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Asymptotic weights of syzygies of toric varieties

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ABSTRACT

The purpose of the paper is to give a sharp asymptotic description of the weights that appear in the syzygies of a smooth toric variety. We prove that as the positivity of the embedding increases, in any strand of syzygies, torus weights after normalization stabilize to the same fixed shape that we explicitly specify.

1. Introduction

The purpose of the paper is to give a sharp asymptotic description of the weights that appear in the syzygies of a smooth toric variety. We prove that as the positivity of the embedding increases, in any strand of syzygies, torus weights after normalization stabilize to the same fixed shape that we explicitly specify.

Let X be a smooth projective toric variety over $\mathbb{C}$ of dimension n throughout the paper, and L be a very ample toric line bundle on X. Then L defines a toric embedding:

$$X \hookrightarrow \mathbb{P}^{r(L)} = \mathbb{P}H^0(X, L) = \text{Proj } S$$

where $r(L) = h^0(X, L) - 1$ and $S = \text{Sym}H^0(X, L)$. Write:

$$R(X; L) = \bigoplus_m H^0(X, mL)$$

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which is viewed as a finitely generated graded S-module. We will be interested in the syzygies of $R(X;L)$ over S. Specifically, R has a graded minimal free resolution

$$\mathbb{F} : \ldots \to F_p \to \ldots \to F_0 \to R \to 0$$

where $F_p = \oplus_j S(-a_{p,j})$ is a free S-module. Write $K_{p,q}(X;L)$ for the finite dimensional vector space of minimal p-th syzygies of degree $(p+q)$, so that:

$$F_p \cong \bigoplus_q K_{p,q}(X;L) \otimes_{\mathbb{C}} S(-p-q)$$

Moreover, in the above setting, the torus action on X induces torus actions on $K_{p,q}(X;L)$. We can naturally ask which torus weights appear in their decompositions.

From an asymptotic perspective, Ein and Lazarsfeld show in [2] that for $1 \leq q \leq n$, if $L \gg 0$, $K_{p,q}(X;L) \neq 0$ for almost all $p \in [1, r_d]$. In this paper, we give a sharp description of the asymptotic distribution of normalized torus weights in syzygies. To give the statement, let Δ be the convex polytope associated to the very ample divisor A ([3], Section 3.4, p. 66, P_A in notation of the book). Let $L_d = A^{\otimes d}$. Then by degree counting, the torus weights of $K_{p,q}(X;L_d)$ correspond to integral points in $(p+q)d \cdot \Delta$. Denote the collection of weights by:

$$\operatorname{wts}(K_{p,q}(X;L_d)) = \{\text{Torus weights of } K_{p,q}(X;L_d)\} \subseteq (p+q)d \cdot \Delta$$

We normalize so that all points lie in Δ:

$$\operatorname{wts}^{\text{nor}}(K_{p,q}(X;L_d)) = \frac{\operatorname{wts}(K_{p,q}(X;L_d))}{(p+q)d} \subseteq \Delta$$

We show the philosophy that asymptotic syzygies are complicated by proving that as $d \to \infty$, the set of all normalized torus weights becomes dense in Δ:

Theorem 4.2. *Fix $1 \leq q \leq n$, then*

$$\bigcup_{\substack{d > 0 \\ 1 \leq p \leq r_d}} \operatorname{wts}^{\text{nor}}(K_{p,q}(X;L_d))$$

is dense in Δ.

The Theorem is illustrated in Fig. 1, which approximates $K_{p,1}(X^2, d)$ for $d = 2$ and $d = 4$.

We can also ask what happens if we focus only on some of the syzygies appearing in the resolution. Is their behavior still as complicated? More specifically, restrict p to lie in a fixed interval relative to r_d, i.e. consider:

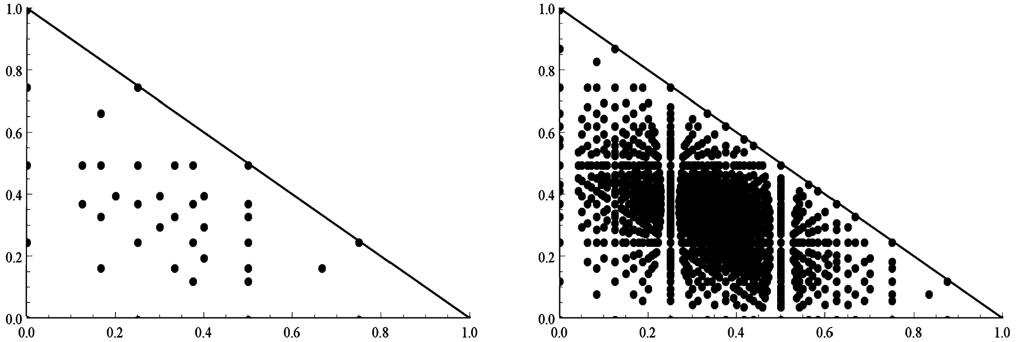

Fig. 1. Normalized torus weights for $K_{p,1}(X^2; d)$ for $d = 2$ and $d = 4$.

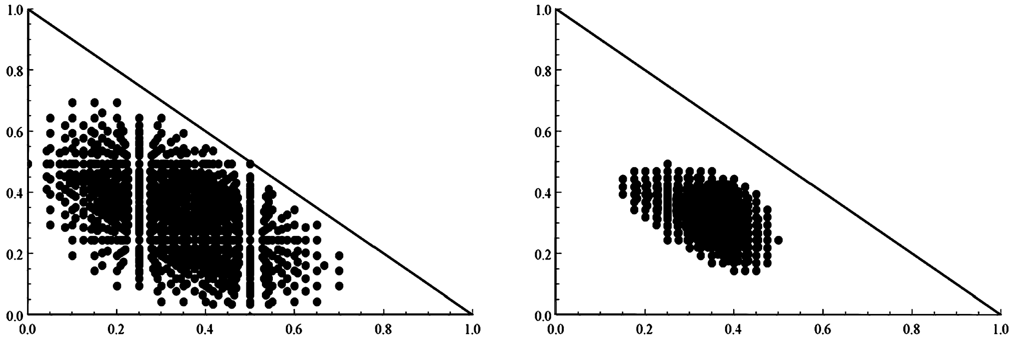

Fig. 2. Closure of normalized weights for $a = 0.33$, $b = 0.66$ and $a = 0.66$, $b = 1$ with $X = X^2$, $d = 4$.

$$\Delta(a,b) \;=\; \overline{\bigcup_{\substack{d \gg 0 \\ a\cdot r_d \,\le\, p \,\le\, b\cdot r_d}} \mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X; L_d))} \subseteq \;\Delta$$

where $0 \le a < b \le 1$. These are no longer necessarily dense inside all of Δ. Fig. 2 shows the normalized weights of $K_{p,1}(X^2; 4)$ for $a = 0.33$, $b = 0.66$ and $a = 0.66$, $b = 1$:

Quite surprisingly, we can explicitly describe this set. The description involves the largest volume of a "nice" region supported at x. More precisely, define:

$$\tau_x = \sup\,\{\mathrm{vol}(S_x) \mid S_x = \text{ finite union of cubes} \subset \Delta, \text{ and center of mass of } S_x = x\}$$

Theorem 4.4. *One has:*

$$\Delta(a,b) = \left\{ x \in \Delta \;\middle|\; \frac{\tau_x}{\mathrm{vol}(\Delta)} \ge a \right\} =: \Delta(a).$$

Note that part of the statement of the theorem is that $\Delta(a,b)$ does not depend on b, so we write $\Delta(a)$ for it. The boundary of $\Delta(a)$ is also explicitly computable.

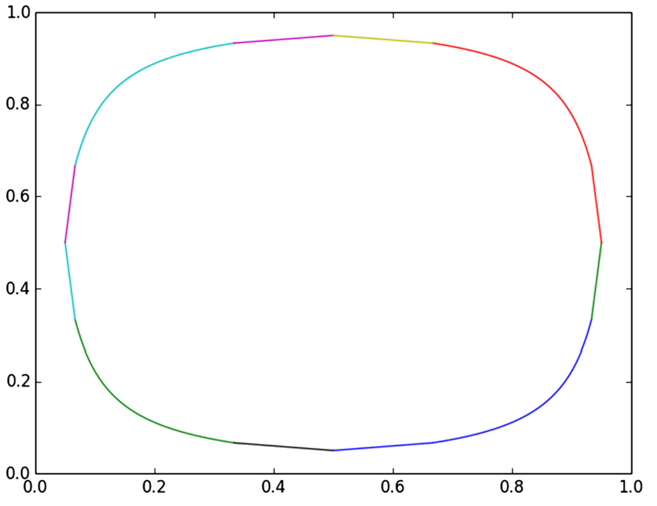

Fig. 3. $\Delta(\frac{1}{10})$ for the unit square.

Example 1.1. For example, let Δ be the unit square. Then boundary of $\Delta(\frac{1}{10})$ consists of 12 pieces, 4 segments of hyperbolas at the corners and 8 line segments in between as illustrated in Fig. 3.

In order to orient the reader, for the rest of the introduction we discuss at some length, the basic strategy of the proof. In the body of the paper, which gives full details, we will refer back to this preview as a roadmap.

Let L be a very ample toric line bundle on a smooth projective toric variety X. As in [5,6] and [2], for L in the evaluation map:

$$\nu_L : H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X(L)$$

we put $M_L = \ker \nu_L$. Thus M_L is a vector bundle sitting in the basic exact sequence:

$$0 \to M_L \to H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X(L) \to 0 \tag{1.1}$$

Thanks to Demazure vanishing (Thm. 9.2.3 [1] and Proposition 2.1), we have Proposition 2.2:

$$K_{p,q}(X; L) = H^q(X, \wedge^{p+q} M_L). \tag{1.2}$$

So the issue is to identify the torus weights appearing in the right hand side of the equality.

To a first approximation, the idea is to find torus equivariant spaces

$$U, \quad W_1, \quad \text{with } \dim U = 1, \quad \dim W_1 \gg 0$$

together with a torus equivariant map:

$$H^q(X, \wedge^{p+q} M_L) \longrightarrow U \otimes \wedge^{p+q} W_1. \tag{1.3}$$

Suppose one knew that (1.3) is surjective. Then we can conclude that every weight appearing in $U \otimes \wedge^{p+q} W_1$ appears in:

$$K_{p,q}(X; L) = H^q(X, \wedge^{p+q} M_L).$$

On the other hand, one can compute combinatorially the weights of $U \otimes \wedge^{p+q} W_1$ from the weights of U and W_1, and the results stated in the previous section would follow.

Strictly speaking, we do not achieve proving surjectivity of (1.3). What we show is that we can find torus stable vector spaces W_0 of small dimension and W_1 of large dimension with W_0 a quotient of W_1 with the following property. Let W be any quotient of W_1 with $\dim W = p + q$ that factors the map to W_0:

$$W_1 \twoheadrightarrow W \twoheadrightarrow W_0.$$

Then there is a surjective mapping:

$$H^q(X, \wedge^{p+q} M_L) \longrightarrow U \otimes \wedge^{p+q} W.$$

As before, this allows us to produce many weights appearing in $K_{p,q}$ and the stated theorem follows.

The next point is to understand how to construct U, W_0 and W_1. Take a w-dimensional torus stable quotient of $H^0(X, L)$ and denote it by W. W defines a toric stable linear subspace:

$$\mathbb{P}(W) \subset \mathbb{P}(H^0(L)).$$

Let $Z \subset X$ be the scheme theoretic intersection:

$$Z = \mathbb{P}(W) \cap X. \tag{1.4}$$

Then there is a natural map:

$$W \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_Z.$$

Taking wedge powers, a local analysis (cf. (2.2)) shows that there is a surjective homomorphism:

$$\wedge^w M_L \longrightarrow \mathcal{I}_{Z/X} \otimes \wedge^w W.$$

Hence, we have a map:

$$H^q(X, \wedge^w M_L) \longrightarrow H^q(X, \mathcal{I}_{Z/X}) \otimes \wedge^w W \tag{1.5}$$

which is also toric equivariant. The goal is to choose W such that:

$$U = H^q(X, \mathcal{I}_{Z/X}) = \mathbb{C} \neq 0. \tag{1.6}$$

Since such W's play the key role of exhibiting non-zero $K_{p,q}$ for us, we define them with the phrase "carrying weight q syzygies" in Section 2 (precise definition in Definition 2.5). In practice, (1.5) is achieved by first choosing a toric stable subspace $Z \subset X$ such that (1.6) holds, and then choosing W to satisfy (1.4). This is carried out in Section 3. Furthermore, we will see in Section 4 that we can take as W quotients of a fixed very large W_1 (Proposition 3.4).

The main technical result of Section 3 is that when we follow this outline, the resulting map (1.5) is surjective (Proposition 2.15). One key point here is that although the map (1.5) is toric, to prove that it is surjective, we do not need to stay in the toric world. Hence, we can follow the inductive arguments in [2] and [7] with essentially no modification.

There is one further asymptotic ingredient. Namely, we are interested in the asymptotics of $K_{p,q}(X; L_d)$ where $L_d = A^{\otimes d}$ and A is very ample. When we go through the constructions just outlined for L_d, we arrive at the following situation.

We have torus stable subspaces $W_{0,d}, W_{1,d}, W_d$:

$$\text{weight}(\wedge^{p+q} W_d) + (\text{some fixed weight}) \in \text{weights}(K_{p,q}(X; L_d))$$

with

$$\dim W_d = p + q, \text{ and } W_{1,d} \twoheadrightarrow W_d \twoheadrightarrow W_{0,d}.$$

Furthermore,

$$\dim(W_{0,d}) \in o(d^n), \text{ and } h^0(X, L_d) - W_{1,d} \in o(d^n).$$

Thus, up to asymptotically insignificant contributions, all the weights of $\wedge^{p+q} W_{1,d}$ appear in $K_{p,q}$. It remains to prove a lemma on asymptotics of normalized weights for wedge powers. This is the content of Section 5. The asymptotic behavior we deduce applies to any sequence of toric quotient spaces W_d asymptotically equal to $H^0(X, L_d)$ in dimension. Then specializing to $W_{1,d}$ gives us a lower bound on the weights that appear by the above discussion. Applying the result to $H^0(X, L_d)$ gives us an upper bound by the definition of Koszul cohomology. Hence, we get our sharp asymptotic description.

I would like to thank my advisor, Rob Lazarsfeld, for his continued suggestions and support. I would also like to thank Daniel Erman, David Speyer and Linquan Ma for helpful discussions and fruitful conversations.

2. Surjectivity of map induced by a secant space

In this section, we adapt the computations in [2] and [7] to the toric case. The reader who is not familiar with the argument in these papers might find it helpful to read the outline of the proof appearing in the end of the previous section. We break the section into two subsections, statement of the key lemma (Lemma 2.3) in Subsection 2.1 and proof of the key lemma in Subsection 2.2. Subsection 2.2 is further divided to two sub-subsections for the construction of Z then W, the two subjects of the key lemma.

2.1. Key lemma

We first recall the key vector bundle used to compute syzygies. Let X be a smooth projective toric variety over $\mathbb{C}$. Let A be a fixed toric very ample line bundle on X. We use L to denote any toric very ample line bundle on X (we will later replace L with $L_d = A^{\otimes d}$).

As in [5,6] and [2], in the evaluation map:

$$\nu_L : H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L$$

we put $M_L = \ker \nu_L$. Then we get the exact sequence mentioned in (1.1):

$$0 \to M_L \to H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0.$$

We will need the following fact in this chapter:

Proposition 2.1. *(Demazure) For any projective toric variety X, and a very ample divisor A, one has:*

$$H^m(X, \mathcal{O}_X(jA)) = 0 \textit{ for } m \geq 1, j \geq 0.$$

Proof. This follows from Demazure vanishing (cf. Thm. 9.2.3 [1]). □

In our setting $L_d = A^{\otimes d}$, we have:

Proposition 2.2. *For $1 \leq q \leq n$, $K_{p,q}(X; L_d) = H^q(X, \wedge^{p+q} M_{L_d})$.*

Proof. The conclusion follows as in [7, Prop. 1.1] and [2, Prop. 3.2, Prop. 3.3] if we know:

$$H^i(X, \mathcal{O}_X(mL_d)) = 0 \text{ for } i > 0, m \geq 0.$$

This follows from the Proposition above. □

Let W be a quotient of $V = H^0(X, L)$ of dimension w. Then we have

$$\mathbb{P}(W) \subset \mathbb{P}(H^0(X, L)).$$

Let

$$Z = \mathbb{P}(W) \cap X$$

be the scheme-theoretic intersection of $\mathbb{P}(W)$ with X. This gives rise to a surjective map of sheaves:

$$W_X = W \otimes \mathcal{O}_X \longrightarrow L \otimes \mathcal{O}_Z,$$

and we denote its kernel by Σ_W. So we get an exact diagram of sheaves:

$$\begin{tikzcd} 0 \arrow[r] & M_L \arrow[r] \arrow[d] & V \otimes \mathcal{O}_X \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Sigma_W \arrow[r] & W \otimes \mathcal{O}_X \arrow[r] & L \otimes \mathcal{O}_Z \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

Through the local analysis of [2] (3.10), we get a diagram:

$$\begin{tikzcd} \bigwedge^w \Sigma_W \arrow[r] \arrow[d] & \bigwedge^w W_X \arrow[d] \\ \mathcal{I}_{Z/X} \otimes \bigwedge^w W_X \arrow[r] & \mathcal{O}_X \otimes \bigwedge^w W_X \end{tikzcd} \tag{2.2}$$

and this induces a surjective map (cf. the map above [2], Def. 3.8):

$$\sigma_\pi : \wedge^w M_L \to \mathcal{I}_{Z/X} \otimes \wedge^w W_X \tag{2.3}$$

Then σ_π induces a map:

$$H^q(X, \wedge^w M_L) \to H^q(X, \mathcal{I}_{Z/X}) \otimes \wedge^w W \tag{2.4}$$

The above works in general without any toric hypothesis. In our setting, when X, L, W are toric, all the above maps are toric equivariant. Following the notations above, the key conclusion of this section is the following lemma:

Lemma 2.3. *For $L = L_d$ with $d \gg 0$ and $1 \leq q \leq n$, there exists a torus stable quotient W with $Z = \mathbb{P}(W) \cap X$ and*

$$H^q(X, I_{Z/X}) \neq 0,$$

such that the induced torus equivariant map:

$$H^q(X, \wedge^w M_L) \to H^q(X, \mathcal{I}_{Z/X}) \otimes \wedge^w W$$

where $w = \dim W$*, is surjective. Therefore, any torus weight in* $H^q(X, \mathcal{I}_{Z/X}) \otimes \wedge^w W$ *also appears in* $K_{w-q,q}(X; L)$*.*

In Section 3, we will show that there are many choices of W, giving many toric weights in $K_{w-q,q}(X; L_d)$.

Remark 2.4. Since torus equivariant has been established as in (2.4), we are only left to prove surjectivity of maps arising through (2.1)–(2.4) as maps of the underlying vector spaces. Hence, we do not write $\otimes \wedge^w W$ any more since $\wedge^w W$ is just $\mathbb{C}$ as a vector space.

Because of Remark 2.4, we will be able to prove the surjectivity of (2.4) by proving the surjectivity of:

$$H^q(X, \wedge^w M_L) \to H^q(X, \mathcal{I}_{Z/X}) \tag{2.5}$$

arising from (2.1). The rest of the proof is technical and follows the same lines of attack as in [7] and [2]. We will give the choice of Z and W later (Lemma 2.7, Proposition 2.15). For now, we introduce some terminology that will help in the induction.

For induction in the proof, we have to add in a twist of the map in 2.5. Let B be a line bundle and consider:

$$H^q(X, \wedge^w M_L(B)) \to H^q(X, \mathcal{I}_{Z/X}(B)). \tag{2.6}$$

At the top level of the induction, B will be trivial and this will reduce from (2.4) to (2.5) because of Remark 2.4.

Definition 2.5. Let W be a quotient of $H^0(X, L)$ as above. We say that W *carries weight* q *syzygies* for B if the map induced by σ_π in equation (2.6) is surjective. (We also say the same for $q = 0$ for notational convenience even though it isn't necessarily directly related to syzygies.)

Remark 2.6. The main use of the property in this definition is its inductivity (Proposition 2.13). Our inductive base case start with the property being trivially true in a low dimension with analogs of $H^q(X, I_{Z/X}(B))$ being 0.

2.2. Proof of Lemma 2.3

2.2.1. Construction of Z

Let us set up some inductive notations to set up Z for Lemma 2.3 first. Take a general divisor $\overline{X} \in |A|$ so that $\overline{X}$ is irreducible and diagram (2.1) remains exact after tensoring with $\mathcal{O}_{\overline{X}}$. For $0 \le i \le q-1$, let

$$X_0 = X, \quad Z_0 = Z, \quad A_0 = A.$$

Having made the definitions for $i-1$, choose a general $X_i \in |A_{i-1}|$ so that X_i is irreducible and the corresponding diagram (2.1) for X_{i-1} remains exact after tensoring with $\mathcal{O}_{X_i}$ (and as previously defined, $\overline{X} = X_1$). Let

$$Z_i = Z_{i-1} \cap X_i, \quad A_i = A_{i-1}|_{X_{i-1}}.$$

Now we construct a toric Z in our smooth projective toric variety X that satisfies the good properties in Definition 2.11. Let $-K_X = e_1 + ... + e_m$ where $\{e_i\}$ are the prime toric invariant divisors. Let $c = n + 1 - q$ with $1 \le q \le n$.

Lemma 2.7. *We can order the e_i such that $Z = e_1 \cap ... \cap e_{c-1} \cap (e_c + ... + e_m)$ is a complete intersection.*

Proof. Choose $e_1, ..., e_n$ such that they generate an n-dimensional cone. Then $F = e_1 \cap ... \cap e_{c-1}$ is a complete intersection. For any $i > c-1$, F either does not meet e_i, or it does so transversely since adding a ray to a cone increase its dimension by at most 1. It meets at least one of them, e_c, since $c \le n$. $\square$

Next, we establish a number of properties of Z.

Proposition 2.8. *With the above choice of Z:*

$$H^q(X, I_{Z/X}) = \mathbb{C} \neq 0$$

Proof. If $q = 1$, then $c = n$, and in this case, since the variety is complete, $e_1, \dots, e_{n-1}$ span an $(n-1)$-dimensional face that sits on two n-dimensional cones in the fan. The intersection of the corresponding $\mathbb{P}^1$ with the other prime T-invariant divisors is the two points corresponding to these cones, so Z consists of two points.

The short exact sequence

$$0 \to \mathcal{I}_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

induces:

$$H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_Z) \to H^1(\mathcal{I}_{Z/X}) \to H^1(\mathcal{O}_X)$$

where $h^0(\mathcal{O}_X) = 1$, $h^0(\mathcal{O}_Z) = 2$ and $h^1(\mathcal{O}_X) = 0$ (since structure sheaves of toric varieties do not have higher cohomology). Hence, $H^1(\mathcal{I}_{Z/X}) = \mathbb{C}$.

Assume $q \geq 2$ and recall that $c = n + 1 - q$. From

$$0 \to \mathcal{I}_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

we get:

$$0 = H^{q-1}(\mathcal{O}_X) \to H^{q-1}(\mathcal{O}_Z) \to H^q(\mathcal{I}_{Z/X}) \to H^q(\mathcal{O}_X) = 0.$$

Then $H^{q-1}(\mathcal{O}_Z) = H^q(\mathcal{I}_{Z/X})$.

Let $F = e_1 \cap e_2 ... \cap e_{c-1}$. From

$$0 \to \mathcal{I}_{Z/F} \to \mathcal{O}_F \to O_Z \to 0.$$

We once again apply Demazure vanishing (Thm. 9.2.3 [1]) with $q - 1 \geq 1$ and noting that the trivial divisor is nef. We have:

$$0 = H^{q-1}(\mathcal{O}_F) \to H^{q-1}(\mathcal{O}_Z) \to H^q(\mathcal{I}_{Z/F}) \to H^q(\mathcal{O}_F) = 0$$

We need to compute $H^q(\mathcal{I}_{Z/F})$. Now $Z = F \cap (e_c + ... + e_m)$ and $e_c + ... + e_m = -K_F$. Since

$$\mathcal{I}_{Z/F} = \mathcal{O}_F(K_F), \quad \dim F = n - c - 1 = n - (n - q) = q,$$

and then Serre duality (cf. Thm. 9.0.9 [1]) applies and we have:

$$H^q(\mathcal{I}_{Z/F}) = H^0(\mathcal{O}_F) = \mathbb{C}. \quad \square$$

Proposition 2.9. *For all $m \geq 1, j \geq i \geq 0$:*

(1) $H^m(X_i, \mathcal{O}_{X_i}(jA)) = 0.$
(2) $H^m(Z_i, \mathcal{O}_{Z_i}(jA)) = 0.$

Proof. We prove the first assertion by induction on i. When $i = 0$, the conclusion follows from Demazure vanishing since X is toric. Suppose the conclusion is true for $i - 1$, then we have:

$$0 \to \mathcal{O}_{X_{i-1}}((j-1)A) \to \mathcal{O}_{X_{i-1}}(jA) \to \mathcal{O}_{X_i}(jA) \to 0.$$

$H^m(\mathcal{O}_{X_{i-1}}(jA)) = H^{m+1}(\mathcal{O}_{X_{i-1}}((j-1)A)) = H^{m+1}(\mathcal{O}_{X_{i-1}}(jA)) = 0$ by inductive assumption, hence $H^m(\mathcal{O}_{X_i}(jA)) = 0$. The second assertion is analogous. $\square$

Proposition 2.10. *For all $i \geq 0$:*

$$H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}((i+1)A)) = 0$$

Proof. Consider on X_i the exact sequence:

$$0 \to \mathcal{I}_{Z_i/X_i}((i+1)A) \to \mathcal{O}_{X_i}((i+1)A) \to \mathcal{O}_{Z_i}((i+1)A) \to 0$$

giving rise to:

$$H^{q-i-1}(\mathcal{O}_{X_i}((i+1)A)) \to H^{q-i-1}(\mathcal{O}_{Z_i}((i+1)A)) \to$$
$$H^{q-i}(\mathcal{I}_{Z_i/X_i}((i+1)A)) \to H^{q-i}(\mathcal{O}_{X_i}((i+1)A))$$

If $q-i=1$, then Z_i has dimension $\dim Z - i = \dim Z - (q-1) = n-(n+1-q)-(q-1) = q-1-q+1 = 0$. Then very ampleness and $H^{q-i}(\mathcal{O}_{X_i}((i+1)A)) = 0$ from Proposition 2.9 implies that $H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}((i+1)A)) = 0$. Assume $q-i-1 \geq 1$, then the two ends in the above sequence are 0 because of Proposition 2.9 and we get:

$$H^{q-i}(\mathcal{I}_{Z_i/X_{i-1}}((i+1)A)) = H^{q-i-1}(\mathcal{O}_{Z_i}((i+1)A)) = 0,$$

as claimed. □

Definition 2.11. We say that Z is *adapted* to the data X, B, A, n, q, if:

(1) $H^q(X, \mathcal{I}_{Z/X}(B)) \neq 0$.
(2) For all $i \geq 0$, $H^{q-i}(X_i, \mathcal{I}_{Z_i/X_i}(B + (i+1)A))) = 0$.
(3) For all $i \geq 0$, Z_i has dimension $q-1-i$.

Putting the computations of cohomologies of Z together, we obtain:

Proposition 2.12. *For any $1 \leq q \leq n$, the scheme Z constructed above is adapted to $X, \mathcal{O}_X, A, n, q$*

Proof. Choosing the divisors as in Proposition 2.7, Definition 2.11 (iii) follows from the complete intersection condition. Definition 2.11 (i), (ii) are checked in Proposition 2.8, 2.10. □

2.2.2. Construction of W

Having constructed Z, we next turn to the construction of quotients W as in Definition 2.5. The issue is to specify inductive conditions that will guarantee that the condition in that definition holds. Recall that

$$V = H^0(X, L)$$

and W is a quotient of V (after Proposition 2.2) with the quotient map:

$$\pi : V \to W.$$

Let

$$V' = H^0(X, I_{\overline{X}/X}(L)) \subseteq V, \quad W' = \pi(V') \tag{2.7}$$

Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & V' = H^0(X, I_{\overline{X}/X}(L)) \arrow[r] \arrow[d, "\pi'"] & V = H^0(X, L) \arrow[r] \arrow[d, "\pi"] & \overline{V} \arrow[r] \arrow[d, "\overline{\pi}"] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & \overline{W} \arrow[r] & 0 \end{tikzcd} \tag{2.8}$$

where we define:

$$\overline{V} = V/V', \ \overline{W} = W/W', \ \overline{L} = L|_{\overline{X}}, \ \overline{B} = B|_{\overline{X}}, \ \overline{Z} = Z \cap \overline{X}. \tag{2.9}$$

As in [2, (3.14)], we get the analogue of (1.3) above for the barred objects and we have the surjection:

$$\overline{\sigma} : \wedge^{\overline{w}} M_{\overline{V}} \to I_{\overline{Z}/\overline{X}},$$

so we can study the behavior of $\overline{W}$ with respect to carrying syzygies.

Lemma 2.13. *Fix $1 \leq q \leq n$. If $\overline{W}$ carries weight $q-1$ syzygies for $\overline{B} + \overline{A}$ on $\overline{X}$ and if*

$$H^q(X, I_{Z/X}(B + A)) = 0,$$

then W carries weight q syzygies for B on X.

Proof. This follows from the same argument as [2], Thm. 3.10 with $(q-1)$ replaced by q and $B \otimes L$ with B in our case. $\square$

Proposition 2.14. *If $d \gg 0$, then the following statements are true and so are their inductive counterparts after cutting down by hyperplanes as above:*

(1) *The map $H^0(X, L_d) \to H^0(Z, L_d)$ is surjective; equivalently:*

$$H^1(X, I_{Z/X}(L_d)) = 0.$$

(2) *The map $H^0(Z, L_d) \to H^0(\overline{Z}, L_d)$ is surjective; equivalently:*

$$H^1(Z, L_d - A) = 0.$$

(3) $H^1(X, I_{Z/X}(L_d - A)) = 0$ *(or equivalently, with W' chosen below, the map $V' \to W'$ is surjective).*

(4) *The map $H^0(X, L_d) \to H^0(\overline{X}, L_d)$ is surjective, or equivalently*

$$H^1(X, L_d - A) = 0.$$

(5) *$I_{Z/X} \otimes \mathcal{O}_X(L_d)$ is globally generated.*

Proof. These all follow from Serre vanishing. □

Proposition 2.15. *Fix $1 \leq q \leq n$. Suppose there exists a subscheme Z of X adapted to X, B, A, n, q. Take $W_d = H^0(X, \mathcal{O}_Z(L_d))$. Then for $d \gg 0$, W_d carries weight q syzygies for B.*

Proof. Start with $W = W_d = H^0(Z, L_d)$. By the definitions in equation (2.7), (2.9) and surjectivity from Proposition 2.14, we have:

$$\begin{tikzcd}
0 \arrow[r] & V' = H^0(X, L_d - A) \arrow[r] \arrow[d] & V = H^0(X, L_d) \arrow[r] \arrow[d] & \overline{V} = H^0(\overline{X}, L_d) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W' \arrow[r] \arrow[d] & W = H^0(Z, L_d) \arrow[r] \arrow[d] & \overline{W} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.10}$$

where

$$W' = H^0(Z, L_d - A), \ \overline{W} = H^0(\overline{Z}, L_d). \tag{2.11}$$

The sheaf $I_{Z/X} \otimes \mathcal{O}_X(L_d)$ is globally generated (Proposition 2.14 (v)) so $Z = \mathbb{P}(W) \cap X$. Moreover, when we cut down by hyperplanes as in Lemma 2.13, we obtain the corresponding diagrams in lower dimensions.

We prove the Proposition by induction on q. Z is always of dimension $q - 1$. When $q = 1, Z$ consists of points. If X is of dimension 1, then surjectivity follows from the fact that sheaf surjective maps imply surjectivity in H^1 since there is no H^2. If the dimension of X is at least 2, then we continue the induction with $\overline{Z} = \phi, \overline{W} = 0$. So the conclusion is trivially true for $q = 0$. Then the conclusion is true for $q = 1$ by Proposition 2.14 (ii) and Lemma 2.13. Then apply Lemma 2.13 repeatedly. □

3. Enlarged secant space

In this section, our key conclusion (Proposition 3.4) is the following. Recall that $L_d = dA$ for some very ample A. Let $\mathrm{wts}(U)$ denote the torus weights in a toric representation U. In the setting of syzygies, we prove that there exist torus stable quotient spaces $W_{0,d}, W_{1,d}$:

$$H^0(X, L_d) \twoheadrightarrow W_{1,d} \twoheadrightarrow W_{0,d},$$

with:

$$\dim(W_{0,d}) \in o(d^n), \text{ and } h^0(X, L_d) - W_{1,d} \in o(d^n),$$

such that for any W_d with:

$$\dim W_d = p + q, \text{ and } W_{1,d} \twoheadrightarrow W_d \twoheadrightarrow W_{0,d},$$

we have:

$$\mathrm{weight}(\wedge^{p+q} W_d) + \text{some fixed weight} \in \mathrm{weights}(K_{p,q}(X; L_d)).$$

As explained in the proof outline in the Introduction, this will let us produce many different weights in $K_{p,q}(X; L_d)$ when we vary W_d.

Lemma 3.1. *Let X be a scheme with A a very ample divisor and $L_d = dA$. Let Z be a subscheme and $\{E_i\}$ a collection of divisors such that $Z = \cap_i E_i$. Then there exists a subspace $\mathcal{J}_d \subset H^0(X, L_d)$ such that $\mathcal{J}_d$ generates $\mathcal{I}_{Z/X}(L_d)$ and $\dim \mathcal{J}_d \in O(1)$.*

Proof. If $\mathcal{J}_d^i \subset H^0(X, L_d)$ generates $\mathcal{I}_{E_i}(L_d)$, then $\sum_i \mathcal{J}_d^i$ generate $\mathcal{I}_{Z/X}(L_d)$, so we can assume $Z = E$ is a divisor. For some large N, $\mathcal{I}_E(L_N)$ is globally generated by sections $F_1, ...F_{m_1}$. Assume A is globally generated by sections $s_1, .., s_{m_2}$. Then for $d > N$, $\mathcal{I}_E(L_d) = \mathcal{O}_X(L_d - E)$ is globally generated by $\mathcal{J}_d =< F_i \cdot s_j^{d-N} >$. It has finitely many vector space generators, so $\dim \mathcal{J}_d$ is finite. □

Remark 3.2. It is straightforward to see that the above lemma is still true if we start with torus equivariant objects and want torus equivariant $\mathcal{J}_d$'s.

Proposition 3.3. *There exist torus equivariant quotients $W_{0,d}, W_{1,d}$ of $H^0(X, L_d)$, such that*

$$\overline{W_{0,d}} = \overline{W_{1,d}} = \overline{W_d}, \quad Z = X \cap \mathbb{P}(W_{0,d}) = X \cap \mathbb{P}(W_{1,d}) \tag{3.1}$$

and the dimensions satisfy

$$\dim W_{0,d} \in o(h^0(X, L_d))$$

and

$$\lim_{d\to\infty} \frac{\dim W_{1,d}}{h^0(X, L_d)} = 1$$

Moreover, $W_{0,d}$ and $W_{1,d}$ carry weight q syzygies for $\mathcal{O}_X$.

Proof. Pick $W_{0,d} = W_d$. This satisfies the conditions. To construct $W_{1,d}$, we will vary W_d while keeping Z and $\overline{W_d}$ the same, i.e. we look for large dimension quotients $W_{1,d}$ of $H^0(X, L_d)$ such that:

$$Z = X \cap \mathbb{P}(W_{1,d}) \tag{3.2}$$

and

$$\overline{W_{1,d}} = \overline{W_{0,d}}. \tag{3.3}$$

By the argument of Proposition 2.15, if the $W_{1,d}$ satisfy the above conditions, then the $W_{1,d}$ also carry weight q syzygies. We first construct $W_{1,d}$ such that they satisfy the conditions in (3.2) and (3.3) and then do a dimension count.

Let $W_{1,d}$ be the quotient of $V_d = H^0(X, L_d)$ by $\mathcal{J}_{1,d}$, i.e.

$$W_{1,d} = V_d/\mathcal{J}_{1,d}.$$

To satisfy (3.2), $\mathcal{J}_{1,d}$ has to generate $\mathcal{I}_{Z/X}(L_d)$. By Lemma 3.1, we can pick torus equivariant $\mathcal{J}_{1,d}$ of bounded dimension. Let

$$V'_d = \ker(V_d \to \overline{V_d}), \quad J_d = \ker(V_d \to W_d).$$

To satisfy (3.3), for the consecutive quotient maps:

$$V_d \to W_{1,d} \to W_d,$$

we need:

$$V'_d + J_d = V'_d + \mathcal{J}_{1,d} \tag{3.4}$$

We start with a toric basis of V_d, then V'_d, J_d both have induced toric basis elements. Denote by $B(V'_d)$ the toric basis elements of V'_d, and $B(J_d)$ those of J_d. Then we can choose $\mathcal{J}_{1,d}$ to have a basis containing $B(V'_d) - B(J_d)$, but contained in $B(V'_d) \cup B(J_d)$. $\mathcal{J}_{1,d}$ will be toric equivariant and we have the dimension count:

$$\begin{aligned}\dim(V'_d + J_d) - \dim V'_d &= \dim V_d - \dim \overline{W_d} - \dim V'_d \\ &= \dim \overline{V_d} - \dim \overline{W_d} \\ &\leq \dim \overline{V_d}.\end{aligned}$$

Hence (3.4) requires the $\mathcal{J}_{1,d}$ to be appropriate subspaces of $V'_d + \mathcal{J}_d$ with the following range of dimensions:

$$\dim \overline{V_d} \ \leq \ \dim \mathcal{J}_{1,d} \ \leq \ \dim(V'_d + J_d).$$

Note that $\dim \overline{V_d} \in o(d^n)$. Therefore, to satisfy both (3.2) and (3.3), we can choose $\mathcal{J}_{1,d}$ such that $\dim \mathcal{J}_{1,d} \in o(d^n) + O(1)$. Since $W_{1,d} = V_d/\mathcal{J}_{1,d}$, $W_{1,d}$ will be torus equivariant and we have $\dim W_{1,d} = \dim V_d - \dim \mathcal{J}_{1,d}$. Then $h^0(X, L_d) \in \Theta(d^n)$ imply that

$$\lim_{d\to\infty} \frac{\dim W_{1,d}}{h^0(X, L_d)} = 1. \quad \square$$

Finally, we conclude:

Proposition 3.4. *For any torus equivariant W_d that fits in the following diagram of consecutive equivariant quotient maps:*

$$W_{1,d} \to W_d \to W_{0,d}$$

we have:

$$\mathrm{wts}(\bigwedge^{\dim W_d} W_d) + \mathrm{wts}(H^q(X, I_{Z/X})) \subseteq \mathrm{wts}(K_{\dim W_d - q, q}(X; L_d)) \tag{3.5}$$

Proof. By the argument of Proposition 2.15, W_d carries weight q syzygies. Then the weight inclusions follow from Lemma 2.3. $\square$

Remark 3.5. Note that dimension-wise, $\bigwedge^{\dim W_d} W_d$ is one dimensional, so $\mathrm{wts}(\bigwedge^{\dim W_d} W_d)$ has one element. One may be able to construct Z's satisfying the assumptions of the proposition with a multidimensional $H^q(X, I_{Z/X})$ and the proposition is true in that generality. In this paper however, the choice of Z makes $H^q(X, I_{Z/X})$ one dimensional (cf. Proposition 2.8), then the above proposition reduces to:

$$\mathrm{weight}(\bigwedge^{\dim W_d} W_d) + \mathrm{weight}(H^q(X, I_{Z/X})) \in \mathrm{weight}(K_{\dim W_d - q, q}(X; L_d))$$

4. Asymptotics of normalized weights

In this section, we prove the main result. As explained in the Introduction, one of the issues is to study torus weights of $\wedge^p W_d$ given the weights of W_d. Recall that we defined in the Introduction:

$$\mathrm{wts}(K_{p,q}(X;L_d)) = \{\text{Torus weights of } K_{p,q}(X;L_d)\} \subseteq (p+q)d\cdot\Delta,$$

where Δ is the convex polytope associated with the very ample divisor A. We are interested in the normalized weights:

$$\mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X;L_d)) = \frac{\mathrm{wts}(K_{p,q}(X;L_d))}{(p+q)d} \subseteq \Delta$$

In this section, we work asymptotically and we are interested in asymptotic closures:

$$\Delta(a,b) \;=\; \overline{\bigcup_{\substack{d\gg 0\\ a\cdot r_d\le p\le b\cdot r_d}} \mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X;L_d))} \subseteq \;\Delta$$

Keeping notation from previous sections, the contributions from weights in $H^q(X,\mathcal{I}_{Z/X})$ and $W_{0,d}$ will be asymptotically insignificant to normalized weights, in other words:

$$\begin{aligned}\Delta(a,b) &\supseteq \overline{\bigcup_{\substack{d\,\gg\,0\\ a\cdot r_d\,\le\,p\,\le\,b\cdot r_d}} \mathrm{wts}^{\mathrm{nor}}\left(\bigwedge^{\dim W_d} W_d + H^q(X,I_{Z/X})\right)} \\ &= \overline{\bigcup_{\substack{d\,\gg\,0\\ a\cdot r_d\,\le\,p\,\le\,b\cdot r_d}} \mathrm{wts}^{\mathrm{nor}}\left(\bigwedge^{p} W_{1,d}\right)}\end{aligned}$$

Hence, to give a lower bound for $\Delta(a,b)$, we can simply work with the sequence $\{W_{1,d}\}$ in (3.5) and for simplicity, we abuse notation and write W_d instead of $W_{1,d}$.

After this simplification, we arrive at the following setting. Let $\Delta \subseteq \mathbb{R}^n$ be the convex polytope associated to a very ample divisor A on X. For $d\in\mathbb{N}$, let

$$W_d \subset d\Delta\cap\mathbb{Z}^n$$

be a subset. Let us use $\bigwedge^{p_d} W_d$ to denote the collection of points in $\mathbb{Z}^n$ expressible as nonrepetitive sums of p_d points in W_d. Assume that for $d\in\mathbb{N}$,

$$\lim_{d\to\infty}\frac{|W_d|}{|d\Delta|} = 1 \tag{4.1}$$

Take any point x inside the polytope Δ. For simplicity, we call the finite union of cubes contained in Δ a cube union. Recall the definition we made before stating Theorem 4.4 in the introduction:

$$\tau_x = \sup\,\{\mathrm{vol}(\mathbb{S}_x)\mid \mathbb{S}_x \text{ is a cube union } \subset\Delta,\ \text{center of mass of } \mathbb{S}_x = x\}$$

The intuition for the two key (Lemmas 4.1, 4.3) in this section are as follows.

(1) If there is a cube union of volume η_x centered at x, we can pick any number between 1 and $(\eta_x - \epsilon)d^n$ such that the average of these weights lie arbitrarily close to x asymptotically.
(2) Any sequence of subsets of lattice weights averaging to x cannot exceed τ_x many elements.

Lemma 4.1. *With the above notations, for any sequence $\{p_d\}$ such that*

$$1 \le p_d \le (\eta_x - \epsilon)d^n$$

and any open set $U_x \subset \mathbb{R}^n$ containing x, there exists d_0 such that for all $d > d_0$, U_x contains a point of

$$\frac{\mathrm{wts}^{\mathrm{nor}}(\bigwedge^{p_d} W_d)}{p_d \cdot d}$$

Sketch of proof. For all d such that

$$p_d \le \frac{1}{2}|(dU_x) \cap \mathbb{Z}^n|,$$

there are at least p_d points in $dU_x \cap W_d$ and the average of these weights normalized will be in U_x. So we can assume $p_d > \frac{1}{2}|(dU_x) \cap \mathbb{Z}^n|$. The exact constant $\frac{1}{2}$ is immaterial, we only need

$$p_d > \mathrm{const} \cdot d^n$$

Choose a finite union of cubes S_x with volume η_x, center of mass x. Note first, by assumption (4.1), W_d contains almost all weights in $d\Delta$ as $d \to \infty$, so let's consider all of the lattice points first. Denote by W'_d, all lattice points in dS_x and $p'_d = |W'_d|$, then we have:

$$\frac{\mathrm{wts}^{\mathrm{nor}}(\bigwedge^{p'_d} W'_d)}{p'_d \cdot d} \to x$$

by the definition of center of mass. It remains to show that if we scale down dS_x such that it has p_d lattice points instead of p'_d points, the above sequence still converges to x. This argument is elementary and not very illuminating, so I will leave it to the interested reader. From these two facts together, we conclude that for any open neighborhood, we can always find a normalized average weight in the neighborhood for large enough d. $\square$

Theorem 4.2. *Fix $1 \le q \le n$. Then*

$$\bigcup_{\substack{d > 0 \\ 1 \le p \le r_d}} \mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X; L_d))$$

is dense in Δ.

Proof. By Proposition 3.3 and 3.4, we know that $\text{wts}^{\text{nor}}(K_{p,q}(\mathbb{P}; L_d))$ contains weights corresponding to nonrepetitive sums of weights in $W_{1,d}$. By Proposition 3.3 and Lemma 4.1, they are dense in Δ. □

Lemma 4.3. *Take $y \in \Delta$. Assume*

$$y = \lim_{d\to\infty} y_d \tag{4.2}$$

where y_d is an average of $w_{d,j} \in (d\cdot \Delta) \cap \mathbb{Z}^n$, i.e.:

$$y_d = \frac{1}{i_d} \sum_{1\le j\le i_d} \frac{w_{d,j}}{d}, \quad w_{d,j} \textit{ distinct}, \quad \textit{and } i_d = |\{w_{d,j}\}|.$$

Then for any subsequence $\{i_{d_j}\}$:

$$\limsup_{j\to\infty} \frac{i_{d_j}}{d^n} \le \tau_y$$

Proof. Suppose there is a subsequence $i_{d'_j}$ such that

$$\liminf_{j\to\infty} \frac{i_{d'_j}}{d^n} \ge \tau.$$

We claim that for any constant $\epsilon > 0$, there is a cube union centered at y with volume at least $\tau - \epsilon$. Assuming the claim, the lemma follows since if we had

$$\limsup_{j\to\infty} \frac{i_{d_j}}{d^n} = \tau > \tau_y,$$

then there is a subsequence $i_{d'_j}$ such that

$$\liminf_{j\to\infty} \frac{i_{d'_j}}{d^n} = \frac{\tau + \tau_y}{2} > \tau_y$$

By the claim, this is a cube union with center of mass at y and volume $\frac{\tau+\tau_y}{2} > \tau_y$. This is a contradiction to τ_y being the biggest volume supported at y.

Let's turn to the claim. For convenience, we write i_j for i_{d_j}. We construct a cube union centered at y with volume at least $\tau - \epsilon$ by taking small cubes centered at $w_{d_j,i}$, then make two adjustments: boundary adjustment and center adjustment.

More specifically, for each $w_{d_j,i}$ with $1 \le i \le d_j$, take the $1 \times ... \times 1$ cubes of dimension n centered at $w_{d_j,i}$. Denote each cube by $C_{d_j,i}$. Suppose the $2 \times ... \times 2$ cube centered at $w_{d_j,i}$ does not intersect the boundary of $d\Delta$ for $1 \le i \le i'_d$. The boundary of $d\Delta$ is bounded by $d(1-\epsilon)\Delta$ and $d(1+\epsilon)\Delta$ for large d and any $\epsilon > 0$. Hence, $i_d - i'_d \in o(d^n)$.

Take the union:

$$\Sigma'_d = \bigcup_{i=1}^{i'_d} C_{d_j,i}$$

and denote by y' the center of mass for Σ'_d. Δ is bounded, so by the previous paragraph and assumption:

$$|y' - y| \leq |y' - y_d| + |y_d - y| \in o(d^n) + O(1).$$

Move Σ'_d by:

$$\frac{y - y'}{i'_d}$$

Then it is straightforward to see that for large d, the cube union above is contained in Δ and has volume arbitrarily close to τ. $\square$

Recall that we define:

$$\Delta(a,b) \; = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X; L_d))} \subseteq \; \Delta$$

Theorem 4.4. *One has:*

$$\Delta(a,b) = \left\{ x \in \Delta \;\middle|\; \frac{\tau_x}{\mathrm{vol}(\Delta)} \geq a \right\} =: \Delta(a)$$

Proof. The inclusion direction, i.e. $\Delta(a,b) \supseteq \left\{ x \in \Delta \;\middle|\; \frac{\tau_x}{\mathrm{vol}(\Delta)} \geq a \right\}$ follows from Lemma 4.1. Apply the lemma with $p_d = a \cdot r_d$, we know that as long as $a \cdot r_d < \tau_x \cdot d^n$, x will be the limit of a sequence of normalized weights. Since we have $\frac{r_d}{d^n} = \mathrm{vol}(\Delta)$, we have the condition on the left for the closure.

The closure $\Delta(a,b) \supseteq \left\{ x \in \Delta \;\middle|\; \frac{\tau_x}{\mathrm{vol}(\Delta)} \geq a \right\}$ follows from a similar argument as in Theorem 4.2 replacing a sphere with cube unions supported at x with volumes arbitrarily close to τ_x. For sequences asymptotically small, we can pick points in a sphere around x. For sequences asymptotically close to τ_x, we can pick points from finite cube unions (possibly further scaled down) approximating τ_x.

As is well known (first in [4], (0.1)–(0.4)), $K_{p,q}(X; L_d)$ can be computed as cohomology in the middle of the following short complex:

$$\begin{aligned} \wedge^{p+1} H^0(X, L_d) \otimes H^0(X, (q-1)L_d) \to \wedge^{p} H^0(X, L_d) \otimes H^0(X, qL_d) \\ \to \wedge^{p-1} H^0(X, L_d) \otimes H^0(X, (q+1)L_d) \end{aligned}$$

Hence all weights of $K_{p,q}(X, L_d)$ correspond to nonrepetitive sums of points in $(p+q)\Delta$. The length of the above Koszul complex is $h^0(X, L_d) = \mathrm{vol}(\Delta) \cdot d^n + O(d^{n-1})$. q is bounded, so:

$$\overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \mathrm{wts}^{\mathrm{nor}}(\wedge^{p+1} H^0(X, L_d) \otimes H^0(X, (q-1)L_d))} = \overline{\bigcup_{\substack{d \gg 0 \\ a \cdot r_d \leq p \leq b \cdot r_d}} \mathrm{wts}^{\mathrm{nor}}(\wedge^{p+1} H^0(X, L_d))},$$

and similarly for the other terms in the Koszul complex. By Lemma 4.3, if a normalized weight is in

$$\overline{\mathrm{wts}^{\mathrm{nor}}(\wedge^{p_d} H^0(X, L_d))},$$

then it satisfies

$$\tau_x \geq \limsup_{d \to \infty} \frac{p_d}{d^n}$$

We are computing for the range $[a\mathrm{vol}(\Delta) d^n, b\mathrm{vol}(\Delta) d^n]$, hence

$$\limsup_{d \to \infty} \frac{p_d}{d^n} \geq b\mathrm{vol}(\Delta),$$

so

$$x \geq a\mathrm{vol}(\Delta).$$

Then by the definition of $K_{p,q}(X, L_d)$ above:

$$\overline{\mathrm{wts}^{\mathrm{nor}}(K_{p,q}(X, L_d))} \subseteq \overline{\mathrm{wts}^{\mathrm{nor}}(\wedge^p H^0(X, L_d) \otimes H^0(X, qL_d))} \subseteq \left\{ x \in \Delta \;\middle|\; \frac{\tau_x}{\mathrm{vol}(\Delta)} \geq a \right\}$$

which gives us the other inclusion. $\square$

Remark 4.5. Note that for any $x \in \Delta$, and $a \in [0, 1]$, the two inclusions of the proof above implies that there is a normalized weight sequence converging to x if and only if

$$a \cdot r_d < \tau_x \cdot d^n.$$

Hence, the area of the closure of all such x decreases as a increases, i.e. the closure gets strictly smaller and smaller as a increases. Therefore, we observe that $\Delta(a, b)$ depends only on a, not on b simply because all the points in the closure can already be found around $a \cdot r_d$. Points closer to $b \cdot r_d$ give rise to a proper subset.

5. Boundary of $\Delta(a)$

In this section, we describe the boundary of $\Delta(a)$.

Fix a convex body $\Delta \subset \mathbb{R}^n$ with a piecewise polynomial boundary, and a constant a in $[0, 1]$. Let v be a unit vector in $\mathbb{R}^n$. Then $H(v, c) := \{x \in \mathbb{R}^n | v.x \leq c\}$ forms a family of parallel half spaces. There is a unique constant c so that:

$$\mathrm{vol}(\Delta \cap H(v, c)) = a\mathrm{vol}(\Delta).$$

Call this constant c_v. Let x_v be the center of gravity of $\Delta \cap H(v, c_v)$.

Proposition 5.1. *The points x_v, as v ranges over all unit vectors, form the boundary of $\Delta(a)$.*

Remark 5.2. We give the intuition but skip the proof of the proposition. Think of Δ as a container holding water of volume a. When one leans it to the direction with v pointing downward, water flows to lower its center of mass (potential energy). In this position, the center of mass, call it x_v, is the extreme of Δ in direction v, hence a boundary point of $\Delta(a)$. The water level in this setting corresponds to the boundary of $H(v, c_v)$ and we can actually see that the tangent of the boundary is the hyperplane perpendicular to v through x_v.

Example 5.3. As in Example 1.1, when Δ is the unit square and $a = \frac{1}{10}$. The interested reader can work out that when the water surface divides the square into a triangle and a pentagon, the center of mass of the water is parametrized by

$$(1 - \frac{1}{15k}, \frac{1}{3}k) \text{ for } \frac{1}{5} \leq k \leq 1$$

and its symmetric images, these account for the 4 hyperbola segments. When the water surface divides the unit square into two trapezoids, the center of mass lies on:

$$(\frac{2}{3} - \frac{5}{3}k, \frac{29}{30} - \frac{1}{6}k) \text{ for } \frac{1}{10} \leq k \leq \frac{1}{5}$$

or its symmetric images. These account for the 8 line segments.

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