



Valuative dimension and monomial orders

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ABSTRACT

The main result from this note provides a constructive characterization of the valuative dimension, which bears a strong analogy to Lombardi's [7] constructive characterization of the Krull dimension. While Lombardi's characterization uses the lexicographic monomial order, ours uses the graded (reverse) lexicographic order or, in fact, any graded rational monomial order. Apart from this, the paper contains some related results and some examples which readers may find illuminating.

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Introduction

In 2002 Lombardi [7] characterized the Krull dimension of a commutative ring \mathbf{A} by means of certain relations between the elements of \mathbf{A} . More precisely, he showed that for a positive integer n , we have $\dim(\mathbf{A}) < n$ if and only if all elements $a_1, \dots, a_n \in \mathbf{A}$

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satisfy a relation $P(a_1, \dots, a_n) = 0$, where $P \in \mathbf{A}[X_1, \dots, X_n]$ is a polynomial whose lexicographically smallest monomial has coefficient 1. Twelve years later, the first author and Trung [5] showed that if \mathbf{A} is Noetherian, then this result extends to any monomial order, not just the lexicographic one.

The research for this note started as an attempt to show that the hypothesis on Noetherianness can be dropped from the results in [5]. However, instead of a proof, we found some counterexamples, which are presented in this paper. These examples seemed to suggest that the valuative dimension had some role to play, and indeed we ended up showing that any graded rational monomial order (which we define, in a rather obvious way, in Section 1) measures the valuative dimension in precisely the same way as the lexicographic order measures the Krull dimension by Lombardi's result. This is the content of Theorem 6 of this paper, providing a constructive characterization of the valuative dimension.

A different constructive characterization of the valuative dimension was given by Coquand [1]. This characterization is in terms of distributive lattices and seems to be much less elementary than the one given in this note. Moreover, Coquand's characterization is restricted to the case of integral domains, whereas ours does not require this hypothesis.

Apart from the characterization of the valuative dimension, this note contains a few other results, which can be summarized as follows, using the language explained in Section 1.

- For a rational monomial order $<$, the maximal number of $<$ -independent elements of a ring lies between its Krull dimension and its valuative dimension (Theorems 4 and 6(a)).
- But if $<$ is only a rational preorder, the maximal number of $<$ -independent elements of a ring may be smaller than its Krull dimension (Proposition 5).
- If $<$ is an irrational monomial order, the maximal number of $<$ -independent elements of a ring may be smaller than the Krull dimension or larger than the valuative dimension (Propositions 5 and 7).
- For a local ring, the maximal number of analytically independent elements does not exceed the valuative dimension (Corollary 9).

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1. Preliminaries

In this note a *ring* is always understood to be a commutative ring with unity.

We will be working with monomial orders on a polynomial ring $\mathbf{A}[X_1, \dots, X_n]$ over a ring \mathbf{A} . Sometimes we will also consider monomial preorders in the sense of [5], which basically are monomial orders where ties between different monomials are allowed. Ac-

cording to [6, Theorem 1.2], every monomial preorder $<$ is given by a matrix $M \in \mathbb{R}^{m \times n}$ for some $m > 0$ in the following way:

$$X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n} < X_1^{f_1} X_2^{f_2} \cdots X_n^{f_n} \iff M \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} <_{\text{lex}} M \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

($e_i, f_i \in \mathbb{N}$), where lex denotes the lexicographic order with $X_1 > X_2 > \cdots > X_n$, applied to exponent vectors in \mathbb{Z}^n . A given matrix M defines a monomial preorder if and only if all columns are nonzero and their first nonzero entry is positive. Notice that different matrices may define the same monomial preorder. For example, adding a multiple of a row of M to a lower row does not change the preorder. By doing this repeatedly, we can achieve that M has nonnegative entries, so from now on we will assume $M \in \mathbb{R}_{\geq 0}^{m \times n}$. Let us call a preorder *rational* if it can be defined by a matrix with rational entries, which can then be assumed to be nonnegative integers. All monomial orders used in practice are rational. A rational preorder is a monomial order (i.e., there are no ties between monomials) if and only if M has rank n , so in this case we may assume $m = n$. By contrast, irrational preorders can be orders even if $m < n$, for example if M consists of a single row of real numbers that are linearly independent over \mathbb{Q} . A monomial preorder is said to be *graded* if the first row of M has only positive entries. It is sometimes convenient to extend a monomial preorder to the monomials in the Laurent polynomial ring $\mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. If $<$ is a monomial ordering and $P \in \mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is a nonzero (Laurent) polynomial, we write $\text{lm}_{<}(P)$ for its smallest monomial, and $\text{lc}_{<}(P)$ for the coefficient of this monomial.

Let us recall the notion of independence according to [5]. Given a monomial order $<$, a sequence $a_1, \dots, a_n \in \mathbf{A}$ is called *dependent* with respect to $<$ (or, for short, *<-dependent*) if there exists $P \in \mathbf{A}[X_1, \dots, X_n]$ with $P(a_1, \dots, a_n) = 0$ and $\text{lc}_{<}(P) = 1$. If $<$ is only a preorder, then a polynomial may have several minimal monomials. In this case it is required that among the minimal monomials of P , at least one has coefficient 1. Let us mention that Lombardi [7] calls a sequence pseudo-singular if it is lex -dependent, and otherwise pseudo-regular. His result can now be stated as follows.

Theorem 1 (Lombardi [7]). *Let \mathbf{A} be a ring and n a positive integer. Then $\dim(\mathbf{A}) < n$ if and only if every sequence of n elements of \mathbf{A} is lex -dependent.*

2. Non-Noetherian rings

Our first example shows that Theorem 1 does not extend to general monomial orders. We start with the monoid ring of $\mathbb{R}_{\geq 0}$ over the rational function field $\mathbb{Q}(u)$. We write this ring as $\mathbb{Q}(u)\{v\}$ (not to be confused with the Puiseux polynomial ring), and we write its elements as sums $\sum_{\alpha \in \mathbb{R}_{\geq 0}} c_{\alpha} v^{\alpha}$ with $c_{\alpha} \in \mathbb{Q}(u)$, where only finitely many c_{α} are nonzero. Consider the prime ideal $\mathfrak{p} \subset \mathbb{Q}(u)\{v\}$ of all elements with $c_0 = 0$, and

set $S := \mathbb{Q}(u)\{v\} \setminus \mathfrak{p}$. Now we form $\mathbf{R} := \mathbb{Q} + S^{-1}\mathfrak{p} \subset \text{Quot}(\mathbb{Q}(u)\{v\})$, which will provide our example. The following result emphasizes the distinguished standing that the lexicographic monomial order enjoys.

Proposition 2. *Let $<$ be a monomial preorder on two variables.*

- (a) *If $<$ is a lexicographic order, then every sequence of two elements of \mathbf{R} is dependent with respect to $<$.*
- (b) *If $<$ is not lexicographic, there exist two elements of \mathbf{R} that are independent with respect to $<$.*
- (c) *\mathbf{R} is a local ring of Krull dimension 1.*

Proof. (a) Let $a = c_0 + \frac{f}{s}v^\alpha$ be an arbitrary element of \mathbf{R} , where $c_0 \in \mathbb{Q}$, $f \in \mathbb{Q}(u)\{v\}$, $s \in S$, and $\alpha \in \mathbb{R}_{>0}$. If $c_0 \neq 0$, then a is invertible since $a^{-1} = c_0^{-1} - \frac{c_0^{-1}f}{c_0s + fv^\alpha}v^\alpha \in \mathbf{R}$, so a satisfies the equation $1 - a^{-1}X = 0$. Since 0 satisfies $X = 0$, we only need to consider two elements $a = \frac{q}{r}v^\alpha$ and $b = \frac{s}{t}v^\beta$ with $q, r, s, t \in S$ and $\alpha, \beta \in \mathbb{R}_{>0}$. With $n := \lfloor \frac{\alpha}{\beta} \rfloor + 1$ we have

$$c := \frac{b^n}{a} = \frac{rs^n}{qt^n}v^{n\beta-\alpha} \in \mathbf{R},$$

so $Y^n - cX$ provides an equation for a, b whose lowest coefficient with respect to the lexicographic order with $X > Y$ is 1. For the lexicographic order with $Y > X$, the roles of a and b need to be interchanged.

- (b) Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a real matrix defining $<$. The entry α must be positive, since otherwise we could assume $(\gamma, \delta) = (1, 0)$ and $<$ would be lexicographic. The same argument shows $\beta > 0$. Now we set $a := v^\alpha$, $b = uv^\beta$ and claim that these elements of \mathbf{R} are independent with respect to $<$.

So let $P \in \mathbf{R}[X, Y]$ with $P(a, b) = 0$. We need to show that no coefficient of P that belongs to a minimal monomial of P can be 1. For a monomial $m = X^iY^j$ we set $\deg(m) = \alpha i + \beta j$. With $\delta \in \mathbb{R}_{\geq 0}$ the minimum degree attained by the monomials of P , it follows that a monomial of P that is minimal needs to have degree δ . Let Q be the sum of all terms of P with degree δ . Then it suffices to show that no coefficient of Q is equal to 1. For a monomial m as above we have $m(a, b) = u^jv^{\deg(m)} = m(1, u)v^{\deg(m)}$. Writing $P = \sum_{m \in \text{Mon}(P)} c_m \cdot m$ with $c_m \in \mathbf{R}$, we obtain

$$0 = P(a, b) = v^\delta \sum_{m \in \text{Mon}(P)} c_m m(1, u) v^{\deg(m)-\delta}.$$

Since \mathbf{R} is a domain, we may divide this equation by v^δ . Applying the homomorphism $\varphi: \mathbf{R} \rightarrow \mathbb{Q}$ that sends an element of \mathbf{R} to its constant coefficient now yields

$$\sum_{m \in \text{Mon}(Q)} \varphi(c_m) m(1, u) = 0.$$

Since the monomials of Q have the same degree, the expressions $m(1, u)$ in the above sum are pairwise distinct powers of u . Since u is algebraically independent over \mathbb{Q} , it follows that $\varphi(c_m) = 0$ for every $m \in \text{Mon}(Q)$. So indeed no coefficient of Q can be equal to 1.

- (c) By Theorem 1, part (a) implies $\dim(\mathbf{R}) \leq 1$. The reverse inequality can perhaps most quickly be seen from the chain $\{0\} \subset S^{-1}\mathfrak{p}$ of primes; it is also easy to see that $v \in \mathbf{R}$ is lex-independent. The calculation in the proof of (a) shows that $\mathbf{R} \setminus S^{-1}\mathfrak{p} \subseteq \mathbf{R}^\times$, so \mathbf{R} is local. \square

We now prove an easy lemma, which will be used several times. For matrix $L = (\alpha_{i,j}) \in \mathbb{Z}^{n \times n}$, we define the homomorphism

$$\varphi_L: \mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \rightarrow \mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}], X_i \mapsto \prod_{j=1}^n X_j^{\alpha_{j,i}}.$$

Lemma 3. *Let \mathbf{A} be a ring and $<$ be a rational monomial order on $\mathbf{A}[X_1, \dots, X_n]$, given by a matrix $M = (\alpha_{i,j}) \in \mathbb{Z}_{\geq 0}^{n \times n}$. Then for $P \in \mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ we have*

$$\text{lc}_<(P) = \text{lc}_{\text{lex}}(\varphi_M(P)).$$

Moreover, let $a_1, \dots, a_n \in \mathbf{A}$ and set $b_i := \prod_{j=1}^n a_j^{\alpha_{j,i}}$. If the b_i form a $<$ -dependent sequence in \mathbf{A} , then $a_1, \dots, a_n \in \mathbf{A}$ are lex-dependent.

Proof. For a monomial $m = \prod_{i=1}^n X_i^{e_i}$ with $e_i \in \mathbb{Z}$ we have

$$m < 1 \iff \prod_{j=1}^n \prod_{i=1}^n X_j^{\alpha_{j,i} e_i} <_{\text{lex}} 1 \iff \varphi_M(m) <_{\text{lex}} 1.$$

Since φ_M is injective on monomials, this implies the first assertion. In the situation of the second assertion there exists $P \in \mathbf{A}[X_1, \dots, X_n]$ with $P(b_1, \dots, b_n) = 0$ and $\text{lc}_<(P) = 1$. So $Q := \varphi_M(P)$ satisfies $Q(a_1, \dots, a_n) = 0$ and $\text{lc}_{\text{lex}}(Q) = \text{lc}_<(P) = 1$. \square

As a first consequence we obtain:

Theorem 4. *Let \mathbf{A} be a ring and $<$ a rational monomial order on n variables. If every sequence of n elements of \mathbf{A} is $<$ -dependent, then $\dim(\mathbf{A}) < n$.*

Proof. The order $<$ is given by a matrix $M = (\alpha_{i,j}) \in \mathbb{Z}_{\geq 0}^{n \times n}$. Let $a_1, \dots, a_n \in \mathbf{A}$ and form the b_i as in Lemma 3. Then the hypothesis and the lemma yield that a_1, \dots, a_n are lex-dependent. From this the assertion follows by Theorem 1. \square

If \mathbf{A} is Noetherian, then by [5, Theorem 3.5], Theorem 4 extends to all monomial preorders and the converse also holds.

We now give an example of a (non-Noetherian) ring such that Theorem 4 fails for all monomial preorders that are not rational orders. It is also an example where the maximal number of independent elements is smaller than the Krull dimension. In contrast, the ring \mathbf{R} constructed above has “too many” independent elements. With u and v indeterminates, consider the subring

$$\mathbf{W} := \mathbb{Q}[u]_{\langle u \rangle} + v \mathbb{Q}(u)[v]_{\langle v \rangle} \subseteq \mathbb{Q}(u, v),$$

of the rational function field, where the subscripts stand for localization at the prime ideal generated by u and v , respectively. It is easy to see that \mathbf{W} consists of the rational functions with denominator not divisible by v , such that the evaluation at $v = 0$ has a denominator not divisible by u .

Recall that a *valuation domain* is an integral domain such that for any two elements, one divides the other. The first assertion of the following proposition will be quite clear for readers who are familiar with valuation domains.

Proposition 5.

- (a) \mathbf{W} is a 2-dimensional valuation domain.
- (b) Let $<$ be a monomial preorder on two variables that is not a rational monomial order. Then every sequence of two elements of \mathbf{W} is $<$ -dependent.

Proof. (a) The sequence

$$\{0\} \subset v \mathbb{Q}(u)[v]_{\langle v \rangle} \subset u \mathbb{Q}[u]_{\langle u \rangle} + v \mathbb{Q}(u)[v]_{\langle v \rangle}$$

of prime ideals shows that $\dim(\mathbf{W}) \geq 2$, and the reverse inequality follows since \mathbf{W} has transcendence degree 2 over \mathbb{Q} (see [4, Theorem 5.5]).

A rational function in \mathbf{W} is invertible in \mathbf{W} if and only if evaluating it at $v = 0$ and then evaluating the result at $u = 0$ yields a nonzero value. Now let $a \in \mathbb{Q}(u, v) = \text{Quot}(\mathbf{W})$ be any nonzero rational function. There is an integer i such that $b := v^i a$ has numerator and denominator not divisible by v , and there is an integer j such that $u^j b(0)$ (the evaluation is at $v = 0$) has numerator and denominator not divisible by u . Therefore $v^i u^j a \in \mathbf{W}^\times$. This shows that the $v^i u^j$ form a system of representatives of $\mathbb{Q}(u, v)^\times / \mathbf{W}^\times$. Moreover, we have $v^i u^j \in \mathbf{W}$ if and only if $\begin{pmatrix} i \\ j \end{pmatrix} \geq_{\text{lex}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So for the above a we have $a \in \mathbf{W}$ or $a^{-1} \in \mathbf{W}$, which shows that \mathbf{W} is a valuation domain.

- (b) The preorder $<$ is given by a real matrix M with two columns and at most two rows. Assuming that M has two rows, we may add a multiple of the first row to the second and then multiply the second row by a positive real number. This way, the second row may be assumed to have entries in $\{0, 1, -1\}$. If $<$ is irrational, it follows

that the first row of M consists of two real numbers with irrational ratio, and in this case the second row can be deleted since the first row completely determines $<$. If, on the other hand, $<$ is not a monomial order, then M has rank 1, and again the second row can be deleted. So in both cases we can assume $M = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R}$ positive.

For showing that all sequences of two elements $a, b \in \mathbf{W}$ are $<$ -dependent, we may assume a and b to be nonzero and replace them by associated elements. So by the above we may assume $a = v^{i_1}u^{j_1}$ and $b = v^{i_2}u^{j_2}$. With $A := \begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}$, we claim that there exists a nonzero vector $\begin{pmatrix} e \\ f \end{pmatrix} \in \mathbb{Q}^2$ such that

$$\alpha e + \beta f = M \cdot \begin{pmatrix} e \\ f \end{pmatrix} \leq 0 \quad \text{but} \quad A \cdot \begin{pmatrix} e \\ f \end{pmatrix} \geq_{\text{lex}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (1)$$

This is clear if the ratios of α and β and of i_1 and i_2 are different. But if the ratios are equal, then any e, f with $i_1e + i_2f = 0$ will satisfy the first inequality in (1), so in addition we need $j_1e + j_2f \geq 0$. If $\text{rank}(A) = 2$, then $j_1e + j_2f > 0$ can be achieved, and if $\text{rank}(A) = 1$, then $j_1e + j_2f = 0$ is true whenever $i_1e + i_2f = 0$. Having proved the claim, we may now assume $e, f \in \mathbb{Z}$ and write $e = e_1 - e_2$ and $f = f_1 - f_2$ with $e_i, f_i \in \mathbb{N}$. Then $X^{e_1}Y^{f_1} \leq X^{e_2}Y^{f_2}$ by the first inequality in (1). Moreover,

$$c := \frac{a^{e_1}b^{f_1}}{a^{e_2}b^{f_2}} = a^e b^f = v^{i_1e+i_2f}u^{j_1e+j_2f} \in \mathbf{W}$$

by the second inequality in (1) and by the above reasoning. So the polynomial $P := X^{e_1}Y^{f_1} - cX^{e_2}Y^{f_2} \in \mathbf{W}[X, Y]$ vanishes at a, b , and the monomial $X^{e_1}Y^{f_1}$ is minimal among the monomials of P . This shows that a and b are $<$ -dependent. \square

3. The valuative dimension

Recall that the *valuative dimension* of a domain \mathbf{A} , denoted by $\dim_v(\mathbf{A})$, is the supremum of the Krull dimensions of all overrings of \mathbf{A} , where an overring of \mathbf{A} is defined to be a subring of $\text{Quot}(\mathbf{A})$ containing \mathbf{A} . It is worth mentioning that, as pointed out by Gilmer [2, Theorem 30.9], $\dim_v(\mathbf{A}) \leq n$ iff for any elements $t_1, \dots, t_n \in \text{Quot}(\mathbf{A})$, $\dim(\mathbf{A}[t_1, \dots, t_n]) \leq n$. In the case of an integral domain, this can be interpreted as a constructive characterization of the valuative dimension, and it is in fact the definition adopted by Lombardi and Quitté in the integral case in their book [8]. If \mathbf{A} is a ring which need not be a domain, $\dim_v(\mathbf{A})$ is defined as the supremum of all $\dim_v(A/\mathfrak{p})$ with $\mathfrak{p} \in \text{Spec}(\mathbf{A})$ a prime ideal (see Jaffard [3, p. 56]). It is clear that $\dim_v(\mathbf{A}) \geq \dim(\mathbf{A})$.

In this section we prove the following theorem, which is the main result of the paper.

Theorem 6. *Let \mathbf{A} be a ring, n a positive integer, and $<$ a rational monomial preorder on n variables.*

- (a) If $\dim_v(\mathbf{A}) < n$, then every sequence of n elements in \mathbf{A} is dependent with respect to $<$.
- (b) If $<$ is a graded monomial order, then the converse of (a) holds.

Proof. (a) Let $a_1, \dots, a_n \in \mathbf{A}$. We start with two reduction steps. First, we can refine $<$ to a rational monomial order by appending some rows of the unit matrix at the bottom of the matrix defining $<$. If we can show that a_1, \dots, a_n are dependent with respect to the order thus obtained, then they are also dependent with respect to the original preorder. Therefore we may assume that $<$ is a rational monomial order. Second, we reduce to the case that \mathbf{A} is a domain. For this, consider the multiplicative set

$$S := \{P(a_1, \dots, a_n) \mid P \in \mathbf{A}[X_1, \dots, X_n] \text{ with } \text{lc}_{<}(P) = 1\} \subseteq \mathbf{A}$$

and assume that we can show part (a) in the domain case. Then for every $\mathfrak{p} \in \text{Spec}(\mathbf{A})$, the sequence a_1, \dots, a_n is $<$ -dependent in \mathbf{A}/\mathfrak{p} , so $S \cap \mathfrak{p} \neq \emptyset$. This means that the localization $S^{-1}\mathbf{A}$ has no prime ideals and is therefore zero. So $0 \in S$, which shows the dependence of a_1, \dots, a_n . Hence indeed we may assume \mathbf{A} to be a domain. We need to show that a_1, \dots, a_n are $<$ -dependent. Since this is clear if an a_i is zero, we may assume the a_i to be nonzero.

By the first reduction, $<$ is given by a matrix $M \in \mathbb{Z}_{\geq 0}^{n \times n}$ of rank n . There is a positive integer k such that $L := k \cdot M^{-1} \in \mathbb{Z}^{n \times n}$. We write $L = (\beta_{i,j})_{i,j=1,\dots,n}$ and set

$$b_i = \prod_{j=1}^n a_j^{\beta_{j,i}} \in \text{Quot}(\mathbf{A}) \quad (i = 1, \dots, n).$$

$\mathbf{B} := \mathbf{A}[b_1, \dots, b_n]$ is an overring of \mathbf{A} , so $\dim(\mathbf{B}) < n$ by hypothesis. It follows by Theorem 1 that the b_i are lex-dependent, so there is a polynomial $P \in \mathbf{B}[X_1, \dots, X_n]$ with $P(b_1, \dots, b_n) = 0$ and $\text{lc}_{\text{lex}}(P) = 1$. Each coefficient c of P can be written as $c = C[b_1, \dots, b_n]$ with $C \in \mathbf{A}[X_1, \dots, X_n]$. Substituting each coefficient c by C yields a polynomial in $\mathbf{A}[X_1, \dots, X_n]$ that also vanishes at b_1, \dots, b_n and has lowest coefficient 1. So we may assume $P \in \mathbf{A}[X_1, \dots, X_n]$. With the notation introduced before Lemma 3, set $Q := \varphi_L(P) \in \mathbf{A}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. We have $\varphi_M(\varphi_L(X_i)) = X_i^k$ for all i , so Lemma 3 shows

$$\text{lc}_{<}(Q) = \text{lc}_{\text{lex}}(\varphi_M(Q)) = \text{lc}_{\text{lex}}(P(X_1^k, \dots, X_n^k)) = \text{lc}_{\text{lex}}(P) = 1.$$

We clearly have $Q(a_1, \dots, a_n) = P(b_1, \dots, b_n) = 0$. By multiplying Q with a suitable monomial, we obtain a polynomial in $\mathbf{A}[X_1, \dots, X_n]$ that also vanishes at a_1, \dots, a_n and has lowest coefficient 1 with respect to $<$. So a_1, \dots, a_n are $<$ -dependent, which finishes the proof of (a).

- (b) We need to show that $\dim_v(\mathbf{A}/\mathfrak{p}) < n$ for every $\mathfrak{p} \in \text{Spec}(\mathbf{A})$. By hypothesis, every sequence of n elements in \mathbf{A} is $<$ -dependent, so it is also $<$ -dependent as a sequence in \mathbf{A}/\mathfrak{p} . Replacing \mathbf{A} by \mathbf{A}/\mathfrak{p} , we may therefore assume that \mathbf{A} is a domain. Given an overring \mathbf{B} of \mathbf{A} and a sequence $b_1, \dots, b_n \in \mathbf{B}$, we need to show that it is lex-dependent; indeed, by Theorem 1, this will imply $\dim(\mathbf{B}) < n$.

Since $\mathbf{B} \subseteq \text{Quot}(\mathbf{A})$, we can choose a nonzero $a \in \mathbf{A}$ such that $ab_i \in \mathbf{A}$ for all i . Our monomial order $<$ is given by a matrix $M = (\alpha_{i,j}) \in \mathbb{Z}_{\geq 0}^{n \times n}$. Since it is graded, we can choose a positive integer k such that $k\alpha_{1,i} \geq \sum_{j=1}^n \alpha_{j,i}$ for all i . Then

$$(a^k b_1)^{\alpha_{1,i}} \cdot \prod_{j=2}^n b_j^{\alpha_{j,i}} = a^{k\alpha_{1,i} - \sum_{j=1}^n \alpha_{j,i}} \cdot \prod_{j=1}^n (ab_j)^{\alpha_{j,i}} \in \mathbf{A}. \quad (2)$$

We claim that it is enough to show that $a^k b_1, b_2, b_3, \dots, b_n$ are lex-dependent. In fact, if they are, then there is a polynomial $P \in \mathbf{B}[X_1, \dots, X_n]$ vanishing at these elements with $\text{lc}_{\text{lex}}(P) = 1$. So $P(a^k X_1, X_2, X_3, \dots, X_n)$ vanishes at b_1, \dots, b_n . If e is the exponent of X_1 in the smallest monomial of P , then all coefficients of $P(a^k X_1, X_2, \dots, X_n)$ are divisible by a^{ke} , so $Q := a^{-ke} P(a^k X_1, X_2, \dots, X_n) \in \mathbf{B}[X_1, \dots, X_n]$. Now $Q(b_1, \dots, b_n) = 0$ and $\text{lc}_{\text{lex}}(Q) = 1$, which yields the desired lex-dependence of the b_i . So the claim is proved.

The claim means that we may replace b_1 by $a^k b_1$. Then by (2), $a_i := \prod_{j=1}^n b_j^{\alpha_{j,i}} \in \mathbf{A}$ ($i = 1, \dots, n$). By hypothesis, a_1, \dots, a_n are $<$ -dependent, so they are also $<$ -dependent when considered as a sequence in \mathbf{B} . Now Lemma 3 shows that the sequence b_1, \dots, b_n is lex-dependent, as desired. \square

The following example shows that Theorem 6(a) does not extend to irrational monomial preorders. Consider $\mathbb{Q}\{v\}$, the monoid ring of $\mathbb{R}_{\geq 0}$ over \mathbb{Q} , and let $\mathbf{V} = \mathbb{Q}\{v\}_{\mathfrak{p}}$ be the localization at the ideal \mathfrak{p} of all elements with constant coefficient equal to 0.

Proposition 7.

- (a) \mathbf{V} is a valuation domain with $\dim(\mathbf{V}) = \dim_v(\mathbf{V}) = 1$.
 (b) Let $<$ be an irrational monomial preorder on two variables. Then there exist two elements of \mathbf{V} that are independent with respect to $<$.

Proof. (a) It is clear that \mathbf{V} is a valuation domain. The chain $\{0\} \subset \mathfrak{p}_{\mathfrak{p}}$ or the lex-independence of v show that $\dim(\mathbf{V}) \geq 1$. The reverse inequality follows by Theorem 1 if we can show that any two elements $a, b \in \mathbf{V}$ are lex-dependent. We may assume a and b to be nonzero and noninvertible, and replace them by associate elements. This yields $a = v^\alpha$ and $b = v^\beta$ with $\alpha, \beta \in \mathbb{R}_{>0}$. With $n := \lceil \frac{\alpha}{\beta} \rceil$ and $c := v^{n\beta - \alpha} \in \mathbf{V}$, the relation $b^n - ca = 0$ shows that a and b are lex-dependent. So $\dim(\mathbf{V}) = 1$, and $\dim_v(\mathbf{V}) = 1$ follows from the fact that for a valuation domain, the valuative and Krull dimensions coincide (see [3, Chap. IV, Prop. 1]).

(b) We may assume that $<$ is given by a matrix $M = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R}_{>0}$ linearly independent over \mathbb{Q} . So with the degree of a monomial defined as in the proof of Proposition 2(b), the smallest monomial of a polynomial is the (unique) monomial with minimal degree. We claim that $a := v^\alpha$ and $b := v^\beta$ are $<$ -independent. So let $P = \sum_{m \in \text{Mon}(P)} c_m \cdot m \in \mathbf{V}[X, Y]$ be a polynomial vanishing at a, b . Then with δ the degree of the smallest monomial of P we have

$$v^\delta \sum_{m \in \text{Mon}(P)} c_m \cdot v^{\deg(m) - \delta} = 0.$$

Dividing this by v^δ and applying the homomorphism $\varphi: \mathbf{V} \rightarrow \mathbb{Q}$ that sends an element of \mathbf{V} to its constant coefficient yields the equation $\varphi(\text{lc}(P)) = 0$, since $\varphi(v^{\deg(m) - \delta}) = 0$ for $m \neq \text{lm}(P)$. So $\text{lc}(P) \neq 1$, which proves the claim. \square

The above proof also shows that if $<$ is a monomial order on n variables given by $M = (\alpha_1, \dots, \alpha_n)$ with the α_i linearly independent over \mathbb{Q} , then there are n elements of \mathbf{V} that are $<$ -independent.

We now present two applications of Theorem 6. The first is a new proof of the well-known but nontrivial fact that for a Noetherian ring, the Krull dimension and the valuative dimension coincide.

Corollary 8 ([3, Chapt. IV, Corollaire 2 to Théorème 5]). *Let \mathbf{A} be a Noetherian ring. Then $\dim_v(\mathbf{A}) = \dim(\mathbf{A})$.*

Proof. Let n be a positive integer and choose a graded rational monomial order $<$ on n variables. Then by [5, Theorem 2.7], the Krull dimension of \mathbf{A} is less than n if and only if every sequence of n elements is $<$ -dependent. But by Theorem 6, this is equivalent to $\dim_v(\mathbf{A}) < n$. \square

Our second application deals with analytic independence, which is defined, according to Matsumura [9], as follows. Some elements $a_1, \dots, a_n \in \mathfrak{m}$ from the maximal ideal of a local ring \mathbf{A} are analytically independent if every homogeneous polynomial in $\mathbf{A}[X_1, \dots, X_n]$ vanishing at a_1, \dots, a_n has all its coefficients lying in \mathfrak{m} . To the best of our knowledge, the following corollary is new.

Corollary 9. *Let \mathbf{A} be a local ring with $\dim_v(\mathbf{A}) < n$ and let $a_1, \dots, a_n \in \mathfrak{m}$ be elements from its maximal ideal. Then the a_i are analytically dependent.*

Proof. Applying Theorem 6(b) to the monomial preorder given by the matrix $M = (1, 1, \dots, 1)$ yields a polynomial $P \in \mathbf{A}[X_1, \dots, X_n]$ vanishing at a_1, \dots, a_n such that the homogeneous part of P of least degree d_0 has a monomial t_0 whose coefficient is 1. We now turn P into a homogeneous polynomial of degree d_0 by “partially evaluating” it. More precisely, we split each monomial (of degree d , say) into monomials of degree d_0

and $d-d_0$, and then evaluate the one of degree $d-d_0$ at the a_i . The resulting homogeneous polynomial also vanishes at the a_i . The process may have changed the coefficient of t_0 , but only by adding an \mathbf{A} -linear combination of nonempty products of the a_i . Since $a_i \in \mathfrak{m}$, the coefficient of t_0 is not in \mathfrak{m} , and corollary follows. \square

The ring \mathbf{R} , constructed at the beginning of Section 2, provides an example showing that in the above result, the valuative dimension cannot be replaced by the Krull dimension. Indeed, Proposition 2(b) shows that there are two elements that are independent with respect to the preorder $<$ given by $M = (1, 1)$. Being independent, they must lie in the maximal ideal of \mathbf{R} , and being $<$ -independent, they are analytically independent. Explicitly, two such elements are v and uv .

On the other hand, the maximal number of analytically independent elements may also be less than the Krull dimension. For example, if \mathbf{A} is a valuation domain, then any two elements are analytically dependent (with an equation of degree 1); but \mathbf{A} may have dimension > 1 , as exemplified by the ring \mathbf{W} from this note. By [9, Theorem 14.5], this cannot happen for Noetherian rings.

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