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The gonality and the Clifford index of curves on a toric surface



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ABSTRACT

We determine the gonality and the Clifford index for curves on a compact smooth toric surface. Moreover, it is shown that their gonality is computed by pencils on the ambient surface. From the geometrical viewpoint, this means that the gonality can be read off from the lattice polygon associated to the curve.

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1. Introduction

In this paper, a *curve* always means a smooth irreducible projective curve over the complex number field unless otherwise mentioned. The *gonality* and the *Clifford index* are significant invariants in the study of linear systems on curves, which are defined by

$$\begin{aligned} \text{gon}(C) &= \min\{\deg f \mid f : C \rightarrow \mathbb{P}^1 \text{ surjective morphism}\} = \min\{k \mid C \text{ has } g_k^1\}, \\ \text{Cliff}(C) &= \min\{\deg D - 2h^0(D) + 2 \mid D : \text{divisor on } C, h^0(D) \geq 2, h^1(D) \geq 2\} \end{aligned}$$

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for a curve C . A curve of gonality k is said to be k -gonal, and a pencil on a curve is called a *gonality pencil* if its degree is equal to the gonality. We cite several basic facts about the gonality and the Clifford index. Clearly, $\text{gon}(C) = 1$ means that C is rational. The three statements $\text{gon}(C) = 2$, $\text{Cliff}(C) = 0$ and C is elliptic or hyperelliptic are equivalent. The equality $\text{Cliff}(C) = 1$ holds if and only if C is trigonal or a smooth plane quintic curve. On the other hand, Brill–Noether theory gives upper bounds $\text{gon}(C) \leq \lfloor (g+3)/2 \rfloor$ and $\text{Cliff}(C) \leq \lfloor (g-1)/2 \rfloor$ for a curve of genus g , and equalities hold if the curve is general in moduli (cf. [1,2]). Lastly, we mention a close relation between these two invariants: $\text{gon}(C) - 3 \leq \text{Cliff}(C) \leq \text{gon}(C) - 2$ (cf. [7]).

Although a considerable amount of work has revealed properties of the gonality and the Clifford index, it is still not easy to determine them for a given curve. Ideally, we would also like to see what kind of gonality pencils does a curve have. In fact, however, it is difficult even to know whether the number of gonality pencils is finite or infinite. There are only two systematic results giving satisfactory answers for these questions: the cases of plane curves and curves on Hirzebruch surfaces (Theorems 1.1 and 1.2). In this paper, we will study more general cases. Concretely, we consider curves on a compact smooth toric surface, and compute the gonality and the Clifford index (Theorem 1.3). From the point of view of the geometry of convex bodies, our result states that the gonality of such a curve coincides with the lattice width (see Definition 3.2) of the lattice polygon associated to the curve. Namely, we can read off the gonality by observing the shape of the lattice polygon. This fact has been conjectured by Castryck and Cools in [3,4], and our result gives an affirmative answer for it except for the case of plane curves. In addition, Theorem 1.3 tells us that apart from a few exceptional cases, a curve on a toric surface has a finite number of gonality pencils, and moreover, they become restrictions of preassigned \mathbb{P}^1 -fibrations of the surface called toric fibrations (see below for definition). On the other hand, in the process to prove the main result, we also obtain the lower bound for the self-intersection number of a curve on a toric surface (Corollary 3.8). This formula by itself is suggestive and of wide application, although which is just a tool in this paper.

Before we state the main theorem, let us review the cases of curves on the projective plane and Hirzebruch surfaces. First, the gonality and the Clifford index of plane curves are computed by the following formula.

Theorem 1.1. (See [14,8].) *Let C be a smooth plane curve of degree d . If $d \geq 2$, then $\text{gon}(C) = d - 1$ and any gonality pencil is cut out by lines passing through a fixed point on C . Furthermore, if $d \geq 5$, then $\text{Cliff}(C) = d - 4$.*

Next, let Σ_e be a Hirzebruch surface of degree e , and $\pi : \Sigma_e \rightarrow \mathbb{P}^1$ be the ruling of Σ_e . Note that if $e = 0$, then Σ_0 has another ruling π' to \mathbb{P}^1 whose fiber is a section of π . In this case, we may assume that $\deg \pi|_C \leq \deg \pi'|_C$. For curves on Hirzebruch surfaces, Martens has computed the gonality.

Theorem 1.2. (See [13].) *Let C be a smooth curve on Σ_e which is not isomorphic to a smooth plane curve. Then $\text{gon}(C) = \deg \pi|_C$. In the case where $e \geq 1$, or $e = 0$ and*

$\deg \pi|_C < \deg \pi'|_C$, $\pi|_C$ is a unique gonality pencil on C . In the case where $e = 0$ and $\deg \pi|_C = \deg \pi'|_C$, C has exactly two gonality pencils $\pi|_C$ and $\pi'|_C$.

In the case of [Theorem 1.2](#), since the set of gonality pencils is finite, we obtain $\text{Cliff}(C) = \text{gon}(C) - 2$ (cf. [\[7\]](#)). Here we recall that the projective plane and Hirzebruch surfaces are simplest examples of toric surfaces. Hence, as a natural continuation of the above results, we expect to determine the gonality and gonality pencils of a curve on a toric surface. In order to give a precise statement, we recall some terminology. Let S be a compact smooth toric surface. Then S contains an algebraic torus T as a nonempty Zariski open set together with an action of T on S , which is a natural extension of the torus action of T on itself. A prime divisor on S is called a *T -invariant divisor* if it is invariant with respect to the above action. Any T -invariant divisor is isomorphic to the projective line. A blowing-down of a T -invariant divisor gives a morphism from S to another toric surface. We call a composition of such morphisms an *equivariant morphism*. It is known that if S is not a projective plane, there exist a finite number of equivariant morphisms from S to Hirzebruch surfaces. Hence, by composing such equivariant morphisms and the rulings of Hirzebruch surfaces, we obtain a finite number of \mathbb{P}^1 -fibrations of S . We call them *toric fibrations*. Now, we state the main theorem of this paper.

Theorem 1.3. *Let S be a compact smooth toric surface, and C be a k -gonal nef curve of genus $g \geq 2$ on S which is not isomorphic to a smooth plane curve. Put $q = \min\{\deg \varphi|_C \mid \varphi : \text{toric fibration of } S\}$. Then q is equal to the lattice width (see [Definition 3.2](#)) of the lattice polygon associated to C , and the following hold.*

- (i) *If $(g, q) \neq (4, 4), (5, 4), (10, 6)$, then any gonality pencil on C is the restriction of a toric fibration of S .*
- (ii) *The equalities $k = \begin{cases} q & ((g, q) \neq (4, 4)) \\ q - 1 & ((g, q) = (4, 4)) \end{cases}$ hold.*
- (iii) *If $(g, q) \neq (10, 6)$, then $\text{Cliff}(C) = k - 2$.*
- (iv) *If $(g, q) = (10, 6)$, then C is a complete intersection of two hypercubics in \mathbb{P}^3 .*

In the case $(g, q) = (4, 4)$, since C is trigonal by (ii), we see that C has one or two gonality pencils. In [Section 5](#), we will show that both cases can occur. In the case $(g, q) = (5, 4)$, the gonality of C achieves the maximum of the upper bound $\text{gon}(C) \leq \lfloor (g + 3)/2 \rfloor$. It follows that C has infinitely many gonality pencils. If $(g, q) = (10, 6)$, by virtue of (iv) and Martens' work [\[12\]](#), we see that $\text{gon}(C) = 6$, $\text{Cliff}(C) = 3$ and C has infinitely many gonality pencils. By a simple consideration, we can rewrite [Theorem 1.3](#) as follows:

Corollary 1.4. *Let S and C be as in [Theorem 1.3](#).*

- (i) *If $(g, k) \neq (4, 3), (5, 4), (10, 6)$, then any gonality pencil on C is the restriction of a toric fibration of S .*

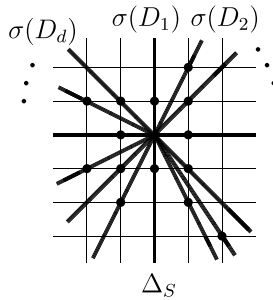


Fig. 1. The fan Δ_S associated to S .

- (ii) If $(g, k) \neq (4, 3)$, then $k = q$.
- (iii) If $(g, k) \neq (10, 6)$, then $\text{Cliff}(C) = k - 2$.
- (iv) If $(g, k) = (10, 6)$, then C is a complete intersection of two hypercubics in \mathbb{P}^3 .

In Section 2, we review the theory of toric surfaces, which is the main stage of our study. The aim of Section 3 is to reveal several properties of the self-intersection number of a curve on a toric surface, which will be utilized to prove the key proposition (Proposition 4.3) and Theorem 1.3 in Section 4. Most proofs in Section 3, however, are just elementary and tedious computations for convex polygons in the affine plane. Hence the reader can skip them without losing the continuity of the paper. In Section 5, as already mentioned after Theorem 1.3, we investigate trigonal curves of genus four. Finally, as an application of our results, we compute Weierstrass gap sequences at ramification points of a trigonal covering of \mathbb{P}^1 in Section 6. In fact, gap sequences at such points are well studied and the classification of them has been already completed [5,6,9,11]. However, by combining Corollary 1.4 with results in [10], we can compute gap sequences in a completely different way and propose a novel geometric interpretation of the reason why the difference between types of gap sequences occurs. This approach can be adapted not only to trigonal curves but also to curves of higher gonality. Hence, as a generalization of the results in Section 6, it is expected that we can classify Weierstrass gap sequences at ramification points of gonality pencils on a curve on a toric surface in the future.

2. Fans and lattice polygons

In this section, we briefly review basic notions in the theory of toric surfaces. For further background and applications of them, we refer the reader to [15] without explicit mention. We henceforth assume that a surface is always compact and smooth.

For a toric surface S , there exists a fan Δ_S , which is the division of \mathbb{R}^2 consisting of a finite number of half-lines starting from the origin called cones (see Fig. 1). Each cone $\sigma(D_i)$ corresponds to a T -invariant divisor D_i , and the lattice point on $\sigma(D_i)$ closest to the origin is called the primitive element. We denote by d the number of cones in Δ_S , by (x_i, y_i) the primitive element of $\sigma(D_i)$, and by $\text{Pr}(S)$ the set consisting of

primitive elements of cones in Δ_S . We assume that $(x_1, y_1) = (0, 1)$. The smoothness of S is equivalent to that $x_{i+1}y_i - y_{i+1}x_i = 1$ hold for $i = 1, \dots, d$, where we formally set $D_{d+1} = D_1$. The Picard group of S is generated by the classes of D_1, \dots, D_d . For instance, the canonical divisor of S can be written as $K_S \sim -\sum_{i=1}^d D_i$. We next define a lattice polygon associated to a divisor on S , which is the essential notion in the study of curves on a toric surface.

Definition 2.1. For a divisor $D = \sum_{i=1}^d n_i D_i$ on S , a *lattice polygon* associated to D is defined by $\square_D = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w \leq n_i \text{ for } 1 \leq i \leq d\}$.

Lastly, we mention the structures of fibers of toric fibrations. We define $M(u, v) = \{(x, y) \in \text{Pr}(S) \mid uy - vx < 0\}$ for integers u and v , and $\text{Pr}^*(S) = \{(x, y) \in \text{Pr}(S) \mid (-x, -y) \in \text{Pr}(S)\}$.

Fact 2.2. For any primitive elements (x_i, y_i) and (x_j, y_j) , we can uniquely write $(x_j, y_j) = \alpha_j(x_i, y_i) + \beta_j(x_{i+1}, y_{i+1})$ with some integers α_j and β_j . We can describe fibers of toric fibrations as follows:

$$\{\text{fibers of toric fibrations of } S\} = \left\{ \sum_{(x_j, y_j) \in M(x_i, y_i)} \beta_j D_j \mid (x_i, y_i) \in \text{Pr}^*(S) \right\}.$$

3. The lower bound for the self-intersection number

In this section, we will find the evaluation formula for the self-intersection number of a curve on a toric surface for later use in the proof of the key proposition ([Proposition 4.3](#)). First, we extend the notion of coprime.

Definition 3.1. For nonnegative integers x and y , we say $(x, y) = 1$ if they satisfy either of the following conditions:

- (i) x and y are positive coprime integers.
- (ii) $(x, y) = (1, 0)$ or $(0, 1)$.

Definition 3.2. Let D be a divisor on a toric surface, and x and y be integers with $(|x|, |y|) = 1$. We denote by $n(x, y)$ the minimal integer n such that $\{(z, w) \mid xz + yw \leq n\} \supset \square_D$, and define

$$\begin{aligned} l(D, (x, y)) &= \{(z, w) \in \mathbb{R}^2 \mid xz + yw = n(x, y)\}, \\ L(D, (x, y)) &= \{(z, w) \in \mathbb{R}^2 \mid xz + yw \leq n(x, y)\}, \\ d(D, (x, y)) &= n(x, y) + n(-x, -y). \end{aligned}$$

In particular, we call $\min\{d(D, (x, y)) \mid x, y \in \mathbb{Z}, (|x|, |y|) = 1\}$ the *lattice width* of \square_D .

Remark 3.3. By definition, if $(z_1, w_1) \in L(D, (x, y))$ and $(z_2, w_2) \in L(D, (-x, -y))$, then $d(D, (x, y)) \geq x(z_1 - z_2) + y(w_1 - w_2)$. In addition, an easy computation shows $d(D, (x, y)) = \sum_{(x_j, y_j) \in M(x, y)} (x_j y - y_j x) D \cdot D_j$.

Let C be a curve on a toric surface S . By [Fact 2.2](#) and [Remark 3.3](#), we see that $q = \min\{d(C, (x_i, y_i)) \mid (x_i, y_i) \in \text{Pr}^*(S)\}$. Without loss of generality, we can assume that $(0, 1) \in \text{Pr}^*(S)$ and $d(C, (0, 1)) = q$. In the case where $\sharp\text{Pr}^*(S) = 2$, that is, Δ_S has only one line passing through the origin, we can assume that $(1, 0) \in \text{Pr}(S)$ and $(x, y) \notin \text{Pr}(S)$ if $y > \min\{0, x\}$. On the other hand, in the case where $\sharp\text{Pr}^*(S) \geq 4$, we can assume that $(1, 0) \in \text{Pr}^*(S)$ and $d(C, (1, y)) \geq d(C, (1, 0))$ for any $(1, y) \in \text{Pr}^*(S)$. In this paper, we will keep the above assumptions for C and Δ_S , and put $q' = d(C, (1, 0))$.

Lemma 3.4. *Let C be a nef curve on a toric surface S . For integers $x (\geq 1)$ and y with $(|x|, |y|) = 1$, the inequality $d(C, (x, y)) \geq q'$ holds.*

Proof. (i) Consider the case where $\sharp\text{Pr}^*(S) = 2$. We denote by P (resp. Q) the vertex of \square_C on $l(C, (0, 1))$ whose z -coordinate is minimal (resp. maximal). By a simple consideration, we see that $Q = P + (q', 0)$. Then by [Remark 3.3](#), we have $d(C, (x, y)) \geq xq'$.

(ii) In the case where $\sharp\text{Pr}^*(S) \geq 4$, we prove only the case where y is positive. One can show the case where y is negative by a similar method. We denote by n the maximal integer such that $(1, n) \in \text{Pr}^*(S)$ and $nx - y \leq 0$, and define

$$P_1 = (z_1, w_1) = \begin{cases} l(C, (1, n)) \cap l(C, (1, 0)) & (n \geq 1), \\ l(C, (0, -1)) \cap l(C, (1, 0)) & (n = 0), \end{cases}$$

$$P_2 = (z_2, w_2) = \begin{cases} l(C, (-1, -n)) \cap l(C, (-1, 0)) & (n \geq 1), \\ l(C, (0, 1)) \cap l(C, (-1, 0)) & (n = 0). \end{cases}$$

Note that $P_1 \in L(C, (x, y))$ and $P_2 \in L(C, (-x, -y))$. In the case where $n \geq 1$, since

$$d(C, (1, n)) = z_1 + nw_1 - z_2 - nw_2 = q' + n(w_1 - w_2) \geq q',$$

we have $w_1 \geq w_2$. Accordingly, we have $d(C, (x, y)) \geq x(z_1 - z_2) + y(w_1 - w_2) \geq xq'$. On the other hand, if $n = 0$, we have $d(C, (x, y)) \geq xq' - yq \geq (x - y)q'$ by [Remark 3.3](#). \square

Lemma 3.5. *For a nef curve C on a toric surface, there exists a nef curve C_0 on a toric surface having the following properties:*

- (i) $d(C_0, (0, 1)) = q$, $d(C_0, (1, 0)) = q'$ and $d(C_0, (1, \pm 1)) \geq q'$.
- (ii) $C_0^2 \leq C^2$.
- (iii) The lattice polygon \square_{C_0} is a triangle or a square, and moreover, each vertex is on one or two of the four lines $l(C_0, (0, \pm 1))$ and $l(C_0, (\pm 1, 0))$.

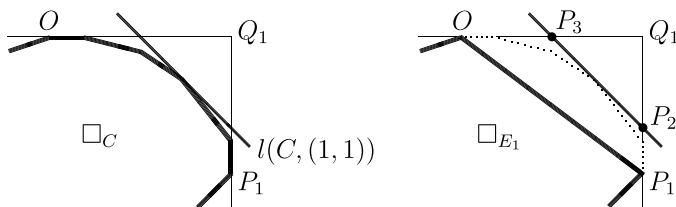


Fig. 2. The shapes of \square_C and \square_{E_1} .

- (iv) If $l(C_0, (0, 1))$ contains two distinct vertices, then they are $l(C_0, (0, 1)) \cap l(C_0, (1, 0))$ and $l(C_0, (0, 1)) \cap l(C_0, (-1, 0))$. A similar property holds for $l(C_0, (0, -1))$.
- (v) If $l(C_0, (1, 0))$ contains two distinct vertices, then at least one of them is $l(C_0, (1, 0)) \cap l(C_0, (0, 1))$ or $l(C_0, (1, 0)) \cap l(C_0, (0, -1))$. A similar property holds for $l(C_0, (-1, 0))$.

Proof. In this proof, we will gradually deform the polygon \square_C toward \square_{C_0} . In the process of this deformation, we construct five polygons \square_{C_i} ($i = 1, \dots, 5$). For simplicity, we abuse notation ‘the properties (i) and (ii)’ for these curves C_i , and move a polygon in parallel as necessary.

We set the vertex of \square_C on $l(C, (0, 1))$ whose z -coordinate is minimal as the origin O , and put $Q_1 = l(C, (0, 1)) \cap l(C, (1, 0))$. We define $\square_{C_1} = \square_C$ if \square_C contains Q_1 . If not, we denote by $P_1 = (z_1, w_1)$ the vertex of \square_C on $l(C, (1, 0))$ whose w -coordinate is minimal, and define a polygon \square_{C_1} by the following procedure. We first make a polygon \square_{E_1} from \square_C by connecting O to P_1 . We put $P_2 = (z_1, w_2) = l(C, (1, 1)) \cap l(C, (1, 0))$ and $P_3 = (z_3, 0) = l(C, (1, 1)) \cap l(C, (0, 1))$ (see Fig. 2). If $z_1 + w_1 \geq 0$ (resp. $z_1 + w_1 < 0$), we define \square_{C_1} as the convex hull of $\square_{E_1} \cup \{P_2\}$ (resp. $\square_{E_1} \cup \{P_3\}$). By definition and Lemma 3.4, C_1 clearly satisfies the property (i). Let us show the inequality $C_1^2 \leq C^2$ in the case where $Q_1 \notin \square_C$. If $z_1 + w_1 \geq 0$, we can take a nonnegative integer a such that the lattice point $P_4 = P_2 + a(-1, 1)$ is contained in $l(C, (1, 1)) \cap \square_C$. We denote by \square_{E_2} the convex hull of $\square_{E_1} \cup \{P_4\}$. Since $E_2^2 \leq C^2$, it is sufficient to verify $C_1^2 \leq E_2^2$. Note that the difference between C_1^2 and E_2^2 is caused only by the two sides P_2P_1 and P_4P_1 . Hence we obtain

$$C_1^2 - E_2^2 = (w_2 - w_1)z_1 - (w_2 + a - w_1)z_1 - aw_1 = -a(z_1 + w_1) \leq 0.$$

Similarly we can check $C_1^2 \leq C^2$ in the case where $z_1 + w_1 < 0$. We should keep in mind that if $z_1 + w_1 \geq 0$, then \square_{C_1} has only one vertex on $l(C_1, (0, 1))$. By adapting a similar operation to other three corners of \square_{C_1} , we make a polygon \square_{C_2} . It is obvious that C_2 satisfies the properties (i) and (ii). Besides, every vertex of \square_{C_2} is on one or two of the four lines $l(C_2, (0, \pm 1))$ and $l(C_2, (\pm 1, 0))$. We put $Q_4 = l(C, (-1, 0)) \cap l(C, (0, 1))$. Let us show that if $l(C_2, (0, 1))$ contains two distinct vertices of \square_{C_2} and one of them is Q_1 (resp. Q_4), then the other one is Q_4 (resp. Q_1). Assume that \square_{C_2} contains Q_1 but not Q_4 . Considering the method of making \square_{C_2} , we deduce that Q_1 is contained in \square_{C_1} also. We

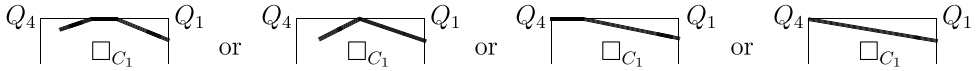


Fig. 3. The upper shape of \square_{C_1} .

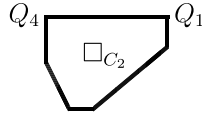


Fig. 4. \square_{C_2} in the case where $Q_1, Q_2 \in \square_{C_2}$.

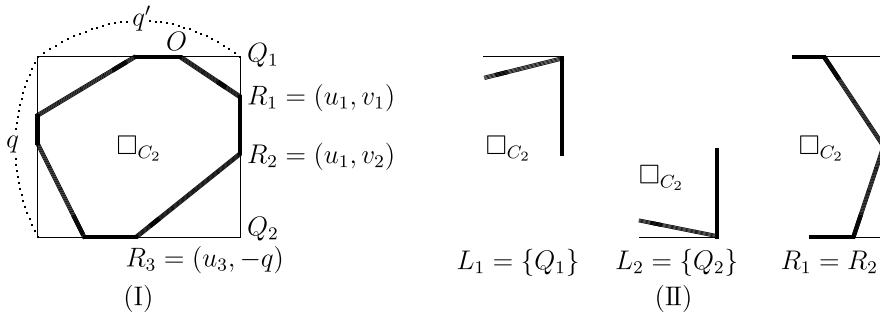


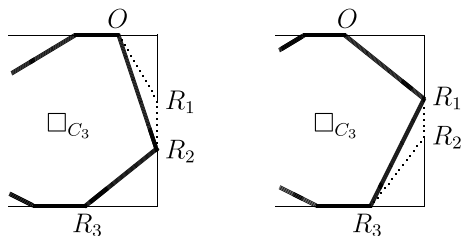
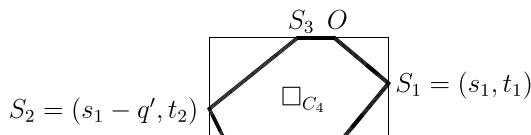
Fig. 5. \square_{C_2} in the case where L_1 (resp. L_2) is not the segment Q_1Q_4 (resp. Q_2Q_3).

denote by Q_0 the vertex of \square_{C_1} on $l(C_1, (-1, 0))$ whose w -coordinate is minimal. Since the slant of the segment Q_1Q_0 is at most one, \square_{C_2} has only one vertex Q_1 on $l(C_2, (0, 1))$. We next consider the case where \square_{C_2} contains Q_4 but not Q_1 . Since Q_1 is not contained in \square_{C_1} also, we obtain the four possibilities for the upper shape of \square_{C_1} as in Fig. 3. By the assumption $Q_4 \in \square_{C_2}$, the first two cases can be excluded. The third case does not occur. Indeed, since Q_4 must be contained in \square_C in this case, we have $z_1 + w_1 \geq 0$. It follows that \square_{C_1} does not have vertices on $l(C_1, (0, 1))$ except for Q_4 . Thus only the last case remains, in which \square_{C_2} has one vertex Q_4 on $l(C_2, (0, 1))$. Similarly, with respect to the points $Q_2 = l(C, (1, 0)) \cap l(C, (0, -1))$ and $Q_3 = l(C, (0, -1)) \cap l(C, (-1, 0))$, we can show that if $l(C_2, (0, -1))$ contains two distinct vertices of \square_{C_2} and one of them is Q_2 (resp. Q_3), then the other one is Q_3 (resp. Q_2).

(a) Consider the case where Q_1 and Q_4 are contained in \square_{C_2} . In this case, \square_{C_2} has at most six vertices (see Fig. 4), where some of the five sides (except Q_1Q_4) may not exist. We denote by Q the vertex of \square_{C_2} on $l(C_2, (0, -1))$ whose z -coordinate is minimal. Then we can finish the proof by defining \square_{C_0} as a triangle Q_1Q_4Q . Indeed, a simple computation shows that C_0 satisfies the properties (i) and (ii).

(b) An argument similar to that in (a) goes through for the case where Q_2 and Q_3 are contained in \square_{C_2} .

(c) We put $L_1 = l(C_2, (0, 1)) \cap \square_{C_2}$ and $L_2 = l(C_2, (0, -1)) \cap \square_{C_2}$, and consider the case where L_1 and L_2 are not the segments Q_1Q_4 and Q_2Q_3 , respectively. In this case, the polygon \square_{C_2} is as in Fig. 5 (I), where we set the point on L_1 whose z -coordinate

Fig. 6. The method of making \square_3 .Fig. 7. The shape of \square_{C_4} .Fig. 8. The upper shape of \square_{C_4} .

is maximal as the origin. Note that some of the eight sides may not exist. We make a polygon \square_{C_3} by the following procedure. If one of the equalities $L_1 = \{Q_1\}$, $L_2 = \{Q_2\}$ and $R_1 = R_2$ holds, we define $\square_{C_3} = \square_{C_2}$ (see Fig. 5 (II)). Assume that $L_1 \neq \{Q_1\}$, $L_2 \neq \{Q_2\}$ and $R_1 \neq R_2$. If $u_1 + v_1 \leq 0$ (resp. $u_1 + v_1 > 0$ and $u_1 - u_3 \leq v_2 + q$), we make \square_{C_3} from \square_{C_2} by connecting O to R_2 (resp. R_1 to R_3) as in Fig. 6. On the other hand, in the case where $u_1 + v_1 > 0$ and $u_1 - u_3 > v_2 + q$, we put $l(-1, -1) \cap l(-1, 0) = (u_1 - q', v_4)$. We note that $v_4 \leq v_1$ by the property (i) for \square_{C_2} . If $v_4 \leq v_2$ (resp. $v_4 > v_2$), we define $R_0 = R_2$ (resp. $R_0 = (u_1, v_4)$), and make \square_{C_3} from \square_{C_2} by connecting R_0 to O and R_3 . In each case, one can easily verify that C_3 satisfies the properties (i) and (ii). By applying a similar operation to the opposite side of \square_{C_3} , we can obtain a lattice polygon \square_{C_4} satisfying the properties (i) and (ii). There exist four types of the shape of \square_{C_4} as in Fig. 7, where we ignore the reflection about z -axis or w -axis or both. We remark that, in the first two cases, the vertical or horizontal sides may not exist. If \square_{C_4} is a triangle or a square, we can finish the proof by putting $C_0 = C_4$. In the case where \square_{C_4} has more than four vertices, we make a polygon \square_{C_5} by the following procedure (see Fig. 8). If $O = S_3$, we define $\square_{C_5} = \square_{C_4}$. In the other case, if $t_1 \geq t_2$ (resp. $t_1 < t_2$), we make \square_{C_5} from \square_{C_4} by connecting S_3 to S_1 (resp. O to S_2). Then an easy computation shows that \square_{C_5} satisfies the properties (i) and (ii). By applying a similar operation to the lower side of \square_{C_5} , we obtain the desired lattice polygon \square_{C_0} . \square

Remark 3.6. As it is apparent from the making method, the equality $C^2 = C_0^2$ holds if and only if $\square_C = \square_{C_0}$.

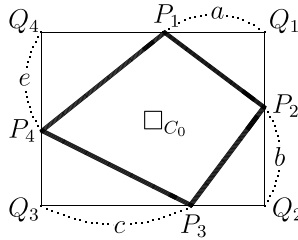


Fig. 9. The shape of \square_{C_0} .

Using Lemma 3.5, we can find the lower bound of the self-intersection number of C .

Proposition 3.7. *Let C be a nef curve on a toric surface, and C_0 be a curve as in Lemma 3.5. Then the inequality $C_0^2 \geq 3q^2/4$ (in particular, $C^2 \geq 3q^2/4$) holds.*

Proof. Recall that when we construct C_0 , we divided the situation into the three cases (a), (b) and (c) in the proof of Lemma 3.5. In the cases (a) and (b), we have $C_0^2 \geq qq' \geq q^2$ by an easy computation. Let us consider the case (c), that is, we assume that both $l(C_0, (0, 1))$ and $l(C_0, (0, -1))$ contain only one vertex. We keep the notation Q_1, \dots, Q_4 as in the proof of Lemma 3.5. Then the polygon \square_{C_0} is drawn as in Fig. 9, where we define

- P_1 : the vertex of \square_{C_0} on $l(C_0, (0, 1))$,
- P_2 : $\begin{cases} \text{the vertex on } l(C_0, (1, 0)) \setminus \{Q_1\} & (\square_{C_0} \text{ has two vertices on } l(C_0, (1, 0)) \setminus \{Q_2\}), \\ \text{the vertex on } l(C_0, (1, 0)) \text{ whose } w\text{-coordinate is maximal} & (\text{otherwise}), \end{cases}$
- P_3 : the vertex of \square_{C_0} on $l(C_0, (0, -1))$,
- P_4 : $\begin{cases} \text{the vertex on } l(C_0, (-1, 0)) \setminus \{Q_3\} & (\square_{C_0} \text{ has two vertices on } l(C_0, (-1, 0)) \setminus \{Q_4\}), \\ \text{the vertex on } l(C_0, (-1, 0)) \text{ whose } w\text{-coordinate is minimal} & (\text{otherwise}). \end{cases}$

Note that $(b, e) \neq (q, 0), (0, q)$. By computing, we obtain the formula

$$C_0^2 = qq' + (a + c - q')(b + e - q).$$

Without loss of generality, we can assume that $b + e \geq q$ and $(q' - a)b \geq (q' - c)e$.

(i) In the case where $a + e \geq q'$, we have $b + c \geq q'$. Indeed, if not, the inequality $eb \geq (q' - a)b \geq (q' - c)e$ gives that $e = 0$ and $b = q$, a contradiction. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point P_3 (resp. P_1), the condition $d(C_0, (1, -1)) \geq q'$ implies that $a + q - (q' - c) \geq q'$. Hence we have $C_0^2 \geq qq' + (q' - q)(b + e - q) \geq qq'$.

(ii) Assume that $a + e < q'$ and $b + c \leq q'$. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point P_2 (resp. P_4), we have $q' - e + q - b \geq q'$. It follows that $b + e = q$ and $C_0^2 = qq'$.

(iii) Assume that $a + e < q'$ and $b + c > q'$. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point P_3 (resp. P_4), we have $q' - e + q - (q' - c) \geq q'$. Hence we have

$$C_0^2 \geq qq' + (a + e - q)(b + e - q) = qq' + \left(e + \frac{a + b - 2q}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2, \quad (1)$$

where the equality holds if and only if $c - e = q' - q$ or $b + e = q$. If $a \geq b$ and $b + e = q$, we have $C^2 = qq'$. On the other hand, if $a \geq b$ and $b + e > q$, we have $a + e > q$ and $C_0^2 > qq'$ by the first inequality of (1). Lastly, if $a < b$, we have $0 < b - a \leq q$ and

$$C_0^2 \geq qq' - \frac{q^2}{4} \geq \frac{3}{4}q^2. \quad \square \quad (2)$$

Proposition 3.7 yields the following interesting corollary, though, which has no direct relation to the subject of this paper.

Corollary 3.8. *For a k -gonal nef curve C on a compact smooth toric surface, the inequality $C^2 \geq 3k^2/4$ holds.*

Considering an irredundant embedding of C , we can obtain a more precise lower bound for C^2 when q is small. Here, the ‘irredundancy’ has the following meaning: If there exists a T -invariant divisor D_i on S such that $D_i^2 = -1$ and $C \cdot D_i \leq 1$, then by blowing it down, we can embed C in another compact smooth toric surface. By carrying out such operation repeatedly, we obtain an embedding satisfying the following condition.

Definition 3.9. Let C be a smooth curve on S . The pair (S, C) (or simply the curve C) is said to be *relatively minimal* if $C \cdot D_i \geq 2$ for any T -invariant divisor D_i on S with self-intersection number -1 .

In the remaining part of this section, we set $O = l(C, (0, 1)) \cap l(C, (1, 0))$. Note that it coincides with the point $l(C_0, (0, 1)) \cap l(C_0, (1, 0))$.

Proposition 3.10. *Let C be a curve as in Theorem 1.3, and assume that (S, C) is relatively minimal. If $q = 2$ (resp. 3), then $C^2 \geq 12$ (resp. 18).*

Proof. We shall prove only the case $q = 3$. Considering the relative minimality of C , the left and right parts of \square_C must be composed of one side containing four lattice points. By using a suitable linear transformation, without loss of generality, we can assume that \square_C is a trapezium (possibly a triangle) as in Fig. 10, where a is a nonnegative integer with $a \geq 3m$. Then $C^2 = 3(2a - 3m)$ becomes less than eighteen only in the cases where $(a, m) = (0, -1), (1, -1), (3, 1), (4, 1)$. In each case, C is isomorphic to a plane curve or not relatively minimal. The case $q = 2$ can be proved similarly by noting $g \geq 2$. \square

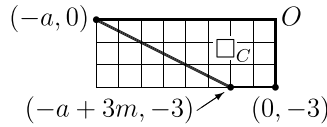


Fig. 10. \square_C in the case where $q = 3$.

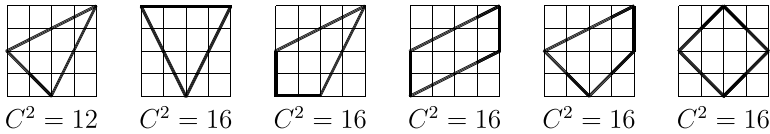


Fig. 11. \square_C in the case where $q = 4$ and $C^2 \leq 16$.

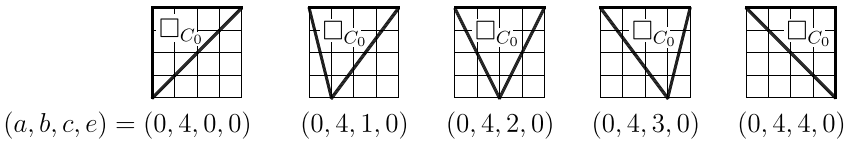


Fig. 12. \square_{C_0} in the case where $C_0^2 = 16$ in the case (a).

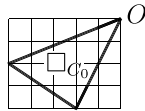


Fig. 13. \square_{C_0} in the case where $q' = 5$ and $C_0^2 = 16$ in the case (c).

Proposition 3.11. *Let C be a curve as in Theorem 1.3, and assume that (S, C) is relatively minimal. If $q = 4$ and $C^2 \leq 16$, the shape of \square_C is one of the six types in Fig. 11, provided that we ignore congruence relations.*

Proof. We take a curve C_0 as in Lemma 3.5. Recall the three cases (a), (b) and (c) in the proof of Lemma 3.5. In the case (a) (that is, the upper side of \square_{C_0} is a horizontal line of length q'), an easy computation gives $C_0^2 \geq 16$. Suppose that $C^2 = 16$. Then, since $C^2 = C_0^2 = 16$, we obtain the five possibilities for the shape of \square_{C_0} as in Fig. 12. Note that $\square_C = \square_{C_0}$ by Remark 3.6. The cases of $(0, 4, 0, 0)$ and $(0, 4, 4, 0)$ are excluded by the assumption that C is not isomorphic to a plane curve. On the other hand, the cases of $(0, 4, 1, 0)$ and $(0, 4, 3, 0)$ contradict the relative minimality of C . By a similar argument, one can show the proposition in the case (b).

Let us consider the case (c). We follow the idea in the proof of Proposition 3.7. Namely, we divide the situation into the three cases (i), (ii) and (iii). In the cases (i) and (ii), we have proved the inequality $C_0^2 \geq 16$. Moreover, if $C^2 = 16$ (that is, the equality holds in the previous inequality), then q' must be equal to four. In the case (iii), by the inequality (2), we have $q' \leq 5$ if $C_0^2 \leq 16$. Suppose that $q' = 5$ and $C^2 = 16$. Then we deduce that $c - e = q' - q$, $e = (2q - a - b)/2$ and $b - a = q$ by (1), that is, $(a, b, c, e) = (0, 4, 3, 2)$ (see Fig. 13). Note that $\square_C = \square_{C_0}$ by Remark 3.6. This contradicts the relative minimality of C . Hence it is sufficient to consider the case $q' = 4$.

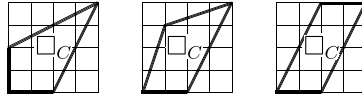


Fig. 14. \square_C in the case where $R_1, R_2 \in \square_C$ and $C^2 \leq 16$.

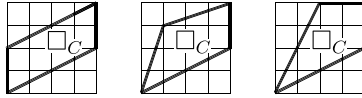


Fig. 15. \square_C in the case where $R_1 \notin \square_C, R_2 \in \square_C$ and $C^2 \leq 16$.

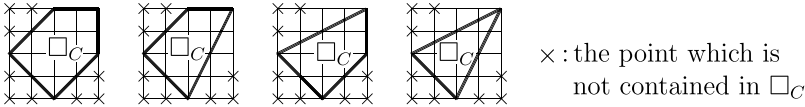


Fig. 16. \square_C in the case where $R_1 \in \square_C, R_2 \notin \square_C$ and $C^2 \leq 16$.

Let us examine the possibility of the shape of \square_C satisfying $q = q' = 4$ and $C^2 \leq 16$. Let P be a vertex of \square_C on $l(C, (0, 1))$. Note that $P \neq (-1, 0), (-3, 0)$. Indeed, if $(-1, 0)$ (resp. $(-3, 0)$) is contained in \square_C , then also O (resp. $(-4, 0)$) is contained in \square_C by the relative minimality. Then C cannot be a relatively minimal curve such that $d(C, (1, -1)) \geq 4$. Since the case $P = (-4, 0)$ is essentially equivalent to the case $P = O$, it is sufficient to consider the two cases $P = O, (-2, 0)$. Assume that $P = O$. In this case, we can assume that none of $(-3, 0), (-4, 0)$ and $(-4, -1)$ is contained in \square_C . Indeed, if not, $(-4, 0)$ is contained in \square_C by the relative minimality, which contradicts the assumption in (c). We can take a unique integer a with $-4 \leq a \leq 0$ such that the line $l(C, (1, -1))$ passes through $(0, a)$. Since $d(C, (1, -1)) \geq 4$, the cases where $a = 0, -1$ do not occur. If $a = -3$ or -4 , then by the relative minimality, the point $(0, -4)$ must be contained in \square_C . Hence, by an argument similar to that in the case (a), we see that \square_C is a triangle with vertices $O, (0, -4)$ and $(-4, -2)$. In the case $a = -2$, since $d(C, (1, -1)) \geq 4$, at least one of the points $(-2, 0), (-3, -1)$ and $(-4, -2)$ is contained in \square_C . We put $R_1 = (-2, -4)$ and $R_2 = (-4, -4)$. Assume that $R_1, R_2 \in \square_C$. Then, by computing, we obtain the three types of \square_C satisfying $C^2 \leq 16$ as in Fig. 14. By the relative minimality, the second type is excluded. Assume that $R_1 \notin \square_C$ and $R_2 \in \square_C$. In this case, by the relative minimality, the lower side of \square_C must be the segment connecting two points $(0, -2)$ and R_2 . Then, by computing, we obtain the three types of \square_C satisfying $C^2 \leq 16$ as in Fig. 15. By the relative minimality, the second type is excluded. Assume that $R_1 \in \square_C$ and $R_2 \notin \square_C$. The relative minimality implies that neither $(-3, -4)$ nor $(-4, -3)$ is contained in \square_C . Hence there exist the four possibilities for the shape of \square_C as in Fig. 16. In the first case, we have $C^2 = 20$. Lastly, we consider the case $P = (-2, 0)$. In order to avoid the duplication, we assume that \square_C contains none of the four corners $O, (0, -4), (-4, -4)$ and $(-4, 0)$. Then only one possibility remains: \square_C is a square with vertices $(-2, 0), (0, -2), (-2, -4)$ and $(-4, -2)$. \square

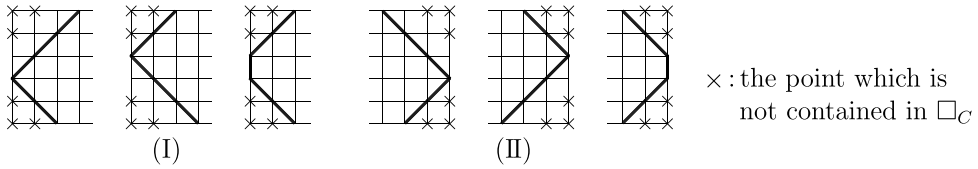


Fig. 17. \square_C in the case where $O, (0, -5), (-5, -5), (-5, 0) \notin \square_C$.

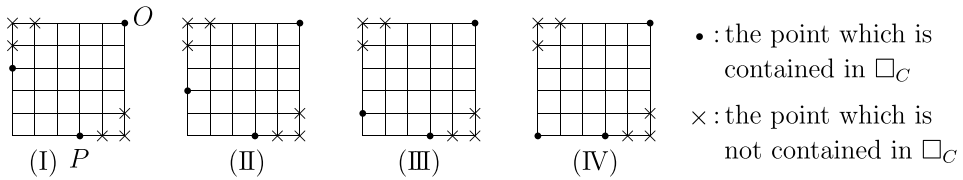


Fig. 18. The case where $O \in \square_C$ and $P = (-2, -5)$.

Proposition 3.12. *Let C be a curve as in Theorem 1.3, and assume that (S, C) is relatively minimal. If $q = 5$, then $C^2 \geq 25$.*

Proof. We take a curve C_0 as in Lemma 3.5. By the same argument as that in the proof of Proposition 3.11, we have $C_0^2 \geq 25$ except for the case (iii) in the proof of Proposition 3.7. Note that $q' \leq 6$, $c - e \geq q' - 5$ and $a < b$ if $C_0^2 \leq 24$. In the case $q' = 6$, we obtain the two possibilities $(a, b, c, e) = (0, 5, 3, 2), (0, 5, 4, 3)$ by computing. In both cases, we have $C_0^2 = 24$. On the other hand, the relative minimality of C implies that \square_C is not equal to \square_{C_0} , which means that $C^2 > C_0^2$.

We next consider the case $q' = 5$. If none of $O, (0, -5), (-5, -5)$ and $(-5, 0)$ is contained in \square_C , then also the eight points $(-1, 0), (0, -1), (0, -5), (-1, -5), (-4, -5), (-5, -4), (-5, -1)$ and $(-4, 0)$ are not contained in \square_C by the relative minimality. Hence the left (resp. right) shape of \square_C is one of three types in Fig. 17 (I) (resp. (II)). By noting the condition $d(C, (1, \pm 1)) \geq 5$, we have $C^2 \geq 25$ in any case. Therefore, considering the reflection, we see that it is sufficient to verify the proposition under the assumption $O \in \square_C$. Since the inequality $C^2 \geq 25$ is obvious if $(0, -5)$ or $(-5, 0)$ is contained in \square_C , we assume that $(0, -5), (-5, 0) \notin \square_C$. It follows that also the four points $(0, -4), (-1, -5), (-5, -1)$ and $(-4, 0)$ are not contained in \square_C . We denote by P the vertex of \square_C on $l(C, (0, -1))$ whose z -coordinate is maximal. Let us consider the case where $P = (-2, -5)$. Then we have the four possibilities as in Fig. 18. In the case (I), by the relative minimality, we see that there exist integers m_1 and m_2 with $-3 \leq m_1, m_2 \leq -1$ such that $(0, m_1)$ and $(m_2, 0)$ is contained in \square_C . An easy computation gives $C^2 \geq 25$ in any case. Consider the case (II). By the relative minimality, we see that either $(0, -1)$ or $(0, -3)$ is contained in \square_C , and likewise either $(-3, -5)$ or $(-5, -5)$ is contained in \square_C . Then it is obvious that the minimum value of C^2 is attained when the lower shape of \square_C is the polygonal line connecting $O, (0, -1), (-2, -5), (-3, -5)$ and $(-5, -3)$. Then we see that C^2 attains its minimum 25 when the upper shape of \square_C is the polygonal line

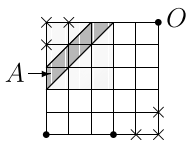


Fig. 19. The latter two cases in Fig. 18.

connecting $(-5, -3)$, $(-4, -2)$ and O . Consider the cases (III) and (IV). We note that the point $(-5, -5)$ is contained in \square_C in the case (III) also. By the condition $d(C, (1, -1)) \geq 5$, \square_C has a lattice point in the domain A in Fig. 19. When \square_C is a square with vertices O , $(-2, -5)$, $(-5, -5)$ and Q , the self-intersection number C^2 achieves its minimum 25, where Q is either $(-5, -3)$, $(-4, -2)$, $(-3, -1)$ or $(-2, 0)$. If $P = (-3, -5)$, we assume that $(-5, -2)$ is not contained in \square_C in order to avoid the duplication. By the relative minimality and the condition $d(C, (1, -1)) \geq 5$, we see that $l(C, (1, -1))$ passes through P and $(-3, 0)$ is contained in \square_C . Then, since the upper shape of \square_C must be the polygonal line connecting $(-5, -4)$, $(-3, 0)$ and O , we obtain $C^2 \geq 25$. We next consider the case where $P = (-4, -5)$. By the relative minimality and the condition $d(C, (1, -1)) \geq 5$, we see that $(-5, -5)$ is contained in \square_C , and moreover, either $(0, -2)$ or $(0, -3)$ is contained in \square_C . Then, considering the reflection and the rotation, this case can be reduced to the case where $P = (-2, -5)$ or $(-3, -5)$. The same argument goes through for the case where $P = (-5, -5)$. \square

4. Proof of the main theorem

To prove Theorem 1.3, we first aim to show that any gonality pencil on C can be extended to a morphism from the ambient surface. Let us prove several lemmas needed later. Also in this section, we use the notion of coprime in the wide sense (see Definition 3.1).

Lemma 4.1. *Let C be a curve as in Theorem 1.3, and assume that $q \geq 3$ and (S, C) is relatively minimal. Let V be an effective divisor on S , and i and j be positive integers with $i \geq 2$ and $i + j \leq 4$. Then $C \cdot V \geq q + i + j - 1$ if $h^0(S, V) \geq i$ and $h^0(S, V + K_S) \geq j$.*

Proof. We write $V = \sum_{i=1}^d n_i D_i$ with nonnegative integer coefficients. We denote by $\sigma(D_{d_0})$ the cone in Δ_S whose primitive element is $(0, -1)$.

(i) Consider the case where $(i, j) = (2, 1)$. By assumption, we can assume that the origin O is contained in \square_V and there exists another lattice point $P = (z, w)$ contained in the interior of \square_V . Without loss of generality, we can assume that $z \geq 0$, $w \leq 0$ and $(z, -w) = 1$. We denote by A_1 the domain drawn in Fig. 20 (I). Since P is contained in the interior of \square_V , the inequality $x_i z + y_i w < n_i$ holds for any $(x_i, y_i) \in A_1 \cap \text{Pr}(S)$. We thus obtain

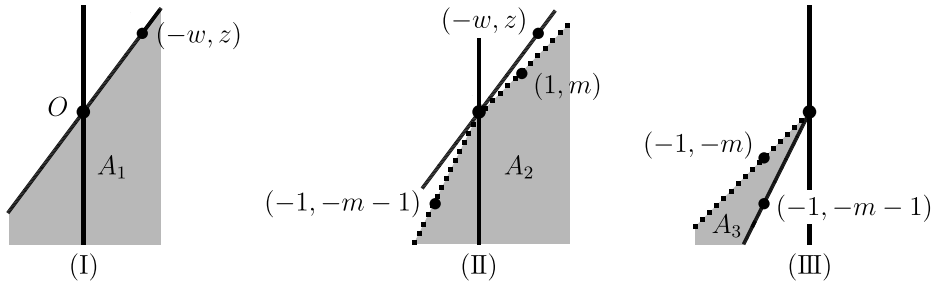


Fig. 20. Three domains A_1 , A_2 and A_3 .

$$\begin{aligned}
 C.V &= \sum_{i=1}^d n_i C.D_i \geq \sum_{\sigma(D_i) \subset A_1} (x_i z + y_i w + 1) C.D_i \\
 &= d(C, (-w, z)) + \sum_{\sigma(D_i) \subset A_1} C.D_i \geq q + \sum_{\sigma(D_i) \subset A_1} C.D_i.
 \end{aligned} \tag{3}$$

Thus it is sufficient to verify $\sum_{\sigma(D_i) \subset A_1} C.D_i \geq 2$. This inequality is true if there exists a cone $\sigma(D_i) \subset A_1$ such that $D_i^2 = -1$. Hence we suppose that there does not exist such a cone (we call this the ‘nonexistence condition’) and $\sum_{\sigma(D_i) \subset A_1} C.D_i = 1$. We can take a cone $\sigma(D_{i_0}) \subset A_1$ such that

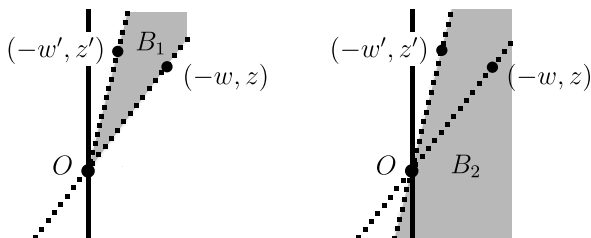
$$C.D_i = \begin{cases} 1 & (i = i_0), \\ 0 & (i \neq i_0, \sigma(D_i) \subset A_1). \end{cases}$$

If there exists only one cone $\sigma(D_j)$ included in $A_1 \setminus \mathbb{R}(-w, z)$, then $d(C, (x_{j-1}, y_{j-1}))$ is equal to one, a contradiction. We thus have $N = \#\{\sigma(D_i) \in \Delta_S \mid \sigma(D_i) \subset A_1 \setminus \mathbb{R}(-w, z)\} \geq 2$. Then, by the nonexistence condition, we deduce that neither z nor w is equal to zero. We denote by m the maximal integer satisfying $z + mw \geq 0$. By the nonexistence condition, there does not exist a cone included in the domain A_2 except for $\sigma(D_{d_0})$ (see Fig. 20 (II)). On the other hand, since $N \geq 2$, there exists a cone $\sigma(D_l) \subset A_1$ such that $x_l \neq 0$. In the case where x_l is positive, we have $(1, m) \in \text{Pr}(S)$ since $\sigma(D_l) \subset A_1 \setminus A_2$ and S is smooth. Thus, it follows from $D_{d_0}^2 \neq -1$ that there does not exist a cone in the domain A_3 (see Fig. 20 (III)). We deduce that

$$\{(x_j, y_j) \in M(x_{i_0-1}, y_{i_0-1}) \mid C.D_j \geq 1\} = \{(x_{i_0}, y_{i_0})\},$$

which implies a contradiction $d(C, (x_{i_0-1}, y_{i_0-1})) = 1$. In the case where x_l is negative, one can obtain a similar contradiction.

(ii) In the case where $(i, j) = (2, 2)$, we can assume that \square_V has two distinct lattice points $(0, 0)$ and (z, w) in its interior, where $z \geq 0$, $w \leq 0$ and $(z, -w) = 1$. As we saw in (i), the inequality $\sum_{\sigma(D_i) \subset A_1} C.D_i \geq 2$ holds. On the other hand, since $(0, 0)$ is contained in the interior of \square_V , the coefficient n_i is positive for every T -invariant divisor

Fig. 21. Two domains B_1 and B_2 .

D_i ($i = 1, \dots, d$). If $C.D_i = 0$ for any $\sigma(D_i) \not\subset A_1$, then we have $d(C, (-w, z)) = 0$, a contradiction. We thus have $C.V \geq q + 2 + \sum_{\sigma(D_i) \not\subset A_1} n_i C.D_i \geq q + 3$.

(iii) In the case where $(i, j) = (3, 1)$, there exist three distinct lattice points $(0, 0)$, (z, w) and (z', w') in \square_V , especially (z, w) is contained in the interior of \square_V . We can assume that $z \geq 0$, $w \leq 0$, $(z, -w) = 1$ and $(|z'|, |w'|) = 1$. Suppose that $C.V = q + 2$, and denote by i_1 (resp. i_2) the minimal (resp. maximal) integer in $\{i \mid \sigma(D_i) \subset A_1, C.D_i \geq 1\}$. By a computation similar to that in (3), we obtain $\sum_{\sigma(D_i) \subset A_1} C.D_i = 2$ and $\sum_{\sigma(D_i) \not\subset A_1} n_i C.D_i = 0$. It follows that $C.D_i = 0$ for $\sigma(D_i) \subset A_1$ except for $i = i_1, i_2$. Moreover, we see that $n_{i_1} = n_{i_2} = C.D_{i_1} = C.D_{i_2} = 1$ (resp. $n_{i_1} = 1$ and $C.D_{i_1} = 2$) if $i_1 \neq i_2$ (resp. $i_1 = i_2$). Let us consider the case where $zw' - wz' > 0$. Let $\sigma(D_j)$ be a cone included in the domain B_1 drawn in Fig. 21. Since (z', w') is contained in $L(V, (x_j, y_j))$, we have the inequalities $n_j \geq x_j z' + y_j w' > 0$. Hence $C.D_j = 0$. Noting that $(z', w') \in L(V, (x_{i_1}, y_{i_1})) \cap L(V, (x_{i_2}, y_{i_2}))$, we have

$$\begin{aligned} d(C, (-w', z')) &= \sum_{\sigma(D_i) \subset B_2} (x_i z' + y_i w') C.D_i \\ &= \begin{cases} (x_{i_1} z' + y_{i_1} w') C.D_{i_1} + (x_{i_2} z' + y_{i_2} w') C.D_{i_2} \leq n_{i_1} C.D_{i_1} + n_{i_2} C.D_{i_2} = 2 & (i_1 \neq i_2), \\ (x_{i_1} z' + y_{i_1} w') C.D_{i_1} \leq n_{i_1} C.D_{i_1} = 2 & (i_1 = i_2). \end{cases} \end{aligned}$$

This contradicts the assumption $q \geq 3$. A similar argument can be carried out for the case where $zw' - wz' \leq 0$. \square

We are now in a position to show the extension of a gonality pencil. In the proof, the following Serrano's result plays an essential role.

Theorem 4.2. (See [16].) *Let X be a smooth curve on a smooth surface Y , and $f : X \rightarrow \mathbb{P}^1$ be a surjective morphism of degree p . Assume that $X^2 > 4p$. If there does not exist an effective divisor V on Y satisfying the following properties (a) and (b), then there exists a morphism from Y to \mathbb{P}^1 whose restriction to X is f .*

- (a) $1 \leq V^2 < (X - V).V \leq p$.
- (b) $X^2 \leq \frac{(p + V^2)^2}{V^2}$.

Proposition 4.3. *Let C be a curve as in Theorem 1.3, and f be a gonality pencil on C . If $(g, q) \neq (4, 4), (5, 4), (10, 6)$, then there exists a morphism from S to \mathbb{P}^1 whose restriction to C is f .*

Proof. If (S, C) is not relatively minimal, by the method explained before Definition 3.9, we can obtain an equivariant morphism ψ from S to another compact smooth toric surface S' such that (S', C) is relatively minimal. Clearly, for a morphism φ from S' to \mathbb{P}^1 , the composite $(\varphi \circ \psi)|_C$ coincides with $\varphi|_C$. Hence, it is sufficient to consider the case where (S, C) is relatively minimal. By using the inequality $q \geq k$, Corollary 3.8 and Propositions 3.10–3.12 and the assumption $(g, q) \neq (4, 4), (5, 4)$, we have

$$C^2 \geq \begin{cases} 12 & (k = 2), \\ 17 & (k = 3, 4), \\ 25 & (k = 5), \\ \frac{3}{4}k^2 & (k \geq 6). \end{cases} \quad (4)$$

In particular, $C^2 > 4k$ holds.

By the condition $g \geq 2$, we have $q \geq 2$. If $q = 2$, then $k = 2$ and C has only one gonality pencil. Thus our proposition is obvious in this case. Let us consider the case where $q \geq 3$. Suppose that, for C , S and f , there exists an effective divisor V satisfying the two properties (a) and (b) in Theorem 4.2. If we put $s = V^2$, the inequalities $1 \leq s < k$ and $C^2 \leq (k + s)^2/s$ hold. Combining them with (4), we see that $k \neq 2, 3$ and

$$\begin{cases} s = 1 & (k = 5 \text{ or } k \geq 9), \\ 1 \leq s \leq 2 & (k = 4, 7, 8), \\ 1 \leq s \leq 3 & (k = 6). \end{cases}$$

We first consider the case where $s \leq 2$. By Riemann–Roch theorem, we have

$$\begin{aligned} 0 &\leq h^1(S, V + K_S) = h^0(S, V + K_S) + h^0(S, -V) - \frac{1}{2}(V + K_S) \cdot V - \chi(\mathcal{O}_S) \\ \frac{1}{2}V \cdot K_S &\leq h^0(S, V + K_S) - \frac{s}{2} - 1, \\ 0 &\leq h^1(S, V) = h^0(S, V) + h^0(S, K_S - V) - \frac{1}{2}V \cdot (V - K_S) - \chi(\mathcal{O}_S) \\ &\leq h^0(S, V) + h^0(S, V + K_S) - s - 2. \end{aligned} \quad (5)$$

Since $h^0(S, V) \geq h^0(S, V + K_S)$, we obtain $2h^0(S, V) \geq s + 2$. Assume that $s = 1$. Then we have $h^0(S, V) \geq 2$ and $C \cdot V \leq k + 1 \leq q + 1$ by the property (a). Hence we have $h^0(S, V + K_S) = 0$ by Lemma 4.1. It follows from (5) that $h^0(S, V) \geq s + 2$. In the case where $s = 2$, since $h^0(S, V) \geq 2$ and $C \cdot V \leq q + 2$, we have $h^0(S, V + K_S) \leq 1$ by Lemma 4.1. We thus have $h^0(S, V) \geq 3$ by (5), and $h^0(S, V + K_S) = 0$ by Lemma 4.1. It follows from (5) that $h^0(S, V) \geq s + 2$. On the other hand, since

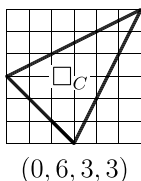


Fig. 22. \square_C in the case where $q = 6$ and $g = 10$.

$$C \cdot (V - C) = C \cdot V - C^2 \leq \begin{cases} k + s - \frac{3}{4}q^2 \leq q + 2 - \frac{3}{4}q^2 < 0 & (q \geq 3), \\ k + s - 12 < 0 & (q = 2) \end{cases}$$

by Proposition 3.7 and Proposition 3.10, we obtain $h^0(S, V - C) = 0$. Therefore, in the case where $s \leq 2$, the cohomology exact sequence

$$0 \rightarrow H^0(S, V - C) \rightarrow H^0(S, V) \rightarrow H^0(C, V|_C) \rightarrow \cdots$$

gives the inequality $h^0(C, V|_C) \geq s + 2$. If we write $g_l^r = |V|_C|$, then $r \geq s + 1$ and $l \leq k + s$. We obtain a net g_{l-r+2}^2 by subtracting $r - 2$ general points of C from it. Since C is not isomorphic to a plane curve, g_{l-r+2}^2 is not very ample. Then we obtain a pencil g_{l-r}^1 such that $l - r \leq k + s - (s + 1) = k - 1$, a contradiction.

Let us prove that the remaining case $(k, s) = (6, 3)$ does not occur. We take a curve C_0 as in Lemma 3.5. By Proposition 3.7,

$$27 = \frac{3}{4}k^2 \leq \frac{3}{4}q^2 \leq C_0^2 \leq C^2 \leq \frac{(k+s)^2}{s} = 27,$$

which implies $q = 6$ and $C_0^2 = C^2 = 27$. Hence we have $\square_C = \square_{C_0}$ by Remark 3.6. By the argument in the proof of Proposition 3.7, we see that C_0 is a curve of type (iii) in it. Then the inequality (2) gives $q' = 6$. Moreover, by the inequality (1), we deduce that $c = e = 6 - (a + b)/2$ and $a - b = -6$. We thus have $(a, b, c, e) = (0, 6, 3, 3)$ and $g = 10$ (see Fig. 22). \square

Lemma 4.4. *Let C be a curve as in Theorem 1.3, and assume that (S, C) is relatively minimal. If $g = 10$ and $q = 6$, then \square_C is a triangle as in Fig. 22.*

Proof. In this proof, we often use the relative minimality of C and the property $d(C, (1, -1)) \geq q'$ (see Lemma 3.4) without further mention. We denote by $\text{Int} \square_C$ the interior of \square_C , and by $l((a_1, b_1), (a_2, b_2))$ the segment connecting two points (a_1, b_1) and (a_2, b_2) . We set the point $l(C, (0, 1)) \cap l(C, (1, 0))$ as the origin O . Now we suppose that none of O , $(0, -6)$, $(-q', -6)$ and $(-q', 0)$ is contained in \square_C . It follows that also the eight points $(-1, 0)$, $(0, -1)$, $(0, -5)$, $(-1, -6)$, $(-q' + 1, -6)$, $(-q', -5)$, $(-q', -1)$ and $(-q' + 1, 0)$ are not contained in \square_C . Assume that $(0, -3)$ is contained in \square_C . Let P be a lattice point contained in $l(-1, 0) \cap \square_C$. We define A as a domain surrounded by the four

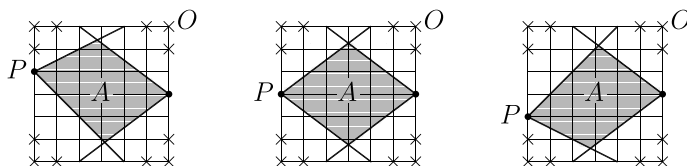


Fig. 23. The case where $(0, -3) \in \square_C$.

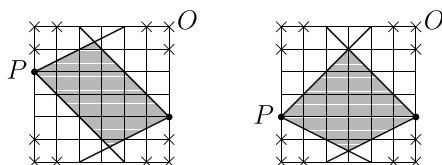


Fig. 24. The case where $(0, -4) \in \square_C$ and $\text{Int } \square_C$ has at most ten points.

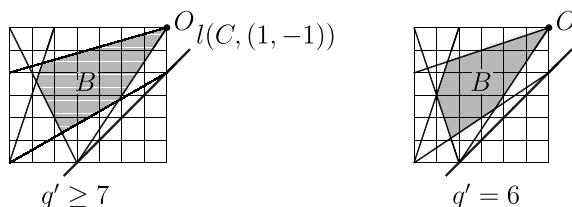
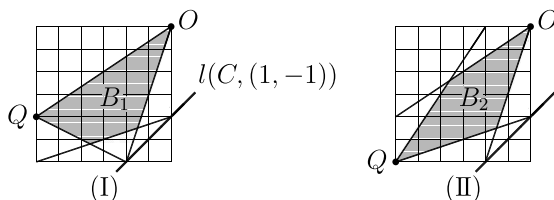
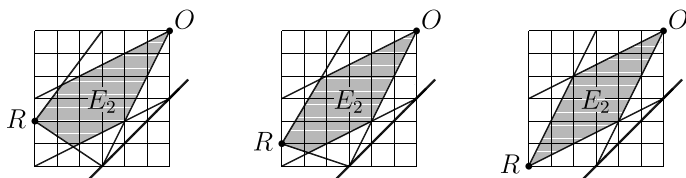


Fig. 25. The case where $a = -2$.

segments $l((0, -3), (-q' + 2, -6))$, $l((-2, -6), P)$, $l(P, (-2, 0))$ and $l((-q' + 2, 0), (0, -3))$. In any case, we see that there exist more than ten lattice points in the interior of A (see Fig. 23). Since \square_C includes A , we obtain $g \geq 11$. Next we assume that $(0, -4)$ is contained in \square_C . Similarly to the previous case, one can check $g \geq 11$ except for the two cases where $q' = 6$ and $P = (-6, -2), (-6, -4)$ (see Fig. 24). If $q' = 6$ and $P = (-6, -2)$, then either $(-3, -1)$ or $(-4, -1)$ must be contained in $\text{Int } \square_C$. Besides, either $(-2, -5)$ or $(-3, -5)$ must be contained in $\text{Int } \square_C$. On the other hand, if $q' = 6$ and $P = (-6, -4)$, then either $(-2, -2)$ or $(-4, -2)$ is contained in $\text{Int } \square_C$. Hence, in each case, we obtain $g \geq 11$.

By the above consideration, we can assume that O is contained in \square_C . There exists an integer a with $-6 \leq a \leq 0$ such that $l(C, (1, -1))$ passes through $(0, a)$. We first remark that the cases where $a = -1, -5$ do not occur by the relative minimality. If $a = 0$, then by the assumption $g = 10$, \square_C must be a triangle with vertices $O, (-6, -6)$ and $(-6, 0)$. This contradicts the assumption that C is not a plane curve. We obtain a similar contradiction if $a = -6$. Let us consider the case where $a = -2$. Then either $(-q' + 2, 0)$ or $(-q', -2)$ is contained in \square_C . We define B as a domain surrounded by the five segments $l(O, (-4, -6))$, $l((0, -2), (-q', -6))$, $l((-4, -6), (-q', 0))$, $l((-q', -6), (-q' + 2, 0))$ and $l((-q', -2), O)$. Since \square_C includes B , we obtain $g \geq 11$ if $q' \geq 7$ (see Fig. 25). In the case where $q' = 6$, we can observe that there are at least eight lattice points in $\text{Int } \square_C$. Note that either $(-2, -3)$ or $(-3, -4)$ must be contained in $\text{Int } \square_C$. Moreover, if $(-4, 0) \in \square_C$

Fig. 26. The case where $a = -4$.Fig. 27. The case where $a = -3$.

(resp. $(-6, -2) \in \square_C$), then $(-3, -1)$ and $(-4, -1)$ (resp. $(-5, -2)$ and $(-5, -3)$) are contained in $\text{Int } \square_C$. Hence, we have $g \geq 11$ in this case also.

We next consider the case where $a = -4$. Let Q be a lattice point contained in $l(C, (-1, 0)) \cap \square_C$. In the cases where $Q = (-q', 0), (-q', -1), (-q', -2), (-q', -3)$ and $(-q', -4)$, we define B_1 as a domain surrounded by the four segments $l(O, (-2, -6)), l((0, -4), (-q', -6)), l((-2, -6), Q)$ and $l(Q, O)$. Then we see that there exist more than ten lattice points in the interior of B_1 (that is, $g \geq 11$) except for the case where $q' = 6$ and $Q = (-6, -4)$ (see Fig. 26 (I)). In this exceptional case, since either $(-3, -5)$ or $(-4, -5)$ must be contained in $\text{Int } \square_C$, we obtain $g \geq 11$. If Q is $(-q', -5)$ or $(-q', -6)$, \square_C includes the domain B_2 surrounded by the five segments $l(O, (-2, -6)), l((0, -4), (-q', -6)), l(Q, (-q', -4)), l(Q, (-q' + 4, 0))$ and $l((-q', -4), O)$. There exist more than ten lattice points in the interior of B_2 except for the case where $q' = 6$ and $Q = (-6, -6)$ (see Fig. 26 (II)). In this exceptional case, since either $(-3, -2)$ or $(-4, -3)$ must be contained in $\text{Int } \square_C$, we obtain $g \geq 11$.

Lastly, we consider the case where $a = -3$. We define E_1 as a domain surrounded by the five segments $l(O, (-3, -6)), l((0, -3), (-q', -6)), l((-3, -6), (-q', 0)), l((-q', -6), (-q' + 3, 0))$ and $l((-q', -3), O)$. Since \square_C includes E_1 , we obtain $g \geq 11$ if $q' \geq 8$. In the case where $q' = 7$, there exist at least ten lattice points in the interior of E_1 . Moreover, if $(-7, 0) \in \square_C$ (resp. $(-7, 0) \notin \square_C$), then we see that $(-5, -2)$ (resp. $(-5, -3)$) is contained in $\text{Int } \square_C$. This means that $g \geq 11$. Let us consider the case where $q' = 6$. We denote by $R = (-6, b)$ the vertex of \square_C on $l(C, (-1, 0))$ whose w -coordinate is maximal. In the cases where $b = -4, -5, -6$, a domain E_2 surrounded by the five segments $l(O, (-3, -6)), l((0, -3), (-6, -6)), l((-3, -6), R), l(R, (-3, 0))$ and $l((-6, -3), O)$ is included in \square_C (see Fig. 27). Note that, in each case, either $(-1, -2)$ or $(-4, -5)$ must be contained in $\text{Int } \square_C$. Moreover, in the case where $b = -6$, either $(-2, -1)$ or $(-5, -4)$ must be contained in $\text{Int } \square_C$. We thus obtain $g \geq 11$. If $-3 \leq b \leq 0$, we define E_3 as a domain surrounded by the four segments $l(O, (-3, -6)), l((0, -3), (-6, -6)), l((-3, -6), R)$

and $l(R, O)$. If $-2 \leq b \leq 0$, then there exist more than eleven lattice points in the interior of E_3 . Assume that $b = -3$. If $(-3, -5)$ is not contained in $\text{Int } \square_C$, then by a simple consideration, we see that at least two points of $(-1, -2)$, $(-1, -3)$ and $(-2, -4)$ are contained in $\text{Int } \square_C$. On the other hand, in the case where $(-3, -5)$ is contained in $\text{Int } \square_C$, it is clear that the equality $g = 10$ holds if and only if \square_C is a triangle with vertices O , $(-3, -6)$ and $(-6, -3)$. \square

Similarly to Lemma 4.4, we obtain the following lemma.

Lemma 4.5. *Let C be a curve as in Theorem 1.3, and assume that (S, C) is relatively minimal. If $g = q = 4$, then \square_C is the first triangle in Fig. 11.*

We are now ready to prove the main theorem.

Proof of Theorem 1.3. As mentioned in the proof of Proposition 4.3, the statements (i) and (ii) are obvious if $q = 2$. Hence we assume that $q \geq 3$ in the following proofs of (i) and (ii).

(i) Let φ be a morphism from S to \mathbb{P}^1 of $\deg \varphi|_C = k$ whose existence is guaranteed by Proposition 4.3. We shall show that φ is a toric fibration of S . We denote by F the fiber of φ . Since $C \cdot (F - C) \leq k - 3k^2/4 < 0$, we have $h^0(C, F - C) = 0$. Hence the cohomology exact sequence

$$0 \rightarrow H^0(S, F - C) \rightarrow H^0(S, F) \rightarrow H^0(C, F|_C) \rightarrow \cdots$$

implies that $h^0(C, F|_C) \geq h^0(S, F)$. Hence we have $h^0(S, F) \leq 2$. Namely, \square_F is a segment connecting two lattice points. We denote these points by $O = (0, 0)$ and $P = (z, w)$, where z and w are integers such that $(|z|, |w|) = 1$. Then the point $(-w, z)$ must be contained in $\text{Pr}^*(S)$. Therefore, by Fact 2.2, we see that F is a fiber of some toric fibration.

In what follows, by reembedding if necessary, we may assume that (S, C) is relatively minimal.

(ii) If $(g, q) \neq (4, 4), (5, 4), (10, 6)$, we have $k = q$ by (i). In the case where $g = q = 4$, we have $k \leq \lfloor (g+3)/2 \rfloor = 3$ and $C^2 \geq 12$ by Proposition 3.11. Suppose that $k = 2$. Then by Theorem 4.2, there exists an effective divisor V on S satisfying the properties (a) and (b) for $X = C$ and $p = 2$. It is, however, impossible that these properties hold at the same time. We can thus conclude that $k = 3$. In the case where $g = 5$ and $q = 4$, similarly to the previous case, we can show that the supposition $k \leq 3$ yields a contradiction. In the case where $g = 10$ and $q = 6$, we obtain $k = 6$ by (iv) which is proved below.

(iii) According to [8], the Clifford dimension $\text{Cliffdim}(C)$ of C is equal to two if and only if C is isomorphic to a plane curve whose degree is at least five. Besides, if $\text{Cliffdim}(C) \geq 3$, then the genus is at least ten. Hence, we have $\text{Cliffdim}(C) = 1$ in the cases $(g, q) = (4, 4), (5, 4)$. In other cases (except for the case $(g, q) = (10, 6)$), we see that the number of gonality pencils on C is finite by (i). It follows that $\text{Cliffdim}(C) = 1$.

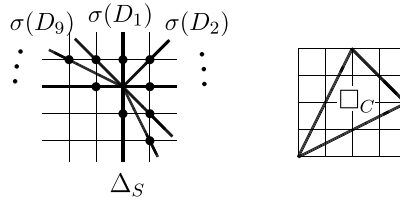


Fig. 28. Δ_S and \square_C in the case where $g = q = 4$.

(iv) By Lemma 4.4 and Fig. 22, in this case, we can see the explicit structures of S and C . First, $|-K_S|$ has no base points, $h^0(S, -K_S) = 4$ and $(-K_S)^2 = 3$. Besides, we can write $C \sim -3K_S$, that is, $C \cdot (-K_S) = 9$. We consider the morphism $\Phi_{|-K_S|} : S \rightarrow \mathbb{P}^3$, and put $Y = \Phi_{|-K_S|}(S)$. Then, by the equality

$$\deg \Phi_{|-K_S|} \cdot \deg Y = (-K_S)^2 = 3,$$

we obtain $\deg \Phi_{|-K_S|} = 1$ and $\deg Y = 3$. We denote by H a hyper plane of Y . The short exact sequence of sheaves $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}(3H) \rightarrow \mathcal{O}_Y(3H) \rightarrow 0$ induces the surjection $H^0(\mathbb{P}^3, 3H) \rightarrow H^0(Y, 3H|_Y) = H^0(Y, C)$, where we abuse notation to denote the image of C under $\Phi_{|-K_S|}$ by the same symbol. Hence we see that C is an irreducible component of $Y \cap Y'$, where Y' is a cubic surface in \mathbb{P}^3 . Since

$$9 = \deg Y \cap Y' \geq \deg C = C \cdot (-K_S) = 9,$$

we can conclude that $C = Y \cap Y'$. \square

5. The case where $(g, q) = (4, 4)$

In this section, we focus on the exceptional curve in Theorem 1.3 (ii), and exhibit its structure. Let S and C be as in Theorem 1.3, and assume $g = q = 4$. By Lemma 4.5, the fan Δ_S and the lattice polygon \square_C are as in Fig. 28. Considering the shape of \square_C , we can take a plane model

$$C' : x^4y^2 + x^2y^4 + ax^2y^2z^2 = z^6 \quad (a \in \mathbb{C})$$

of C . We shall denote the pull-backs on S of functions x , y and z by same symbols. Since $K_S \sim -\sum_{i=1}^9 D_i$ and $C \sim -2K_S$, we obtain $h^0(S, K_S) = h^1(S, K_S) = 0$, which implies that $H^0(C, K_C) \simeq H^0(S, -K_S) = \langle x^2y, xy^2, xyz, z^3 \rangle$. Hence the restriction of the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ (x : y : z) &\longmapsto (x^2y : xy^2 : xyz : z^3) \end{aligned}$$

to C' gives the canonical embedding of C . Let $(s : t : u : v)$ be a homogeneous coordinate system in \mathbb{P}^3 . Then the canonical curve of C lies on a quadric surface $Y : s^2 + t^2 + au^2 = v^2$.

Thus if $a \neq 0$, then two families of lines on Y cut out two distinct pencils g_3^1 and h_3^1 on C . On the other hand, if $a = 0$, then Y is a quadric cone, and one family of lines cuts out a unique g_3^1 on C . In sum, we can conclude that

- (i) If $a \neq 0$, C is a curve of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.
- (ii) If $a = 0$, C is linearly equivalent to $3\Delta_0 + 6F$ on Σ_2 , where Δ_0 and F denote the minimal section and the fiber of the ruling of Σ_2 , respectively.

Unfortunately, however, we cannot distinguish the above difference from the information of the lattice polygon.

6. Application

By combining [Corollary 1.4](#) with results in [\[10\]](#), we can compute Weierstrass gap sequences at ramification points (with high ramification indexes) of a gonality pencil. For example, in this section, we consider trigonal curves and provide a geometric interpretation of the structure of gap sequences at ramification points. Let us review the preliminary results. First, it is known that a trigonal covering of \mathbb{P}^1 has four types of gap sequences.

Theorem 6.1. (See [\[5, 6\]](#).) *Let C be a smooth trigonal curve of genus g and Maroni invariant m , and P be a ramification point of a trigonal covering from C to \mathbb{P}^1 . Then the Weierstrass gap sequence at P is one of the following types.*

In the case where P is a total ramification point:

- type I $\{1, 2, 4, 5, \dots, 3m+1, 3m+2, 3m+4, 3m+7, \dots, 3(g-m)-5\}$,
- type II $\{1, 2, 4, 5, \dots, 3m+1, 3m+2, 3m+5, 3m+8, \dots, 3(g-m)-4\}$.

In the case where P is an ordinary ramification point:

- type I $\{1, 2, 3, \dots, 2m+1, 2m+2, 2m+3, 2m+5, \dots, 2(g-m)-3\}$,
- type II $\{1, 2, 3, \dots, 2m+1, 2m+2, 2m+4, 2m+6, \dots, 2(g-m)-2\}$.

Besides, Kato and Horiuchi presented the following criterion for distinguishing the above types.

Theorem 6.2. (See [\[9\]](#).) *Let C be a trigonal curve of genus $g \geq 5$ and Maroni invariant m . Then C has a plane model defined by*

$$y^3 + x^\mu A(x)y + x^\nu B(x) = 0, \quad (6)$$

where $\deg A(x) + \mu = 2m + 4$, $\deg B(x) + \nu = 3m + 6$ and $A(0)B(0) \neq 0$.

- (i) If $\mu \geq \nu = 1$, there exists a total ramification point of type I over $x = 0$.

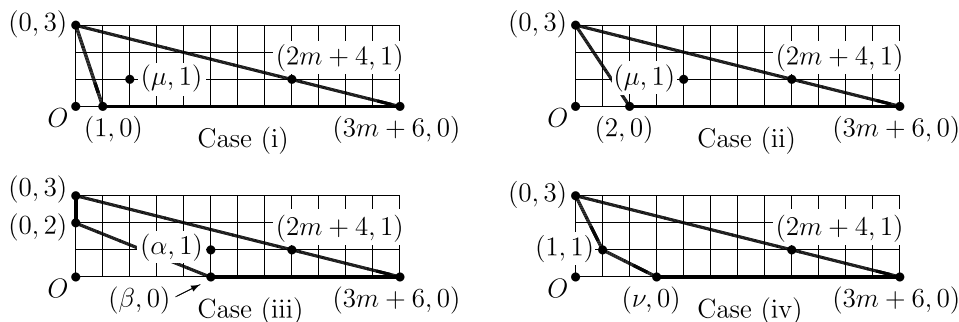


Fig. 29. Polygons associated to a trigonal curve.

- (ii) If $\mu \geq \nu = 2$, there exists a total ramification point of type II over $x = 0$.
- (iii) If $\mu = \nu = 0$ and the order of zero of $4A(x)^3 + 27B(x)^2$ at $x = 0$ is odd, there exists an ordinary ramification point of type I over $x = 0$.
- (iv) If $\nu > \mu = 1$, there exists an ordinary ramification point of type II over $x = 0$.
- (v) Otherwise, there exist no ramification points over $x = 0$.

In order to interpret Theorem 6.2 geometrically, we transform the defining equation (6) in the case (iii). Note that we need the condition $m < (g - 2)/2$ in this case, since $2(g - m) - 3$ must be at least g . We can write two polynomials as $A(x) = \sum_{i=1}^{2m+4} a_i x^i - 3k^2$ and $B(x) = \sum_{i=1}^{3m+6} b_i x^i + 2k^3$, where $k \neq 0$. We formally set $a_{2m+5} = \dots = a_{3m+6} = 0$, and define $\alpha = \min\{i \mid a_i \neq 0\}$, $\beta = \min\{i \mid ka_i + b_i \neq 0\}$, $A_1(x) = \sum_{i=\beta}^{3m+6} a_i x^i$, $B_1(x) = \sum_{i=\beta}^{3m+6} b_i x^i$ and $E(x) = A(x) - A_1(x) + 3k^2$. Then we have $\mindeg(kA_1(x) + B_1(x)) = \beta$, where the notation ‘mindeg’ denotes the minimal degree of a polynomial. Since $m > g/2$, we see that β is less than $2m + 4$. Let us check that $\mindeg(4A(x)^3 + 27B(x)^2) = \beta < 2\alpha$. By a simple computation, we have

$$4A(x)^3 + 27B(x)^2 = 4(E(x) + A_1(x))^3 + 27(-kE(x) + B_1(x))^2 - 36k^2(E(x) + A_1(x))^2 + 108k^3(kA_1(x) + B_1(x)). \quad (7)$$

In the case where $E(x) = 0$, β is equal to α by definition, and the equality $\mindeg(4A(x)^3 + 27B(x)^2) = \beta$ follows from (7). On the other hand, if $E(x) \neq 0$, we have $\mindeg(4A(x)^3 + 27B(x)^2) = \min\{2\mindeg E(x), \beta\} = \beta$ by (7) and its oddness. Note that $\mindeg E(x) = \alpha$ in this case. Consequently, if we perform a coordinate transformation $y' = y - k$, and put $y = y'$ again, then the defining equation (6) is rewritten as

$$y^3 + 3ky^2 + x^\alpha C(x)y + x^\beta D(x) = 0 \quad (8)$$

in the case (iii), where α and β are positive integers such that β is odd and $\beta < \min\{2\alpha, 2m + 4\}$, $\deg C(x) + \alpha = 2m + 4$, $\deg D(x) + \beta = 3m + 6$ and $C(0)D(0) \neq 0$.

Now we are in a position to translate Theorem 6.2 in terms of the geometry of lattice polygons. If we embed C in a toric surface by blowing up repeatedly, the lattice polygon

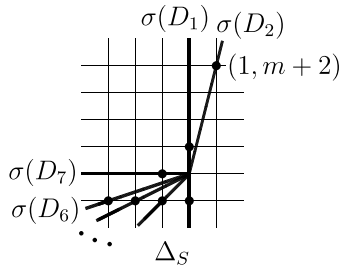


Fig. 30. Δ_S in Case (i) in Fig. 29.

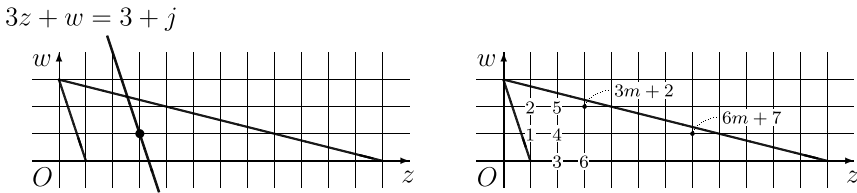


Fig. 31. The gap sequence in Case (i) in Fig. 29.

\square_C associated to C is drawn as in Fig. 29. In Case (i), the defining fan Δ_S of S is as in Fig. 30. Considering the process of blowing-ups, we see that an intersection point $P = C \cap D_6$ is a unique point over the origin of the plane model (6). On the other hand, Corollary 1.4 and Fact 2.2 show that the fiber F of a trigonal covering from C to \mathbb{P}^1 is $F \sim D_4 + 2D_5 + 3D_6 + D_7$, which implies that P is a total ramification point. By applying Corollary 1.6 in [10], we can determine the gap sequence at P as

$$\{j \mid \text{the line } 3z + w = 3 + j \text{ has a lattice point in } \text{Int} \square_C\},$$

where z and w are the coordinates of the plane in which \square_C lies. For better understanding, we attach to each lattice point an integer j such that the line $3z + w = 3 + j$ passes through it (see Fig. 31). Then we can find the gap sequence

$$\{1, 2, 4, 5, \dots, 3m+1, 3m+2, 3m+4, \dots, 6m+7\}$$

as a set of integers contained in $\text{Int} \square_C$. Since the genus of C is equal to the number of lattice points contained in $\text{Int} \square_C$, we have $g = 3m+4$ in this case. Hence the above gap sequence at a total ramification point P is truly of type I in Theorem 6.1. Similarly, for the remaining cases in Fig. 29, we obtain the gap sequence at a ramification point and the genus of C as follows.

- (ii) $\{j \mid \text{the line } 3z + 2w = 6 + j \text{ has a lattice point in } \text{Int} \square_C\}$, $g = 3m+3$,
- (iii) $\{j \mid \text{the line } 2z + \beta w = 2\beta + j \text{ has a lattice point in } \text{Int} \square_C\}$, $g = 3m - \frac{\beta-9}{2}$,
- (iv) $\{j \mid \text{the line } 2z + w = 3 + j \text{ has a lattice point in } \text{Int} \square_C\}$, $g = 3m+3$.

Consequently, in each case, the result of [Theorem 6.1](#) can be visualized in a similar way as in [Fig. 31](#).

This idea is applicable for the cases of higher gonality. By [Theorem 1.3](#), a lattice polygon associated to a k -gonal curve C can be drawn as a polygon with height k and sufficiently large width. Assume that there exists an oblique side which has no lattice points except for two end points, and denote by D the T -invariant divisor corresponding to this side. In this case, a point $P = C \cap D$ is a total ramification point of a gonality pencil on C , and moreover, P satisfies the assumption in Corollary 1.6 in [\[10\]](#). Hence one can determine the Weierstrass gap sequence at P by moving the oblique side similarly to [Fig. 31](#). This fact suggests the possibility of the classification of gap sequences at total ramification points of a curve on a toric surface. We will deal with this prospective problem in future work.

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