Journal of Algebra 562 (2020) 621–626

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Elimination ideals and Bézout relations

Andre Galligo [a], Zbigniew Jelonek [b,*]

[a] Université Côte d'Azur, LJAD, INRIA, France
[b] Instytut Matematyczny PAN, Śniadeckich 8, 00-656 Warszawa, Poland

ARTICLE INFO

Article history:
Received 27 October 2018
Available online 13 July 2020
Communicated by Steven Dale Cutkosky

MSC:
14D06
14Q20

Keywords:
Nullstellensatz
Polynomials
Elimination
Affine variety

ABSTRACT

Let k be an infinite field and $I \subset k[x_1, \dots, x_n]$ be a non-zero ideal such that $\dim V(I) = q \geq 0$. Denote by $(f_1, \dots, f_s)$ a set of generators of I. One can see that in the set $I \cap k[x_1, ..., x_{q+1}]$ there exist non-zero polynomials, depending only on these $q+1$ variables. We aim to bound the minimal degree of the polynomials of this type, and of a Bézout (i.e. membership) relation expressing such a polynomial as a combination of the f_i. In particular we show that if $\deg f_i = d_i$ where $d_1 \geq d_2 ... \geq d_s$, then there exist a non-zero polynomial $\phi(x) \in k[x_1, ..., x_{q+1}] \cap I$, such that $\deg \phi \leq d_s \prod_{i=1}^{n-q-1} d_i$. We also give a relative version of this theorem.

1. Introduction

Let $I \subset k[x_1, ..., x_n]$ be a non-zero ideal such that $\dim V(I) = q \geq 0$. Using Hilbert Nullstellensatz we can easily see, that in the elimination ideal $I \cap k[x_1, ..., x_{q+1}]$ there exist non-zero polynomials. It is interesting to know the minimal degree of the polynomials in this ideal. Here, performing a generic change of coordinates, and continuing the approach presented in [5], we get a sharp estimate for the degree of such a minimal polynomial

* Corresponding author.
E-mail addresses: galligo@unice.fr (A. Galligo), najelone@cyf-kr.edu.pl (Z. Jelonek).

https://doi.org/10.1016/j.jalgebra.2020.06.022
0021-8693/

(and also for a corresponding generalized Bezout identity), in terms of the degrees of generators of the ideal I. Then, using a deformation arguments we solve the stated problem. We show that if $\deg f_i = d_i$ where $d_1 \geq d_2 ... \geq d_s$, then there exist polynomials $g_j \in k[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi(x) \in k[x_1, ..., x_{q+1}]$ such that

(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,

(b) $\phi(x) = \sum_{j=1}^{s} g_j f_j$.

Note that our result works also in the case $\dim V(I) = -1$ (i.e. in the case when $V(I) = \emptyset$) if we put $k[x_0] := k$ and $d_i = 1$ for $i > s$ (however our result in this case is a little bit worse than these in [5], [6]). Hence, from this point of view, we can treat our result as a generalization of the Effective Nullstellensatz. We also give a relative version of this theorem.

Effective versions of Nullstellensatz and Membership problems have a long story and several variants, going back to G. Hermann [4]. The interested reader can consult e.g. the references listed by Brownawell [2], Brownawell [3], Kollar [6], and D′ Andrea et al. [1].

2. Main result

In this section we present a geometric construction and establish degree bounds, relying on generic changes of coordinates. Let us recall (see [5]) two important tools that we will use in the proof of the main theorem of this section.

Theorem 1. *(Perron Theorem) Let $\mathbb{L}$ be a field and let $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[x_1, \ldots, x_n]$ be non-constant polynomials with $\deg Q_i = d_i$. If the mapping $Q = (Q_1, \ldots, Q_{n+1}) : \mathbb{L}^n \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that*

(a) $W(Q_1, \ldots, Q_{n+1}) = 0$,

(b) $\deg\, W(T_1^{d_1}, T_2^{d_2}, \ldots, T_{n+1}^{d_{n+1}}) \leq \prod_{j=1}^{n+1} d_j$.

Lemma 2. *Let $\mathbb{K}$ be an algebraic closed field and let $k \subset \mathbb{K}$ be its infinite subfield. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension n. For sufficiently general numbers $a_{ij} \in k$ the mapping*

$$\pi : X \ni (x_1, \ldots, x_m) \to \Big(\sum_{j=1}^{m} a_{1j} x_j, \sum_{j=2}^{m} a_{2j} x_j, \ldots, \sum_{j=n}^{m} a_{1j} x_j \Big) \in \mathbb{K}^n$$

is finite. □

In the sequel for a given ideal $I \subset k[x_1, ..., x_n]$ by $V(I)$ we mean the set of algebraic zeros of I, i.e., the zeroes of I in $\mathbb{K}^n$, where $\mathbb{K}$ is an algebraic closure of k. Now we can formulate our first main result:

Theorem 3. *Let k be an infinite field and let $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 ... \geq d_s$. Assume that $I = (f_1, \dots, f_s) \in k[x_1, \dots, x_n]$ is a non-zero ideal, such that $V(I)$ has dimension $q \geq 0$. If we take a sufficiently general system of coordinates $(x_1, \dots, x_n)$, then there exist polynomials $g_j \in k[x_1, \dots, x_n]$ and a non-zero polynomial $\phi(x) \in k[x_1, ..., x_{q+1}]$ such that*

(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,

(b) $\phi(x) = \sum_{j=1}^{s} g_j f_j$.

Proof. Let $\mathbb{K}$ be the algebraic closure of k. Take $F_{n-q} = f_s$ and $F_i = \sum_{j=i}^{s} \alpha_{ij} f_j$ for $i = 1, ..., n-q-1$, where $\alpha_{ij} \in k$ are sufficiently general. Take $J = (F_1, ..., F_{n-q})$. Then $\deg F_{n-q} = d_s$ and $\deg F_i = d_i$ for $i = 1, ..., n-q-1$. Moreover, $V(J)$ has pure dimension q and $J \subset I$. The mapping

$$\Phi : \mathbb{K}^n \times \mathbb{K} \ni (x, z) \to (F_1(x)z, \dots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n$$

is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma = \mathrm{cl}(\Phi(\mathbb{K}^n \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection defined over the field k. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 2 we can assume that

$$\Psi = (\sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \dots, \sum_{j=n-q}^{n-q} \gamma_{n-qj} F_j z + l_{n-q}(x), l_{n-q+1}(x), ..., l_{n+1}(x)),$$

where $l_1, \dots, l_{n+1}$ are generic linear form. In particular we can assume that l_{n-q+i}, $i = 1, .., q+1$ is the variable x_i in a new generic system of coordinates (of $\mathbb{K}^n$).

Apply Theorem 1 to $\mathbb{L} = k(z)$, and to the polynomials $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{L}[x]$. Thus there exists a non-zero polynomial $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$ such that

$$W(\Psi_1, \dots, \Psi_{n+1}) = 0 \text{ and } \deg W(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}) \leq d_s \prod_{j=1}^{n-q-1} d_j,$$

where $k = n - q$. Since the coefficients of W are in $k(z)$, there is a non-zero polynomial $\tilde{W} \in k[T_1, \dots, T_{n+1}, Y]$ such that

(a) $\tilde{W}(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z), z) = 0$,

(b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$, where $\deg_T$ denotes the degree with respect to the variables $T = (T_1, \dots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}^{n+1}$ is locally finite outside the set $V(J) \times \mathbb{K}$. Consider $\mathbb{K}^{n+1}$ as a product $\mathbb{K}^{n-q} \times \mathbb{K}^{q+1}$, and let us consider in this product coordinates $(y_{q+2}, ..., y_{n+1}, y_1, ..., y_{q+1})$. Hence Ψ restricted to $V(J) \times \mathbb{K}$ coincides with the mapping: $(x, z) \mapsto (l_1(x), ..., l_{n-q}(x), x_1, ..., x_{q+1})$ (recall that we consider a new generic system of coordinates). Let $\phi' = 0$ describes the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.$$

Put $S = \{T \in \mathbb{K}^{n+1} : \phi'(T) = 0\}$. Hence $V(J) \times \mathbb{K}$ is contained in $\Psi^{-1}(S)$. Consequently the mapping Ψ is proper outside the hypersurface S and thus the set of non-properness of the mapping Ψ is contained in the S.

Since the mapping Ψ is finite outside S, for every $H \in \mathbb{K}[x_1, \ldots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \ldots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \ldots, \Psi_{n+1}, H) = \sum_{i=0}^{r} b_i(\Psi_1, \ldots, \Psi_{n+1})H^{r-i} = 0$ and the coefficient b_0 satisfies $\{T : b_0(T) = 0\} \subset S$. In particular b_0 depends only on variables $x_1, ..., x_{q+1}$. Note that P_H describes a hypersurface given by parametric equation $(\Psi_1, ..., \Psi_{n+1}, H)$. Hence if $H \in k[x_1, \ldots, x_n, z]$, then by Gröbner base computation we see that we can assume $P_H(T, Y) \in k[T_1, \ldots, T_{n+1}][Y]$. Now set $H = z$.

We have

$$\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \ldots, T_n^{d_n}, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j$$

and consequently we obtain the equality $b_0(x_1, ..., x_{q+1}) + \sum_{i=1}^{n-q} F_i g_i = 0$, where deg $F_i g_i \leq d_s \prod_{j=1}^{n-q-1} d_j$. Set $\phi = b_0$. By the construction the polynomial ϕ has zeros only on the image of the projection

$$\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}. \quad \square$$

Remark 4. Simple application of the Bezout theorem shows that our bound on the degree of ϕ is sharp.

Corollary 5. *Let k and I and system of coordinates be as above. If $V(I)$ has pure dimension q and I has not embedded components, then there is a polynomial $\phi_1 \in k[x_1, ..., x_{q+1}]$ which describes the image of the projection*

$$\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}$$

such that

(a) $\phi_1 \in I$,

(b) $\deg \phi_1 \leq d_s \prod_{i=1}^{n-q-1} d_i$.

Proof. The set $V(J) = q$ has pure dimension q. Consequently $\pi(V(J))$ and $\pi(V(I))$ are hypersurfaces. Moreover, by Gröbner bases computation the set $\pi(V(I))$ is described by a polynomial ψ from $k[x_1, ..., x_{q+1}]$. Let ϕ be a polynomial as above which vanishes exactly on $\pi(V(J))$. Let ϕ_1 be a product of all irreducible factors of ϕ (over the field k) which divides ψ. Hence $\phi = \phi_1\phi_2,\ \phi_1, \phi_2 \in k[x_1, ..., x_{q+1}]$, where ϕ_2 does not vanish on any component of $V(I)$. Let $I = \bigcap^r I_s$ be a primary decomposition of I. Consequently $\phi_1 \in I_j$ for every s (by properties of primary ideals) and consequently $\phi_1 \in I$. But ϕ_1 describes the image of the projection

$$\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}. \quad \square$$

3. A deformation argument

In this section, we improve Theorem 3 by releasing the necessity of a generic change of coordinates, so conditions (a) and (b) will be satisfied in the initial system of coordinates.

Theorem 6. *Let k be an infinite field and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \geq ... \geq d_s$. Assume that $I = (f_1, \ldots, f_s) \subset k[x_1, \ldots, x_n]$ is a non zero ideal, such that $V(I)$ has dimension $q \geq 0$. There exist polynomials $g_j \in k[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi \in k[x_1, ..., x_{q+1}]$ such that*
(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,
(b) $\phi = \sum_{j=1}^{s} g_j f_j$.

Proof. We use Theorem 3, but over the field $\mathbb{L} = k(t)$. We consider a new generic change of coordinates using generic values $a_{i,j}$ in the infinite field k, together with the inverse change of coordinates

$$X_i = x_i + t \sum_{j=i+1}^{n} a_{i,j} x_j \ ; \ x_i = X_i + t \sum_{j=i+1}^{n} b_{i,j}(t) X_j,$$

where $b_{i,j}(t) \in k[t]$.

As in the proof of Theorem 3, we obtain some polynomials $G_j \in \mathbb{L}[X_1, \ldots, X_n]$ and a non-zero polynomial $b_0 \in \mathbb{L}[X_{n-q}, ..., X_n]$ such that, after chasing the denominators,

$$b_0(X,t) = \sum_{j=1}^{n-q} G_j(X,t) \bar{F}_j(X,t),$$

where $b_0(X,t), G_j(X,t), \bar{F}_j(X,t) \in k[t][X_1, ..., X_n]$.

We cannot just simplify this equality by t and then set $t = 0$, because we cannot exclude the possibility that there will be a remaining factor t^p in the left hand side with p strictly positive. To rule out this possibility, we need to perform several reduction steps. Consider the sub-module $M = \{H(x) = (H_1(x), \ldots, H_{n-q}(x))\}$ of $k[x]^{n-q}$ formed by the relations (first syzygies) between the polynomials $F_1(x), \ldots, F_{n-q}(x)$. To each element $H(x)$ in M corresponds via the change of coordinates a relation $\bar{H}(X,t)$ between the polynomials $\bar{F}_1(X,t), \ldots, \bar{F}_{n-q}(X,t)$, such that $\bar{H}(X,t) - H(X)$ is divisible by t. Rewriting in (x,t), we obtain that

$$b_0(X,t) = \sum_{j=1}^{n-q} (G_j(X,t) - \bar{H}_j(X,t)) \bar{F}_j(X,t).$$

We may assume that in the previous equality $b_0(X,t)$ has the form $b_0(X,t) = t^p(\phi(x) + t\phi_1(x,t))$; notice that the $x-$ degree of $\phi(x)$ is bounded by the X-degree of $b_0(X,t)$).

Each reduction step will produce a similar equality (with the same degree in x bounds) but with a strictly smaller power p.

Assume $p > 0$ and let $t = 0$, we obtain a non trivial relation $0 = \sum_{j=1}^{n-q} G_j(x, 0)F_j(x)$, hence $H = (G_1(x, 0), \dots, G_{n-q}(x, 0))$, a non trivial element of M. Notice that the $x-$ degree of $G_j(x, 0)$ is bounded by the X-degree of $G_j(X, t)$. To which we associate its $\bar{H}$ as above with the same degree bound in X (equivalently in x by linearity) and notice that now $\sum_{j=1}^{n-q}(G_j(X, t) - \bar{H}_j(X, t))\bar{F}_j(X)$ vanishes for $t = 0$, hence admits a factor t. We can simplify the two sides of the previous equality by t and obtain

$$t^{p-1}(\phi(x) + t\phi_1(x, t)) = \sum_{j=1}^{n-q} \frac{(G_j(X, t) - \bar{H}_j(X, t))}{t} \bar{F}_j(X, t).$$

After at most p such reduction steps, we get rid of the initial factor t^p and setting $t = 0$, we obtain the announced equality with the announced bounds. □

Using the more general Theorem 3.3 from [5] instead of our Theorem 1 we can prove in the same way a more general result:

Theorem 7. *Let $\mathbb{K}$ be an algebraically closed field and $k \subset \mathbb{K}$ be its infinite subfield. Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n and of degree D, such that its ideal $I(X)$ is generated by polynomials from $k[x_1, \dots, x_m]$. Let $f_1, \dots, f_s \in k[x_1, \dots, x_m]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \dots \geq d_s$. Assume that $I = (f_1, \dots, f_s) \subset \mathbb{K}[x_1, \dots, x_m]$ is an ideal, such that $V(I) \cap X$ has dimension $q \geq 0$ and the ideal $I\mathbb{K}[X]$ is non-zero in $\mathbb{K}[X]$. There exist polynomials $g_j \in k[x_1, \dots, x_m]$ and a polynomial $\phi \in k[x_1, \dots, x_{q+1}]$, which does not vanish on X identically, such that*

(a) $\deg g_j f_j \leq D d_s \prod_{i=1}^{n-q-1} d_i$*,*

(b) $\phi = \sum_{j=1}^{s} g_j f_j$ *on* X*.*

References

[1] Carlos D'Andrea, Teresa Krick, Martín Sombra, Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze, Ann. Sci. Éc. Norm. Supér. (4) 46 (4) (2013) 549–627.
[2] W.D. Brownawell, Bounds for the degrees in the Nullstellensatz, Ann. Math. 126 (1987) 577–591.
[3] W.D. Brownawell, The Hilbert Nullstellensatz, inequalities for polynomials, and algebraic independence, in: Introduction to Algebraic Independence Theory, in: Lecture Notes in Math., vol. 1752, Springer, 2001, pp. 239–248.
[4] G. Hermann, Die Frage der endlich vielen Schritte in der Theorie der Polynomideale, Math. Ann. 95 (1926) 736–788.
[5] Z. Jelonek, On the effective Nullstellensatz, Invent. Math. 162 (2005) 1–17.
[6] J. Kollár, Sharp effective Nullstellensatz, J. Am. Math. Soc. 1 (1988) 963–975.