometry [24], Lie groups and representation theory, noncommutative algebraic/differential geometry [35], (super)integrable systems [29], quantum field theory, vertex operator algebras, quantum groups [11] and so on. The theory of Poisson algebras experienced a new upsurge in the last twenty years in connection to the development of noncommutative geometry. One of the problems in this context is the right definition of what should be a *non-commutative* Poisson algebra. From a purely algebraic point of view a non-commutative Poisson algebra means a Poisson algebra whose underlying algebra structure is non-commutative [22,23], etc. This view point is also adopted in this work: more precisely, throughout this paper we do not assume the underlying associative algebra P of a Poisson algebra to be commutative.

Beyond the remarkable applications in the above mentioned fields, Poisson algebras are objects of study in their own right, from a purely algebraic viewpoint [7–10,14,16,21,22,27,28,34] etc. In this spirit a tempting question arises: *for a given positive integer n , classify up to an isomorphism all Poisson algebras of dimension n over a field k* . Having in mind the duality established by the functor $C^\infty(-)$, the question can be viewed as the algebraic version of its geometric counterpart initiated in [17] where the first steps towards the classification of low dimensional Poisson structures is given using differential geometry tools. The geometrical approach is also exposed in [24, Chapter 9] where the classification of Poisson structures of two or three-dimensional manifolds over a field of

characteristic zero is given. The classification of non-commutative Poisson algebras of a given dimension is a problem far from being trivial as it contains as subsequent problems the classification of finite dimensional Lie algebras – known up to dimension 7 [32] – as well as the classification of finite dimensional associative algebras – known up to dimension 5 [30]. In both cases the classification holds over an algebraically closed field of characteristic $\neq 2$. The first steps for the algebraic approach of the classification of Poisson algebras were taken in [16, Section 2], while [23] classifies the finite-dimensional non-commutative Poisson algebras with Jacobson radical of zero square. The classification of *finite objects*, such as groups of a given order or associative (resp. Lie, Poisson, Hopf, etc.) algebras of a given dimension, relies on the famous *extension problem* initiated at the level of groups by Hölder and studied later on for Lie algebras [12], associative algebras [20], quantum groups [5], Poisson algebras [9], etc. For the relevance of the extension problem in differential geometry we refer to [4,25]. A generalization of the extension problem, called the *global extension problem*, was introduced in [31] at the level of Leibniz algebras, as the categorical dual of what we have called the *extending structures problem* [1]. The aim of this paper is the study of the global extension (GE) problem for Poisson algebras which consists of the following question:

The global extension problem

Let P be a Poisson algebra, E a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces. Describe and classify all Poisson algebra structures $(\cdot, [-, -])$ that can be defined on E such that $\pi : E \rightarrow P$ is a morphism of Poisson algebras.

We start by explaining the meaning of the word ‘global’. We recall the classical extension problem formulated in the setting of Poisson algebras in [9]. Let P and Q be two fixed Poisson algebras. The extension problem asks for the classification of all Poisson algebras \mathfrak{E} which are extensions of P by Q , i.e. all Poisson algebras \mathfrak{E} that fit into an exact sequence of Poisson algebras:

$$0 \longrightarrow Q \xrightarrow{i} \mathfrak{E} \xrightarrow{P} P \longrightarrow 0 . \quad (1)$$

The classification is up to an isomorphism of Poisson algebras that stabilizes Q and co-stabilizes P . To the best of our knowledge, the problem was studied so far only for commutative Poisson algebras and moreover Q was considered to be the abelian Poisson algebra (i.e. the multiplication and the bracket on Q are both the trivial maps). In this case the isomorphism classes of all extensions of P by Q are parameterized by the second Poisson cohomology group $H^2(P, Q)$ [9, Proposition 6.1]. Now, if $(E, \cdot_E, [-, -]_E)$ is a Poisson algebra structure on E such that $\pi : (E, \cdot_E, [-, -]_E) \rightarrow P$ is a morphism of Poisson algebras, then $(E, \cdot_E, [-, -]_E)$ is an extension of P by $V := \text{Ker}(\pi)$, which is a Poisson subalgebra of $(E, \cdot_E, [-, -]_E)$. However, this Poisson algebra structure of V is not fixed from the input data as in the case of the extension problem: it depends essentially on the Poisson algebra structures on E which we are looking for. Thus, we can

conclude that the classical extension problem is the ‘local’ version of the GE-problem: namely, the case when the Poisson algebra structures on $\text{Ker}(\pi)$ are fixed. Secondly, we will explain what we mean by classification in the GE-problem and its geometrical interpretation. We denote by $\mathcal{C}_\pi(E, P)$ the small category whose objects are all Poisson algebra structures $(\cdot_E, [-, -]_E)$ on E such that $\pi : (E, \cdot_E, [-, -]_E) \rightarrow P$ is a Poisson algebra map. A morphism $\varphi : (\cdot_E, [-, -]_E) \rightarrow (\cdot'_E, [-, -]'_E)$ in the category $\mathcal{C}_\pi(E, P)$ is a Poisson algebra map $\varphi : (E, \cdot_E, \{-, -\}_E) \rightarrow (E, \cdot'_E, \{-, -\}'_E)$ which stabilizes V and co-stabilizes P , i.e. the following diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{i} & E & \xrightarrow{\pi} & P \\
 Id \downarrow & & \downarrow \varphi & & \downarrow Id \\
 V & \xrightarrow{i} & E & \xrightarrow{\pi} & P
 \end{array} \tag{2}$$

is commutative. In this case we say that the Poisson algebra structures $(\cdot_E, [-, -]_E)$ and $(\cdot'_E, [-, -]'_E)$ on E are *cohomologous* and we denote this by $(E, \cdot_E, \{-, -\}_E) \approx (E, \cdot'_E, \{-, -\}'_E)$. The category $\mathcal{C}_\pi(E, P)$ is a groupoid, i.e. any morphism is an isomorphism. In particular, we obtain that \approx is an equivalence relation on the set of objects of $\mathcal{C}_\pi(E, P)$ and we denote by $\text{Gext}(P, E)$ the set of all equivalence classes via \approx , i.e. $\text{Gext}(P, E) := \mathcal{C}_\pi(E, P) / \approx$. The answer to the GE-problem will be provided by explicitly computing $\text{Gext}(P, E)$ for a given Poisson algebra P and a vector space E . From geometrical view point this means to give the decomposition of the groupoid $\mathcal{C}_\pi(E, P)$ into connected components and to indicate a ‘point’ in each such component. This will be the main result of the paper: we shall prove that $\text{Gext}(P, E)$ is parameterized by a global cohomological object denoted by $\mathcal{GPH}^2(P, V)$, where $V = \text{Ker}(\pi)$. The explicit bijection between $\mathcal{GPH}^2(P, V)$ and $\text{Gext}(P, E)$ is also indicated. Moreover, we shall prove that $\mathcal{GPH}^2(P, V)$ is the coproduct in the category of sets of all classifying objects of all ‘local’ extension problems, which are the *non-abelian* cohomological objects denoted by $\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V))$ – for any Poisson algebra structure $(\cdot_V, [-, -]_V)$ on V . The second classical Poisson cohomology group $H^2(P, V)$ [9,15] appears as the most elementary piece among all components of $\mathcal{GPH}^2(P, V)$. The results are proved in the general case, leaving aside the ‘abelian’ case which is the traditional setting for the extension problem for groups or Lie (resp. associative, Poisson) algebras.

The paper is organized as follows: in Section 1 we recall the basic concepts that will be used throughout the article following the terminology of [2,3]. Section 2 contains the main results of the paper: the crossed product of Poisson algebras is introduced as the key tool of our approach. The crossed product, denoted by $P \# V$ is associated to two Poisson algebras P and V connected by three actions $(\rightarrow, \triangleleft, \triangleright)$ and two non-abelian cocycles $\vartheta : P \times P \rightarrow V$ and $f : P \times P \rightarrow V$. The datum $(\rightarrow, \triangleleft, \triangleright, \vartheta, f)$ must satisfy several axioms in order to obtain a crossed product of Poisson algebras as given in Theorem 2.2. Let $\pi : E \rightarrow P$ be an epimorphism of vector spaces between a vector space E and a Poisson algebra P with $V = \text{Ker}(\pi)$. We prove that any Poisson algebra structure

$(\cdot_E, [-, -]_E)$ on E such that $\pi : (E, \cdot_E, [-, -]_E) \rightarrow P$ is a morphism of Poisson algebras is cohomologous to a crossed product $P \sharp V$. Based on this, the theoretical answer to the GE-problem is given in [Theorem 2.9](#): the classifying set $\text{Gext}(P, E)$ is parameterized by an explicitly constructed global cohomological object $\mathcal{GPH}^2(P, V)$ and the bijection between the elements of $\mathcal{GPH}^2(P, V)$ and $\text{Gext}(P, E)$ is given. The relation ‘global’ vs ‘local’ for the extension problem is given in [Corollary 2.10](#). For a fixed Poisson algebra $V = (V, \cdot_V, [-, -]_V)$ the classifying object $\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V))$ of all extensions of P by V is constructed as an answer to the classical extension problem for Poisson algebras leaving aside the abelian case. Then we show that $\mathcal{GPH}^2(P, V)$ is the coproduct in the category of sets of all local non-abelian cohomological groups $\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V))$, the coproduct being made over all possible Poisson algebra structures $(\cdot_V, \{-, -\}_V)$ on V . The abelian case, corresponding to the trivial Poisson structure on V is derived as a special case. The concept of metabelian Poisson algebra is introduced and their structure is explicitly described. In [Section 3](#) we shall identify a way of computing $\mathcal{GPH}^2(P, V)$ for what we have called co-flag Poisson algebras over P as defined in [Definition 3.1](#). All co-flag Poisson structures over P can be completely described by a recursive reasoning, the key step being treated in [Theorem 3.6](#). For instance, [Example 3.7](#) describes and classifies the 4-dimensional co-flag algebras over the Heisenberg Poisson algebra. At the end of the paper, as an application of our constructions, we take the first steps toward the classification of co-flag Poisson algebras. [Corollary 3.8](#) provides the description and classification of all 2-dimensional co-flag Poisson algebras. Using the recursive method introduced in [Section 3](#) we can then describe all 3-dimensional co-flag Poisson algebras: a relevant example is given in [Corollary 3.9](#).

1. Preliminaries

For two sets X and Y we shall denote by $X \sqcup Y$ their coproduct in the category of sets, i.e. $X \sqcup Y$ is the disjoint union of X and Y . All vector spaces, Lie or associative algebras, linear or bilinear maps are over an arbitrary field k . A map $f : V \rightarrow W$ between two vector spaces is called the *trivial* map if $f(v) = 0$, for all $v \in V$. Let $\pi : E \rightarrow P$ be a linear map between two vector spaces and $V := \text{Ker}(\pi)$. We say that a linear map $\varphi : E \rightarrow E$ *stabilizes* V (resp. *co-stabilizes* P) if the left square (resp. the right square) of diagram [\(2\)](#) is commutative. Throughout this paper, by an algebra $P = (P, m_P)$ we will always mean an associative, not necessarily commutative or unital, algebra over k . The bilinear multiplication $m_P : P \times P \rightarrow P$ will be denoted by $m_P(p, q) = pq$, for all $p, q \in P$. When $P = (P, m_P)$ is unital the unit will be denoted by 1_P . The following convention will be used: when defining the Poisson algebra structure we only write down the non-zero values of the multiplication and the bracket. For an algebra P we shall denote by ${}_P\mathcal{M}_P$ the category of all P -bimodules, i.e. triples $(V, \rightharpoonup, \triangleleft)$ consisting of a vector space V and two bilinear maps $\rightharpoonup : P \times V \rightarrow V$, $\triangleleft : V \times P \rightarrow V$ such that (V, \rightharpoonup) is a left P -module, (V, \triangleleft) is a right P -module and $p \rightharpoonup (x \triangleleft q) = (p \rightharpoonup x) \triangleleft q$, for all $p, q \in P$ and $x \in V$. Representations of a Lie algebra $P = (P, [-, -])$ will be viewed as

left modules over P . Explicitly, a left Lie P -module is a vector space V together with a bilinear map $\triangleright : P \times V \rightarrow V$ such that for any $p, q \in P$ and $x \in V$ we have:

$$[p, q] \triangleright x = p \triangleright (q \triangleright x) - q \triangleright (p \triangleright x). \quad (3)$$

The category of left Lie P -modules will be denoted by ${}^P\mathcal{M}$. A *Poisson algebra* is a triple $P = (P, m_P, [-, -])$, where (P, m_P) is an associative algebra, $(P, [-, -])$ is a Lie algebra such that any Hamiltonian $[-, r] : P \rightarrow P$ is a derivation of the associative algebra (P, m_P) , i.e. the Leibniz law:

$$[pq, r] = [p, r]q + p[q, r] \quad (4)$$

holds for any $p, q, r \in P$. Usually, a Poisson algebra P is by definition assumed to be commutative, as the main examples are in classical differential geometry. However, throughout this paper we do not impose this restriction. A morphism between two Poisson algebras P and P' is a linear map $\varphi : P \rightarrow P'$ that is a morphism of associative algebras as well as of Lie algebras.

Remarks 1.1. (1) Let P be a unitary Poisson algebra. By applying (4) for $p = q = 1_P$ we obtain:

$$[1_P, r] = 0 = [r, 1_P] \quad (5)$$

for all $r \in P$. Any non-unital Poisson algebra embeds into a unital Poisson algebra by adding a unit 1_P to the associative algebra (P, m_P) and considering (5) as the defining relation for the bracket evaluated in the unit 1_P .

(2) Let $P = (P, m_P)$ be an algebra and $u \in k$. Then $(P, m_P, [-, -]_u)$ is a Poisson algebra, where $[a, b]_u := u(ab - ba)$, for all $a, b \in P$ [22]. In particular, any associative algebra $P = (P, m_P)$ is a Poisson algebra with the abelian Lie algebra structure, i.e. $[a, b] := 0$, for all $a, b \in P$. Any Lie algebra $P = (P, [-, -])$ is a Poisson algebra with the trivial algebra structure, i.e. $pq := 0$, for all $p, q \in P$. Any vector space V has a Poisson algebra structure with the trivial algebra structure and the abelian Lie algebra structure. This trivial Poisson algebra structure on V will be called *abelian* and will be denoted from now on by V_0 .

(3) If P is a Poisson algebra that is non-commutative and *prime* as an associative algebra, then [14, Theorem 2.1] shows that the bracket on P has the form $[p, q] = \lambda(pq - qp)$, for some $\lambda \in \mathcal{Z}^+(P)$, where $\mathcal{Z}^+(P)$ is the center of the ring of quotients of P . In particular, any Poisson algebra structure on the Weyl algebra A_1 has the Lie bracket of the form $[p, q] = \lambda(pq - qp)$, for some $\lambda \in k$. For further results on the structure of a bracket on a non-commutative Poisson algebra we refer to [22].

Examples 1.2. (1) Up to an isomorphism, there exist two 1-dimensional Poisson algebras: the first one is k_0 , i.e. k viewed with the abelian Poisson algebra structure and the second

one, denoted by k_1 , is the vector space k , with the unital algebra structure ($1 \cdot 1 := 1$) and the trivial bracket.

(2) Any unital 2-dimensional Poisson algebra has the trivial bracket, thanks to the compatibility (5). Therefore, their classification follows from [3, Corollary 4.5] where all 2-dimensional associative, unital algebras over an arbitrary field are classified. We point out that if k has characteristic 2 then the number of types of 2-dimensional Poisson algebras can be infinite.

Poisson bimodules will play a key role in constructing the cohomological object $\mathcal{PH}^2(P, V_0)$. A *Poisson bimodule* [15,23] over a Poisson algebra P is a system $(V, \rightharpoonup, \triangleleft, \triangleright)$ consisting of a vector space V and three bilinear maps such that $(V, \rightharpoonup, \triangleleft) \in {}_P\mathcal{M}_P$ is a P -bimodule, $(V, \triangleright) \in {}^P\mathcal{M}$ is a left Lie P -module satisfying the following compatibility conditions for any $p, q \in P$ and $x \in V$:

$$(pq) \triangleright x = p \rightharpoonup (q \triangleright x) + (p \triangleright x) \triangleleft q \tag{6}$$

$$[p, q] \rightharpoonup x = p \rightharpoonup (q \triangleright x) - q \triangleright (p \rightharpoonup x) \tag{7}$$

$$x \triangleleft [p, q] = (q \triangleright x) \triangleleft p - q \triangleright (x \triangleleft p). \tag{8}$$

We denote by ${}^P_P\mathcal{M}_P$ the category of Poisson bimodules over P having as morphisms all linear maps which are compatible with the actions. Any vector space V has a trivial Poisson bimodule structure over P , where $(\rightharpoonup, \triangleleft, \triangleright)$ are all the trivial maps. Using the Leibniz law (4), we can easily see that $P := (P, \rightharpoonup = \triangleleft := m_P, \triangleright := [-, -])$ is a Poisson bimodule over P .

1.1. The Hochschild product for associative algebras

We recall the construction of the Hochschild product for associative algebras following the terminology used in [3].

Definition 1.3. Let P be an algebra and V a vector space. A *Hochschild (or pre-crossed) datum* of P by V is a system $\Theta(P, V) = (\rightharpoonup, \triangleleft, \vartheta, \cdot)$ consisting of four bilinear maps

$$\rightharpoonup : P \times V \rightarrow V, \quad \triangleleft : V \times P \rightarrow V, \quad \vartheta : P \times P \rightarrow V, \quad \cdot = \cdot_V : V \times V \rightarrow V.$$

Let $\Theta(P, V) = (\rightharpoonup, \triangleleft, \vartheta, \cdot)$ be a Hochschild datum and we denote by $P \star V = P_{\star(\triangleleft, \rightharpoonup, \vartheta, \cdot)} V$ the vector space $P \times V$ with the multiplication given by

$$(p, x) \star (q, y) := (pq, \vartheta(p, q) + p \rightharpoonup y + x \triangleleft q + x \cdot y) \tag{9}$$

for all $p, q \in P, x, y \in V$. $P \star V$ is called the *Hochschild (or crossed) product* associated to $\Theta(P, V)$ if it is an associative algebra with the multiplication given by (9). In this case the Hochschild datum $\Theta(P, V) = (\rightharpoonup, \triangleleft, \vartheta, \cdot)$ is called a *Hochschild (or crossed) system* of P by V and we denote by $\mathcal{HS}(P, V)$ the set of all Hochschild systems of P by V .

It was proved in [3, Proposition 1.2] that $\Theta(P, V)$ is a Hochschild system if and only if the following compatibility conditions hold for any $p, q, r \in P$ and $x, y \in V$:

- (H0) (V, \cdot) is an associative algebra
- (H1) $(x \cdot y) \triangleleft p = x \cdot (y \triangleleft p)$
- (H2) $(x \triangleleft p) \cdot y = x \cdot (p \rightharpoonup y)$
- (H3) $p \rightharpoonup (x \cdot y) = (p \rightharpoonup x) \cdot y$
- (H4) $(p \rightharpoonup x) \triangleleft q = p \rightharpoonup (x \triangleleft q)$
- (H5) $\vartheta(p, q) \triangleleft r = \vartheta(p, qr) - \vartheta(pq, r) + p \rightharpoonup \vartheta(q, r)$
- (H6) $(pq) \rightharpoonup x = p \rightharpoonup (q \rightharpoonup x) - \vartheta(p, q) \cdot x$
- (H7) $x \triangleleft (pq) = (x \triangleleft p) \triangleleft q - x \cdot \vartheta(p, q)$.

The Hochschild product was introduced in [20, Theorem 6.2] in the special case when \cdot is the trivial multiplication on V (i.e. $x \cdot y = 0$, for all $x, y \in V$). We point out that in this case axioms (H0)–(H7) reduce to $(V, \rightharpoonup, \triangleleft)$ being an P -bimodule and $\vartheta : P \times P \rightarrow V$ being a 2-cocycle.

If $P \star V$ is a Hochschild product, then the map $\pi_P : P \star V \rightarrow P$, $\pi_P(p, x) := p$ is a surjective algebra morphism and $\text{Ker}(\pi) = 0 \times V \cong V$. Conversely, [3, Proposition 1.4] proves the following: if P is an algebra, E a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces with $V = \text{Ker}(\pi)$, then any algebra structure \cdot_E which can be defined on the vector space E such that $\pi : (E, \cdot_E) \rightarrow P$ becomes a morphism of algebras is isomorphic to a Hochschild product $P \star V$ and moreover, the isomorphism of algebras $(E, \cdot_E) \cong P \star V$ can be chosen such that it stabilizes V and co-stabilizes P .

1.2. The crossed product of Lie algebras

We shall recall the construction of the crossed product for Lie algebras following the terminology of [2, Section 3].

Definition 1.4. Let $P = (P, [-, -])$ be a Lie algebra and V a vector space. A *pre-crossed data* of P by V is a system $\Lambda(P, V) = (\triangleright, f, [-, -]_V)$ consisting of three bilinear maps

$$\triangleright : P \times V \rightarrow V, \quad f : P \times P \rightarrow V, \quad [-, -]_V : V \times V \rightarrow V.$$

Let $\Lambda(P, V) = (\triangleright, f, [-, -]_V)$ be a pre-crossed data of P by V and we denote by $P \# V = P \#_{\triangleright, f}^V$ the vector space $P \times V$ with the bracket given for any $p, q \in P$ and $x, y \in V$ by:

$$\{(p, x), (q, y)\} := ([p, q], f(p, q) + p \triangleright y - q \triangleright x + [x, y]_V). \tag{10}$$

Then $P \# V$ is called the *crossed product* associated to $\Lambda(P, V) = (\triangleright, f, [-, -]_V)$ if it is a Lie algebra with the bracket (10). In this case the pre-crossed data $\Lambda(P, V) = (\triangleright, f, [-, -]_V)$ is called a *crossed system* of P by V and we denote by $\mathcal{LS}(P, V)$ the set of all crossed systems of the Lie algebra P by V .

As a special case of [2, Theorem 2.2] or by a straightforward computation it is easy to see that $\Lambda(P, V) = (\triangleright, f, [-, -]_V)$ is a crossed system of P by V if and only if the following compatibilities hold for any $p, q, r \in P$ and $x, y \in V$:

(L0) $(V, [-, -]_V)$ is a Lie algebra

(L1) $f(p, p) = 0$

(L2) $p \triangleright [x, y]_V = [p \triangleright x, y]_V + [x, p \triangleright y]_V$

(L3) $[p, q] \triangleright x = p \triangleright (q \triangleright x) - q \triangleright (p \triangleright x) + [x, f(p, q)]_V$

(L4) $f(p, [q, r]) + f(q, [r, p]) + f(r, [p, q]) + p \triangleright f(q, r) + q \triangleright f(r, p) + r \triangleright f(p, q) = 0.$

Example 1.5. Let P be a Lie algebra having $\{e_i \mid i \in I\}$ as a basis. Then there exists a bijection between the set $\mathcal{LS}(P, k)$ of all crossed systems of P by k and the set of pairs (λ, f) consisting of a linear map $\lambda : P \rightarrow k$ and a bilinear map $f : P \times P \rightarrow k$ satisfying the following compatibility conditions for all $p, q, r \in P$:

$$f(p, p) = \lambda([p, q]) = 0$$

$$f(p, [q, r]) + f(q, [r, p]) + f(r, [p, q]) + \lambda(p)f(q, r) + \lambda(q)f(r, p) + \lambda(r)f(p, q) = 0.$$

Under this bijection the crossed product $P \#_{\lambda}^f k$ corresponding to (λ, f) is the Lie algebra having $\{x, e_i \mid i \in I\}$ as a basis and the bracket $[-, -]_{\lambda}^f$ defined for any $i, j \in I$ by:

$$[e_i, e_j]_{\lambda}^f := [e_i, e_j] + f(e_i, e_j)x, \quad [e_i, x]_{\lambda}^f := \lambda(e_i)x.$$

Indeed, since $V := k$ the Lie bracket on k is the trivial map $[-, -] = 0$. Moreover, any bilinear map $\triangleright : P \times k \rightarrow k$ is implemented by a unique linear map $\lambda : P \rightarrow k$ such that $p \triangleright 1_k = \lambda(p)$. The rest of the proof is straightforward.

2. Crossed products and the global extension problem

In this section we shall give the theoretical answer to the GE-problem. First we introduce the following:

Definition 2.1. Let P be a Poisson algebra and V a vector space. A *pre-crossed datum* of P by V is a system $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ consisting of seven bilinear maps

$$\begin{aligned} \dashv : P \times V &\rightarrow V, & \triangleleft : V \times P &\rightarrow V, & \vartheta : P \times P &\rightarrow V, & \cdot_V : V \times V &\rightarrow V \\ \triangleright : P \times V &\rightarrow V, & f : P \times P &\rightarrow V, & [-, -]_V : V \times V &\rightarrow V. \end{aligned}$$

Let $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ be a pre-crossed datum of P by V . We denote by $P \#_{\Omega(P, V)} V = P \# V$ the vector space $P \times V$ together with the multiplication \star given by (9) and the bracket $\{-, -\}$ given by (10), i.e.

$$(p, x) \star (q, y) := (pq, \vartheta(p, q) + p \rightharpoonup y + x \triangleleft q + x \cdot_V y) \tag{11}$$

$$\{(p, x), (q, y)\} := ([p, q], f(p, q) + p \triangleright y - q \triangleright x + [x, y]_V). \tag{12}$$

for all $p, q \in P, x, y \in V$. The object $P \sharp V = (P \times V, \star, \{-, -\})$ is called the *crossed product* of P and $\Omega(P, V)$ if it is a Poisson algebra. In this case the pre-crossed datum $\Omega(P, V)$ is called a *crossed system* of the Poisson algebra P by V . The maps $\leftarrow, \triangleleft, \triangleright$ are called the *actions* of $\Omega(P, V)$ while ϑ and f are called the *cocycles* of $\Omega(P, V)$.

The next theorem provides the necessary and sufficient conditions that need to be fulfilled by a pre-crossed datum $\Omega(P, V)$ such that $P \sharp V$ is a Poisson algebra.

Theorem 2.2. *Let $\Omega(P, V) = (\leftarrow, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ be a pre-crossed datum of a Poisson algebra P by a vector space V . Then $P \sharp V$ is a Poisson algebra if and only if the following compatibilities hold:*

- (P0) $(\leftarrow, \triangleleft, \vartheta, \cdot_V)$ is a Hochschild system of the algebra P by V , $(\triangleright, f, [-, -]_V)$ is a crossed system of the Lie algebra P by V and $(V, \cdot_V, [-, -]_V)$ is a Poisson algebra
- (P1) $f(pq, r) - f(p, r) \triangleleft q - p \rightharpoonup f(q, r) = r \triangleright \vartheta(p, q) + \vartheta([p, r], q) + \vartheta(p, [q, r])$
- (P2) $(pq) \triangleright x = p \rightharpoonup (q \triangleright x) + (p \triangleright x) \triangleleft q - [\vartheta(p, q), x]_V$
- (P3) $[p, q] \rightharpoonup x = p \rightharpoonup (q \triangleright x) - q \triangleright (p \rightharpoonup x) - f(p, q) \cdot_V x$
- (P4) $p \rightharpoonup [x, y]_V = [p \rightharpoonup x, y]_V - (p \triangleright y) \cdot_V x$
- (P5) $x \triangleleft [p, q] = (q \triangleright x) \triangleleft p - q \triangleright (x \triangleleft p) - x \cdot_V f(p, q)$
- (P6) $[x, y]_V \triangleleft p = [x \triangleleft p, y]_V - x \cdot_V (p \triangleright y)$
- (P7) $p \triangleright (x \cdot_V y) = (p \triangleright x) \cdot_V y + x \cdot_V (p \triangleright y)$

for all $p, q, r \in P$ and $x, y \in V$.

Proof. We have already noticed in Preliminaries that $(P \sharp V, \star)$ is an associative algebra if and only if $(\leftarrow, \triangleleft, \vartheta, \cdot_V)$ is a Hochschild system of the associative algebra P by V and $(P \sharp V, \{-, -\})$ is a Lie algebra if and only $(\triangleright, f, [-, -]_V)$ is a crossed system of the Lie algebra P by V . These are the first two assumptions from (P0) which from now on we assume to be fulfilled. Then, $(P \sharp V, \star, \{-, -\})$ is a Poisson algebra if and only if the Leibniz identity holds, i.e.

$$\{(p, x) \star (q, y), (r, z)\} = \{(p, x), (r, z)\} \star (q, y) + (p, x) \star \{(q, y), (r, z)\} \tag{13}$$

for all $p, q, r \in P$ and $x, y, z \in V$. The rest of the proof relies on a detailed analysis of the Leibniz identity (13): since in $P \times V$ we have $(p, x) = (p, 0) + (0, x)$ it follows that (13) holds if and only if it holds for all generators of $P \times V$, i.e. for the set $\{(p, 0) \mid p \in P\} \cup \{(0, x) \mid x \in V\}$. However, since the computations are rather long but straightforward we will only indicate the main steps of the proof, the details being left to the reader.

We will start by proving that (13) holds for the triple $(p, 0), (q, 0), (r, 0)$ if and only if (P1) holds. Indeed, we can easily see that the left hand side of (13), evaluated at $(p, 0), (q, 0), (r, 0)$ is equal to $([pq, r], f(pq, r) - r \triangleright \vartheta(p, q))$ while the right hand side of (13) is

$$([p, r]q + p[q, r], \vartheta([p, r], q) + f(p, r) \triangleleft q + \vartheta(p, [q, r]) + p \dashv f(q, r)).$$

Since P is a Poisson algebra we obtain that (13) holds for the triple $(p, 0), (q, 0), (r, 0)$ if and only if (P1) hold.

In the same manner we can prove the following: (13) holds for the triple $(p, 0), (q, 0), (0, x)$ if and only if (P2) holds; (13) holds for the triple $(p, 0), (0, x), (q, 0)$ if and only if (P3) holds; (13) holds for the triple $(p, 0), (0, x), (0, y)$ if and only if (P4) holds; (13) holds for the triple $(0, x), (p, 0), (q, 0)$ if and only if (P5) holds; (13) holds for the triple $(0, x), (p, 0), (0, y)$ if and only if (P6) holds and (13) holds for the triple $(0, x), (0, y), (p, 0)$ if and only if (P7) holds. Finally, one can see that (13) holds for the triple $(0, x), (0, y), (0, z)$ if and only if

$$[x \cdot_V y, z]_V = [x, z]_V \cdot_V y + x \cdot_V [y, z]_V$$

i.e. the Leibniz law holds for $(V, \cdot_V, [-, -]_V)$, that is $(V, \cdot_V, [-, -]_V)$ is a Poisson algebra and this finishes the proof. \square

From now on, a crossed system of a Poisson algebra P by a vector space V will be viewed as a pre-crossed datum $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ satisfying the compatibility conditions (P0)–(P7) of Theorem 2.2. We denote by $\mathcal{PS}(P, V)$ the set of all crossed systems of P by V . We also use the following convention: if one of the maps of a crossed system $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ is the trivial one, then we will omit it from the system $\Omega(P, V)$.

Example 2.3. Let $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ be a pre-crossed datum of a Poisson algebra P by V such that $\dashv, \triangleleft, \vartheta, \triangleright, f$ are all the trivial maps. Then $\Omega(P, V)$ is a crossed system of P by V if and only if $(V, \cdot_V, [-, -]_V)$ is a Poisson algebra. The associated crossed product $P \sharp V$ is just the direct product $P \times V$ of Poisson algebras.

Any crossed product $P \sharp V$ is an extension of the Poisson algebra P by V via the canonical maps

$$0 \longrightarrow V \xrightarrow{i_V} P \sharp V \xrightarrow{\pi_P} P \longrightarrow 0 \tag{14}$$

where $i_V(v) = (0, v)$ and $\pi_P(p, x) := p$. Conversely, the crossed product is the tool to answer the global extension problem for Poisson algebras. More precisely, the next result provides the answer to the description part of the GE-problem and can be seen as a generalization at the level of Poisson algebras of [20, Theorem 5.2]:

Proposition 2.4. *Let P be a Poisson algebra, E a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces with $V = \text{Ker}(\pi)$. Then any Poisson algebra structure $(\cdot_E, [-, -]_E)$ which can be defined on E such that $\pi : (E, \cdot_E, [-, -]_E) \rightarrow P$ is a morphism of Poisson algebras is isomorphic to a crossed product $P \sharp V$ and moreover, the isomorphism of Poisson algebras $(E, \cdot_E, [-, -]_E) \cong P \sharp V$ can be chosen such that it stabilizes V and co-stabilizes P .*

In particular, any Poisson algebra extension of P by V is cohomologous to a crossed product extension (14).

Proof. Indeed, let $(\cdot_E, [-, -]_E)$ be a Poisson algebra structure of E such that $\pi : (E, \cdot_E, [-, -]_E) \rightarrow P$ is a morphism of Poisson algebras. Let $s : P \rightarrow E$ be a k -linear section of π , i.e. $\pi \circ s = \text{Id}_P$. Using this section s we define a pre-crossed datum $\Omega(P, V) = (\rightharpoonup = \rightharpoonup_s, \triangleleft = \triangleleft_s, \vartheta = \vartheta_s, \cdot_V = \cdot_{V,s}, \triangleright = \triangleright_s, f = f_s, [-, -]_V = [-, -]_{V,s})$ of P by V by the following formulas:

$$\begin{aligned} \rightharpoonup : P \times V &\rightarrow V, & p \rightharpoonup x &:= s(p) \cdot_E x, & \triangleleft : V \times P &\rightarrow V, & x \triangleleft p &:= x \cdot_E s(p) \\ \vartheta : P \times P &\rightarrow V, & \vartheta(p, q) &:= s(p) \cdot_E s(q) - s(pq), & \cdot_V : V \times V &\rightarrow V, & x \cdot_V y &:= x \cdot_E y \\ \triangleright : P \times V &\rightarrow V, & p \triangleright x &:= [s(p), x]_E, & [-, -]_V : V \times V &\rightarrow V, & [x, y]_V &:= [x, y]_E \\ f : P \times P &\rightarrow V, & f(p, q) &:= [s(p), s(q)]_E - s([p, q]) \end{aligned}$$

for all $p, q \in P$ and $x, y \in V$. Then

$$\varphi : P \times V \rightarrow E, \quad \varphi(p, x) := s(p) + x$$

is an isomorphism of vector spaces with the inverse $\varphi^{-1}(y) = (\pi(y), y - s(\pi(y)))$, for all $y \in E$. The key step is the following: the unique Poisson algebra structure $(\star, \{-, -\})$ that can be defined on the direct product of vector spaces $P \times V$ such that $\varphi : P \times V \rightarrow (E, \cdot_E, [-, -]_E)$ is an isomorphism of Poisson algebras has the multiplication \star given by (11) and the bracket $\{-, -\}$ given by (12) associated to the system $(\rightharpoonup, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ as defined above arising from s . Indeed, [3, Proposition 1.4] proves that the unique multiplication \star that can be defined on $P \times V$ such that $\varphi : P \times V \rightarrow (E, \cdot_E)$ is an isomorphism of algebras is the one given by (11) associated to the Hochschild system $(\rightharpoonup_s, \triangleleft_s, \vartheta_s, \cdot_{V,s})$. In a similar manner we can show that the unique bracket $\{-, -\}$ that can be defined on $P \times V$ such that $\varphi : P \times V \rightarrow (E, [-, -]_E)$ is an isomorphism of Lie algebras is the one given by (12) associated to the crossed system $(\triangleright_s, f_s, [-, -]_{V,s})$ of the Lie algebra P by V . Thus, $\varphi : P \sharp V \rightarrow (E, \cdot_E, [-, -]_E)$ is an isomorphism of Poisson algebras that stabilizes V and co-stabilizes P . \square

Example 2.5. Among all special cases of crossed products the most important is the semidirect product since, exactly as in the case of groups, Lie algebras or associative algebras, the semidirect products will describe the split epimorphisms in the category of

Poisson algebras. Let P and $V = (V, \cdot_V, [-, -]_V)$ be two given Poisson algebras. Then a *semi-direct system* of P by V is a system consisting of three bilinear maps $(\dashv, \triangleleft, \triangleright)$ such that $(\dashv, \triangleleft, \vartheta := 0, \cdot_V, \triangleright, f := 0, [-, -]_V) = (\dashv, \triangleleft, \cdot_V, \triangleright, [-, -]_V)$ is a crossed system of P by V . If $(\dashv, \triangleleft, \triangleright)$ is a semi-direct system between P and V , then the associated crossed product $P \sharp V$ is denoted by $P \rtimes V$ and is called the *semi-direct product* of P and V . Thus, $P \rtimes V$ is the vector space $P \times V$ with the Poisson algebra structure defined by:

$$(p, x) \star (q, y) := (pq, p \dashv y + x \triangleleft q + x \cdot_V y) \tag{15}$$

$$\{(p, x), (q, y)\} := ([p, q], p \triangleright y - q \triangleright x + [x, y]_V) \tag{16}$$

for all $p, q \in P, x, y \in V$. If $P \rtimes V$ is a semidirect product of Poisson algebras then the canonical projection $\pi_P : P \rtimes V \rightarrow P, \pi_P(p, x) := p$ is a morphism of Poisson algebras that has a section $s_P : P \rightarrow P \rtimes V, s_P(p) := (p, 0)$ that is also a morphism of Poisson algebras. Conversely, the semidirect product of Poisson algebras describes the split epimorphisms in the category of Poisson algebras. Indeed, let $\pi : E \rightarrow P$ be a morphism of Poisson algebras which has a section that is a Poisson algebra map. Then there exists an isomorphism of Poisson algebras $E \cong P \rtimes V$, where $P \rtimes V$ is the semidirect product between P and $V = \text{Ker}(\pi)$. The result follows from the proof of [Proposition 2.4](#): if $s : P \rightarrow E$ is a morphism of Poisson algebras, then the cocycles $f = f_s$ and $\vartheta = \vartheta_s$ constructed there are both the trivial maps, i.e. the corresponding crossed product $P \sharp V$ is a semidirect product $P \rtimes V$.

Based on [Proposition 2.4](#) the answer to the classification part of the GE-problem reduces to the classification of all crossed products associated to all crossed systems between P and V . This is what we do next by explicitly constructing a classification object, denoted by $\mathcal{GPH}^2(P, V)$. First we need the following technical result:

Lemma 2.6. *Let P be a Poisson algebra, $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ and $\Omega'(P, V) = (\dashv', \triangleleft', \vartheta', \cdot'_V, \triangleright', f', [-, -]'_V)$ two crossed systems of P by a vector space V and $P \sharp V$, respectively $P \sharp' V$, the corresponding crossed products. Then there exists a bijection between the set of all morphisms of Poisson algebras $\psi : P \sharp V \rightarrow P \sharp' V$ which stabilize V and co-stabilize P and the set of all linear maps $r : P \rightarrow V$ satisfying the following compatibilities for all $p, q \in P, x, y \in V$:*

- (M1) $x \cdot_V y = x \cdot'_V y$
- (M2) $x \triangleleft p = x \triangleleft' p + x \cdot'_V r(p)$
- (M3) $p \dashv x = p \dashv' x + r(p) \cdot'_V x$
- (M4) $\vartheta(p, q) = \vartheta'(p, q) + p \dashv' r(q) + r(p) \triangleleft' q - r(pq) + r(p) \cdot'_V r(q)$
- (M5) $[x, y]_V = [x, y]'_V$
- (M6) $p \triangleright x = p \triangleright' x + [r(p), x]'_V$
- (M7) $f(p, q) = f'(p, q) + p \triangleright' r(q) - q \triangleright' r(p) + [r(p), r(q)]'_V - r([p, q])$.

Under the above bijection the morphism of Poisson algebras $\psi = \psi_r : P \sharp V \rightarrow P \sharp' V$ corresponding to $r : P \rightarrow V$ is given by $\psi(p, x) = (p, r(p) + x)$, for all $p \in P$ and $x \in V$. Moreover, $\psi = \psi_r$ is an isomorphism with the inverse $\psi_r^{-1} = \psi_{-r}$.

Proof. A linear map $\psi : P \sharp V \rightarrow P \sharp' V$ stabilizes V and co-stabilizes P if and only if there exists a uniquely determined linear map $r : P \rightarrow V$ such that $\psi(p, x) = (p, r(p) + x)$, for all $p \in P, x \in V$. Let $\psi = \psi_r$ be such a linear map. Then $\psi : P \sharp V \rightarrow P \sharp' V$ is a morphism of associative algebras if and only if the compatibility conditions (M1)–(M4) hold [3, Lemma 1.5]. By a straightforward computation we can also prove that $\psi = \psi_r : P \sharp V \rightarrow P \sharp' V$ is a morphism of Lie algebras if and only if (M5)–(M7) hold. For this it is enough to check the compatibility condition $\psi(\{(p, x), (q, y)\}) = \{\psi(p, x), \psi(q, y)\}'$ on the set of generators, i.e. for the set $\{(p, 0) \mid p \in P\} \cup \{(0, x) \mid x \in V\}$. \square

Lemma 2.6 leads to the following definition:

Definition 2.7. Let P be a Poisson algebra and V a vector space. Two crossed systems $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ and $\Omega'(P, V) = (\dashv', \triangleleft', \vartheta', \cdot'_V, \triangleright', f', [-, -]'_V)$ are called *cohomologous*, and we denote this by $\Omega(P, V) \approx \Omega'(P, V)$, if and only if $\cdot_V = \cdot'_V, [-, -] = [-, -]'_V$ and there exists a linear map $r : P \rightarrow V$ such that for any $p, q \in P, x, y \in V$ we have:

$$p \dashv x = p \dashv' x + r(p) \cdot_V x \tag{17}$$

$$x \triangleleft p = x \triangleleft' p + x \cdot_V r(p) \tag{18}$$

$$\vartheta(p, q) = \vartheta'(p, q) + p \dashv' r(q) + r(p) \triangleleft' q - r(pq) + r(p) \cdot_V r(q) \tag{19}$$

$$p \triangleright x = p \triangleright' x + [r(p), x]_V \tag{20}$$

$$f(p, q) = f'(p, q) + p \triangleright' r(q) - q \triangleright' r(p) + [r(p), r(q)]_V - r([p, q]). \tag{21}$$

Example 2.8. Let P and V be two Poisson algebras and $\Omega(P, V) = (\cdot_V, [-, -]_V)$ be the trivial crossed system of the Poisson algebra P by V from Example 2.3. A crossed system $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ cohomologous with the trivial crossed system is called a *coboundary*. Hence, $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ is a coboundary if and only if there exists a linear map $r : P \rightarrow V$ such that $\dashv, \triangleleft, \vartheta, \triangleright$ and f are implemented by r via the formulas:

$$p \dashv x = r(p) \cdot_V x, \quad x \triangleleft p = x \cdot_V r(p), \quad p \triangleright x = [r(p), x]_V$$

$$\vartheta(p, q) = r(p) \cdot_V r(q) - r(pq), \quad f(p, q) = [r(p), r(q)]_V - r([p, q])$$

for all $p, q \in P, x, y \in V$. Lemma 2.6 shows that any crossed product $P \sharp V$ associated to a coboundary is isomorphic to the usual direct product $P \times V$ of Poisson algebras.

As a conclusion, we obtain the theoretical answer to the GE-problem for Poisson algebras:

Theorem 2.9. *Let P be a Poisson algebra, E a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces with $V = \text{Ker}(\pi)$. Then \approx is an equivalence relation on the set $\mathcal{PS}(P, V)$ of all crossed systems of P by V . If we denote by $\mathcal{GPH}^2(P, V) := \mathcal{PS}(P, V)/\approx$, then the map*

$$\mathcal{GPH}^2(P, V) \rightarrow \text{Gext}(E, P), \quad \overline{(\dashv, \triangleleft, \vartheta, \triangleright, f, [-, -]_V)} \longmapsto P \sharp_{(\dashv, \triangleleft, \vartheta, \triangleright, f, [-, -]_V)} V$$

is a bijection between $\mathcal{GPH}^2(P, V)$ and $\text{Gext}(E, P)$.

Proof. Follows from [Theorem 2.2](#), [Proposition 2.4](#) and [Lemma 2.6](#). \square

2.1. The global vs. the local extension problem

Computing the classifying object $\mathcal{GPH}^2(P, V)$ is a highly nontrivial problem. We explain below the details as well as the connection with the classical (i.e. local) extension problem. The extension problem has as input data two given Poisson algebras P and $V = (V, \cdot_V, [-, -]_V)$ and it asks for the classification of all extensions of P by the fixed Poisson algebra V . We denote by $\text{Ext}(P, (V, \cdot_V, [-, -]_V))$ the isomorphism classes of all extensions of P by V , i.e. up to an isomorphism of Poisson algebras that stabilizes V and co-stabilizes P . We should point out that we work in the general case, with no further assumptions on the Poisson algebras (e.g. commutative, abelian etc.). The answer to the extension problem follows as a special case of [Theorem 2.9](#):

Let $\mathcal{LPS}(P, (V, \cdot_V, [-, -]_V))$ be the set of all *local crossed systems* of P by $(V, \cdot_V, [-, -]_V)$, i.e. $\mathcal{LPS}(P, (V, \cdot_V, [-, -]_V))$ contains the set of all 5-uples $(\dashv, \triangleleft, \vartheta, \triangleright, f)$ of bilinear maps

$$\begin{aligned} \dashv : P \times V &\rightarrow V, & \triangleleft : V \times P &\rightarrow V, & \vartheta : P \times P &\rightarrow V, \\ \triangleright : P \times V &\rightarrow V, & f : P \times P &\rightarrow V, \end{aligned}$$

satisfying axioms (H1)–(H7), (L1)–(L4) and (P1)–(P7). Two local crossed systems $(\dashv, \triangleleft, \vartheta, \triangleright, f)$ and $(\dashv', \triangleleft', \vartheta', \triangleright', f')$ are *local cohomologous* and we denote this by $(\dashv, \triangleleft, \vartheta, \triangleright, f) \approx_l (\dashv', \triangleleft', \vartheta', \triangleright', f')$ if there exists a linear map $r : P \rightarrow V$ satisfying the compatibility conditions [\(17\)–\(21\)](#). The answer in the general case to the extension problem for Poisson algebras and the connection *local vs. global* is given below. In particular, we obtain the formula for computing $\mathcal{GPH}^2(P, V)$:

Corollary 2.10. *Let P be a Poisson algebra. Then:*

(1) *If $V = (V, \cdot_V, [-, -]_V)$ is a fixed Poisson algebra, then \approx_l is an equivalence relation on the set $\mathcal{LPS}(P, (V, \cdot_V, [-, -]_V))$ of all local crossed systems of P by $(V, \cdot_V, [-, -]_V)$. If*

we denote by $\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V))$ the quotient set $\mathcal{LPS}(P, (V, \cdot_V, [-, -]_V)) / \approx_i$, then the map

$$\mathcal{PH}^2(P, (V, \cdot_V, [-, -]_V)) \rightarrow \text{Ext}(P, (V, \cdot_V, [-, -]_V)), \quad \overline{(\dashv, \triangleleft, \vartheta, \triangleright, f)} \mapsto P \sharp_{(\dashv, \triangleleft, \vartheta, \triangleright, f)} V$$

is a bijection.

(2) Let E be a vector space and $\pi : E \rightarrow P$ an epimorphism of vector spaces with $V = \text{Ker}(\pi)$. Then:

$$\mathcal{GPH}^2(P, V) = \sqcup_{(\cdot_V, \{-, -\}_V)} \mathcal{PH}^2(P, (V, \cdot_V, \{-, -\}_V)) \tag{22}$$

where the coproduct in the right hand side is in the category of sets over all possible Poisson algebra structures $(\cdot_V, \{-, -\}_V)$ on the vector space V .

Proof. The first part follows from the above considerations and [Theorem 2.9](#) while the second part follows from relations (M1) and (M5) of [Lemma 2.6](#). \square

2.2. The abelian case. Metabelian Poisson algebras

The formula (22) for computing $\mathcal{GPH}^2(P, V)$ highlights the difficulty of the GE-problem. In fact, even computing every cohomology object $\mathcal{PH}^2(P, (V, \cdot_V, \{-, -\}_V))$ from the right hand side, for a given Poisson algebra $(V, \cdot_V, \{-, -\}_V)$ is a problem far from being trivial. Traditionally, regardless if we consider the case of groups [33], associative algebras [20] or Lie algebras [12], the extension problem and therefore the cohomology groups are considered only in the abelian case. In other words, only one of the elements in the coproduct from the right hand side of (22) is well understood, namely the one corresponding to the abelian case – which for Poisson algebras comes down to V being an abelian Poisson algebra, i.e. $x \cdot_V y = [x, y]_V = 0$, for all $x, y \in V$. We recall that we have denoted by V_0 the vector space V with the abelian Poisson algebra structure.

In what follows we will describe the object $\mathcal{PH}^2(P, V_0)$, for the abelian Poisson algebra V_0 . In this case the set $\mathcal{LPS}(P, V_0)$ of local crossed systems of P by V_0 consists of the set of all 5-uples $(\dashv, \triangleleft, \vartheta, \triangleright, f)$ of bilinear maps

$$\begin{aligned} \dashv : P \times V &\rightarrow V, & \triangleleft : V \times P &\rightarrow V, & \vartheta : P \times P &\rightarrow V, \\ \triangleright : P \times V &\rightarrow V, & f : P \times P &\rightarrow V, \end{aligned}$$

satisfying the following compatibilities for all $p, q, r \in P$ and $x \in V$:

- (Ab1) $(V, \dashv, \triangleleft, \triangleright) \in {}^P_P\mathcal{M}_P$ is a Poisson bimodule over P
- (Ab2) $\vartheta(p, q) \triangleleft r = \vartheta(p, qr) - \vartheta(pq, r) + p \dashv \vartheta(q, r)$
- (Ab3) $f(p, p) = 0$

- (Ab4) $f(p, [q, r]) + f(q, [r, p]) + f(r, [p, q]) + p \triangleright f(q, r) + q \triangleright f(r, p) + r \triangleright f(p, q) = 0$
- (Ab5) $f(pq, r) - f(p, r) \triangleleft q - p \dashv f(q, r) = r \triangleright \vartheta(p, q) + \vartheta([p, r], q) + \vartheta(p, [q, r]).$

These are the axioms that remain from (H1)–(H7), (L1)–(L4) and (P1)–(P7) in the case that $\cdot_V = [-, -]_V = 0$. In this context Definition 2.7 takes the following simplified form: two local crossed systems $(\dashv, \triangleleft, \vartheta, \triangleright, f)$ and $(\dashv', \triangleleft', \vartheta', \triangleright', f')$ of P by V_0 are local cohomologous and we denote this by $(\dashv, \triangleleft, \vartheta, \triangleright, f) \approx_{l,0} (\dashv', \triangleleft', \vartheta', \triangleright', f')$ if and only if $\dashv = \dashv', \triangleleft = \triangleleft', \triangleright = \triangleright'$ and there exists a linear map $r : P \rightarrow V$ such that

$$\vartheta(p, q) = \vartheta'(p, q) + p \dashv r(q) + r(p) \triangleleft q - r(pq) \tag{23}$$

$$f(p, q) = f'(p, q) + p \triangleright r(q) - q \triangleright r(p) - r([p, q]) \tag{24}$$

for all $p, q \in P$. The equalities $\dashv = \dashv', \triangleleft = \triangleleft', \triangleright = \triangleright'$ show that the object $\mathcal{PH}^2(P, V_0)$ is also a coproduct in the category of sets over all triples $(\dashv, \triangleleft, \triangleright)$ such that $(V, \dashv, \triangleleft, \triangleright) \in {}^P\mathcal{M}_P$ is a Poisson bimodule. We will highlight this fact by fixing the Poisson module structure $(V, \dashv, \triangleleft, \triangleright) \in {}^P\mathcal{M}_P$ on V and denoting by $\mathcal{C}_{(\dashv, \triangleleft, \triangleright)}(P, V)$ the set of all pairs (ϑ, f) consisting of two bilinear maps $\vartheta : P \times P \rightarrow V$ and $f : P \times P \rightarrow V$ satisfying the compatibility conditions (Ab2)–(Ab5). Two pairs (ϑ, f) and $(\vartheta', f') \in \mathcal{C}_{(\dashv, \triangleleft, \triangleright)}(P, V)$ are local cohomologous if there exists a linear map $r : P \rightarrow V$ satisfying the compatibility conditions (23)–(24). If we denote by $\mathcal{PH}^2_{(\dashv, \triangleleft, \triangleright)}(P, V)$ the quotient set of $\mathcal{C}_{(\dashv, \triangleleft, \triangleright)}(P, V)$ via this equivalence relation we obtain the following:

Corollary 2.11. *Let P be a Poisson algebra and V a vector space with the abelian Poisson algebra structure V_0 . Then:*

$$\mathcal{PH}^2(P, V_0) = \sqcup_{(\dashv, \triangleleft, \triangleright)} \mathcal{PH}^2_{(\dashv, \triangleleft, \triangleright)}(P, V) \tag{25}$$

where the coproduct in the right hand side is in the category of sets over all possible Poisson bimodule structures $(\dashv, \triangleleft, \triangleright)$ on V .

Now we shall take a step forward and we will consider P as well with the abelian Poisson algebra structure, i.e. $P = P_0$. Having in mind the theory of groups² we introduce the following:

Definition 2.12. A Poisson algebra Q is called *metabelian* if Q is an extension of an abelian Poisson algebra by another abelian Poisson algebra.

Using Proposition 2.4 and then Theorem 2.2 we obtain that any metabelian Poisson algebra Q is isomorphic to a crossed product $P_0 \# V_0$, for some vector spaces $P = P_0$

² We recall that a group is called metabelian if it is an extension of an abelian group by an abelian group.

and $V = V_0$, i.e. $P_0 \# V_0$ has the multiplication and the bracket given for any $p, q \in P$, $x, y \in V$ by:

$$(p, x) \star (q, y) := (0, \vartheta(p, q) + p \rightharpoonup y + x \triangleleft q) \tag{26}$$

$$\{(p, x), (q, y)\} := (0, f(p, q) + p \triangleright y - q \triangleright x) \tag{27}$$

for some bilinear maps

$$\begin{aligned} \rightharpoonup : P \times V &\rightarrow V, & \triangleleft : V \times P &\rightarrow V, & \vartheta : P \times P &\rightarrow V, \\ \triangleright : P \times V &\rightarrow V, & f : P \times P &\rightarrow V \end{aligned}$$

satisfying the following compatibility conditions for any $p, q, r \in P$ and $x \in V$:

$$(p \rightharpoonup x) \triangleleft q = p \rightharpoonup (x \triangleleft q), \quad \vartheta(p, q) \triangleleft r = p \rightharpoonup \vartheta(q, r) \tag{28}$$

$$p \rightharpoonup (q \rightharpoonup x) = (x \triangleleft p) \triangleleft q = 0 \tag{29}$$

$$f(p, p) = 0, \quad p \triangleright (q \triangleright x) = q \triangleright (p \triangleright x) \tag{30}$$

$$p \triangleright f(q, r) + q \triangleright f(r, p) + r \triangleright f(p, q) = 0 \tag{31}$$

$$p \triangleright \vartheta(p, q) + f(p, r) \triangleleft q + p \rightharpoonup f(q, r) = 0 \tag{32}$$

$$p \rightharpoonup (q \triangleright x) = -(p \triangleright x) \triangleleft q = q \triangleright (p \rightharpoonup x) \tag{33}$$

$$(q \triangleright x) \triangleleft p = q \triangleright (x \triangleleft p) \tag{34}$$

which are the ones remaining from the axioms (P0)–(P7) in this context. For two vector spaces P and V we denote by $\mathcal{MA}(P, V)$ the set of all *metabelian systems of P by V* , that is all bilinear maps $(\rightharpoonup, \triangleleft, \vartheta, \triangleright, f)$ satisfying the compatibility conditions (28)–(34). With these notations we obtain the following result that classifies all metabelian Poisson algebras that are extensions of P_0 by V_0 :

Corollary 2.13. *Let P and V be two vector spaces with the abelian Poisson algebra structure. Then:*

$$\mathcal{PH}^2(P_0, V_0) \cong \mathcal{MA}(P, V) / \approx_m^a \tag{35}$$

where \approx_m^a is the following relation on $\mathcal{MA}(P, V)$: $(\rightharpoonup, \triangleleft, \vartheta, \triangleright, f) \approx_m^a (\rightharpoonup', \triangleleft', \vartheta', \triangleright', f')$ if and only if $\rightharpoonup = \rightharpoonup'$, $\triangleleft = \triangleleft'$, $\triangleright = \triangleright'$ and there exists a linear map $r : P \rightarrow V$ such that for any $p, q \in P$ we have:

$$\begin{aligned} \vartheta(p, q) &= \vartheta'(p, q) + p \rightharpoonup r(q) + r(p) \triangleleft q, \\ f(p, q) &= f'(p, q) + p \triangleright r(q) - q \triangleright r(p). \end{aligned} \tag{36}$$

Proof. Follows from the above considerations once we observe that the equivalence relation from [Definition 2.7](#) takes the form given in the statement for abelian Poisson algebras P_0 and V_0 . \square

Remark 2.14. The way we defined the equivalence relation \approx_m^a in [Corollary 2.13](#) indicates the decomposition of $\mathcal{PH}^2(P_0, V_0)$ into a coproduct over all triples $(\dashv, \triangleleft, \triangleright)$ such that $(V, \dashv, \triangleleft, \triangleright)$ is a Poisson bimodule over the abelian Poisson algebra P_0 . The decomposition is just the special case of the one given in [Corollary 2.11](#) by taking $P = P_0$.

The classification of all metabelian Poisson algebras of a given dimension seems to be a very challenging problem which will be addressed elsewhere. However we also include here two examples: the first one classifies all crossed products $k_0 \# k_0$ by computing the object $\mathcal{PH}^2(k_0, k_0)$; in particular, we will classify all 2-dimensional metabelian Poisson algebras.

Example 2.15. Let $P = V := k_0$ be the abelian Poisson algebra of dimension 1. Then $\mathcal{PH}^2(k_0, k_0) \cong \{(a, b) \in k^2 \mid ab = 0\}$. The explicit bijection between $\{(a, b) \in k^2 \mid ab = 0\}$ and the set of all equivalence classes of all non-cohomologous extensions of k_0 by k_0 is given by: $(a, b) \mapsto k_{a,b}^2$, where $k_{a,b}^2$ is the Poisson algebra with the basis $\{e_1, e_2\}$ and the structure: $e_1 \star e_1 := ae_2$ and $[e_1, e_2] := be_2$, for all $(a, b) \in \{(a, b) \in k^2 \mid ab = 0\}$.

In particular, up to an isomorphism of Poisson algebras, there exist three 2-dimensional metabelian Poisson algebras: k_0^2 , $k_{(1,0)}^2$ and $k_{(0,1)}^2$.

Indeed, in the first step we can easily see that there exists a bijection $\mathcal{MA}(k_0, k_0) \cong \{(a, b) \in k^2 \mid ab = 0\}$ given such that the metabelian system $(\dashv, \triangleleft, \vartheta, \triangleright, f)$ associated to the pair (a, b) with $ab = 0$ is given by:

$$\dashv := 0, \quad \triangleleft := 0, \quad f := 0, \quad \vartheta(1, 1) := a, \quad 1 \triangleright 1 := b.$$

Then, we observe that the equivalence relation of [Corollary 2.13](#) written equivalently on $\{(a, b) \in k^2 \mid ab = 0\}$ becomes $(a, b) \approx_m^a (a', b')$ if and only if $a = a'$ and $b = b'$. Thus, $\mathcal{PH}^2(k_0, k_0) = \mathcal{MA}(k_0, k_0)$. The Poisson algebra $k_{a,b}^2$ is just the crossed product $k_0 \# k_0$ with the structures given by [\(26\)–\(27\)](#) associated to above metabelian system. We have considered the canonical basis in $k \times k$.

Let n be a positive integer. We describe all crossed products $k_0 \# k_0^n$ (these are $(n + 1)$ -dimensional metabelian Poisson algebras) and then we shall classify all extension of k_0 by k_0^n by computing $\mathcal{PH}^2(k_0, k_0^n)$. The tool for both questions is an interesting set of matrices defined as follows: Let $\mathfrak{C}(n)$ be the set of all 4-uples $(A, B, C, \theta_0) \in M_n(k) \times M_n(k) \times M_n(k) \times k^n$ satisfying the following compatibilities:

$$\begin{aligned} AB = BA, \quad AC = CA = -BC = -CB, \quad A^2 = B^2 = 0, \\ A\theta_0 = B\theta_0, \quad C\theta_0 = 0. \end{aligned} \tag{37}$$

Two 4-uples (A, B, C, θ_0) and (A', B', C', θ'_0) are cohomologous and we denote this by $(A, B, C, \theta_0) \approx_m^a (A', B', C', \theta'_0)$ if and only if $A = A', B = B', C = C'$ and there exists $r \in k^n$ such that $\theta_0 - \theta'_0 = (A + B)r$. With these notations we have:

Example 2.16. Any crossed product $k_0 \# k_0^n$ is isomorphic to the Poisson algebra denoted by $k_{(A,B,C,\theta_0)}^{n+1}$ which is the vector space with basis $\{E_1, E_2, \dots, E_{n+1}\}$, multiplication \star and bracket given by:

$$E_i \star E_{n+1} := \sum_{j=1}^n b_{ji} E_j, \quad E_{n+1} \star E_i := \sum_{j=1}^n a_{ji} E_j \tag{38}$$

$$E_{n+1} \star E_{n+1} := \sum_{j=1}^n \theta_{0j} E_j, \quad \{E_{n+1}, E_i\} := \sum_{j=1}^n c_{ji} E_j \tag{39}$$

for all $i = 1, \dots, n$ and $(A, B, C, \theta_0) \in \mathfrak{C}(n)$ – all undefined operations are zero and $A = (a_{ij}), B = (b_{ij}), C = (c_{ij})$ and $\theta_0 = (\theta_{0j})$. Furthermore,

$$\mathcal{PH}^2(k_0, k_0^n) \cong \mathfrak{C}(n) / \approx_m^a.$$

Indeed, by a straightforward computation we can show that there exists a bijection between the set $\mathcal{MA}(k_0, k_0^n)$ of all metabelian systems of k_0 by k_0^n and the set of all 4-tuples $(\lambda, A, \gamma, \theta_0)$ consisting of three linear maps $\lambda, A, \gamma : k^n \rightarrow k^n$ and a vector $\theta_0 \in k^n$ satisfying the following compatibility conditions:

$$A \circ \lambda = \lambda \circ A, \quad A(\theta_0) = \lambda(\theta_0), \quad \lambda^2 = A^2 = 0 \tag{40}$$

$$\gamma(\theta_0) = 0, \quad \lambda \circ \gamma = \gamma \circ \lambda = -A \circ \gamma = -\gamma \circ A. \tag{41}$$

The bijection is given such that the metabelian system $(\rightarrow, \triangleleft, \vartheta, \triangleright, f)$ corresponding to $(\lambda, A, \gamma, \theta_0)$ is given by:

$$\begin{aligned} a \rightarrow x &:= a\lambda(x), & x \triangleleft a &:= aA(x), & \vartheta(a, b) &:= ab\theta_0, \\ a \triangleright x &:= a\gamma(x), & f(a, b) &:= 0 \end{aligned}$$

for all $a, b \in k$ and $x \in k^n$. We denote by $\{e_1, \dots, e_n\}$ the canonical basis of k^n and let A, B, C be the matrices associated to λ, A and respectively γ with respect to this basis. If we take as a basis in $k \times k^n$ the vectors $E_1 = (0, e_1), \dots, E_n = (0, e_n)$ and $E_{n+1} = (1, 0)$ then the Poisson algebra $k_{(A,B,C,\theta_0)}^{n+1}$ is just the crossed product $k_0 \# k_0^n$ with the structures given by (26)–(27) associated to the above metabelian system written equivalently with matrices. Finally, the equivalence relation of Corollary 2.13 rephrased using the elements of $\mathfrak{C}(n)$ is precisely the one given before Example 2.16.

Similar computations to the ones performed in Example 2.16 lead to the description and classification of all crossed products $k_0^n \# k_0$:

Example 2.17. Any crossed product $k_0^n \# k_0$ is isomorphic to one of the two families of Poisson algebras $k_{(\theta,f)}^{n+1}$ or $k_{(\gamma,f)}^{n+1}$ described below.

$$k_{(\theta,f)}^{n+1} : \quad E_i \star E_j := \theta(e_i, e_j)E_{n+1} \quad \{E_i, E_j\} := f(e_i, e_j)E_{n+1} \quad (42)$$

for all bilinear maps $\theta : k^n \times k^n \rightarrow k, f : k^n \times k^n \rightarrow k$ satisfying the condition $f(x, x) = 0$, for all $x \in k^n$. We denote by $\mathcal{MA}_1(k^n, k)$ the set of all such pairs (θ, f) .

$$k_{(\gamma,f)}^{n+1} : \quad \{E_i, E_j\} := f(e_i, e_j)E_{n+1}, \quad \{E_i, E_{n+1}\} := \gamma(e_i)E_{n+1} \quad (43)$$

for all pairs (γ, f) consisting of a non-trivial linear map $\gamma : k^n \rightarrow k$ and a bilinear map $f : k^n \times k^n \rightarrow k$ satisfying the following two compatibility conditions:

$$f(x, x) = 0, \quad \gamma(x)f(y, z) + \gamma(y)f(z, x) + \gamma(z)f(x, y) = 0$$

for all $x, y, z \in k^n$. We denote by $\mathcal{MA}_2(k^n, k)$ the set of all such pairs (γ, f) . Furthermore, we have that

$$\mathcal{PH}^2(k_0^n, k_0) \cong \mathcal{MA}_1(k^n, k) \sqcup \mathcal{MA}_2(k^n, k).$$

The details are left to the reader. We just mention that the set $\mathcal{MA}(k^n, k)$ of all metabelian systems of k_0^n by k_0 is parameterized by the set of all triples (θ, γ, f) consisting of two bilinear maps $\theta : k^n \times k^n \rightarrow k, f : k^n \times k^n \rightarrow k$ and a linear map $\gamma : k^n \rightarrow k$ satisfying the following compatibility conditions:

$$f(x, x) = 0, \quad \gamma(x)f(y, z) + \gamma(y)f(z, x) + \gamma(z)f(x, y) = 0, \quad \theta(x, y)\gamma(z) = 0 \quad (44)$$

for all $x, y, z \in k^n$. The bijection is given such that the metabelian system $(\rightarrow, \triangleleft, \vartheta, \triangleright, f)$ corresponding to (θ, γ, f) is given by:

$$\rightarrow := 0, \quad \triangleleft := 0, \quad \vartheta(x, y) := \theta(x, y), \quad x \triangleright a := a\gamma(x), \quad f(x, y) := f(x, y)$$

for all $a \in k$ and $x, y \in k^n$. The last equation of (44) leads us to consider two cases (depending on whether λ is the trivial map or not) and to write the set $\mathcal{MA}(k^n, k)$ as a coproduct of $\mathcal{MA}_1(k^n, k)$ and $\mathcal{MA}_2(k^n, k)$. The Poisson algebras $k_{(\theta,f)}^{n+1}$ and $k_{(\gamma,f)}^{n+1}$ are just the corresponding crossed products $k_0^n \# k_0$. The equivalence relation written on each of the sets $\mathcal{MA}_i(k^n, k)$ becomes an equality and this proves our last assertion.

The description of all 3-dimensional metabelian Poisson algebras follows just by taking $n = 2$ in Example 2.16 and Example 2.17:

Corollary 2.18. Any 3-dimensional metabelian Poisson algebra is isomorphic to one of the following Poisson algebras: $k_{(A,B,C,\theta_0)}^3$, for all $(A, B, C, \theta_0) \in \mathfrak{C}(2)$, or $k_{(\theta,f)}^3$, for all $(\theta, f) \in \mathcal{MA}_1(k^2, k)$, or $k_{(\gamma,f)}^3$, for all $(\gamma, f) \in \mathcal{MA}_2(k^2, k)$.

3. Co-flag Poisson algebras

Corollary 2.10 shows that computing the global cohomological object $\mathcal{GPH}^2(P, V)$ is a highly non-trivial task. In this section we deal with a special class of Poisson algebras for which we provide a recursive method to compute this classification object. These are the so-called co-flag Poisson algebras as defined below:

Definition 3.1. Let P be a Poisson algebra and E a vector space. A Poisson algebra structure $(\cdot_E, [-, -]_E)$ on E is called a *co-flag Poisson algebra over P* if there exists a positive integer n and a finite chain of epimorphisms of Poisson algebras

$$P_n := (E, \cdot_E, [-, -]_E) \xrightarrow{\pi_n} P_{n-1} \xrightarrow{\pi_{n-1}} P_{n-2} \cdots \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 := P \longrightarrow 0 \quad (45)$$

such that $\dim_k(\text{Ker}(\pi_i)) = 1$, for all $i = 1, \dots, n$. A finite dimensional Poisson algebra is called a *co-flag Poisson algebra* if it is a co-flag Poisson algebra over $\{0\}$.

The recursive method we introduce in this section relies on the first step, namely $n = 1$. Therefore we will start by describing and classifying all co-flag Poisson algebra structures over P of dimension $1 + \dim_k(P)$ or, equivalently, all crossed products between P and a vector space V_1 of dimension 1. This process can be iterated by replacing the initial Poisson algebra P by such a crossed product between P and V_1 . Before proving the main theoretical result which allows us to develop this iteration process we need to introduce a few pieces of terminology:

Definition 3.2. Let P be a Poisson algebra. An *abelian co-flag datum of P* is a 5-tuple $(\lambda, \Lambda, \theta, \gamma, f)$, where $\lambda, \Lambda, \gamma : P \rightarrow k$ are linear maps, $\theta, f : P \times P \rightarrow k$ are bilinear maps satisfying the following compatibilities for any $p, q, r \in P$:

- (AF1) λ and $\Lambda : P \rightarrow k$ are morphisms of associative algebras
- (AF2) $\theta(p, qr) - \theta(pq, r) = \theta(p, q)\Lambda(r) - \theta(q, r)\lambda(p)$
- (AF3) $f(p, p) = \gamma([p, q]) = 0$
- (AF4) $f(p, [q, r]) + f(q, [r, p]) + f(r, [p, q]) + \gamma(p)f(q, r) + \gamma(q)f(r, p) + \gamma(r)f(p, q) = 0$
- (AF5) $f(pq, r) - \Lambda(q)f(p, r) - \lambda(p)f(q, r) = \gamma(r)\theta(p, q) + \theta([p, r], q) + \theta(p, [q, r])$
- (AF6) $\gamma(pq) = \gamma(p)\Lambda(q) + \lambda(p)\gamma(q)$
- (AF7) $\lambda([p, q]) = \Lambda([p, q]) = 0$.

We denote by $\mathcal{AF}(P)$ the set of all abelian co-flag data of P .

Remark 3.3. Definition 3.2 simplifies considerably in the case of perfect Poisson algebras. Indeed, let P be a perfect Poisson algebra, i.e. P is generated as a vector space by all brackets $[x, y]$ with $x, y \in P$. Then, it follows from the compatibilities (AF3) and (AF7) that $\lambda = \Lambda = \gamma \equiv 0$. Thus, an abelian co-flag datum reduces to a pair of bilinear maps (θ, f) satisfying the following compatibilities for all $p, q, r \in P$:

$$\begin{aligned}
 f(p, p) &= 0, & \theta(p, qr) &= \theta(pq, r) \\
 f(p, [q, r]) + f(q, [r, p]) + f(r, [p, q]) &= 0 \\
 f(pq, r) &= \theta([p, r], q) + \theta(p, [q, r]).
 \end{aligned}$$

Definition 3.4. Let P be a Poisson algebra. A *non-abelian co-flag datum* of P is a triple (λ, θ, u) , where $u \in k \setminus \{0\}$, $\lambda : P \rightarrow k$ is a linear map, $\theta : P \times P \rightarrow k$ is a bilinear map satisfying the following two compatibilities for any $p, q, r \in P$:

$$\begin{aligned}
 \text{(NF1)} \quad & \lambda(pq) = \lambda(p)\lambda(q) - u\theta(p, q) \\
 \text{(NF2)} \quad & \theta(p, qr) - \theta(pq, r) = \theta(p, q)\lambda(r) - \theta(q, r)\lambda(p).
 \end{aligned}$$

We denote by $\mathcal{NF}(P)$ the set of all non-abelian co-flag data of P and by $\mathcal{F}(P) := \mathcal{AF}(P) \sqcup \mathcal{NF}(P)$ the disjoint union of the two sets. The elements of $\mathcal{F}(P)$ will be called *co-flag data* of P . In light of the following result co-flag data are the key tools for describing co-flag Poisson algebras:

Proposition 3.5. *Let P be a Poisson algebra and V a vector space of dimension 1 with a basis $\{x\}$. Then there exists a bijection between the set $\mathcal{PS}(P, V)$ of all crossed systems of P by V and the set $\mathcal{F}(P) = \mathcal{AF}(P) \sqcup \mathcal{NF}(P)$ of all co-flag data of P .*

The bijection is such that the crossed system $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ corresponding to $(\lambda, \Lambda, \theta, \gamma, f) \in \mathcal{AF}(P)$ is given by:

$$p \dashv x = \lambda(p)x, \quad x \triangleleft p = \Lambda(p)x, \quad \vartheta(p, q) = \theta(p, q)x \tag{46}$$

$$p \cdot_V x = [x, x]_V = 0, \quad p \triangleright x = \gamma(p)x, \quad f(p, q) = f(p, q)x \tag{47}$$

while the crossed system $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ corresponding to $(\lambda, \theta, u) \in \mathcal{NF}(P)$ is given by:

$$p \dashv x = \lambda(p)x, \quad x \triangleleft p = \lambda(p)x, \quad \vartheta(p, q) = \theta(p, q)x \tag{48}$$

$$x \cdot_V x = ux, \quad p \triangleright x = 0, \quad f(p, q) = -u^{-1}\lambda([p, q])x, \quad [x, x]_V = 0 \tag{49}$$

for all $p, q \in P$.

Proof. We have to compute all crossed systems $\Omega(P, V) = (\dashv, \triangleleft, \vartheta, \cdot_V, \triangleright, f, [-, -]_V)$ between P and V , i.e. all bilinear maps which satisfy the compatibilities (H0)–(H7), (L0)–(L4) and (P0)–(P7). Since V has dimension 1 and $(V, [-, -]_V)$ is a Lie algebra it follows that $[-, -]_V = 0$, the trivial map. Once again by the fact that V has dimension 1 we obtain that any crossed system from $\Omega(P, V)$ is uniquely determined by three linear maps $\lambda, \Lambda, \gamma : P \rightarrow k$, two bilinear maps $\theta, f : P \times P \rightarrow k$ and a scalar $u \in k$ via the following formulas:

$$\begin{aligned}
 p \dashv x &= \lambda(p)x, & x \triangleleft p &= \Lambda(p)x, & \vartheta(p, q) &= \theta(p, q)x \\
 x \vee p &= ux, & p \triangleright x &= \gamma(p)x, & f(p, q) &= f(p, q)x
 \end{aligned}$$

for all $p \in P$. We are left to derive the axioms that need to be fulfilled by the above maps in order for compatibilities (H0)–(H7), (L0)–(L4) and (P0)–(P7) to hold.

We start by writing down the compatibility (H2) which implies $u\Lambda(p) = u\lambda(p)$, for all $p \in P$. Therefore we distinguish two cases, namely $u = 0$ or $u \neq 0$. Suppose first that $u = 0$. We will prove that in this case the axioms (H0)–(H7), (L0)–(L4) and (P0)–(P7) hold if and only if $(\lambda, \Lambda, \theta, \gamma, f)$ is an abelian co-flag datum. It is easy to see that the compatibilities (H0), (H1), (H3) and (H4) are trivially fulfilled. (H6) and (H7) are equivalent to the fact that λ and Λ are algebra maps and thus (AF1) holds while (H5) collapses to (AF2). Furthermore, (L0) and (L2) are trivially fulfilled, (L1) together with (L3) collapses to (AF3) while (L4) is equivalent to (AF4). (P1) collapses to (AF5), (P2) is equivalent to (AF6) while (P4), (P6) and (P7) are trivially fulfilled. Finally, (AF7) is derived from (P3) and (P5).

Suppose now that $u \neq 0$ and thus $\Lambda = \lambda$. In this case (P3) comes down to $f(p, q) = -u^{-1}\lambda([p, q])$ while (P4) implies that $\gamma = 0$. It can be easily seen that (H5) collapses to (NF2) while (H6) is equivalent to (NF1). Based on these facts it is straightforward to see that the other compatibilities are trivially fulfilled. We will only check (P1). Indeed, the left hand side of (P1) gives:

$$\begin{aligned}
 LHS(P1) &= f(pq, r) - \lambda(q)f(p, r) - \lambda(p)f(q, r) \\
 &= (-\lambda(\underline{[pq, r]}) + \lambda(q)\lambda([p, r]) + \lambda(p)\lambda([q, r]))u^{-1} \\
 &\stackrel{(4)}{=} (-\lambda(\underline{[p, r]q}) - \lambda(\underline{p[q, r]}) + \lambda(q)\lambda([p, r]) + \lambda(p)\lambda([q, r]))u^{-1} \\
 &\stackrel{(NF1)}{=} (-\lambda([p, r])\lambda(q) - \lambda(p)\lambda([q, r]) + \theta([p, r], q)u + \theta(p, [q, r])u \\
 &\quad + \lambda(q)\lambda([p, r]) + \lambda(p)\lambda([q, r]))u^{-1} \\
 &= \theta([p, r], q) + \theta(p, [q, r]) = RHS(P1)
 \end{aligned}$$

and the proof is now finished. \square

Let $(\lambda, \Lambda, \theta, \gamma, f) \in \mathcal{AF}(P)$. The crossed product $P \#_{(\lambda, \Lambda, \theta, \gamma, f)} V$ associated to the crossed system given by (46)–(47) will be denoted by $P_{\mathcal{A}}(x | (\lambda, \Lambda, \theta, \gamma, f))$ and has the Poisson algebra structure defined by:

$$\begin{aligned}
 (p, 0) \star (q, 0) &= (pq, \theta(p, q)x), & \{(p, 0), (q, 0)\} &= ([p, q], f(p, q)x) \\
 (p, 0) \star (0, x) &= (0, \lambda(p)x), & \{(p, 0), (0, x)\} &= (0, \gamma(p)x) \\
 (0, x) \star (p, 0) &= (0, \Lambda(p)x), & \{(0, x), (p, 0)\} &= (0, -\gamma(p)x).
 \end{aligned}$$

On the other hand, for $(\lambda, \theta, u) \in \mathcal{NF}(P)$, the crossed product $P \#_{(\lambda, \theta, u)} V$ associated to the crossed system given by (48)–(49) will be denoted by $P_{\mathcal{N}}(x | (\lambda, \theta, u))$ and has the Poisson algebra structure defined by:

$$\begin{aligned} (p, 0) \star (q, 0) &= (pq, \theta(p, q)x), & (p, 0) \star (0, x) &= (0, \lambda(p)x) \\ (0, x) \star (0, x) &= (0, ux), & (0, x) \star (p, 0) &= (0, \lambda(p)x) \\ \{(p, 0), (q, 0)\} &= ([p, q], -u^{-1}\lambda([p, q])x). \end{aligned}$$

The first explicit classification result for the GE-problem follows: it is also the key step in the classification of all co-flag Poisson algebras over P .

Theorem 3.6. *Let P be a Poisson algebra. Then:*

$$\mathcal{GP}\mathcal{H}^2(P, k) \cong (\mathcal{AF}(P)/\equiv_1) \sqcup (\mathcal{NF}(P)/\equiv_2) \quad \text{where:}$$

\equiv_1 is the equivalence relation on the set $\mathcal{AF}(P)$ defined as follows: $(\lambda, \Lambda, \theta, \gamma, f) \equiv_1 (\lambda', \Lambda', \theta', \gamma', f')$ if and only if $\lambda = \lambda', \Lambda = \Lambda', \gamma = \gamma'$ and there exists a linear map $r : P \rightarrow k$ such that for any $p, q \in P$:

$$\theta(p, q) = \theta'(p, q) + r(q)\lambda'(p) + r(p)\Lambda'(q) - r(pq) \tag{50}$$

$$f(p, q) = f'(p, q) + r(q)\gamma'(p) - r(p)\gamma'(q) - r([p, q]), \tag{51}$$

\equiv_2 is the equivalence relation on $\mathcal{NF}(P)$ given by: $(\lambda, \theta, u) \equiv_2 (\lambda', \theta', u')$ if and only if $u = u'$ and there exists a linear map $r : P \rightarrow k$ such that for any $p, q \in P$:

$$\lambda(p) = \lambda'(p) + r(p)u \tag{52}$$

$$\theta(p, q) = \theta'(p, q) + r(q)\lambda'(p) + r(p)\lambda'(q) - r(pq) + r(p)r(q)u. \tag{53}$$

Proof. By applying Proposition 3.5 for $V = k$ we know that the set $\mathcal{PS}(P, k)$ of all crossed systems of P by k is in bijection to the set $\mathcal{F}(P) = \mathcal{AF}(P) \sqcup \mathcal{NF}(P)$ of all co-flag data of P . Thus, the problem reduces to computing the set $(\mathcal{AF}(P) \sqcup \mathcal{NF}(P))/\approx$. Based on these facts, a little computation shows that the compatibility conditions from Definition 2.7, imposed for the crossed systems (46)–(47) and respectively (48)–(49), take precisely the form given in the statement of the theorem. To this end we should notice that an abelian co-flag datum is never cohomologous to a non-abelian co-flag datum thanks to the compatibility condition (M1) from Lemma 2.6. \square

We provide a first example which relies on Theorem 3.6:

Example 3.7. Let \mathfrak{h} be the Heisenberg Lie algebra with the basis $\{h_1, h_2, h_3\}$ and the bracket defined by $[h_1, h_2] = h_3$. \mathfrak{h} admits a Poisson algebra structure [16] with the associative multiplication given by $h_1^2 = h_3$. Then $\mathcal{GP}\mathcal{H}^2(\mathfrak{h}, k) \cong k^4 \sqcup k^* \sqcup k^* \sqcup (k^* \times k^*) \sqcup k^*$

and the equivalence classes of all non-cohomologous 4-dimensional co-flag algebras over \mathfrak{H} are represented by the following Poisson algebras with basis $\{e_1, e_2, e_3, e_4\}$, multiplication and bracket given by:

$$\begin{aligned} \mathfrak{H}_{\lambda, \zeta, \xi}^\tau : \quad & e_1 \star e_1 = e_3, \quad e_2 \star e_2 = \lambda e_4, \quad e_3 \star e_3 = \zeta e_4, \quad e_4 \star e_4 = \xi e_4, \\ & \{e_1, e_2\} = e_3 + \tau e_4, \quad \text{where } \tau, \lambda, \zeta, \xi \in k \\ \mathfrak{H}^v : \quad & e_1 \star e_1 = e_3, \quad \{e_1, e_2\} = e_3, \quad \{e_2, e_4\} = v e_4, \quad \text{where } v \in k^* \\ \widetilde{\mathfrak{H}}^v : \quad & e_1 \star e_1 = e_3, \quad \{e_1, e_2\} = e_3, \quad \{e_1, e_4\} = v e_4, \quad \text{where } v \in k^* \\ \mathfrak{H}^{v, w} : \quad & e_1 \star e_1 = e_3, \quad \{e_1, e_2\} = e_3, \quad \{e_1, e_4\} = v e_4, \quad \{e_2, e_4\} = w e_4, \quad \text{where } v, w \in k^* \\ \widetilde{\mathfrak{H}}^v : \quad & e_1 \star e_1 = e_3 + v e_4, \quad \{e_1, e_2\} = e_3, \quad \text{where } v \in k^*. \end{aligned}$$

To this end we use first [Proposition 3.5](#). However, as the computations are rather long but straightforward we will only indicate the main steps of the proof. It can be shown that the abelian co-flag data of \mathfrak{H} are given as follows:

$$\begin{aligned} (\lambda_1, A_1, \theta_1, \gamma_1, f_1) : \quad & \lambda_1 = A_1 = \gamma_1 \equiv 0, \quad f_1(e_1, e_2) = -f_1(e_2, e_1) = \sigma, \\ & \theta_1(e_1, e_1) = \beta_1, \quad \theta_1(e_2, e_2) = \beta_2, \quad \theta_1(e_1, e_2) = \beta_3, \quad \theta_1(e_2, e_1) = \beta_4 \\ & \text{with } \sigma, \beta_i \in k, i \in \{1, 2, 3\} \\ (\lambda_2, A_2, \theta_2, \gamma_2, f_2) : \quad & \lambda_2 = A_2 \equiv 0, \quad \gamma_2(e_2) = v, \quad \theta_2(e_1, e_1) = \omega, \\ & f_2(e_1, e_2) = -f_2(e_2, e_1) = \eta, \quad f_2(e_3, e_2) = -f_2(e_2, e_3) = v\omega, \\ & \text{with } v \in k^*, \omega, \eta \in k \\ (\lambda_3, A_3, \theta_3, \gamma_3, f_3) : \quad & \lambda_3 = A_3 \equiv 0, \quad \gamma_3(e_1) = w, \quad \theta_3(e_1, e_1) = \zeta, \\ & f_3(e_1, e_2) = -f_3(e_2, e_1) = \mu, \quad f_3(e_3, e_1) = -f_3(e_1, e_3) = -w\zeta, \\ & \text{with } w \in k^*, \mu, \zeta \in k \\ (\lambda_4, A_4, \theta_4, \gamma_4, f_4) : \quad & \lambda_4 = A_4 \equiv 0, \quad \gamma_4(e_1) = \bar{v}, \quad \gamma_4(e_2) = \bar{w}, \quad \theta_4(e_1, e_1) = \nu, \\ & f_4(e_1, e_2) = -f_4(e_2, e_1) = \delta, \quad f_4(e_3, e_1) = -f_4(e_1, e_3) = \bar{v}\nu, \\ & f_4(e_3, e_2) = -f_4(e_2, e_3) = \bar{w}\nu, \quad \text{with } \bar{v}, \bar{w} \in k^*, \nu, \delta \in k. \end{aligned}$$

By [Theorem 3.6](#) it follows that two abelian co-flag data $(\lambda_i, A_i, \theta_i, \gamma_i, f_i)$ and $(\lambda_j, A_j, \theta_j, \gamma_j, f_j)$ with $i \neq j$ are never equivalent since γ_i is obviously different from γ_j . Using again [Theorem 3.6](#) we can easily notice that an abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by $\sigma, \beta_i \in k, i \in \{1, \dots, 4\}$ is equivalent to an abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by $\sigma - \beta_1, 0, \beta_2, \beta_3, \beta_4$. The Poisson algebras denoted by $\mathfrak{H}_{\lambda, \zeta, \xi}^\tau$ are obtained from these type of co-flag data. Moreover, it is straightforward to see that an abelian co-flag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by $v \in k^*, \omega, \eta \in k$ is equivalent to an abelian co-flag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by $v \in k^*$ and $\omega = \eta = 0$ which gives rise to the Poisson algebra denoted by \mathfrak{H}^v . In the same manner any

abelian co-flag datum $(\lambda_3, A_3, \theta_3, \gamma_3, f_3)$ implemented by $w \in k^*$, $\mu, \zeta \in k$ is equivalent to an abelian co-flag datum $(\lambda_3, A_3, \theta_3, \gamma_3, f_3)$ implemented by $w \in k^*$ and $\mu = \zeta = 0$ while any abelian co-flag datum $(\lambda_4, A_4, \theta_4, \gamma_4, f_4)$ implemented by $\bar{v}, \bar{w} \in k^*$, $\nu, \delta \in k$ is equivalent to an abelian co-flag datum $(\lambda_4, A_4, \theta_4, \gamma_4, f_4)$ implemented by $\bar{v}, \bar{w} \in k^*$ and $\nu = \delta = 0$. The latter two abelian co-flag data give rise to the Poisson algebras denoted by $\tilde{\mathfrak{H}}^v$ and $\mathfrak{H}^{v,w}$ respectively.

On the other hand, the non-abelian co-flag data of \mathfrak{H} are given as follows:

$$(\lambda, \theta, u) : \quad \lambda(h_i) = \alpha_i \in k, \quad i \in \{1, 2, 3\}, \quad u \in k^*,$$

θ	h_1	h_2	h_3
h_1	$(\alpha_1^2 - \alpha_3)u^{-1}$	$\alpha_1\alpha_2u^{-1}$	$\alpha_1\alpha_3u^{-1}$
h_2	$\alpha_1\alpha_2u^{-1}$	$\alpha_2^2u^{-1}$	$\alpha_2\alpha_3u^{-1}$
h_3	$\alpha_3\alpha_1u^{-1}$	$\alpha_3\alpha_2u^{-1}$	$\alpha_3^2u^{-1}$

By a routine computation based on [Theorem 3.6](#) we obtain that the non-abelian co-flag datum (λ, θ, u) implemented by $u \in k^*$, $\alpha_1, \alpha_2, \alpha_3$ is equivalent to the non-abelian co-flag datum (λ, θ, u) implemented by $u \in k^*$, $0, 0, 0$ which gives rise to the Poisson algebra denoted by $\tilde{\mathfrak{H}}^u$. Thus $\mathcal{GPH}^2(\mathfrak{H}, k) \cong k^4 \sqcup k^* \sqcup k^* \sqcup k^{*2} \sqcup k^*$.

Next we will highlight the efficiency of [Theorem 3.6](#) in classifying co-flag Poisson algebras of small dimension. We start by computing all 2-dimensional co-flag Poisson algebras. Since there are two non-isomorphic Poisson algebra structures on a vector space of dimension one, namely k_0 and k_1 , we have to consider the cases $P = k_0$ and $P = k_1$.

Corollary 3.8. *Let k be a field. Then $\mathcal{GPH}^2(k_0, k) \cong k^* \sqcup k \sqcup k^*$ and $\mathcal{GPH}^2(k_1, k) \cong \{*\} \sqcup \{*\} \sqcup k \sqcup k \sqcup k^*$ where $\{*\}$ denotes the singleton set. Explicitly, any 2-dimensional co-flag Poisson algebra is cohomologous to one of the following Poisson algebras with basis $\{e_1, e_2\}$, multiplication and bracket defined by:*

$$\begin{aligned}
 k_{0,\delta}^2 : \quad & e_1 \star e_1 = \delta e_2, \quad \text{where } \delta \in k^* \\
 {}_\mu k_0^2 : \quad & \{e_1, e_2\} = \mu e_2, \quad \text{where } \mu \in k \\
 k_{0,u}^2 : \quad & e_2 \star e_2 = ue_2, \quad \text{where } u \in k^* \\
 k_1^2 : \quad & e_1 \star e_1 = e_1 \\
 \bar{k}_1^2 : \quad & e_1 \star e_1 = e_1, \quad e_1 \star e_2 = e_2 \star e_1 = e_2 \\
 {}_\mu k_1^2 : \quad & e_1 \star e_1 = e_1, \quad e_1 \star e_2 = e_2, \quad \{e_1, e_2\} = \mu e_2, \quad \text{where } \mu \in k \\
 {}_\mu \bar{k}_1^2 : \quad & e_1 \star e_1 = e_1, \quad e_2 \star e_1 = e_2, \quad \{e_1, e_2\} = \mu e_2, \quad \text{where } \mu \in k \\
 k_u^2 : \quad & e_1 \star e_1 = e_1, \quad e_2 \star e_2 = ue_2, \quad \text{where } u \in k^*.
 \end{aligned}$$

Proof. In what follows we consider $\{y\}$ as a basis in $P = k$. Suppose first that $P = k_0$. Then the abelian co-flag data of k_0 are given as follows:

$$\begin{aligned}
 (\lambda_1, A_1, \theta_1, \gamma_1, f_1) : \quad & \lambda_1 = A_1 = \gamma_1 \equiv 0, \theta_1(y, y) = \delta, f_1 \equiv 0, \text{ with } \delta \in k \\
 (\lambda_2, A_2, \theta_2, \gamma_2, f_2) : \quad & \lambda_2 = A_2 \equiv 0, \theta_2 = f_2 \equiv 0, \gamma_2(y) = \mu, \text{ with } \mu \in k^*.
 \end{aligned}$$

An abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by $\delta \in k$ is equivalent to another abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by $\delta' \in k$ if and only if $\delta = \delta'$. In the same manner it follows that an abelian co-flag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by $\mu \in k^*$ is equivalent to another abelian co-flag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by $\mu' \in k^*$ if and only if $\mu = \mu'$. Moreover, as the abelian co-flag data $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ are implemented by a non-zero scalar $\mu \in k^*$ it follows that $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ is never equivalent to a co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$. The two non-equivalent abelian co-flag data give rise to the Poisson algebras denoted by $k_{0,\delta}^2$ and respectively ${}_{\mu}k_0^2$, $\delta \in k$, $\mu \in k^*$.

On the other hand, the non-abelian co-flag data of k_0 are given as follows:

$$(\lambda, \theta, u) : \quad \lambda(y) = \alpha, \theta(y, y) = \alpha^2 u^{-1}, \text{ with } \alpha \in k, u \in k^*.$$

Again by [Theorem 3.6](#) it follows that two non-abelian co-flag data corresponding to scalars $(\alpha, u) \in k \times k^*$ and respectively $(\alpha', u') \in k \times k^*$ are equivalent if and only if $u = u'$. Therefore any non-abelian co-flag datum implemented by (α, u) is equivalent to a non-abelian co-flag datum implemented by $(0, u)$ which gives rise to the Poisson algebra denoted by $k_{0,u}^2$. Thus we obtain $\mathcal{GPH}^2(k_0, k) \cong k^* \sqcup k \sqcup k^*$.

Now we turn to the second case, namely $P = k_1$. The abelian co-flag data of k_1 are given as follows:

$$\begin{aligned}
 (\lambda_1, A_1, \theta_1, \gamma_1, f_1) : \quad & \lambda_1 = A_1 = \gamma_1 \equiv 0, \theta_1(y, y) = \delta, f_1 \equiv 0, \text{ with } \delta \in k \\
 (\lambda_2, A_2, \theta_2, \gamma_2, f_2) : \quad & \lambda_2(y) = A_2(y) = 1, \theta_2(y, y) = \zeta, \gamma_2 \equiv 0, f_2 \equiv 0, \text{ with } \zeta \in k \\
 (\lambda_3, A_3, \theta_3, \gamma_3, f_3) : \quad & \lambda_3(y) = 1, A_3 \equiv 0, \gamma_3(y) = \mu, \theta_3 = f_3 \equiv 0, \text{ with } \mu \in k \\
 (\lambda_4, A_4, \theta_4, \gamma_4, f_4) : \quad & \lambda_4 \equiv 0, A_4(y) = 1, \gamma_4(y) = \mu, \theta_4 = f_4 \equiv 0, \text{ with } \mu \in k.
 \end{aligned}$$

Since by [Theorem 3.6](#) two equivalent abelian co-flag data need to have the same three maps λ , A and γ it easily follows that an abelian co-flag datum $(\lambda_i, A_i, \theta_i, \gamma_i, f_i)$ is never equivalent to $(\lambda_j, A_j, \theta_j, \gamma_j, f_j)$ if $i \neq j$. Moreover, for any $\delta \in k$ the abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by δ is equivalent to the abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by 0 which give rise to the Poisson algebra denoted by k_1^2 . In the same manner for any $\zeta \in k$ the abelian co-fag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by ζ is equivalent to the abelian co-fag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by 0 which gives rise to the Poisson algebra denoted by \bar{k}_1^2 . The situation changes if we look at the abelian co-flag data $(\lambda_3, A_3, \theta_3, \gamma_3, f_3)$: two such data implemented by

μ and respectively μ' are equivalent if and only if $\mu = \mu'$. The abelian co-flag datum $(\lambda_3, \Lambda_3, \theta_3, \gamma_3, f_3)$ implemented by $\mu \in k$ gives rise to the Poisson algebra denoted by ${}_{\mu}k_1^2$. Finally, the abelian co-flag datum $(\lambda_4, \Lambda_4, \theta_4, \gamma_4, f_4)$ gives rise to the Poisson algebra denoted by ${}_{\mu}\bar{k}_1^2$.

The non-abelian co-flag data of k_1 are given as follows:

$$(\lambda, \theta, u) : \quad \lambda(y) = \alpha, \theta(y, y) = (\alpha^2 - \alpha)u^{-1}, \text{ with } \alpha \in k, u \in k^*.$$

Two non-abelian co-flag data corresponding to scalars $(\alpha, u) \in k \times k^*$ and respectively $(\alpha', u') \in k \times k^*$ are equivalent if and only if $u = u'$. Therefore any non-abelian co-flag datum implemented by (α, u) is equivalent to a non-abelian co-flag datum implemented by $(0, u)$ which gives rise to the Poisson algebra denoted by k_u^2 . \square

We can now apply our recursive method to the Poisson algebras described in [Corollary 3.8](#): we will obtain the description and classification of all 3-dimensional co-flag Poisson algebras. In what follows we will only list the 3-dimensional co-flag Poisson algebras over ${}_1k_1^2$.

Corollary 3.9. *Any 3-dimensional co-flag Poisson algebra over ${}_1k_1^2$ is cohomologous to one of the following Poisson algebras with basis $\{e_1, e_2, e_3\}$:*

$$\begin{aligned} {}_1k_1^3 : \quad & e_1 \star e_1 = e_1, e_1 \star e_2 = e_2, \{e_1, e_2\} = e_2 \\ {}_1\bar{k}_1^3 : \quad & e_1 \star e_1 = e_1, e_1 \star e_2 = e_2, e_1 \star e_3 = e_3 \star e_1 = e_3, \{e_1, e_2\} = e_2 \\ {}_{\nu}k_1^3 : \quad & e_1 \star e_1 = e_1, e_1 \star e_2 = e_2, e_1 \star e_3 = e_3, \{e_1, e_3\} = e_3, \\ & \{e_1, e_2\} = e_2 + \nu e_3, \text{ where } \nu \in k \\ {}_{\omega}\tilde{k}_1^3 : \quad & e_1 \star e_1 = e_1, e_1 \star e_2 = e_2, e_1 \star e_3 = e_3, \{e_1, e_3\} = \omega e_3, \{e_1, e_2\} = e_2, \\ & \text{where } \omega \in k - \{1\} \\ {}_{\tau}\bar{k}_1^3 : \quad & e_1 \star e_1 = e_1, e_1 \star e_2 = e_2, e_3 \star e_1 = e_3, \{e_1, e_3\} = \tau e_3, \{e_1, e_2\} = e_2, \\ & \text{where } \tau \in k \\ k_{1,u}^3 : \quad & e_1 \star e_1 = e_1, e_3 \star e_3 = u e_3, \{e_1, e_2\} = e_2, \text{ where } u \in k^*. \end{aligned}$$

Furthermore, we have $\mathcal{GPH}^2({}_1k_1^2, k) \cong \{*\} \sqcup \{*\} \sqcup k \sqcup (k - \{1\}) \sqcup k \sqcup k^*$.

Proof. We denote by $\{y_1, y_2\}$ a k -basis of ${}_1k_1^2$. The abelian co-flag data of ${}_1k_1^2$ are given as follows:

$$\begin{aligned} (\lambda_1, \Lambda_1, \theta_1, \gamma_1, f_1) : \quad & \lambda_1 = \Lambda_1 = \gamma_1 \equiv 0, \theta_1(y_1, y_1) = \alpha, \theta_1(y_1, y_2) = \beta, \\ & f_1(y_1, y_2) = -f_1(y_2, y_1) = \beta, \text{ with } \alpha, \beta \in k \\ (\lambda_2, \Lambda_2, \theta_2, \gamma_2, f_2) : \quad & \lambda_2(y_1) = \Lambda_2(y_1) = 1, \gamma_2 \equiv 0, \theta_2(y_1, y_1) = \zeta, \theta_2(y_2, y_1) = \delta, \end{aligned}$$

$$\begin{aligned}
 & f_2(y_2, y_1) = -f_2(y_1, y_2) = \delta, \text{ with } \zeta, \delta \in k \\
 (\lambda_3, A_3, \theta_3, \gamma_3, f_3) : & \quad \lambda_3(y_1) = 1, A_3 \equiv 0, \theta_3 \equiv 0, \gamma_3(y_1) = \omega, \\
 & \quad f_3(y_1, y_2) = -f_3(y_2, y_1) = \nu, \text{ with } \omega, \nu \in k \\
 (\lambda_4, A_4, \theta_4, \gamma_4, f_4) : & \quad \lambda_4 \equiv 0, A_4(y_1) = 1, \theta_4 \equiv 0, \gamma_4(y_1) = \tau, f_4 \equiv 0 \text{ with } \tau \in k.
 \end{aligned}$$

To start with we should notice that in the light of [Theorem 3.6](#) two equivalent abelian co-flag data need to have the same three maps λ, A and γ and thus an abelian co-flag datum $(\lambda_i, A_i, \theta_i, \gamma_i, f_i)$ is never equivalent to $(\lambda_j, A_j, \theta_j, \gamma_j, f_j)$ if $i \neq j$. Using again [Theorem 3.6](#) we obtain that an abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by (α, β) is equivalent to the abelian co-flag datum $(\lambda_1, A_1, \theta_1, \gamma_1, f_1)$ implemented by $(0, 0)$. The latter gives rise to the Poisson algebra ${}_1k_1^3$. A co-flag datum $(\lambda_2, A_2, \theta_2, \gamma_2, f_2)$ implemented by (ζ, δ) is equivalent to the abelian co-flag datum implemented by $(0, 0)$. This gives rise to the Poisson algebras denoted by ${}_1\bar{k}_1^3$. Consider now two co-flag data $(\lambda_3, A_3, \theta_3, \gamma_3, f_3)$ implemented by (ω, ν) and respectively (ω', ν') . In order for the two co-flag data to be equivalent we need to have $\omega = \omega'$. If $\omega = \omega' = 1$ then the co-flag data are equivalent if and only if $\nu = \nu'$; we denote the corresponding Poisson algebra by ${}_\nu k_1^3$. On the other hand, if $\omega = \omega' \neq 1$ then the two co-flag data are always equivalent and therefore any such co-flag datum is equivalent to the co-flag datum implemented by $(\omega, 0)$; the corresponding Poisson algebra will be denoted by ${}_\omega \tilde{k}_1^3$. Finally two co-flag data $(\lambda_4, A_4, \theta_4, \gamma_4, f_4)$ corresponding to τ and respectively τ' are equivalent if and only if $\tau = \tau'$. The corresponding Poisson algebras are denoted by ${}_\tau \bar{k}_1^3$.

On the other hand, the non-abelian co-flag data of ${}_1k_1^2$ are given as follows:

$$(\lambda, \theta, u) : \quad \lambda(y_i) = \alpha_i \in k, \quad i \in \{1, 2\}, \quad u \in k^*, \quad \begin{array}{c|cc} \theta & y_1 & y_2 \\ \hline y_1 & (\alpha_1^2 - \alpha_1)u^{-1} & (\alpha_1\alpha_2 - \alpha_2)u^{-1} \\ y_2 & \alpha_1\alpha_2u^{-1} & \alpha_2^2u^{-1} \end{array}$$

A straightforward computation based on [Theorem 3.6](#) proves that the non-abelian co-flag datum (λ, θ, u) implemented by $u \in k^*, \alpha_1, \alpha_2$ is equivalent to the non-abelian co-flag datum (λ, θ, u) implemented by $u \in k^*, 0, 0$ which gives rise to the Poisson algebra denoted by $k_{1,u}^3$. The proof is now finished. \square

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