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Semigroup graded algebras and codimension growth of graded polynomial identities [☆]



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ABSTRACT

We show that if T is any of four semigroups of two elements that are not groups, there exists a finite dimensional associative T -graded algebra over a field of characteristic 0 such that the codimensions of its graded polynomial identities have a non-integer exponent of growth. In particular, we provide an example of a finite dimensional graded-simple semigroup graded algebra over an algebraically closed field of characteristic 0 with a non-integer graded PI-exponent, which is strictly less than the dimension of the algebra. However, if T is a left or right zero band and the T -graded algebra is unital, or T is a cancellative semigroup, then the T -graded algebra satisfies the graded analog of Amitsur's conjecture, i.e. there exists an integer graded PI-exponent. Moreover, in the first case it turns out that the ordinary and the graded PI-exponents coincide. In addition, we consider related problems on the structure of semigroup graded algebras.

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The notion of a semigroup graded algebra is a natural generalization of the notion of a group graded algebra, however the first notion is much less restricting: e.g. if an algebra is the direct sum of its left ideals or if an algebra is the direct sum of a subalgebra and an ideal, this can be expressed in the language of semigroup gradings.

In 2010–2011 E. Aljadeff, A. Giambruno, and D. La Mattina [1,2,5] proved that if an associative PI-algebra is graded by a finite group, then there exists an integer exponent of codimensions of its graded polynomial identities, i.e. the graded analog of Amitsur's conjecture holds. In [7, Theorem 1] and [9, Theorem 3] the author proved the same for finite dimensional associative and Lie algebras graded by any groups. In [10] A.V. Kelarev studied semigroup graded PI-algebras.

The next question that naturally arises in this investigation is as to whether the results on codimension growth of graded polynomial identities hold for semigroup graded associative algebras.

In the associative case the main properties that we use in order to prove the graded analog of Amitsur's conjecture (Theorem 2) are the gradedness (or homogeneity) of the Jacobson radical and the graded version of the Wedderburn–Artin theorem. We consider these properties in Sections 2 and 3 and obtain the graded analog of Amitsur's conjecture for algebras graded by cancellative semigroups (Theorem 6) and unital algebras graded by left or right zero bands (Theorem 7). In the first case we use Kelarev and Plant's result on gradedness of Jacobson radicals in algebras graded by cancellative groupoids [11, Corollary 4.1].

Until now, there were no examples known of an associative algebra with a non-integer PI-exponent of any kind (graded, Hopf, etc.). In 1999 S.P. Mishchenko and M.V. Zaicev gave an example of an infinite dimensional Lie algebra with a non-integer PI-exponent [12] (see the proof in [13]). Here we use their ideas to present a finite dimensional semigroup graded associative algebra with a non-integer exponent of codimension growth of graded polynomial identities (Theorems 3–5) for each of four semigroups of two elements that are not groups.

The PI-exponent of a finite dimensional graded-simple group graded Lie or associative algebra over an algebraically closed field of characteristic 0 equals the dimension of the algebra [7, Example 12] and [9, Theorem 4]. In Theorem 5 we provide an example of a finite dimensional graded-simple semigroup graded algebra over an algebraically closed field of characteristic 0 with a non-integer graded PI-exponent, which is strictly less than the dimension of the algebra.

1. Semigroups of two elements

First we describe all the five non-isomorphic semigroups of two elements.

Let $T_1 = \{0, 1\}$ be the multiplicative semigroup of the field \mathbb{Z}_2 .

Let $T_2 = \{0, v\}$ be the semigroup defined by relations $v^2 = 0^2 = 0 \cdot v = v \cdot 0 = 0$.

Recall that a semigroup T is a *left zero band* if $t_1 t_2 = t_1$ for every $t_1, t_2 \in T$ and a *right zero band* if $t_1 t_2 = t_2$ for every $t_1, t_2 \in T$.

Let T_3 be the right zero band of two elements.

Proposition 1. *Let T be a semigroup that consists of two elements. Then T is isomorphic to one of semigroups from the list $\{T_1, T_2, T_3, T_3^{\text{op}}, (\mathbb{Z}_2, +)\}$ and each two semigroups from this list are non-isomorphic. (Here T_3^{op} is anti-isomorphic to T_3 .)*

Proof. First, consider the case when $T = \{a, a^2\}$ for some $a \in T$. Then if $a^3 = a$, we have $a^4 = a^2$, a^2 is the identity element of T , and $T \cong (\mathbb{Z}_2, +)$. If $a^3 = a^2$, then $a^4 = a^3 = a^2$, a^2 is the zero element of T , and $T \cong T_2$.

Now consider the case when $T \neq \{a, a^2\}$ for all $a \in T$. Then $T = \{a, b\}$, $a^2 = a$, $b^2 = b$. If $ab = ba$, then $T \cong T_1$. If $ab \neq ba$, then $T \cong T_3$ for $ab = b$, $ba = a$, and $T \cong T_3^{\text{op}}$ for $ab = a$, $ba = b$. \square

2. Gradedness of the Jacobson radical

Let T be a semigroup. An algebra A is T -graded if $A = \bigoplus_{t \in T} A^{(t)}$ (direct sum of subspaces) and $A^{(h)}A^{(t)} \subseteq A^{(ht)}$ for all $h, t \in T$. A subspace V of A is *graded* (or *homogeneous*) if $V = \bigoplus_{t \in T} V \cap A^{(t)}$.

In particular, T_1 -graded algebras are exactly the algebras with a fixed decomposition into the direct sum of a two-sided ideal and a subalgebra. If T is a right zero band, then T -graded algebras are algebras with a fixed decomposition into the direct sum of left ideals indexed by the elements of T .

It is known [11, Example 4.2], that the Jacobson radical is not necessarily graded. (See also the survey of positive results in [11, Section 4.4].) Here we provide examples and results related to semigroups of two elements and left and right zero bands.

Denote by $M_k(F)$ is the full $k \times k$ matrix algebra over a field F and $\text{UT}_k(F)$ is the algebra of upper triangular $k \times k$ matrices. In $M_k(F)$ we fix the basis of matrix units $e_{i\ell}$, $1 \leq i, \ell \leq k$.

Example 1. Let $A = M_k(F) \oplus \text{UT}_k(F)$ (direct sum of ideals) where F is a field, $k \geq 2$. Define a T_1 -grading on A by $A^{(0)} = (M_k(F), 0)$, $A^{(1)} = \{(\varphi(a), a) \mid a \in \text{UT}_k(F)\}$ where $\varphi: \text{UT}_k(F) \hookrightarrow M_k(F)$ is the natural embedding. Then

$$J(A) = \{(0, e_{ij}) \mid 1 \leq i < j \leq k\} \subset (0, \text{UT}_k(F)),$$

$J(A) \cap A^{(0)} = J(A) \cap A^{(1)} = 0$, and $J(A)$ is not a graded ideal.

Example 2. Let $A = M_k(F) \oplus V$ (direct sum of ideals) where $V \cong M_k(F)$ as a vector space, $k \in \mathbb{N}$, $V^2 = 0$, and F is a field. Denote by $\varphi: V \xrightarrow{\sim} M_k(F)$ the corresponding linear isomorphism. Define a T_2 -grading on A by $A^{(0)} = (M_k(F), 0)$, $A^{(v)} = \{(\varphi(a), a) \mid a \in V\}$. Then

$$J(A) = (0, V), \quad J(A) \cap A^{(0)} = J(A) \cap A^{(v)} = 0,$$

and $J(A)$ is not a graded ideal.

Example 3. Let $A = M_k(F) \oplus V$ (direct sum of left ideals) where $V \cong M_k(F)$ as a left $M_k(F)$ -module, $k \in \mathbb{N}$, $V^2 = VM_k(F) = 0$, and F is a field. Denote by $\varphi: V \xrightarrow{\sim} M_k(F)$ the corresponding isomorphism. Define a T_3 -grading on A by $A^{(e_1)} = (M_k(F), 0)$,

$$A^{(e_2)} = \{(\varphi(a), a) \mid a \in V\}.$$

Then

$$J(A) = (0, V), \quad J(A) \cap A^{(e_1)} = J(A) \cap A^{(e_2)} = 0,$$

and $J(A)$ is not a graded ideal.

Remark. One can use the opposite example to show that the Jacobson radical is not necessarily T_3^{op} -graded.

However, if an algebra is unital and T_3 - or T_3^{op} -graded, then the Jacobson radical is graded. In fact, a more general result holds.

Proposition 2. *Let A be a T -graded associative algebra with 1 over a field F for some left or right zero band T . Then every ideal of A is graded.*

Proof. Consider the case when $t_1 t_2 = t_2$ for every $t_1, t_2 \in T$. (Another case is considered analogously.) Then all $A^{(t)}$, $t \in T$, are left ideals of A and $1 = \sum_{t \in T} e_t$ for some $e_t \in A^{(t)}$. Let I be an ideal. Then for every $a \in I$ we have $a = \sum_{t \in T} a e_t$ where $a e_t \in I \cap A^{(t)}$ for every $t \in T$. Hence $I = \bigoplus_{t \in T} I \cap A^{(t)}$ is a graded ideal. \square

3. Graded analogs of the Wedderburn theorems and T -graded simplicity

Now we study whether the graded analogs of the Wedderburn theorems hold for T -graded algebras where T is a semigroup. Recall that a T -graded algebra A is a *graded-simple algebra* if $A^2 \neq 0$ and A has no graded ideals other than A and 0.

Example 4. Let $B = M_k(F) \oplus M_k(F)$ (direct sum of ideals), $k \in \mathbb{N}$, where F is a field. Define a T_1 -grading on A by $A^{(0)} = (M_k(F), 0)$, $A^{(1)} = \{(a, a) \mid a \in M_k(F)\}$. Then B cannot be presented as the direct sum of T_1 -graded ideals that are T_1 -graded-simple algebras, i.e. the T_1 -graded analog of the Wedderburn–Artin theorem does not hold.

Proof. Note that the semisimple algebra B has only four ideals: 0, B , $(0, M_k(F))$, and $(M_k(F), 0)$. Three of them are T_1 -graded, namely, 0, B , and $(M_k(F), 0)$, and only $(M_k(F), 0)$ is a T_1 -graded-simple algebra. \square

Remark. Since $B^{(0)}$ is always a graded ideal, every T_1 -graded-simple algebra B has the trivial grading, i.e. $B = B^{(0)}$. Therefore, every T_1 -graded-simple algebra is simple as an ordinary algebra.

Proposition 3. *Let B be a finite dimensional associative T_2 -graded semisimple algebra over a field F . Then $B = B^{(0)}$, and by the ordinary Wedderburn–Artin theorem, B is the direct sum of T_2 -graded ideals that are simple algebras (with the trivial grading). Moreover, every T_2 -graded-simple algebra is simple as an ordinary algebra. In particular, the T_2 -graded analog of the Wedderburn–Artin theorem holds.*

Proof. Suppose $B \neq B^{(0)}$. Note that $B^{(0)}$ is an ideal and, by the ordinary Wedderburn–Artin theorem, $B = B^{(0)} \oplus I$ for some semisimple ideal I of B . However, $(B/B^{(0)})^2 = 0$ since $(B^{(1)})^2 \subseteq B^{(0)}$, and $I \cong B/B^{(0)}$ cannot be semisimple. Hence $B = B^{(0)}$, the algebra B has the trivial T_2 -grading, and we can apply to B the ordinary Wedderburn–Artin theorem. \square

Proposition 4. *Let B be a finite dimensional associative T -graded semisimple algebra over a field F for some left or right zero band T . Then B is the direct sum of T -graded ideals that are simple algebras. In particular, the T -graded analog of the Wedderburn–Artin theorem holds and every finite dimensional semisimple T -graded-simple algebra is simple as an ordinary algebra.*

Proof. Consider the case when T is a right zero band. The other case is considered analogously. By the ordinary Wedderburn–Artin theorem,

$$B = B_1 \oplus B_2 \oplus \dots \oplus B_s \text{ (direct sum of ideals)}$$

for some simple algebras B_i . By Proposition 2, each B_i is a T -graded ideal. Now the theorem follows. \square

However there exist non-semisimple T_3 -graded-simple algebras. (See Proposition 6 below.)

By Proposition 2, if a T -graded algebra, where T is a left or right zero band, contains unity, the its Jacobson radical is graded (as well as all the other ideals). Therefore, one may ask whether the T -graded analog of the Wedderburn–Mal’cev theorem holds for such algebras. In fact, the answer is true.

Theorem 1. *Let A be a finite dimensional associative T -graded algebra with unity over a field F where T is a left or right zero band and $A/J(A)$ is a separable algebra. (E.g., F is a perfect field.) Then there exists a graded maximal semisimple subalgebra B such that $A = B \oplus J$ (direct sum of graded spaces) where $J := J(A)$.*

Proof. Without loss of generality, we may assume that T is a right zero band. First we consider the case $J^2 = 0$.

Note that $1_A = \sum_{t \in T} e_t$ for some $e_t \in A^{(t)}$. Moreover, since $1_A e_t = \sum_{r \in T} e_r e_t$ and $e_r e_t \in A^{(t)}$ for every $t \in T$, we have $e_t^2 = e_t$ and $e_r e_t = 0$ for all $r \neq t$.

Using the ordinary Wedderburn–Mal’cev theorem we choose a maximal semisimple subalgebra B such that $A = B \oplus J$ (direct sum of subspaces). Let $\pi: A \twoheadrightarrow A/J$ be the natural projection which is a graded map since J is graded. Let $\varphi: A/J \hookrightarrow A$ be a homomorphic embedding such that $\varphi(A/J) = B$ and $\pi\varphi = \text{id}_{A/J}$. Note that $\pi(1_A) = \sum_{t \in T} \pi(e_t)$ is the unity of A/J . In addition, $1_A = 1_B = \varphi\pi(1_A)$.

Let $T = \{t_1, \dots, t_s\}$. If $\varphi\pi(e_{t_i}) = e_{t_i}$ for all $1 \leq i \leq s$, then $Be_{t_i} \subseteq B$ for all $1 \leq i \leq s$ and $B = \bigoplus_{t \in T} Be_t$ is a graded subalgebra and the theorem is proved.

Suppose $\varphi\pi(e_{t_i}) \neq e_{t_i}$ for at least one $1 \leq i \leq s$. Choose $0 \leq k \leq s-1$ such that $\varphi\pi(e_{t_i}) = e_{t_i}$ for all $1 \leq i \leq k$ and $\varphi\pi(e_{t_{k+1}}) \neq e_{t_{k+1}}$. Note that $\pi(\varphi\pi(e_{t_{k+1}}) - e_{t_{k+1}}) = 0$ and $\varphi\pi(e_{t_{k+1}}) = e_{t_{k+1}} + j$ for some $j \in J$. In addition, $je_{t_i} = (\varphi\pi(e_{t_{k+1}}) - e_{t_{k+1}})e_{t_i} = \varphi\pi(e_{t_{k+1}}e_{t_i}) - e_{t_{k+1}}e_{t_i} = 0$ for all $1 \leq i \leq k$. Analogously, $e_{t_i}j = 0$ for all $1 \leq i \leq k$. Moreover, since $(e_{t_{k+1}} + j)^2 = e_{t_{k+1}} + j$, we have $j = e_{t_{k+1}}j + je_{t_{k+1}}$ and $e_{t_{k+1}}je_{t_{k+1}} = 0$.

Let $\tilde{\varphi}: A/J \hookrightarrow A$ be the homomorphic embedding defined by

$$\begin{aligned}\tilde{\varphi}(a) &= (1_A + e_{t_{k+1}}j - je_{t_{k+1}})\varphi(a)(1_A + e_{t_{k+1}}j - je_{t_{k+1}})^{-1} = \\ &= (1_A + e_{t_{k+1}}j - je_{t_{k+1}})\varphi(a)(1_A - e_{t_{k+1}}j + je_{t_{k+1}}).\end{aligned}$$

Note that $\pi\tilde{\varphi} = \text{id}_{A/J}$ and

$$\tilde{\varphi}\pi(e_{t_i}) = (1_A + e_{t_{k+1}}j - je_{t_{k+1}})e_{t_i}(1_A - e_{t_{k+1}}j + je_{t_{k+1}}) = e_{t_i} \text{ for all } 1 \leq i \leq k.$$

Moreover

$$\begin{aligned}\tilde{\varphi}\pi(e_{t_{k+1}}) &= (1_A + e_{t_{k+1}}j - je_{t_{k+1}})\varphi\pi(e_{t_{k+1}})(1_A - e_{t_{k+1}}j + je_{t_{k+1}}) = \\ &= (1_A + e_{t_{k+1}}j - je_{t_{k+1}})(e_{t_{k+1}} + j)(1_A - e_{t_{k+1}}j + je_{t_{k+1}}) = \\ &= (e_{t_{k+1}} + j - je_{t_{k+1}})(1_A - e_{t_{k+1}}j + je_{t_{k+1}}) = \\ &= e_{t_{k+1}} + j - je_{t_{k+1}} - e_{t_{k+1}}j = e_{t_{k+1}}.\end{aligned}$$

Therefore $\tilde{B} = \tilde{\varphi}(A/J)$ is a maximal semisimple subalgebra such that $A = \tilde{B} \oplus J$ (direct sum of subspaces) and $\tilde{\varphi}\pi(e_{t_i}) = e_{t_i}$ for all $1 \leq i \leq k+1$. Thus, using the induction argument, we may assume that $e_t = \tilde{\varphi}\pi(e_t) \in \tilde{B}$ for all $t \in T$. Hence $\tilde{B} = \bigoplus_{t \in T} \tilde{B}e_t$ is a graded subalgebra of A .

We have proved the theorem for the case $J^2 = 0$. The general case is proved by induction on $\dim A$. Suppose $J^2 \neq 0$. Then $A/J^2 = B_0 \oplus J/J^2$ (direct sum of graded subspaces) for some graded maximal semisimple subalgebra B_0 of A/J^2 . Note that $1_{A/J^2} \in B_0$. Consider the preimage B_1 of B_0 in A under the natural map $\pi_1: A \twoheadrightarrow A/J^2$. Then $1_A \in B_1$. Since $B_0 \cong A/J$ is semisimple, $J(B_1) = J^2$. Moreover $\dim B_1 < \dim A$ and, by the induction assumption, we have $B_1 = B \oplus J^2$ (direct sum of graded subspaces) for some graded maximal semisimple subalgebra B in A . Hence $A = B \oplus J$ (direct sum of graded subspaces) and the theorem is proved. \square

Recall that a semigroup T is *cancellative* if for every $a, b, c \in T$ each of the conditions $ac = bc$ and $ca = cb$ implies $a = b$.

Proposition 5. *Let B be a finite dimensional associative T -graded semisimple algebra over a field F for some cancellative semigroup T . Then B is the direct sum of T -graded ideals that are T -graded-simple algebras. In particular, the T -graded analog of the Wedderburn–Artin theorem holds.*

Proof. By the ordinary Wedderburn–Artin theorem, $B = B_1 \oplus \dots \oplus B_s$ (direct sum of ideals) for some simple algebras B_i . Let I be a minimal T -graded ideal in B . Then $I = B_{i_1} \oplus \dots \oplus B_{i_k}$ for some i_1, \dots, i_k . Define $N := \bigoplus_{i \in \{1, \dots, s\} \setminus \{i_1, \dots, i_k\}} B_i$. Since all B_i are semisimple, we have $N = \{b \in B \mid ba = 0 \text{ for all } a \in I\}$. Since T is cancellative and I is graded, the ideal N is graded too. Hence $B = I \oplus N$ (direct sum of graded ideals) where I is a T -graded-simple algebra. Applying to N the inductive argument, we get the proposition. \square

4. Graded polynomial identities, their codimensions and cocharacters

Let T be a semigroup and let F be a field. Denote by $F\langle X^{T\text{-gr}} \rangle$ the free T -graded associative algebra over F on the countable set

$$X^{T\text{-gr}} := \bigcup_{t \in T} X^{(t)},$$

$X^{(t)} = \{x_1^{(t)}, x_2^{(t)}, \dots\}$, i.e. the algebra of polynomials in non-commuting variables from $X^{T\text{-gr}}$. The indeterminates from $X^{(t)}$ are said to be homogeneous of degree t . The T -degree of a monomial $x_{i_1}^{(t_1)} \dots x_{i_s}^{(t_s)} \in F\langle X^{T\text{-gr}} \rangle$ is defined to be $t_1 t_2 \dots t_s$, as opposed to its total degree, which is defined to be s . Denote by $F\langle X^{T\text{-gr}} \rangle^{(t)}$ the subspace of the algebra $F\langle X^{T\text{-gr}} \rangle$ spanned by all the monomials having T -degree t . Notice that

$$F\langle X^{T\text{-gr}} \rangle^{(t)} F\langle X^{T\text{-gr}} \rangle^{(h)} \subseteq F\langle X^{T\text{-gr}} \rangle^{(th)},$$

for every $t, h \in T$. It follows that

$$F\langle X^{T\text{-gr}} \rangle = \bigoplus_{t \in T} F\langle X^{T\text{-gr}} \rangle^{(t)}$$

is a T -grading. Let $f = f(x_{i_1}^{(t_1)}, \dots, x_{i_s}^{(t_s)}) \in F\langle X^{T\text{-gr}} \rangle$. We say that f is a *graded polynomial identity* of a T -graded algebra $A = \bigoplus_{t \in T} A^{(t)}$ and write $f \equiv 0$ if $f(a_{i_1}^{(t_1)}, \dots, a_{i_s}^{(t_s)}) = 0$ for all $a_{i_j}^{(t_j)} \in A^{(t_j)}$, $1 \leq j \leq s$. The set $\text{Id}^{T\text{-gr}}(A)$ of graded polynomial identities of A is a graded ideal of $F\langle X^{T\text{-gr}} \rangle$.

Example 5. Let $T = (\mathbb{Z}_2, +) = \{\bar{0}, \bar{1}\}$, $M_2(F) = M_2(F)^{(\bar{0})} \oplus M_2(F)^{(\bar{1})}$ where $M_2(F)^{(\bar{0})} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $M_2(F)^{(\bar{1})} = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$. Then $x^{(\bar{0})}y^{(\bar{0})} - y^{(\bar{0})}x^{(\bar{0})} \in \text{Id}^{T\text{-gr}}(M_2(F))$.

Let $P_n^{T\text{-gr}} := \langle x_{\sigma(1)}^{(t_1)} x_{\sigma(2)}^{(t_2)} \dots x_{\sigma(n)}^{(t_n)} \mid t_i \in T, \sigma \in S_n \rangle_F \subset F\langle X^{T\text{-gr}} \rangle$, $n \in \mathbb{N}$. Then the number

$$c_n^{T\text{-gr}}(A) := \dim \left(\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \text{Id}^{T\text{-gr}}(A)} \right)$$

is called the *nth codimension of graded polynomial identities* or the *nth graded codimension* of A .

The analog of Amitsur's conjecture for graded codimensions can be formulated as follows.

Conjecture. *There exists $\text{PIexp}^{T\text{-gr}}(A) := \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \in \mathbb{Z}_+$.*

If T is the trivial (semi)group of one element, we get the notion of ordinary polynomial identities, ordinary codimensions $c_n(A)$, and the ordinary PI-exponent $\text{PIexp}(A)$.

As we shall see in [Theorems 3–5](#) below, the analog of Amitsur's conjecture fails for all semigroups T of two elements that are not groups.

However, in [Theorem 2](#) below we provide sufficient conditions for a graded algebra to satisfy the analog of Amitsur's conjecture. As a consequence, we prove that if T is a cancellative semigroup or T is a left or right zero band, and a finite dimensional T -graded algebra A contains 1, then A satisfies the graded analog of Amitsur's conjecture ([Theorems 6 and 7](#)).

5. Polynomial H -identities and their codimensions

In our case, instead of working with graded codimensions directly, it is more convenient to replace the grading with the corresponding dual structure and study the asymptotic behavior of polynomial H -identities.

Let H be an arbitrary associative algebra with 1 over a field F . We say that an associative algebra A is an algebra with a *generalized H -action* if A is endowed with a homomorphism $H \rightarrow \text{End}_F(A)$ and for every $h \in H$ there exist $k \in \mathbb{N}$ and $h'_i, h''_i, h'''_i, h''''_i \in H$, $1 \leq i \leq k$, such that

$$h(ab) = \sum_{i=1}^k ((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a)) \text{ for all } a, b \in A. \quad (1)$$

Remark. We use the term “generalized H -action” in order to distinguish from the case when an algebra is an H -module algebra for some Hopf algebra H which is a particular case of the generalized H -action.

Let $F\langle X \rangle$ be the free associative algebra without 1 on the set $X := \{x_1, x_2, x_3, \dots\}$. Then $F\langle X \rangle = \bigoplus_{n=1}^{\infty} F\langle X \rangle^{(n)}$ where $F\langle X \rangle^{(n)}$ is the linear span of all monomials of total degree n . Consider the algebra

$$F\langle X|H \rangle := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\langle X \rangle^{(n)}$$

with the multiplication $(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\langle X \rangle^{(j)}$, $w_2 \in F\langle X \rangle^{(k)}$. We use the notation

$$x_{i_1}^{h_1} x_{i_2}^{h_2} \dots x_{i_n}^{h_n} := (h_1 \otimes h_2 \otimes \dots \otimes h_n) \otimes x_{i_1} x_{i_2} \dots x_{i_n}.$$

Here $h_1 \otimes h_2 \otimes \dots \otimes h_n \in H^{\otimes n}$, $x_{i_1} x_{i_2} \dots x_{i_n} \in F\langle X \rangle^{(n)}$.

Note that if $(\gamma_\beta)_{\beta \in \Lambda}$ is a basis in H , then $F\langle X|H \rangle$ is isomorphic to the free associative algebra over F with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$. We refer to the elements of $F\langle X|H \rangle$ as *associative H -polynomials*. Note that here we do not consider any H -action on $F\langle X|H \rangle$.

Let A be an associative algebra with a generalized H -action. Any map $\psi: X \rightarrow A$ has the unique homomorphic extension $\bar{\psi}: F\langle X|H \rangle \rightarrow A$ such that $\bar{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H -polynomial $f \in F\langle X|H \rangle$ is an H -identity of A if $\bar{\psi}(f) = 0$ for all maps $\psi: X \rightarrow A$. In other words, $f(x_1, x_2, \dots, x_n)$ is an H -identity of A if and only if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_i \in A$. In this case we write $f \equiv 0$. The set $\text{Id}^H(A)$ of all H -identities of A is an ideal of $F\langle X|H \rangle$.

We denote by P_n^H the space of all multilinear H -polynomials in x_1, \dots, x_n , $n \in \mathbb{N}$, i.e.

$$P_n^H = \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \dots x_{\sigma(n)}^{h_n} \mid h_i \in H, \sigma \in S_n \rangle_F \subset F\langle X|H \rangle.$$

Then the number $c_n^H(A) := \dim \left(\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)} \right)$ is called the n th *codimension of polynomial H -identities* or the n th *H -codimension* of A .

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group S_n acts on the space $\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}$ by permuting the variables. Irreducible FS_n -modules are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and their Young diagrams D_λ . The character $\chi_n^H(A)$ of the FS_n -module $\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}$ is called the n th *cocharacter* of polynomial H -identities of A . We can rewrite it as a sum

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m(A, H, \lambda) \chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\text{sign } \sigma) \sigma$, be Young symmetrizers corresponding to a Young tableau T_λ . Then $M(\lambda) = FS_n e_{T_\lambda} \cong FS_n e_{T_\lambda}^*$ is an irreducible FS_n -module

corresponding to a partition $\lambda \vdash n$. We refer the reader to [3,4,6] for an account of S_n -representations and their applications to polynomial identities.

6. Generalized $(FT)^*$ -action on T -graded algebras

In this section we show that every finite dimensional semigroup graded algebra is an algebra with a generalized H -action for a suitable associative algebra H .

For an arbitrary semigroup T one can consider the *semigroup algebra* FT over a field F which is the vector space with the formal basis $(t)_{t \in T}$ and the multiplication induced by the one in T .

Consider the vector space $(FT)^*$ dual to FT . Then $(FT)^*$ is an algebra with the multiplication defined by $(hw)(t) = h(t)w(t)$ for $h, w \in (FT)^*$ and $t \in T$. The identity element is defined by $1_{(FT)^*}(t) = 1$ for all $t \in T$. In other words, $(FT)^*$ is the algebra dual to the coalgebra FT .

Let $\Gamma: A = \bigoplus_{t \in T} A^{(t)}$ be a grading on an algebra A . We have the following natural $(FT)^*$ -action on A : $ha^{(t)} := h(t)a^{(t)}$ for all $h \in (FT)^*$, $a^{(t)} \in A^{(t)}$ and $t \in T$.

Remark. If T is a finite group, then A is an FT -comodule algebra for the Hopf algebra FT and an $(FT)^*$ -module algebra for the Hopf algebra $(FT)^*$.

For every $t \in T$ define $h_t \in (FT)^*$ by $h_t(g) = \begin{cases} 0 & \text{if } g \neq t, \\ 1 & \text{if } g = t \end{cases}$ for $g \in T$.

If A is finite dimensional, the set $\text{supp } \Gamma := \{t \in T \mid A^{(t)} \neq 0\}$ is finite and

$$h_t(ab) = \sum_{\substack{g, w \in \text{supp } \Gamma, \\ gw = t}} h_g(a)h_w(b) \text{ for all } a, b \in A.$$

Note that $ha = \sum_{t \in \text{supp } \Gamma} h(t)h_t a$ for all $a \in A$ and

$$x^h - \sum_{t \in \text{supp } \Gamma} h(t)x^{h_t} \in \text{Id}^{(FT)^*}(A) \quad (2)$$

for all $h \in (FT)^*$. By linearity, we get (1). Therefore, A is an algebra with a generalized $(FT)^*$ -action.

Lemma 1. *Let A be a finite dimensional algebra over a field F graded by a semigroup T . Then $c_n^{T\text{-gr}}(A) = c_n^{(FT)^*}(A)$ for all $n \in \mathbb{N}$.*

Proof. Denote the grading $A = \bigoplus_{t \in T} A^{(t)}$ by Γ .

Let

$$\xi: F\langle X \mid (FT)^* \rangle \rightarrow F\langle X^{T\text{-gr}} \rangle$$

be the homomorphism of algebras defined by $\xi(x_i^h) = \sum_{t \in \text{supp } \Gamma} h(t)x_i^{(t)}$, $i \in \mathbb{N}$, $h \in (FT)^*$.

Suppose $f \in \text{Id}^{(FT)^*}(A)$. Consider an arbitrary graded homomorphism $\psi: F\langle X^{T\text{-gr}} \rangle \rightarrow A$. Then the homomorphism of algebras $\psi\xi: F\langle X \mid (FT)^* \rangle \rightarrow A$ satisfies the condition

$$\psi\xi(x_i^h) = \sum_{t \in \text{supp } \Gamma} h(t)\psi(x_i^{(t)}) = h\left(\sum_{t \in \text{supp } \Gamma} \psi(x_i^{(t)})\right) = h\psi\xi(x_i).$$

Thus $\psi\xi(f) = 0$ and $\xi(f) \in \text{Id}^{T\text{-gr}}(A)$. Hence $\xi(\text{Id}^{(FT)^*}(A)) \subseteq \text{Id}^{T\text{-gr}}(A)$. Denote by

$$\tilde{\xi}: F\langle X \mid (FT)^* \rangle / \text{Id}^{(FT)^*}(A) \rightarrow F\langle X^{T\text{-gr}} \rangle / \text{Id}^{T\text{-gr}}(A)$$

the homomorphism induced by ξ .

Let

$$\eta: F\langle X^{T\text{-gr}} \rangle \rightarrow F\langle X \mid (FT)^* \rangle$$

be the homomorphism defined by $\eta(x_i^{(t)}) = x_i^{ht}$ for all $i \in \mathbb{N}$ and $t \in T$. Consider an arbitrary graded polynomial identity $f \in F\langle X^{T\text{-gr}} \rangle$. Let $\psi: F\langle X \mid (FT)^* \rangle \rightarrow A$ be a homomorphism satisfying the condition $\psi(x_i^h) = h\psi(x_i)$ for every $i \in \mathbb{N}$ and $h \in (FT)^*$. Then for any $i \in \mathbb{N}$ and $g, t \in T$ we have

$$h_g\psi\eta(x_i^{(t)}) = h_g\psi(x_i^{h_t}) = h_g h_t \psi(x_i) = \begin{cases} 0 & \text{if } g \neq t, \\ \psi\eta(x_i^{(t)}) & \text{if } g = t. \end{cases}$$

Thus $\psi\eta(x_i^{(t)}) \in A^{(t)}$ and $\psi\eta$ is a graded homomorphism. Therefore, $\psi\eta(f) = 0$ and $\eta(\text{Id}^{T\text{-gr}}(A)) \subseteq \text{Id}^{(FT)^*}(A)$. Denote by $\tilde{\eta}: F\langle X^{T\text{-gr}} \rangle / \text{Id}^{T\text{-gr}}(A) \rightarrow F\langle X \mid (FT)^* \rangle / \text{Id}^{(FT)^*}(A)$ the induced homomorphism.

Now we use the notation $\bar{f} = f + \text{Id}^{(FT)^*}(A) \in F\langle X \mid (FT)^* \rangle / \text{Id}^{(FT)^*}(A)$ for $f \in F\langle X \mid (FT)^* \rangle$ and $\bar{f} = f + \text{Id}^{T\text{-gr}}(A) \in F\langle X^{T\text{-gr}} \rangle / \text{Id}^{T\text{-gr}}(A)$ for $f \in F\langle X^{T\text{-gr}} \rangle$. We have

$$\tilde{\eta}\tilde{\xi}(\bar{x}_i^h) = \tilde{\eta}\left(\sum_{t \in \text{supp } \Gamma} h(t)\bar{x}_i^{(t)}\right) = \sum_{t \in \text{supp } \Gamma} h(t)\bar{x}_i^{h_t} = \bar{x}_i^h$$

for every $h \in (FT)^*$ and $i \in \mathbb{N}$. (Here we use (2).) Thus $\tilde{\eta}\tilde{\xi} = \text{id}_{F\langle X \mid (FT)^* \rangle / \text{Id}^{(FT)^*}(A)}$. Moreover $\tilde{\xi}\tilde{\eta}(\bar{x}_i^{(t)}) = \tilde{\xi}(\bar{x}_i^{h_t}) = \bar{x}_i^{(t)}$ for every $t \in T$ and $i \in \mathbb{N}$. Therefore, $\tilde{\xi}\tilde{\eta} = \text{id}_{F\langle X^{T\text{-gr}} \rangle / \text{Id}^{T\text{-gr}}(A)}$ and $F\langle X^{T\text{-gr}} \rangle / \text{Id}^{T\text{-gr}}(A) \cong F\langle X \mid (FT)^* \rangle / \text{Id}^{(FT)^*}(A)$ as algebras. The restriction of $\tilde{\xi}$ provides the isomorphism of $\frac{P_n^{(FT)^*}}{P_n^{(FT)^*} \cap \text{Id}^{(FT)^*}(A)}$ and $\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \text{Id}^{T\text{-gr}}(A)}$. Hence

$$c_n^{(FT)^*}(A) = \dim \frac{P_n^{(FT)^*}}{P_n^{(FT)^*} \cap \text{Id}^{(FT)^*}(A)} = \dim \frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \text{Id}^{T\text{-gr}}(A)} = c_n^{T\text{-gr}}(A). \quad \square$$

Now we can provide a sufficient condition for a graded algebra to satisfy the graded analog of Amitsur's conjecture.

Theorem 2. *Let A be a finite dimensional non-nilpotent T -graded associative algebra over an algebraically closed field F of characteristic 0 for some semigroup T . Suppose that the Jacobson radical $J := J(A)$ is a graded ideal. Let*

$$A/J = B_1 \oplus \dots \oplus B_q \text{ (direct sum of graded ideals)}$$

where B_i are graded-simple algebras and let $\varkappa: A/J \rightarrow A$ be any homomorphism of algebras (not necessarily graded) such that $\pi\varkappa = \text{id}_{A/J}$ where $\pi: A \rightarrow A/J$ is the natural projection. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leq c_n^{T\text{-gr}}(A) \leq C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$ where

$$d = \max \dim \left(B_{i_1} \oplus B_{i_2} \oplus \dots \oplus B_{i_r} \mid r \geq 1, \right.$$

$$\left. ((FT)^*\varkappa(B_{i_1}))A^+ ((FT)^*\varkappa(B_{i_2}))A^+ \dots ((FT)^*\varkappa(B_{i_{r-1}}))A^+ ((FT)^*\varkappa(B_{i_r})) \neq 0 \right)$$

and $A^+ := A + F \cdot 1$.

Proof. The theorem is an immediate consequence of Lemma 1 and [8, Theorem 1]. \square

Remark. The existence of the map \varkappa follows from the ordinary Wedderburn–Mal'cev theorem.

Remark. If A is nilpotent, i.e. $x_1 \dots x_p \equiv 0$ for some $p \in \mathbb{N}$, then $P_n^{T\text{-gr}} \subseteq \text{Id}^{T\text{-gr}}(A)$ and $c_n^{T\text{-gr}}(A) = 0$ for all $n \geq p$.

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

7. Partitions restricted to convex polytopes

Here we apply ideas from [13] and prove auxiliary results that we use in the construction of algebras with non-integer graded PI-exponents.

In this section we show that if all the partitions $\lambda \vdash n$ that correspond to irreducible FS_n -modules with nonzero multiplicities $m(A, H, \lambda)$ belong to a convex polyhedron, then $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{c_n^H(A)}$ is bounded by the maximum of a particular function Φ on the “continuous” version of the polyhedron.

Fix $q \in \mathbb{N}$. Let $\Phi(\alpha_1, \dots, \alpha_q) = \frac{1}{\alpha_1^{\alpha_1} \dots \alpha_q^{\alpha_q}}$. Define $0^0 := 1$. Then Φ is continuous on the segment $\{(\alpha_1, \dots, \alpha_q) \mid \alpha_i \geq 0\}$.

Suppose we have some numbers $\gamma_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$, $0 \leq j \leq q$ and $\theta_k \in \mathbb{Z}_+$ for $q < k \leq r$ where $m, r \in \mathbb{Z}_+$, $r \geq q$. Define

$$\Omega = \left\{ (\alpha_1, \dots, \alpha_q) \in \mathbb{R}^q \mid \sum_{i=1}^q \alpha_i = 1, \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_q \geq 0, \right. \\ \left. \sum_{j=1}^q \gamma_{ij} \alpha_j \geq 0 \text{ for } 1 \leq i \leq m \right\}.$$

For every $n \in \mathbb{N}$ we define

$$\Omega_n = \left\{ \lambda \vdash n \mid \sum_{j=1}^q \gamma_{ij} \lambda_j + \gamma_{i0} \geq 0 \text{ for } 1 \leq i \leq m, \lambda_i \leq \theta_i \text{ for } q < i \leq r, \lambda_{r+1} = 0 \right\}.$$

We treat Ω and Ω_n as the “continuous” and the “discrete” version of the same polyhedron.

Denote by d the maximum of Φ on the compact set Ω . (We assume Ω to be non-empty.)

Lemma 2. *Let A be an algebra with a generalized H -action where H is an associative algebra with unity over a field F of characteristic 0. Suppose $m(A, H, \lambda) = 0$ for all $\lambda \vdash n$, $\lambda \notin \Omega_n$, $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^H(A)} \leq d$.*

Proof. Let $\lambda \vdash n$ such that $m(A, H, \lambda) \neq 0$. By the hook formula, $\dim M(\lambda) = \frac{n!}{\prod_{i,j} h_{ij}}$ where h_{ij} is the length of the hook with the edge in (i, j) in the Young diagram D_λ . Hence $\dim M(\lambda) \leq \frac{n!}{\lambda_1! \dots \lambda_r!}$. Note that $(x^x)' = (e^{x \ln x})' = (\ln x + 1)e^{x \ln x}$ and x^x is decreasing for $x \leq \frac{1}{e}$. By the Stirling formula, for all sufficiently large n we have

$$\dim M(\lambda) \leq \frac{C_1 n^{r_1} \left(\frac{n}{e}\right)^n}{\left(\frac{\lambda_1}{e}\right)^{\lambda_1} \dots \left(\frac{\lambda_r}{e}\right)^{\lambda_r}} = C_1 n^{r_1} \left(\frac{1}{\left(\frac{\lambda_1}{n}\right)^{\frac{\lambda_1}{n}} \dots \left(\frac{\lambda_r}{n}\right)^{\frac{\lambda_r}{n}}} \right)^n \leq \\ C_1 n^{r_1} \left(\Phi \left(\frac{\lambda_1}{n}, \dots, \frac{\lambda_q}{n} \right) \right)^n \frac{n^{\theta_{q+1} + \dots + \theta_r}}{\theta_{q+1}^{\theta_{q+1}} \dots \theta_r^{\theta_r}} = C_2 n^{r_2} \left(\Phi \left(\frac{\lambda_1}{n}, \dots, \frac{\lambda_q}{n} \right) \right)^n \quad (3)$$

for some $C_1, C_2 > 0$ and $r_1, r_2 \in \mathbb{R}$ that do not depend on λ_i .

Let $\varepsilon > 0$. Since Φ is continuous, there exists $\delta > 0$ such that for every x from the domain of Φ such that the distance between x and Ω is less than δ , we have $\Phi(x) < d + \varepsilon$.

Therefore, by (3), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\lambda \vdash n$ such that $m(A, H, \lambda) \neq 0$ we have $\dim M(\lambda) \leq C_2 n^{r_2} (d + \varepsilon)^n$.

By [8, Theorem 5], there exist $C_3 > 0$, $r_3 \in \mathbb{Z}_+$ such that

$$\sum_{\lambda \vdash n} m(A, H, \lambda) \leq C_3 n^{r_3} \text{ for all } n \in \mathbb{N}.$$

Hence

$$c_n^H(A) = \sum_{\lambda \vdash n} m(A, H, \lambda) \dim M(\lambda) \leq C_2 C_3 n^{r_2+r_3} (d + \varepsilon)^n$$

and $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{c_n^H(A)} \leq d + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get the lemma. \square

Lemma 3. Let $q \in \mathbb{N}$, $q \geq 4$,

$$\Omega = \left\{ (\alpha_1, \dots, \alpha_q) \in \mathbb{R}^q \mid \sum_{i=1}^q \alpha_i = 1, \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_q \geq 0, \alpha_q + \alpha_{q-1} \leq \alpha_1 \right\}.$$

Then $d := \max_{x \in \Omega} \Phi(x) = (q-3) + 2\sqrt{2} = q - 0.1716\dots$

Proof. We express α_1 in terms of $\alpha_2, \dots, \alpha_q$ and consider

$$\Phi_0(\alpha_2, \dots, \alpha_q) := \Phi\left(1 - \sum_{i=2}^q \alpha_i, \alpha_2, \dots, \alpha_q\right) = \frac{1}{(1 - \sum_{i=2}^q \alpha_i)^{(1 - \sum_{i=2}^q \alpha_i)} \alpha_2^{\alpha_2} \dots \alpha_q^{\alpha_q}}$$

on the segment

$$\Omega_0 = \left\{ (\alpha_2, \dots, \alpha_q) \mid \alpha_2 \geq 0, \dots, \alpha_q \geq 0, \alpha_2 + \dots + \alpha_{q-2} + 2\alpha_{q-1} + 2\alpha_q \leq 1 \right\}.$$

Note that we have weakened the restrictions on α_i . However, we will see that $\max_{x \in \Omega} \Phi(x) = \max_{x \in \Omega_0} \Phi_0(x)$.

We have $\Phi_0(\alpha_2, \dots, \alpha_q) = e^{-(1 - \sum_{i=2}^q \alpha_i) \ln(1 - \sum_{i=2}^q \alpha_i) - \sum_{i=2}^q (\alpha_i \ln \alpha_i)}$ and

$$\frac{\partial \Phi_0}{\partial \alpha_k}(\alpha_2, \alpha_3, \dots, \alpha_q) = \left(\ln\left(1 - \sum_{i=2}^q \alpha_i\right) - \ln \alpha_k \right) e^{-(1 - \sum_{i=2}^q \alpha_i) \ln(1 - \sum_{i=2}^q \alpha_i) - \sum_{i=2}^q (\alpha_i \ln \alpha_i)}.$$

Hence the only critical point is $(\alpha_2, \alpha_3, \dots, \alpha_q) = \left(\frac{1}{q}, \dots, \frac{1}{q}\right) \notin \Omega_0$. Therefore, Φ_0 takes its maximal values on the border $\partial\Omega_0$ of Ω_0 . Note that $\partial\Omega_0 = \Upsilon \cup \bigcup_{i=2}^q \Omega_i$ where $\Omega_i = \{(\alpha_2, \dots, \alpha_q) \in \Omega_0 \mid \alpha_i = 0\}$ and $\Upsilon = \{(\alpha_2, \dots, \alpha_q) \in \Omega_0 \mid \alpha_2 + \dots + \alpha_{q-2} + 2\alpha_{q-1} + 2\alpha_q = 1\}$. Determining the critical points once again, we get $\Phi_0(x) \leq q-1$ for all $x \in \bigcup_{i=2}^q \Omega_i$.

Consider Φ_0 on Υ . We express α_2 in terms of $\alpha_3, \dots, \alpha_q$ and define

$$\begin{aligned} \Phi_1(\alpha_3, \dots, \alpha_q) &= \Phi_0(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q, \alpha_3, \dots, \alpha_q) = \\ &= \frac{1}{(\alpha_{q-1} + \alpha_q)^{\alpha_{q-1} + \alpha_q} (1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q)^{1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q} \alpha_3^{\alpha_3} \dots \alpha_q^{\alpha_q}} = \\ &= e^{-(\alpha_{q-1} + \alpha_q) \ln(\alpha_{q-1} + \alpha_q) - (1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) \ln(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) - \alpha_3 \ln \alpha_3 - \dots - \alpha_q \ln \alpha_q} \end{aligned}$$

on

$$\Upsilon_1 = \{(\alpha_3, \dots, \alpha_q) \mid 1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q \geq 0, \alpha_3 \geq 0, \dots, \alpha_q \geq 0\}.$$

Then

$$\frac{\partial \Phi_1}{\partial \alpha_i}(\alpha_3, \dots, \alpha_q) = (\ln(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) - \ln \alpha_i) \cdot e^{-(\alpha_{q-1} + \alpha_q) \ln(\alpha_{q-1} + \alpha_q) - (1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) \ln(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) - \alpha_3 \ln \alpha_3 - \dots - \alpha_q \ln \alpha_q}$$

for $i = 3, \dots, q-2$ and

$$\frac{\partial \Phi_1}{\partial \alpha_i}(\alpha_3, \dots, \alpha_q) = (2 \ln(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) - \ln(\alpha_{q-1} + \alpha_q) - \ln \alpha_i) \cdot e^{-(\alpha_{q-1} + \alpha_q) \ln(\alpha_{q-1} + \alpha_q) - (1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) \ln(1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q) - \alpha_3 \ln \alpha_3 - \dots - \alpha_q \ln \alpha_q},$$

for $i = q-1, q$. Therefore, $(\tilde{\alpha}_3, \dots, \tilde{\alpha}_q)$ where $\tilde{\alpha}_3 = \dots = \tilde{\alpha}_{q-2} = \frac{\sqrt{2}}{4+(q-3)\sqrt{2}}$ and $\tilde{\alpha}_{q-1} = \tilde{\alpha}_q = \frac{1}{4+(q-3)\sqrt{2}}$, is the critical point of Φ_1 . Hence $\Phi_1(\tilde{\alpha}_3, \dots, \tilde{\alpha}_q) = (q-3) + 2\sqrt{2} = q - 0.1716\dots$ is the maximum of Φ_1 on Υ_1 .

Now we return to the original variables. Since $\tilde{\alpha}_1 = \tilde{\alpha}_{q-1} + \tilde{\alpha}_q = \frac{2}{4+(q-3)\sqrt{2}}$ and

$$\tilde{\alpha}_2 = 1 - \alpha_3 - \dots - \alpha_{q-2} - 2\alpha_{q-1} - 2\alpha_q = \frac{\sqrt{2}}{4 + (q-3)\sqrt{2}},$$

we have $\tilde{\alpha}_1 \geq \tilde{\alpha}_2 \geq \dots \geq \tilde{\alpha}_q$. Therefore $d = \Phi(\tilde{\alpha}_1, \dots, \tilde{\alpha}_q) = (q-3) + 2\sqrt{2} = q - 0.1716\dots$ is the maximum of Φ on Ω . \square

Lemma 4. Let $A = M_2(F) \oplus W$ (direct sum of left ideals) be an algebra over a field F of characteristic 0 and let $T = \{t_1, t_2\}$ be a semigroup of two elements. Suppose there exists a linear isomorphism $\varphi: W \xrightarrow{\sim} \langle \mathcal{B}_0 \rangle_F \subset M_2(F)$ where $\mathcal{B}_0 \subseteq \{e_{11}, e_{12}, e_{22}\}$ is a subset such that $e_{12} \in \mathcal{B}_0$. Define the T -grading on A by $A^{(t_1)} = (M_2(F), 0)$ and $A^{(t_2)} = \{(\varphi(a), a) \mid a \in W\}$. Suppose $WM_2(F) = 0$ and one of the following three conditions holds:

- (1) $M_2(F)W = 0$ and φ is a homomorphism of algebras;
- (2) $M_2(F)W = 0$ and $W^2 = 0$;
- (3) $W^2 = 0$ and φ is a homomorphism of left $M_2(F)$ -modules.

Then if $m(A, (FT)^*, \lambda) \neq 0$ for some $\lambda \vdash n$, $n \in \mathbb{N}$, we have $\lambda_{q+1} = 0$ and $\lambda_{q-1} + \lambda_q \leq \lambda_1 + 1$ where $q := \dim A$.

Proof. In order to prove that $m(A, (FT)^*, \lambda) = 0$ for some $\lambda \vdash n$, it is sufficient to show that $e_{T_\lambda}^* f \equiv 0$ for every $f \in P_n^{(FT)^*}$ and a Young tableau T_λ . Note that $e_{T_\lambda}^* f$ is alternating in the variables of each column of T_λ . Since f is multilinear, it is sufficient to substitute only basis elements. However $\dim A = q$ and if $\lambda_{q+1} > 0$, then at least two of

the basis elements corresponding to the variables of the first column coincide and $e_{T_\lambda}^* f$ vanish. Hence if $\lambda_{q+1} > 0$, we have $e_{T_\lambda} f \equiv 0$.

Consider in A the homogeneous basis

$$\mathcal{B} = \{(e_{11}, 0), (e_{12}, 0), (e_{21}, 0), (e_{22}, 0)\} \cup \{(a, \varphi^{-1}(a) \mid a \in \mathcal{B}_0\}.$$

Note that the product of any two elements of \mathcal{B} is either 0 or again an element of \mathcal{B} . Define the function $\theta: \mathcal{B} \rightarrow \mathbb{Z}$ by $\theta(e_{ij}, \varphi^{-1}(e_{ij})) = \theta(e_{ij}, 0) = j - i$. Let $a_1, \dots, a_k \in \mathcal{B}$. If $a_1 \dots a_k \neq 0$, then

$$-1 \leq \sum_{i=1}^k \theta(a_i) = \theta(a_1 \dots a_k) \leq 1. \quad (4)$$

Note that $\sum_{b \in \mathcal{B}} \theta(b) = 1$ and $\sum_{i=1}^{q-1} \theta(a_i) \geq 0$ for any different $a_i \in \mathcal{B}$. If $m(A, (FT)^*, \lambda) \neq 0$, then $e_{T_\lambda}^* f \neq 0$ for some $f \in P_n^{(FT)^*}$ and $e_{T_\lambda}^* f$ does not vanish under substitution of some basis elements a_1, \dots, a_n . Again, $e_{T_\lambda}^* f$ is alternating in the variables of each column. In each of the first λ_q columns we have q boxes and in each of the next $(\lambda_{q-1} - \lambda_q)$ columns we have $(q-1)$ boxes. Therefore, the impact in $\sum_{i=1}^n \theta(a_i)$ of the basis elements corresponding to the first λ_{q-1} columns is at least λ_q . Since $\theta(a) = -1$ only for $a = (e_{21}, 0)$ and we cannot substitute more than one such element for the variables of the same column, by (4) there must be at least $(\lambda_q - 1)$ other columns. Hence $\lambda_1 - \lambda_{q-1} \geq \lambda_q - 1$ and we get the lemma. \square

Lemma 5. Let A be an algebra from Lemma 4. Suppose that for every $\lambda \vdash n$, $n \in \mathbb{N}$, such that $\lambda_{q-1} + \lambda_q \leq \lambda_1$, we have $m(A, (FT)^*, \lambda) \neq 0$. Then there exists

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} = (q-3) + 2\sqrt{2} = q - 0.1716 \dots$$

Proof. Let

$$\Omega = \left\{ (\alpha_1, \dots, \alpha_q) \in \mathbb{R}^q \mid \sum_{i=1}^q \alpha_i = 1, \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_q \geq 0, \alpha_q + \alpha_{q-1} \leq \alpha_1 \right\}.$$

By Lemma 3, $d := \max_{x \in \Omega} \Phi(x) = (q-3) + 2\sqrt{2} = q - 0.1716 \dots$. Denote by $(\alpha_1, \dots, \alpha_q) \in \Omega$ such a point that $\Phi(\alpha_1, \dots, \alpha_q) = d$. For every $n \in \mathbb{N}$ define $\mu \vdash n$ by $\mu_i = [\alpha_i n]$ for $2 \leq i \leq q$ and $\mu_1 = n - \sum_{i=2}^q \mu_i$. For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\Phi(\frac{\mu_1}{n}, \dots, \frac{\mu_q}{n}) > d - \varepsilon$. By the assumptions of the lemma, $m(A, (FT)^*, \mu) \neq 0$ and by the hook and the Stirling formulas, there exist $C_1 > 0$ and $r_1 \in \mathbb{R}$ such that we have

$$c_n^{(FT)^*}(A) \geq \dim M(\mu) = \frac{n!}{\prod_{i,j} h_{ij}} \geq \frac{n!}{(\mu_1 + q - 1)! \dots (\mu_q + q - 1)!} \geq$$

$$\frac{n!}{n^{q(q-1)}\mu_1!\dots\mu_q!} \geq \frac{C_1 n^{r_1} \left(\frac{n}{e}\right)^n}{\left(\frac{\mu_1}{e}\right)^{\mu_1} \dots \left(\frac{\mu_q}{e}\right)^{\mu_q}} \geq C_1 n^{r_1} \left(\frac{1}{\left(\frac{\mu_1}{n}\right)^{\frac{\mu_1}{n}} \dots \left(\frac{\mu_q}{n}\right)^{\frac{\mu_q}{n}}}\right)^n \geq C_1 n^{r_1} (d - \varepsilon)^n. \quad (5)$$

Hence $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{(FT)^*}}(A) \geq d - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{(FT)^*}}(A) \geq d$. Now [Lemmas 1, 2, 3, and 4](#) finish the proof. \square

8. A T_1 -graded algebra with a non-integer graded PI-exponent

Theorem 3. Let $A = M_2(F) \oplus \text{UT}_2(F)$ (direct sum of ideals) where F is a field of characteristic 0. Define a T_1 -grading on A by $A^{(0)} = (M_2(F), 0)$, $A^{(1)} = \{(\varphi(a), a) \mid a \in \text{UT}_2(F)\}$ where $\varphi: \text{UT}_2(F) \hookrightarrow M_2(F)$ is the natural embedding. In other words, A is an algebra from [Example 1](#) for $k = 2$. Then there exists $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T_1\text{-gr}}}(A) = 4 + 2\sqrt{2} = 6.8284\dots$

To prove [Theorem 3](#), we need the following lemma. We omit φ for shortness and write (e_{ij}, e_{ij}) instead of $(e_{ij}, \varphi^{-1}(e_{ij}))$.

Lemma 6. Let $\lambda \vdash n$, $n \in \mathbb{N}$, $\lambda_8 = 0$, and $\lambda_6 + \lambda_7 \leq \lambda_1$. Then $m(A, (FT_1)^*, \lambda) \neq 0$.

Proof. It is sufficient to show that for some $f \in P_n$ and some T_λ we have $e_{T_\lambda} f \neq 0$ on A .

Note that each $e_{T_\lambda} f$ is alternating in λ_7 disjoint sets of variables, each of 7 variables. By the proof [Lemma 4](#), each column will have the impact at least 1 to the sum of the values of θ on the elements substituted for the variables of $e_{T_\lambda} f$. Therefore, we need a compensation.

Let $\beta_2 = \lambda_6 - \lambda_7$. Fix numbers $\beta_3, \dots, \beta_{12} \geq 0$ such that $\beta_3 + \beta_5 + \beta_7 + \beta_9 + \beta_{11} = \lambda_7$, $\beta_3 + \beta_4 = \lambda_5 - \lambda_6$, $\beta_5 + \beta_6 = \lambda_4 - \lambda_5$, $\beta_7 + \beta_8 = \lambda_3 - \lambda_4$, $\beta_9 + \beta_{10} = \lambda_2 - \lambda_3$, and $\beta_{11} + \beta_{12} = \lambda_1 - \lambda_2$. In other words, we have

$$D_\lambda = \begin{array}{cccccccccccccc} \lambda_7 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & & & \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & & & & & & \\ \hline \dots & \dots & \dots & \dots & & & & & & & & \\ \hline \dots & \dots & & & & & & & & & & \\ \hline \dots & & & & & & & & & & & \\ \hline \dots & & & & & & & & & & & \\ \hline \end{array} \end{array}.$$

(Here β_i denotes the number of columns in each block of columns.)

Each of the first λ_7 columns will give an impact 1 to θ , which will be compensated by the columns that are marked by $\beta_3, \beta_5, \beta_7, \beta_9$, and β_{11} . The columns that are marked by $\beta_2, \beta_3, \beta_4, \beta_6, \beta_8, \beta_{10}$, and β_{12} will give zero impact to θ .

We fix some Young tableau T_λ of the shape λ filled in with the numbers from 1 to n . For each column of T_λ we define a multilinear alternating polynomial depending on the variables with the indexes from the column. For shortness, we denote the polynomials corresponding to different columns in the i th block by the same letter f_i . By (i_1, \dots, i_ℓ) we denote the ℓ -tuple of numbers from a column (from up to down). By $S\{i_1, \dots, i_\ell\}$ we denote the symmetric group on i_1, \dots, i_ℓ . We define

$$\begin{aligned} f_1 &:= \sum_{\sigma \in S\{i_1, \dots, i_7\}} (\text{sign } \sigma) x_{\sigma(i_3)}^{h_0} x_{\sigma(i_2)}^{h_1} x_{\sigma(i_6)}^{h_0} x_{\sigma(i_4)}^{h_1} x_{\sigma(i_5)}^{h_0} x_{\sigma(i_1)}^{h_0} x_{\sigma(i_7)}^{h_1}, \\ f_2 &:= \sum_{\sigma \in S\{i_1, \dots, i_6\}} (\text{sign } \sigma) x_{\sigma(i_3)}^{h_0} x_{\sigma(i_2)}^{h_1} x_{\sigma(i_6)}^{h_0} x_{\sigma(i_4)}^{h_1} x_{\sigma(i_5)}^{h_0} x_{\sigma(i_1)}^{h_0}, \\ f_3 &:= \sum_{\sigma \in S\{i_1, \dots, i_5\}} (\text{sign } \sigma) x_{\sigma(i_5)}^{h_0} x_{\sigma(i_4)}^{h_1} x_{\sigma(i_1)}^{h_0} x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_0}, \\ f_4 &:= \sum_{\sigma \in S\{i_1, \dots, i_5\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_0} x_{\sigma(i_5)}^{h_1} x_{\sigma(i_4)}^{h_1} x_{\sigma(i_1)}^{h_0}, \\ f_5 &:= \sum_{\sigma \in S\{i_1, \dots, i_4\}} (\text{sign } \sigma) x_{\sigma(i_4)}^{h_1} x_{\sigma(i_1)}^{h_0} x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_0}, \\ f_6 &:= \sum_{\sigma \in S\{i_1, \dots, i_4\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_1} x_{\sigma(i_4)}^{h_1} x_{\sigma(i_1)}^{h_0}, \\ f_7 &:= \sum_{\sigma \in S\{i_1, i_2, i_3\}} (\text{sign } \sigma) x_{\sigma(i_1)}^{h_0} x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_0}, & f_8 &:= \sum_{\sigma \in S\{i_1, i_2, i_3\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_1} x_{\sigma(i_3)}^{h_1} x_{\sigma(i_1)}^{h_0}, \\ f_9 &:= \sum_{\sigma \in S\{i_1, i_2\}} (\text{sign } \sigma) x_{\sigma(i_1)}^{h_0} x_{\sigma(i_2)}^{h_1}, & f_{10} &:= \sum_{\sigma \in S\{i_1, i_2\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_0} x_{\sigma(i_1)}^{h_0}, \\ f_{11} &:= x_{i_1}^{h_0}, & f_{12} &:= x_{i_1}^{h_1}. \end{aligned}$$

Define the polynomial

$$f = (f_1 f_3)^{\beta_3} (f_1 f_5)^{\beta_5} (f_1 f_7)^{\beta_7} (f_1 f_9)^{\beta_9} (f_1 f_{11})^{\beta_{11}} f_2^{\beta_2} f_4^{\beta_4} f_6^{\beta_6} f_8^{\beta_8} f_{10}^{\beta_{10}} f_{12}^{\beta_{12}} \in P_n.$$

As we have already mentioned, here different copies of f_i depend on different variables.

The copies of f_1 are alternating polynomials of degree 7 corresponding to the first λ_7 columns of height 7.

The copies of f_2 are alternating polynomials of degree 6 corresponding to the next β_2 columns of height 6.

...

The copies of f_{12} are polynomials of degree 1 with the indexes from the last β_{12} columns of height 1.

λ_7	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	(e_{11}, e_{11})
(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	(e_{11}, e_{11})	$(e_{12}, 0)$	
$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	(e_{12}, e_{12})	$(e_{11}, 0)$	(e_{12}, e_{12})				
(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})						
$(e_{22}, 0)$	$(e_{22}, 0)$	$(e_{22}, 0)$	(e_{12}, e_{12})								
$(e_{12}, 0)$	$(e_{12}, 0)$										
(e_{12}, e_{12})											

(Here in the i th block we have β_i columns with the same values in all cells of a row. For shortness, we depict each value for each block only once. The tableau τ is still of the shape λ .)

Fig. 1. Substitution for the variables of $e_{T_\lambda} f$, grading semigroup T_1 .

We claim that $e_{T_\lambda} f \neq 0$. In order to verify this, we fill D_λ with specific homogeneous elements and denote the tableau obtained by τ . (See Fig. 1.)

Now for each variable we substitute the element from the corresponding box in τ . Note that f does not vanish under this substitution.

Recall that $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ where a_{T_λ} is the symmetrization in the variables of each row and b_{T_λ} is the alternation in the variables of each column. Since all f_i are alternating polynomials, $b_{T_\lambda} f$ is a nonzero multiple of f .

Two sets of variables correspond to the second row of T_λ . For the variables of the first group we substitute $(e_{11}, e_{11}) \in A^{(1)}$, for the second one, we substitute $(e_{12}, 0) \in A^{(0)}$. Thus if an item in a_{T_λ} mixes variables from these two groups, at least one variable from the second group, i.e. in f_{11} , is replaced with a variable from the first one. However, f_{10} vanishes if at least one variable of it is replaced with an element of $A^{(1)}$ since h_0 is applied for both variables of f_{10} . Thus all items in $a_{T_\lambda} b_{T_\lambda} f$ where variables from these two groups are mixed, vanish.

Therefore, if an item in a_{T_λ} replaces a variable from the first two columns with a variable with a different value from the tableau τ , we will have too many elements from $A^{(1)}$ substituted for the variables of f_1 and f_2 and the result is zero in virtue of the action of h_0 . Therefore all items in $a_{T_\lambda} b_{T_\lambda} f$ where variables from the first two columns having different values are mixed, vanish. We continue this procedure and finally show that if an item in a_{T_λ} does not stabilize the sets of variables with the same values from the tableau τ , the corresponding item in $a_{T_\lambda} b_{T_\lambda} f$ vanishes. Hence the value of $a_{T_\lambda} b_{T_\lambda} f$ is a nonzero multiple of the value of $b_{T_\lambda} f$, i.e. is nonzero. The lemma is proved. \square

Proof of Theorem 3. We use Lemmas 5 and 6. \square

9. A T_2 -graded algebra with a non-integer graded PI-exponent

Theorem 4. Let $A_2 = M_2(F) \oplus Fj_{11} \oplus Fj_{12} \oplus Fj_{22}$ (direct sum of ideals) where F is a field of characteristic 0 and $j_{11}^2 = j_{12}^2 = j_{22}^2 = 0$. Define a T_2 -grading on A_2 by $A_2^{(0)} = (M_2(F), 0)$, $A_2^{(v)} = \langle (e_{11}, j_{11}), (e_{12}, j_{12}), (e_{22}, j_{22}) \rangle$. Then there exists

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T_2\text{-gr}}(A_2)} = 4 + 2\sqrt{2} = 6.8284 \dots$$

Proof. Let A be the algebra from [Theorem 3](#). Define a linear isomorphism $\psi: A \xrightarrow{\sim} A_2$ by $\psi(e_{ij}, e_{kl}) = (e_{ij}, j_{kl})$ and $\psi(e_{ij}, 0) = (e_{ij}, 0)$. Then $\psi(A^{(0)}) = A_2^{(0)}$ and $\psi(A^{(1)}) = A_2^{(v)}$. Define an isomorphism $\Theta: F\langle X^{T_1\text{-gr}} \rangle \rightarrow F\langle X^{T_2\text{-gr}} \rangle$ of algebras by $\Theta(x_i^{(0)}) = x_i^{(0)}$ and $\Theta(x_i^{(1)}) = x_i^{(v)}$. We claim that

$$\Theta(\text{Id}^{T_1\text{-gr}}(A)) = \text{Id}^{T_2\text{-gr}}(A_2). \quad (6)$$

Since F is of characteristic 0, both $\text{Id}^{T_1\text{-gr}}(A)$ and $\text{Id}^{T_2\text{-gr}}(A_2)$ are generated by multilinear polynomials. (The proof is completely analogous to [\[6, Theorem 1.3.8\]](#).) In other words, in order to prove (6), it is sufficient to show that if $f \in F\langle X^{T_1\text{-gr}} \rangle$ is multilinear as an ordinary polynomial in variables $x_1^{(t_1)}, x_2^{(t_2)}, \dots, x_n^{(t_n)}$ where $t_i \in T_1$, then $f \in \text{Id}^{T_1\text{-gr}}(A)$ if and only if $\Theta(f) \in \text{Id}^{T_2\text{-gr}}(A_2)$. We substitute only homogeneous elements. Note that $\pi\psi(a) = \pi(a)$ where π is the projection on the first component: $\pi(a, b) = a$ for all $(a, b) \in A$ and $(a, b) \in A_2$. Moreover, if $a \in A^{(0)} \cup A^{(1)}$, then $\pi(a) = 0$ if and only if $a = 0$. Since T_1 and T_2 are commutative, the value of f under the substitution of homogeneous elements is again a homogeneous element. Applying π , we show that $f \in P_n^{T_1\text{-gr}}$ vanishes under the substitution of homogeneous elements $a_i^{(t_i)} \in A^{(t_i)}$, $t_i \in T_1$, if and only if $\Theta(f)$ vanishes under the substitution of $\psi(a_i^{(\tilde{t}_i)}) \in A_2^{(\tilde{t}_i)}$. (Here $\tilde{0} = 0$ and $\tilde{1} = v$.) Hence (6) holds and

$$c_n^{T_1\text{-gr}}(A) = c_n^{T_2\text{-gr}}(A_2) \text{ for all } n \in \mathbb{N}.$$

Now we apply [Theorem 3](#). \square

10. A T_3 -graded algebra with a non-integer graded PI-exponent

Theorem 5. *Let F be a field of characteristic 0. Denote by I the irreducible left $M_2(F)$ -module isomorphic to the minimal left ideal $\langle e_{12}, e_{22} \rangle_F \subset M_2(F)$. Let $A = M_2(F) \oplus I$ (direct sum of left ideals) where $IM_2(F) := 0$ and $I^2 := 0$. Define a T_3 -grading on A by $A^{(e_1)} = (M_2(F), 0)$, $A^{(e_2)} = \{(\varphi(a), a) \mid a \in I\}$ where $\varphi: I \hookrightarrow M_2(F)$ is the natural embedding which is a homomorphism of $M_2(F)$ -modules. Then there exists*

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T_3\text{-gr}}(A)} = 3 + 2\sqrt{2} = 5.8284 \dots$$

[Theorem 5](#) is proved at the end of the section.

Remark. Similarly, $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^{T_3^{\text{op}}\text{-gr}}(A^{\text{op}})} = 3 + 2\sqrt{2} = 5.8284 \dots$

Remark. Note that A does not contain unity. If a T_3 -graded algebra is unital, its graded PI-exponent is integer. (See [Theorem 7](#) below.)

Remark. The algebra A is T_3 -graded-simple (see [Proposition 6](#) below), however $3 + 2\sqrt{2} = \text{PIexp}^{T_3\text{-gr}}(A) < \dim A = 6$ even if F is algebraically closed.

Proposition 6. *The algebra A is T_3 -graded-simple.*

Proof. First, we notice that $J(A) = I$. Suppose $W \neq 0$ is a graded ideal of A . Then there exists a nonzero homogeneous element $(a_1, b_1) \in W$ where $a_1 \in M_2(F)$ and $b_1 \in I$. Since I does not contain homogeneous elements, we have $a_1 \neq 0$. However $(a_1, b_1)(E, 0) = (a_1, 0) \in (M_2(F), 0) \cap W$. Since W is a two sided ideal and $M_2(F)$ is simple, we get $(M_2(F), 0) \subseteq W$. Furthermore $(0, I) = (M_k(F), 0)(0, I) \subseteq W$. Thus $W = A$, and A is T_3 -graded-simple. \square

To prove [Theorem 5](#), we need [Lemma 7](#) which is an analog of [Lemma 6](#). We omit φ for shortness and write (e_{ij}, e_{ij}) instead of $(e_{ij}, \varphi^{-1}(e_{ij}))$.

Lemma 7. *Let $\lambda \vdash n$, $n \in \mathbb{N}$, $\lambda_7 = 0$, and $\lambda_5 + \lambda_6 \leq \lambda_1$. Then $m(A, (FT_3)^*, \lambda) \neq 0$.*

Proof. It is sufficient to show that for some $f \in P_n$ and some T_λ we have $e_{T_\lambda} f \neq 0$ on A .

Let $\beta_2 = \lambda_5 - \lambda_6$. Fix numbers $\beta_3, \dots, \beta_{10} \geq 0$ such that $\beta_3 + \beta_5 + \beta_7 + \beta_9 = \lambda_6$, $\beta_3 + \beta_4 = \lambda_4 - \lambda_5$, $\beta_5 + \beta_6 = \lambda_3 - \lambda_4$, $\beta_7 + \beta_8 = \lambda_2 - \lambda_3$, and $\beta_9 + \beta_{10} = \lambda_1 - \lambda_2$. In other words, we have

$$D_\lambda = \begin{array}{cccccccccccc} \lambda_6 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & & & & \\ \hline \dots & \dots & \dots & \dots & & & & & & \\ \hline \dots & \dots & & & & & & & & \\ \hline \dots & & & & & & & & & \\ \hline \end{array} & . \end{array}$$

(Here β_i denotes the number of columns in each block of columns.)

We fix some Young tableau T_λ of the shape λ filled in with the numbers from 1 to n . Like in [Lemma 6](#), for each column of T_λ we define a multilinear alternating polynomial depending on the variables with the indexes from the column. For shortness, we denote the polynomials corresponding to different columns in the i th block by the same letter f_i . By (i_1, \dots, i_ℓ) we denote the ℓ -tuple of numbers from a column (from up to down). By $S\{i_1, \dots, i_\ell\}$ we denote the symmetric group on i_1, \dots, i_ℓ . We define

$$\begin{aligned} f_1 &:= \sum_{\sigma \in S\{i_1, \dots, i_6\}} (\text{sign } \sigma) x_{\sigma(i_3)}^{h_{e_1}} x_{\sigma(i_5)}^{h_{e_2}} x_{\sigma(i_4)}^{h_{e_1}} x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_6)}^{h_{e_1}}, \\ f_2 &:= \sum_{\sigma \in S\{i_1, \dots, i_5\}} (\text{sign } \sigma) x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_3)}^{h_{e_1}} x_{\sigma(i_5)}^{h_{e_2}} x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_4)}^{h_{e_1}}, \\ f_3 &:= \sum_{\sigma \in S\{i_1, \dots, i_4\}} (\text{sign } \sigma) x_{\sigma(i_4)}^{h_{e_1}} x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_3)}^{h_{e_1}}, \end{aligned}$$

	λ_7	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	$(e_{21}, 0)$	(e_{22}, e_{22})
	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	(e_{22}, e_{22})	
$\tau =$	$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	$(e_{11}, 0)$	(e_{12}, e_{12})				
	$(e_{22}, 0)$	$(e_{22}, 0)$	$(e_{22}, 0)$	(e_{12}, e_{12})						
	(e_{12}, e_{12})	(e_{12}, e_{12})								
	$(e_{12}, 0)$									

(Here in the i th block we have β_i columns with the same values in all cells of a row. For shortness, we depict each value for each block only once. The tableau τ is still of the shape λ .)

Fig. 2. Substitution for the variables of $e_{T_\lambda} f$, grading semigroup T_3 .

$$f_4 := \sum_{\sigma \in S\{i_1, \dots, i_4\}} (\text{sign } \sigma) x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_3)}^{h_{e_1}} x_{\sigma(i_4)}^{h_{e_2}} x_{\sigma(i_2)}^{h_{e_2}},$$

$$f_5 := \sum_{\sigma \in S\{i_1, i_2, i_3\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_3)}^{h_{e_1}}, \quad f_6 := \sum_{\sigma \in S\{i_1, i_2, i_3\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_3)}^{h_{e_2}},$$

$$f_7 := \sum_{\sigma \in S\{i_1, i_2\}} (\text{sign } \sigma) x_{\sigma(i_2)}^{h_{e_2}} x_{\sigma(i_1)}^{h_{e_1}}, \quad f_8 := \sum_{\sigma \in S\{i_1, i_2\}} (\text{sign } \sigma) x_{\sigma(i_1)}^{h_{e_1}} x_{\sigma(i_2)}^{h_{e_1}},$$

$$f_9 := x_{i_1}^{h_{e_1}}, \quad f_{10} := x_{i_1}^{h_{e_2}}.$$

Define the polynomial

$$f = (f_3 f_1)^{\beta_3} (f_5 f_1)^{\beta_5} (f_7 f_1)^{\beta_7} (f_9 f_1)^{\beta_9} f_2^{\beta_2} f_4^{\beta_4} f_6^{\beta_6} f_8^{\beta_8} f_{10}^{\beta_{10}} \in P_n.$$

As we have already mentioned, here different copies of f_i depend on different variables.

The copies of f_1 are alternating polynomials of degree 6 corresponding to the first λ_6 columns of height 6.

The copies of f_2 are alternating polynomials of degree 5 corresponding to the next β_2 columns of height 5.

...

The copies of f_{10} are polynomials of degree 1 with the indexes from the last β_{10} columns of height 1.

We claim that $e_{T_\lambda} f \neq 0$. In order to verify this, we fill D_λ with specific homogeneous elements and denote the tableau obtained by τ . (See Fig. 2.)

Now for each variable we substitute the element from the corresponding box in τ . Note that f does not vanish under this substitution.

Recall that $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ where a_{T_λ} is the symmetrization in the variables of each row and b_{T_λ} is the alternation in the variables of each column. Since all f_i are alternating polynomials, $b_{T_\lambda} f$ is a nonzero multiple of f .

Two sets of variables correspond to the second row of T_λ . For the variables of the first group we substitute $(e_{22}, e_{22}) \in A^{(e_2)}$, for the second one, we substitute $(e_{12}, 0) \in A^{(e_1)}$. Thus if an item in a_{T_λ} mixes variables from these two groups, at least one variable from the second group, i.e. in f_8 , is replaced with a variable from the first one. However, f_8 vanishes if at least one variable of it is replaced with an element of $A^{(e_2)}$ since h_{e_1} is

applied for both variables of f_8 . Thus all items in $a_{T_\lambda} b_{T_\lambda} f$ where variables from these two groups are mixed, vanish.

Therefore, if an item in a_{T_λ} replaces a variable from the first three columns with a variable with a different value from the tableau τ , we have too many elements from $A^{(e_2)}$ substituted for the variables of f_1 , f_2 , and f_3 and the result is zero in virtue of the action of h_{e_1} . Therefore all items in $a_{T_\lambda} b_{T_\lambda} f$ where variables from the first three columns having different values are mixed, vanish. We continue this procedure and finally show that if an item in a_{T_λ} does not stabilize the sets of variables with the same values from the tableau τ , the corresponding item in $a_{T_\lambda} b_{T_\lambda} f$ vanishes. Hence the value of $a_{T_\lambda} b_{T_\lambda} f$ is a nonzero multiple of the value of $b_{T_\lambda} f$, i.e. is nonzero. The lemma is proved. \square

Proof of Theorem 5. We use [Lemmas 5 and 7](#). \square

11. Positive results on the analog of Amitsur's conjecture for polynomial T -graded identities

Theorem 6. *Let A be a finite dimensional non-nilpotent T -graded associative algebra with 1 over a field F of characteristic 0 for some cancellative semigroup T . Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$, such that*

$$C_1 n^{r_1} d^n \leq c_n^{T\text{-gr}}(A) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$

Corollary. *The graded analog of Amitsur's conjecture holds for such codimensions.*

Proof of Theorem 6. Note that graded codimensions do not change upon an extension of the base field. The proof is analogous to the case of ordinary codimensions [[6, Theorem 4.1.9](#)]. Hence we may assume F to be algebraically closed.

By [[11, Corollary 4.1](#)], $J(A)$ is a graded ideal. By [Proposition 5](#), $A/J(A)$ is the sum of graded ideals that are T -graded-simple algebras. Now we apply [Theorem 2](#). \square

Theorem 7. *Let A be a finite dimensional non-nilpotent T -graded associative algebra with 1 over a field F of characteristic 0 for some left or right zero band T . Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, such that*

$$C_1 n^{r_1} d^n \leq c_n^{T\text{-gr}}(A) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}$$

where $d = \text{PIexp}(A)$ is the ordinary PI-exponent of A .

Corollary. *The graded analog of Amitsur's conjecture holds for such codimensions.*

Proof of Theorem 7. Since A is finite dimensional, we may assume that T is finite. Again, without loss of generality, we may assume F to be algebraically closed.

By Propositions 2 and 4, the Jacobson radical of A is a graded ideal and $A/J(A)$ is the sum of graded ideals that are simple algebras. Therefore, by Theorem 2 there exists an integer $\text{PIexp}^{T\text{-gr}}(A)$. By Theorem 1, we can choose a graded maximal semisimple subalgebra B such that $A = B \oplus J(A)$ (direct sum of graded subspaces). Then we may define an embedding $\varkappa: A/J \xrightarrow{\sim} B$ (see Theorem 2) to be graded. Let $B = B_1 \oplus \dots \oplus B_s$ (direct sum of ideals) for some simple algebras B_i . Then by Proposition 2 the ideals B_i are graded, $A/J = \varkappa^{-1}(B_1) \oplus \dots \oplus \varkappa^{-1}(B_s)$ (direct sum of graded ideals), and

$$\begin{aligned} \text{PIexp}^{T\text{-gr}}(A) &= \max \dim \left(B_{i_1} \oplus B_{i_2} \oplus \dots \oplus B_{i_r} \mid r \geq 1, \right. \\ &\quad \left. ((FT)^* B_{i_1}) A^+ ((FT)^* B_{i_2}) A^+ \dots ((FT)^* B_{i_{r-1}}) A^+ ((FT)^* B_{i_r}) \neq 0 \right) = \\ &= \max \dim \left(B_{i_1} \oplus B_{i_2} \oplus \dots \oplus B_{i_r} \mid r \geq 1, B_{i_1} A^+ B_{i_2} A^+ \dots B_{i_{r-1}} A^+ B_{i_r} \neq 0 \right) = \\ &= \max \dim \left(B_{i_1} \oplus B_{i_2} \oplus \dots \oplus B_{i_r} \mid r \geq 1, \right. \\ &\quad \left. B_{i_1} J(A) B_{i_2} J(A) \dots B_{i_{r-1}} J(A) B_{i_r} \neq 0 \right) = \text{PIexp}(A) \end{aligned}$$

since B_i are simple as ordinary algebras. \square

Remark. The equality $\text{PIexp}^{T\text{-gr}}(A) = \text{PIexp}(A)$ does not imply the equality of codimensions. Indeed, let $k \in \mathbb{N}$, $k \geq 2$, $A = M_k(F)$, $T = \{t_1, \dots, t_k\}$ where $t_i t_\ell = t_\ell$ for all $1 \leq i, \ell \leq k$, $A^{(t_i)} = \langle e_{1i}, \dots, e_{ki} \rangle_F$. Then $x_1^{(t_i)}$ are linearly independent modulo $\text{Id}^{T\text{-gr}}(A)$. In order to check this, it is sufficient to substitute $x_1^{(t_i)} = e_{ii}$. Thus $c_1(A) = 1 < c_1^{T\text{-gr}}(A) = k$.

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