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Noncommutative ampleness for multiple divisors

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Abstract

The twisted homogeneous coordinate ring is one of the basic constructions of the noncommutative projective geometry of Artin, Van den Bergh, and others. Chan generalized this construction to the multi-homogeneous case, using a concept of right ampleness for a finite collection of invertible sheaves and automorphisms of a projective scheme. From this he derives that certain multi-homogeneous rings, such as tensor products of twisted homogeneous coordinate rings, are right noetherian. We show that right and left ampleness are equivalent and that there is a simple criterion for such ampleness. Thus we find under natural hypotheses that multi-homogeneous coordinate rings are noetherian and have integer GK-dimension.

Keywords: Vanishing theorems; Invertible sheaves; Noetherian graded rings; Noncommutative projective geometry

1. Introduction

Let R be an $\mathbb{N}$-graded algebra over an algebraically closed field k such that $\dim R_i < \infty$ for all i. One of the main techniques of noncommutative projective geometry is to study a graded ring R via a category $\mathcal{C}$ of graded R-modules. More specifically, one usually examines $\operatorname{QGr} R$, the quotient category of graded right R-modules modulo the full subcategory of torsion modules; one hopes that $\operatorname{QGr} R$ will have geometric properties, since the Serre Correspondence Theorem says that if R is commutative and generated

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in degree one, then there is a category equivalence $\operatorname{QGr} R \cong \operatorname{Qch} X$, where $\operatorname{Qch} X$ is the category of quasi-coherent sheaves on $X = \operatorname{Proj} R$ [1, Exercise II.5.9].

The twisted homogeneous coordinate rings are the most basic class of rings in noncommutative geometry. Such a ring R is constructed from a commutative projective scheme X, an automorphism σ of X, and an invertible sheaf $\mathcal{L}$. When the pair $(\mathcal{L}, \sigma)$ satisfies "right σ-ampleness," then R is right noetherian and has $\operatorname{QGr} R \cong \operatorname{Qch} X$ [2, Theorems 1.3, 1.4]. These rings were first used to show that Artin–Schelter regular algebras of dimension 3 are noetherian domains [3–5] and their basic properties were studied in [2]. Further, any domain of GK-dimension 2, generated in degree one, is a twisted homogeneous coordinate ring for some curve X [6].

A simple criterion for right σ-ampleness was found in [7]. From this criterion one sees that right and left σ-ampleness are equivalent. Hence the associated ring R is noetherian. One also sees that the GK-dimension of R is an integer. (While this paper and [7] work over an algebraically closed field, we note that [8] generalized these results to the case of a commutative noetherian base ring.)

Chan introduced twisted multi-homogeneous coordinate rings in [9], which are constructed from a finite collection $\{(\mathcal{L}_i, \sigma_i)\}$ of invertible sheaves and automorphisms on a projective scheme X. When the set $\{(\mathcal{L}_i, \sigma_i)\}$ is "right ample," then the category $\operatorname{QGr} R$ of multi-graded right R-modules modulo torsion modules again has $\operatorname{QGr} R \cong \operatorname{Qch} X$. With some natural extra hypotheses, R will be right noetherian. Via these methods, Chan shows that some rings associated to twisted homogeneous coordinate rings, like tensor products of two such coordinate rings, are right noetherian.

In this paper, we will generalize the results of [7] to the multi-homogeneous case and thereby strengthen [9]. More specifically, we show

Theorem 1.1 (see Theorem 2.7, Corollary 2.8). *Let X be a projective scheme and let $\{(\mathcal{L}_i, \sigma_i)\}$ be a finite set of pairs of invertible sheaves and automorphisms. Then there is a simple criterion for $\{(\mathcal{L}_i, \sigma_i)\}$ to be right ample. This criterion shows that right and left ampleness are equivalent.*

We then immediately have, in Corollary 3.5, that the tensor product $B \otimes_k B'$ is noetherian, where B, B' are twisted homogeneous coordinate rings associated to ample pairs $(\mathcal{L}, \sigma), (\mathcal{L}', \sigma')$. If B is generated in degree one and I is the irrelevant ideal of B, then the Rees algebra $B[It]$ is noetherian; see Corollary 3.4.

We also show

Theorem 1.2 (see Theorem 4.6). *Let B be a twisted multi-homogeneous coordinate ring under suitable hypotheses* (*Hypothesis* 4.1). *Then* $\operatorname{GKdim} B$ *is an integer with geometrically defined bounds.*

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2. Right ampleness is left ampleness

Because of the notational difficulties associated with handling the ampleness of arbitrarily many pairs $(\mathcal{L}_i, \sigma_i)$, we will use the concept of an invertible bimodule $\mathcal{L}_\sigma$. In this paper it will only be important to know how invertible bimodules act on a coherent sheaf $\mathcal{F}$, so we will treat $\mathcal{L}_\sigma$ as a notational convenience where

$$\mathcal{F} \otimes \mathcal{L}_\sigma = \sigma_*(\mathcal{F} \otimes \mathcal{L}), \qquad \mathcal{L}_\sigma \otimes \mathcal{F} = \mathcal{L} \otimes \sigma^* \mathcal{F}$$

and the right-hand side of the above equations are just $\mathcal{O}_X$-modules. For a formal definition of invertible bimodule see [2, §2]. Given two invertible bimodules $\mathcal{L}_\sigma$ and $\mathcal{M}_\tau$, one finds the tensor product to be

$$\mathcal{L}_\sigma \otimes \mathcal{M}_\tau = (\mathcal{L} \otimes \sigma^* \mathcal{M})_{\tau\sigma}, \tag{2.1}$$

where the second tensor product is the usual product on quasi-coherent sheaves [2, Lemma 2.14]. We will sometimes denote the product of invertible bimodules by juxtaposition if the meaning is clear. The $\mathcal{O}_X$-module underlying a product of bimodules $\mathcal{L}_\sigma \otimes \mathcal{M}_\tau$ will be denoted $|\mathcal{L}_\sigma \otimes \mathcal{M}_\tau|$; in this particular case $|\mathcal{L}_\sigma \otimes \mathcal{M}_\tau| = \mathcal{L} \otimes \sigma^* \mathcal{M}$.

We will also use the notation $\mathcal{L}^\sigma = \sigma^* \mathcal{L}$. The automorphism σ induces a natural isomorphism

$$\mathcal{F} \otimes \mathcal{L}_\sigma = \sigma_*(\mathcal{F} \otimes \mathcal{L}) \cong \mathcal{L}^{\sigma^{-1}} \otimes \mathcal{F}^{\sigma^{-1}} = \mathcal{L}^{\sigma^{-1}}_{\sigma^{-1}} \otimes \mathcal{F} \tag{2.2}$$

for any coherent sheaf $\mathcal{F}$.

We now sketch the construction of a twisted multi-homogeneous coordinate ring; for details see [9, §2]. Let $\{(\mathcal{L}_i)_{\sigma_i}\}$ be a collection of s invertible bimodules, possibly with repetitions. For notational convenience, we will write $\mathcal{L}_{(i,\sigma_i)} = (\mathcal{L}_i)_{\sigma_i}$. Given these s invertible bimodules, one wishes to form an associated twisted multi-homogeneous coordinate ring $B = B(X; \{\mathcal{L}_{(i,\sigma_i)}\})$. For an s-tuple $\bar{n} = (n_1, \ldots, n_s)$ we define the multi-graded piece $B_{\bar{n}}$ as

$$B_{\bar{n}} = H^0\big(X, \mathcal{L}^{n_1}_{(1,\sigma_1)} \cdots \mathcal{L}^{n_s}_{(s,\sigma_s)}\big), \tag{2.3}$$

where the cohomology of an invertible bimodule is just cohomology of the underlying sheaf. Multiplication should be given by

$$a \cdot b = a\sigma^{\bar{m}}(b), \tag{2.4}$$

when $a \in B_{\bar{m}}$ and $b \in B_{\bar{n}}$. Here $\sigma^{\bar{m}}(b) = \sigma_1^{m_1}\sigma_2^{m_2} \cdots \sigma_s^{m_s}(b)$, where the action of an automorphism on a global section is induced by pullback.

However, to make the ring construction work, [9] shows that we need the invertible bimodules to commute with each other. Examining (2.1), we see that two bimodules $\mathcal{L}_\sigma$, $\mathcal{M}_\tau$ commute when

$$\mathcal{L} \otimes \sigma^* \mathcal{M} \cong \mathcal{M} \otimes \tau^* \mathcal{L} \quad \text{and} \quad \sigma\tau = \tau\sigma. \tag{2.5}$$

Thus we need sheaf isomorphisms $\varphi_{ij} : \mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(i,\sigma_i)} \to \mathcal{L}_{(i,\sigma_i)}\mathcal{L}_{(j,\sigma_j)}$ for each $1 \leqslant i < j \leqslant s$. It is further noted in [9] that when there are three or more bimodules, these isomorphisms must be compatible on "overlaps" in the sense of Bergman's Diamond Lemma. In terms of the isomorphism φ_{ij} this means [9, p. 444]

$$\begin{aligned}&(\varphi_{ij} \otimes 1_{\mathcal{L}_{(k,\sigma_k)}}) \circ (1_{\mathcal{L}_{(j,\sigma_j)}} \otimes \varphi_{ik}) \circ (\varphi_{jk} \otimes 1_{\mathcal{L}_{(i,\sigma_i)}})\\ &\quad = (1_{\mathcal{L}_{(i,\sigma_i)}} \otimes \varphi_{jk}) \circ (\varphi_{ik} \otimes 1_{\mathcal{L}_{(j,\sigma_j)}}) \circ (1_{\mathcal{L}_{(k,\sigma_k)}} \otimes \varphi_{ij})\end{aligned} \tag{2.6}$$

in $\operatorname{Hom}(\mathcal{L}_{(k,\sigma_k)}\mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(i,\sigma_i)}, \mathcal{L}_{(i,\sigma_i)}\mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(k,\sigma_k)})$. We will always assume that we have this compatibility when forming the ring B. Summarizing, we have

Proposition 2.1. *Let $\{\mathcal{L}_{(i,\sigma_i)}\}$ be a finite collection of commuting invertible bimodules. Assume that these bimodules have compatible pairwise commutation relations in the sense of* (2.6). *Then there is a multi-graded ring B with multi-graded pieces given by* (2.3) *and multiplication given by* (2.4).

To study these rings, a multi-graded version of σ-ampleness is introduced. Since we will be interested in both this version of ampleness and the usual commutative one, we will call this (right) NC-*ampleness*, whereas [9] uses the terminology (right) ampleness. We define the ordering on s-tuples to be the standard one, i.e., $(n'_1, \dots, n'_s) \geqslant (n_1, \dots, n_s)$ if $n'_i \geqslant n_i$ for all i. For simplicity we write $\mathcal{L}_{\bar{\sigma}}^{\bar{m}} = \mathcal{L}_{(1,\sigma_1)}^{m_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{m_s}$.

Definition 2.2. Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$.

(1) If for any coherent sheaf $\mathcal{F}$, there exists an $\bar{m}_0$ such that

$$H^q\big(X, \mathcal{F} \otimes \mathcal{L}_{\bar{\sigma}}^{\bar{m}}\big) = 0$$

for $q > 0$ and $\bar{m} \geqslant \bar{m}_0$, then the set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is called *right* NC-*ample*.

(2) If for any coherent sheaf $\mathcal{F}$, there exists an $\bar{m}_0$ such that

$$H^q\big(X, \mathcal{L}_{\bar{\sigma}}^{\bar{m}} \otimes \mathcal{F}\big) = 0$$

for $q > 0$ and $\bar{m} \geqslant \bar{m}_0$, then the set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is called *left* NC-*ample*.

As in the case of one invertible bimodule, right and left NC-ampleness are related.

Lemma 2.3 (cf. [7, Lemma 2.3]). *Let X be a projective scheme with s commuting invertible bimodules $\{(\mathcal{L}_i)_{\sigma_i}\}$. Then the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ commutes pairwise. Also, the set $\{(\mathcal{L}_i)_{\sigma_i}\}$ is right* NC-*ample if and only if the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ is left* NC-*ample.*

Proof. Let $\mathcal{L}_\sigma, \mathcal{M}_\tau$ be two commuting invertible bimodules. Then (2.5) holds. Obviously $\sigma^{-1}\tau^{-1} = \tau^{-1}\sigma^{-1}$. Now since $\mathcal{L} \otimes \sigma^*\mathcal{M} \cong \mathcal{M} \otimes \tau^*\mathcal{L}$, pulling back by $\sigma^{-1}\tau^{-1}$ we have

$$\left(\tau^{-1}\right)^*\left(\sigma^{-1}\right)^*\mathcal{L} \otimes \left(\tau^{-1}\right)^*\mathcal{M} \cong \left(\sigma^{-1}\right)^*\left(\tau^{-1}\right)^*\mathcal{M} \otimes \left(\sigma^{-1}\right)^*\mathcal{L}.$$

So $\mathcal{L}^{\sigma^{-1}}_{\sigma^{-1}} = ((\sigma^{-1})^*\mathcal{L})_{\sigma^{-1}}$ and $\mathcal{M}^{\tau^{-1}}_{\tau^{-1}} = ((\tau^{-1})^*\mathcal{M})_{\tau^{-1}}$ commute.

Now using (2.2) and the fact that the bimodules commute, we see that

$$H^q\left(X, \mathcal{F} \otimes (\mathcal{L}_1)^{m_1}_{\sigma_1} \cdots (\mathcal{L}_s)^{m_s}_{\sigma_s}\right) = H^q\left(X, \left(\mathcal{L}_1^{\sigma_1^{-1}}\right)^{m_1}_{\sigma_1^{-1}} \cdots \left(\mathcal{L}_s^{\sigma_s^{-1}}\right)^{m_s}_{\sigma_s^{-1}} \otimes \mathcal{F}\right)$$

for all q, m_i. Thus right NC-ampleness of $\{(\mathcal{L}_i)_{\sigma_i}\}$ is equivalent to left NC-ampleness of $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$. $\square$

Lemma 2.4. *Let X be a projective scheme over k with s commuting invertible bimodules $\{(\mathcal{L}_i)_{\sigma_i}\}$. Assume that the commutation relations of $\{(\mathcal{L}_i)_{\sigma_i}\}$ and of $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ are compatible in the sense of* (2.6). *If $B' = B(X; \{(\mathcal{L}_i)_{\sigma_i}\})$ and $B = B(X; \{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\})$, then $B \cong (B')^{\mathrm{op}}$.*

Proof. Let $\tau : B \to (B')^{\mathrm{op}}$ be given by $\tau(a) = \sigma_1^{n_1} \cdots \sigma_s^{n_s}(a)$ for $a \in B_{(n_1,\dots,n_s)}$. Extend τ linearly so it is a vector space map. It is obviously a vector space isomorphism.

Let $\cdot$ be multiplication in B and $*$ be multiplication in $(B')^{\mathrm{op}}$. For $a \in B_{\bar{n}}, b \in B_{\bar{m}}$,

$$\begin{aligned}\tau(a \cdot b) &= \tau\left(a\sigma^{-\bar{n}}(b)\right) = \sigma^{\bar{n}+\bar{m}}(a)\sigma^{\bar{m}}(b),\\ \tau(a) * \tau(b) &= \sigma^{\bar{n}}(a) * \sigma^{\bar{m}}(b) = \sigma^{\bar{m}}(b)\sigma^{\bar{n}+\bar{m}}(a).\end{aligned}$$

Thus $\tau(a \cdot b) = \tau(a) * \tau(b)$, as required. $\square$

As in [7, Proposition 2.3], we have simpler equivalent conditions for a set of bimodules to be right NC-ample.

Proposition 2.5. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. Then the following are equivalent*:

(1) *The set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right* NC*-ample.*
(2) *For any coherent sheaf $\mathcal{F}$, there exists an $\bar{m}_0$ such that $\mathcal{F} \otimes \mathcal{L}^{\bar{m}}_{\bar{\sigma}}$ is generated by global sections for $\bar{m} \geqslant \bar{m}_0$.*
(3) *For any invertible sheaf $\mathcal{H}$, there exists an $\bar{m}_0$ such that $|\mathcal{H}^{-1} \otimes \mathcal{L}^{\bar{m}}_{\bar{\sigma}}|$ is very ample for $\bar{m} \geqslant \bar{m}_0$.*
(4) *For any invertible sheaf $\mathcal{H}$, there exists an $\bar{m}_0$ such that $|\mathcal{H}^{-1} \otimes \mathcal{L}^{\bar{m}}_{\bar{\sigma}}|$ is ample for $\bar{m} \geqslant \bar{m}_0$.*

A similar statement holds for left NC*-ample.*

Proof. This is a special case of [8, Theorem 1.3, Proposition 6.9]. □

We can now give a connection between right NC-ampleness and the concept of σ-ampleness for one invertible sheaf $\mathcal{L}$.

Lemma 2.6. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. Suppose that $\bar{n} = (n_1, \ldots, n_s) \in (\mathbb{N}^+)^s$ and set $\tau = \sigma_1^{n_1} \cdots \sigma_s^{n_s}$. If the set of bimodules is right* NC*-ample, then $|\mathcal{L}_{(1,\sigma_1)}^{n_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_s}|$ is τ-ample.*

Proof. Let $\mathcal{H}$ be an invertible sheaf and let $\bar{m}_0$ be such that for all $\bar{m} \geqslant \bar{m}_0$, the sheaf $|\mathcal{H}^{-1} \otimes \mathcal{L}_{\bar{\sigma}}^{\bar{m}}|$ is ample by Proposition 2.5(4).

Now there exists an integer l_0 such that for all $l \geqslant l_0$, we have $l\bar{n} \geqslant \bar{m}_0$. So $|\mathcal{H}^{-1} \otimes (\mathcal{L}_{\bar{\sigma}}^{\bar{n}})^l|$ is ample. Thus by [7, Proposition 2.3(4)], $|\mathcal{L}_{\bar{\sigma}}^{\bar{n}}|$ is τ-ample. □

Recall that the Picard group of X modulo numerical equivalence, $A^1_{\mathrm{Num}}(X) = \operatorname{Pic} X/\equiv$, is a finitely generated free abelian group [10, p. 305, Remark 3]. Thus the action of σ on $A^1_{\mathrm{Num}}(X)$ is given by some $P \in \mathrm{GL}_\rho(\mathbb{Z})$ for some $\rho > 0$. We say that σ is *unipotent* if all the eigenvalues of P equal 1 and that σ is *quasi-unipotent* if all the eigenvalues of P are roots of unity. This is a well-defined notion [8, Proposition 7.12]. We then have a new version of [7, Theorem 1.3].

Theorem 2.7. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. The set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is* (*right*) NC*-ample if and only if every σ_i is quasi-unipotent and there exists $\bar{m}_0 \in \mathbb{N}^s$ such that $|\mathcal{L}_{\bar{\sigma}}^{\bar{m}}|$ is ample for all $\bar{m} \geqslant \bar{m}_0$.*

Proof. Suppose that $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample. Then by Proposition 2.5(4), there exists $\bar{m}_0 \in \mathbb{N}^s$ such that $|\mathcal{L}_{\bar{\sigma}}^{\bar{m}}|$ is ample for all $\bar{m} \geqslant \bar{m}_0$. Further, by the previous lemma, $\mathcal{L}_{(1,\sigma_1)}^{n_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_s}$ is τ-ample when $\tau = \sigma_1^{n_1} \cdots \sigma_s^{n_s}$ and each $n_i > 0$. Now recall that all the automorphisms commute and hence their actions on $A^1_{\mathrm{Num}}(X)$ are commuting matrices. Thus the eigenvalues of the product $\sigma_1^{n_1} \cdots \sigma_s^{n_s}$ are products of eigenvalues from each σ_i. So if σ_1 were not quasi-unipotent, then either $\tau_1 = \sigma_1\sigma_2 \cdots \sigma_s$ or $\tau_2 = \sigma_1^2\sigma_2 \cdots \sigma_s$ would not be quasi-unipotent. But τ_1 and τ_2 must be quasi-unipotent by [7, Theorem 1.3] since the corresponding sheaves $\mathcal{L}_{(1,\sigma_1)}^1 \cdots \mathcal{L}_{(s,\sigma_s)}^1$ and $\mathcal{L}_{(1,\sigma_1)}^2 \cdots \mathcal{L}_{(s,\sigma_s)}^1$ are τ_1-ample and τ_2-ample respectively. Thus each σ_i must be quasi-unipotent.

Now suppose that every σ_i is quasi-unipotent and there exists $\bar{m}_0 \in \mathbb{N}^s$ such that $|\mathcal{L}_{\bar{\sigma}}^{\bar{m}}|$ is ample for all $\bar{m} \geqslant \bar{m}_0$. As the σ_i commute, $\tau = \sigma_1 \cdots \sigma_s$ is quasi-unipotent. Then by [7, Theorem 1.3], the invertible bimodule $\mathcal{L}_{(1,\sigma_1)} \cdots \mathcal{L}_{(s,\sigma_s)}$ is τ-ample. So given any invertible sheaf $\mathcal{H}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\mathcal{H}^{-1} \otimes \left(\mathcal{L}_{(1,\sigma_1)} \cdots \mathcal{L}_{(s,\sigma_s)}\right)^n\right| = \left|\mathcal{H}^{-1} \otimes \mathcal{L}_{(1,\sigma_1)}^n \cdots \mathcal{L}_{(s,\sigma_s)}^n\right|$$

is ample for $n \geqslant n_0$ by [7, Proposition 2.3(4)]. Then we have that for all $\bar{m} \geqslant (n_0, n_0, \ldots, n_0) + \bar{m}_0$ the invertible sheaf

$$\left|\mathcal{H}^{-1} \otimes \mathcal{L}_{\bar{\sigma}}^{\bar{m}}\right| = \left|\mathcal{H}^{-1} \otimes \mathcal{L}_{(1,\sigma_1)}^{n_0} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_0}\right| \otimes \left|\mathcal{L}_{(1,\sigma_1)}^{m_1-n_0} \cdots \mathcal{L}_{(s,\sigma_s)}^{m_s-n_0}\right|^{\sigma_1^{n_0} \cdots \sigma_s^{n_0}}$$

is the tensor product of two ample invertible sheaves. Hence it is ample and so the set of invertible bimodules is right NC-ample by Proposition 2.5(4). □

Corollary 2.8. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. Then $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right* NC*-ample if and only if it is left* NC*-ample.*

Proof. Suppose that $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample. Then each σ_i is quasi-unipotent and there exists $\bar{m}_0$ such that $|\mathcal{L}_{(1,\sigma_1)}^{m_1}\cdots\mathcal{L}_{(s,\sigma_s)}^{m_s}|$ is ample for $(m_1,\ldots,m_s)\geqslant\bar{m}_0$. Pulling back by $\sigma_1^{-m_1}\cdots\sigma_s^{-m_s}$, we have that the invertible sheaf

$$\bigl|\bigl(\mathcal{L}_1^{\sigma_1^{-1}}\bigr)_{\sigma_1^{-1}}^{m_1}\cdots\bigl(\mathcal{L}_s^{\sigma_s^{-1}}\bigr)_{\sigma_s^{-1}}^{m_s}\bigr|$$

is ample. Thus by Theorem 2.7, the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ is right NC-ample. So the original set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is left NC-ample by Lemma 2.3. The argument is clearly reversible. □

Thus we may now refer to a set of bimodules as being simply NC-ample.

Note the difference between [7, Theorem 1.3] and Theorem 2.7. The former requires only that $|\mathcal{L}_\sigma^m|$ is ample for one value of m, while the latter requires the product of bimodules to be ample for all $\bar{m}\geqslant\bar{m}_0$. To see this stronger requirement is necessary, let X be any projective scheme with $\mathcal{L}$ any ample invertible sheaf. We need to rule out the pair $\mathcal{L},\mathcal{L}^{-1}$ where the bimodule action is the usual commutative one. In this particular case, of course $\mathcal{L}^1\otimes(\mathcal{L}^{-1})^0$ is ample. But $\mathcal{L}^{m_1}\otimes(\mathcal{L}^{-1})^{m_2}$ is not ample for all (m_1,m_2) sufficiently large; just fix m_1 and let m_2 go to infinity.

It is not necessary for one of the $\mathcal{L}_{(i,\sigma_i)}^m$ to be ample for $m\gg 0$, since on $\mathbb{P}^1\times\mathbb{P}^1$, the pair $\mathcal{O}(1,0),\mathcal{O}(0,1)$ is NC-ample, where again these bimodules act only as commutative invertible sheaves.

3. Ring theoretic consequences

Unlike the case of only one bimodule, the multi-graded ring B may not be noetherian when $\{\mathcal{L}_{(i,\sigma_i)}\}$ is NC-ample. In fact, [9, Example 5.1] gives a simple commutative (and hence not finitely generated) counterexample. However, Chan introduces an additional property for an invertible bimodule $\mathcal{L}_\sigma$ on X to guarantee the noetherian condition.

Hypothesis 3.1. *There exists a projective scheme Y with automorphism σ and a σ-equivariant morphism $f : X\to Y$. That is $\sigma_Y\circ f=f\circ\sigma_X$. There also exists an invertible sheaf $\mathcal{L}'$ on Y such that $\mathcal{L}=f^*\mathcal{L}'$ and such that $\mathcal{L}'_\sigma$ is σ-ample.* ([9] *labels this property* $(*)$.)

This property (Hypothesis 3.1) is saying that for $m\gg 0$, $|\mathcal{L}_\sigma^m|$ is generated by global sections, since it is a pullback of $|(\mathcal{L}')_\sigma^m|$, which is eventually very ample by [7, Proposition 2.3(3)]. Note in particular that if $\mathcal{L}$ is already σ-ample, then $\mathcal{L}_\sigma$ satisfies Hypothesis 3.1 trivially. Using this property, one determines

Theorem 3.2 [9, Theorem 5.2]. *Let X be a projective scheme with commuting invertible bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$. Suppose that the pair is* (*right*) NC-*ample and each bimodule satisfies Hypothesis* 3.1, *possibly for different Y. Then $B(X; \mathcal{L}_\sigma, \mathcal{M}_\tau)$ is right noetherian.*

Then combining Corollary 2.8, Lemma 2.4, and the theorem above, we have

Theorem 3.3. *Let X be a projective scheme with commuting invertible bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$. Suppose that the pair is* NC-*ample and each bimodule satisfies Hypothesis* 3.1, *possibly for different Y. Then $B(X; \mathcal{L}_\sigma, \mathcal{M}_\tau)$ is noetherian.*

Now we can prove that two particularly interesting twisted multi-homogeneous coordinate rings, a Rees ring and a tensor product, are noetherian, strengthening the results of [9, Corollaries 5.7, 5.8]. In the latter case, we may replace his proof, based on spectral sequences, by an easier one since the criterion of Theorem 2.7 simplifies testing the NC-ampleness of the relevant pair of bimodules.

Corollary 3.4. *Let $\mathcal{L}_\sigma$ be σ-ample on a projective scheme X. Let the ring $B = B(X; \mathcal{L}_\sigma)$ be generated in degree one. Then the Rees ring $B[It] = \bigoplus_{r=0}^{\infty} I^r t^r$ of B is noetherian, where $I = B_{>0}$ is the irrelevant ideal.*

Proof. The ring $B[It]$ has bigraded pieces

$$B_{(i,j)} = H^0\big(X, \mathcal{L}_\sigma^i \mathcal{L}_\sigma^j\big)t^j$$

since $I^j = \bigoplus_{l=j}^{\infty} B_l$ when B is generated in degree one. The pair $\mathcal{L}_\sigma, \mathcal{L}_\sigma$ is obviously NC-ample and satisfies Hypothesis 3.1. Thus Theorem 3.3 applies. □

Corollary 3.5. *Let $\mathcal{L}_\sigma$ be σ-ample on a projective scheme X and let $\mathcal{M}_\tau$ be τ-ample on a projective scheme Y. Then $B(X; \mathcal{L}_\sigma) \otimes B(Y; \mathcal{M}_\tau)$ is noetherian.*

Proof. It is argued in [9, Example 4.3] that

$$B(X; \mathcal{L}_\sigma) \otimes B(Y; \mathcal{M}_\tau) \cong B\big(X \times Y; \big(\pi_1^*\mathcal{L}\big)_{\sigma\times 1}, \big(\pi_2^*\mathcal{M}\big)_{1\times\tau}\big),$$

where the π_i are the natural projections. These two invertible bimodules on $X \times Y$ obviously satisfy Hypothesis 3.1.

Since $\mathcal{L}_\sigma$ is σ-ample and $\mathcal{M}_\tau$ is τ-ample, there is an m_0 such that $|\mathcal{L}_\sigma^m|$ and $|\mathcal{M}_\tau^m|$ is ample for all $m \geqslant m_0$. Note that $(\sigma \times 1)^*\pi_1^*\mathcal{L} = \pi_1^*\sigma^*\mathcal{L}$ and a similar formula holds for $\mathcal{M}_\tau$. Then

$$\big|\big(\pi_1^*\mathcal{L}\big)_{\sigma\times 1}^{m_1}\big(\pi_2^*\mathcal{M}\big)_{1\times\tau}^{m_2}\big|$$

is ample for all $(m_1, m_2) \geqslant (m_0, m_0)$ by [1, p. 125, Exercise 5.11].

Now σ is quasi-unipotent and we wish to show $\sigma \times 1$ is as well. It is tempting to think that as a matrix acting on $A^1_{\mathrm{Num}}(X \times Y)$ one has $\sigma \times 1 = \sigma \oplus 1$. However, this may not be the case, since in general $A^1_{\mathrm{Num}}(X \times Y)$ has larger rank than $A^1_{\mathrm{Num}}(X) \oplus A^1_{\mathrm{Num}}(Y)$ [1, p. 367, Exercise 1.6]. But let $\mathcal{H}_X$ and $\mathcal{H}_Y$ be ample invertible sheaves on X and Y, respectively. If $\sigma \times 1$ is not quasi-unipotent, then by [7, Lemma 3.2], there exists $r > 1$, $c > 0$, and an integral curve C on $X \times Y$ such that

$$\big(\big((\sigma \times 1)^*\big)^m \big(\pi_1^*\mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y\big).C\big) \geqslant cr^m \quad \text{for all } m \geqslant 0. \tag{3.1}$$

But

$$\big((\sigma \times 1)^*\big)^m \big(\pi_1^*\mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y\big) = \pi_1^*\big(\sigma^*\big)^m \mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y.$$

Since σ is quasi-unipotent, the intersection numbers of the right-hand side with any curve C must be bounded by a polynomial. This contradicts (3.1). So $\sigma \times 1$ must be quasi-unipotent. Similarly, $1 \times \tau$ is quasi-unipotent. Thus by Theorem 2.7, the pair $(\pi_1^*\mathcal{L})_{\sigma\times 1}, (\pi_2^*\mathcal{M})_{1\times\tau}$ is NC-ample and thus the ring of interest is noetherian. □

4. Gel'fand–Kirillov dimension

In this section we generalize the results of [7, §6], showing that a noetherian twisted multi-homogeneous coordinate ring has integer GK-dimension. We first fix hypotheses on the ring B.

Hypothesis 4.1. *Let X be a projective scheme with s commuting* NC*-ample bimodules* $\{\mathcal{L}_{(i,\sigma_i)}\}$. *Assume that the commutation relations of* $\{(\mathcal{L}_i)_{\sigma_i}\}$ *are compatible in the sense of* (2.6). *Let* $B = B(X; \{(\mathcal{L}_i)_{\sigma_i}\})$ *and suppose that B is right noetherian.*

If B is the twisted multi-homogeneous coordinate ring associated to an NC-ample set of invertible bimodules, then the vanishing of cohomology in Definition 2.2 allows one to control the dimension of $B_{\bar\imath}$ for $\bar\imath \geqslant \bar\imath_0$ for some $\bar\imath_0 \in \mathbb{N}^s$. We are not guaranteed such control on the "edges" $\bigoplus_j B_{(0,\dots,j,\dots,0)}$. Thus, it will be easier to study the GK-dimension of the ideal $B_{\geqslant \bar\imath_0}$ rather then the GK-dimension of B.

Lemma 4.2. *Let B satisfy Hypothesis* 4.1, *and let $\bar\imath \in \mathbb{N}^s$. Then*

$$\operatorname{GKdim} B = \operatorname{GKdim}\big(B_{\geqslant \bar\imath}\big)_B.$$

Proof. If $\bar\jmath \geqslant \bar\imath$, then $B_{\geqslant \bar\jmath} \subseteq B_{\geqslant \bar\imath}$ and $\operatorname{GKdim} B_{\geqslant \bar\jmath} \leqslant \operatorname{GKdim} B_{\geqslant \bar\imath}$. So we may assume that $\bar\imath$ is sufficiently large so that $\mathcal{L}^{j_1}_{(1,\sigma_1)} \cdots \mathcal{L}^{j_s}_{(s,\sigma_s)}$ is generated by global sections for $(j_1, \dots, j_s) \geqslant \bar\imath$ by Proposition 2.5(2). So $B_{\bar\jmath} \subseteq B_{\bar\jmath+\bar\imath}$ for all $\bar\jmath \in \mathbb{N}^s$.

We may grade B by $\{B_{\leqslant(n,n,\dots,n)}/B_{\leqslant(n-1,n-1,\dots,n-1)}\colon n \in \mathbb{N}\}$ and grade $B_{\geqslant\bar\imath}$ by $\{B_{\leqslant(n,n,\dots,n)+\bar\imath}/B_{\leqslant(n-1,n-1,\dots,n-1)+\bar\imath}\colon n \in \mathbb{N}\}$. Then

$$\operatorname{GKdim} B = \overline{\lim}_n \log_n \dim B_{\leqslant(n,n,\dots,n)} \leqslant \overline{\lim}_n \log_n \dim B_{\leqslant(n,n,\dots,n)+\bar\imath} = \operatorname{GKdim} B_{\geqslant\bar\imath}$$

since $B_{\geqslant \bar{i}}$ is finitely generated [11, Lemma 6.1]. That $\operatorname{GKdim} B_{\geqslant \bar{i}} \leqslant \operatorname{GKdim} B$ is trivial. $\square$

We will need to use multi-Veronese subrings and also generalize a standard lemma for graded rings to the multi-graded case.

Definition 4.3. Let B be a k-algebra, finitely multi-graded by $\mathbb{N}^s$ (that is, each multi-graded piece is finite dimensional). Then the subring

$$B^{(n_1,\dots,n_s)} = \bigoplus_{(i_1,\dots,i_s)\in\mathbb{N}^s} B_{(n_1 i_1,\dots,n_s i_s)}$$

is a *multi-Veronese subring* of B.

Lemma 4.4. *Let B be a k-algebra, finitely multi-graded by $\mathbb{N}^s$.*

(1) *If B has ACC on multi-graded right ideals, then B is right noetherian.*
(2) *If B is right noetherian, then the multi-Veronese subring $A = B^{(n_1,\dots,n_s)}$ is right noetherian for any $(n_1,\dots,n_s) \in (\mathbb{N}^+)^s$.*

Proof. Both claims are simple generalizations of the graded case. For (1), one may see that the conclusion is implicit in the proof that a right multi-filtered ring is right noetherian if its associated multi-graded ring is right noetherian [12, Theorem 1.5]. The proof of (2) is as in [13, Proposition 5.10(1)], noting that if I is a multi-graded ideal of A, then $I = IB \cap A$. $\square$

Now we may replace B with a multi-Veronese.

Lemma 4.5. *Let B satisfy Hypothesis* 4.1*, and let $\bar{n} \in (\mathbb{N}^+)^s$. Then $B^{(\bar{n})}$ satisfies Hypothesis* 4.1 *and* $\operatorname{GKdim} B = \operatorname{GKdim} B^{(\bar{n})}$.

Proof. Let $\bar{n} = (n_1,\dots,n_s)$ and $A = B^{(\bar{n})}$. For the first claim, we have already seen in Lemma 4.4 that A is right noetherian. The bimodules $\{\mathcal{L}_{(i,\sigma_i)}^{n_i}\}$ commute compatibly because their commutation relations are compositions of the commutation relations for $\{\mathcal{L}_{(i,\sigma_i)}\}$. The bimodules $\{\mathcal{L}_{(i,\sigma_i)}^{n_i}\}$ are also NC-ample by Theorem 2.7.

Now choose $\bar{m} = (m_1,\dots,m_s) \in \mathbb{N}^s$ such that $\mathcal{L}_{(1,\sigma_1)}^{j_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{j_s}$ is generated by global sections for $(j_1,\dots,j_s) \geqslant \bar{m}$ by Proposition 2.5(2). Then for $m_i \leqslant j_i < n_i + m_i$ there are short exact sequences

$$0 \to \mathcal{K}_{(j_1,\dots,j_s)} \to B_{(j_1,\dots,j_s)} \otimes \mathcal{O}_X \to \mathcal{L}_{(1,\sigma_1)}^{j_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{j_s} \to 0.$$

Then tensoring with $\mathcal{L}_{(1,\sigma_1)}^{n_1 a_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_s a_s}$ and taking cohomology, we have

$$\begin{aligned} B_{(j_1,\dots,j_s)} \otimes H^0\big(\mathcal{L}_{(1,\sigma_1)}^{n_1 a_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_s a_s}\big) &\to B_{(j_1+n_1 a_1,\dots,j_s+n_s a_s)} \\ &\to H^1\big(\mathcal{K}_{(j_1,\dots,j_s)} \otimes \mathcal{L}_{(1,\sigma_1)}^{n_1 a_1} \cdots \mathcal{L}_{(s,\sigma_s)}^{n_s a_s}\big). \end{aligned}$$

For $(a_1, \dots, a_s)$ sufficiently large, the rightmost cohomology group vanishes. So there exists $\bar{b}$ such that

$$B_{\geqslant \bar{b}} \subseteq \sum_{1 \leqslant i \leqslant s} \sum_{0 \leqslant j_i < n_i + m_i} B_{(j_1, \dots, j_s)} A.$$

Hence $B_{\geqslant \bar{b}}$ is a finite A-module, so

$$\mathrm{GKdim}\, B = \mathrm{GKdim}\big(B_{\geqslant \bar{b}}\big)_B = \mathrm{GKdim}\big(B_{\geqslant \bar{b}}\big)_A \leqslant \mathrm{GKdim}\, A$$

by Lemma 4.2 and [11, Corollary 5.4]. □

We may now generalize [7, Theorem 6.1] to the multi-homogeneous case.

Theorem 4.6. *Let B satisfy Hypothesis* 4.1. *Then* GKdim B *is an integer and*

$$\dim X + 1 \leqslant \mathrm{GKdim}\, B \leqslant s\big((\ell + 1) \dim X + 1\big),$$

where s is the number of commuting bimodules, $\rho = \rho(X)$ is the Picard number of X, and $\ell = 2\lfloor \frac{\rho - 1}{2} \rfloor$.

Proof. By Lemma 4.5, we may replace B with a multi-Veronese; hence, replacing each σ_i with $\sigma_i^{m_i}$ for some m_i, we may assume each σ_i is unipotent. That is, up to numerical equivalence, $\sigma_i^{-1} \equiv I + N_i \in \mathrm{GL}_\rho(\mathbb{Z})$. We know $N_i^{\ell+1} = 0$ for all i [7, Lemma 6.12]. (We choose to use σ_i^{-1} since we will use Cartier divisors and if $\mathcal{L} \cong \mathcal{O}_X(D)$, then $\mathcal{L}^\sigma \cong \mathcal{O}_X(\sigma^{-1} D)$.)

Since the set of bimodules is NC-ample, we may again replace B with a multi-Veronese and assume that $H^q(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}}) = 0$ for all $q > 0$, $n_i > 0$ where $\bar{n} = (n_1, \dots, n_s)$. Thus $\dim H^0(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}}) = \chi(\mathcal{L}_{\bar{\sigma}}^{\bar{n}})$ for $n_i > 0$. So by the Riemann–Roch Theorem [14, p. 361, Example 18.3.6],

$$\dim H^0\big(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}}\big) = \sum_{j=0}^{\dim X} \frac{1}{j!} \int_X \big((\mathcal{L}_{\bar{\sigma}}^{\bar{n}})^{\bullet j}\big) \cap \tau_{X,j}(\mathcal{O}_X), \tag{4.1}$$

where $\bullet j$ denotes jth self-intersection and the $\tau_{X,j}(\mathcal{O}_X)$ are constant j-cycles. By Lemma 4.2, we may ignore $\dim H^0(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}})$ when some $n_i = 0$.

Let D_i be a Cartier divisor such that $\mathcal{L}_i \cong \mathcal{O}_X(D_i)$. The action of σ^{-n_i} on Cartier divisors modulo numerical equivalence is given by [7, (4.2), (4.3)]

$$\sigma_i^{-n_i} \equiv \sum_{c=0}^{\ell} \binom{n_i}{c} N_i^c,$$

$$\sum_{m=0}^{n_i - 1} \sigma_i^{-m} \equiv \sum_{d=0}^{\ell} \binom{n_i}{d+1} N_i^d.$$

So up to numerical equivalence,

$$\mathcal{L}_{\bar{\sigma}}^{\bar{n}} \equiv \sum_{a=1}^{s} \left[\left(\prod_{b=1}^{a-1} \left(\sum_{c=0}^{\ell} \binom{n_b}{c} N_b^c \right) \right) \cdot \left(\sum_{d=0}^{\ell} \binom{n_a}{d+1} N_a^d D_a \right) \right].$$

Thus $\dim H^0(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}})$ is a polynomial in n_i, $i = 1, \ldots, s$, with the degree of n_i at most $(\ell + 1) \dim X$, since one has at most a $(\dim X)$th self-intersection.

Now let $B_{\geqslant(1,1,\ldots,1)}$ have the filtration given by assuming each $n_i \leqslant n$. Then $f(n) = \dim B_{(1,1,\ldots,1)\leqslant(n_1,\ldots,n_s)\leqslant(n,\ldots,n)}$ is a polynomial in n of degree at most $s((\ell+1)\dim X + 1)$. This is because summing over each $i = 1, \ldots, s$ adds 1 to the degree of n_i. Then the degree $f(n)$ is maximized if $\dim H^0(X, \mathcal{L}_{\bar{\sigma}}^{\bar{n}})$ has a term of the form $n_1^{(\ell+1)\dim X} \cdots n_s^{(\ell+1)\dim X}$, since in this case, $(\ell + 1)\dim X + 1$ is added to itself s times.

Thus $\operatorname{GKdim} B$ is an integer with the desired upper bound by Lemma 4.2. Now by Lemma 2.6, B has a twisted homogeneous coordinate ring C as a subring. Now $\dim X + 1 \leqslant \operatorname{GKdim} C$ [8, Theorem 7.17], so the lower bound on $\operatorname{GKdim} B$ holds. □

Examining [7, Theorem 6.1], [8, Theorem 7.17] we see that these bounds on $\operatorname{GKdim} B$ are not optimal for the case $s = 1$. However, the notational difficulties of repeating the arguments of [7, §6] for s bimodules seem to outweigh the benefits, given that exact results can be given in the following specific cases.

Proposition 4.7. *Let $\mathcal{L}_\sigma$ be a σ-ample invertible bimodule on a projective scheme X. Let $B = B(X; \mathcal{L}_\sigma)$ be generated in degree one. Then*

$$\operatorname{GKdim} B[It] = \operatorname{GKdim} B + 1,$$

where $I = B_{>0}$ is the irrelevant ideal.

Proof. By Lemma 4.5, we may replace σ with some σ^m and assume that σ is unipotent, $\dim B_m = \chi(\mathcal{L}_\sigma^m)$ for $m \geqslant 1$, and $\dim B_m \leqslant \dim B_{m+1}$ for $m \geqslant 0$. Let $f(m) = \dim B_m$. Then $\operatorname{GKdim} B = \deg f + 1$ [7, (6.4)]. Filter $B[It]$ by $(B[It])_{(i,j)\leqslant(n,n)}$, $n \in \mathbb{N}$. Now $\dim(B[It])_{(i,j)} = \dim B_{i+j}$, so

$$g(n) = \sum_{i=0}^{n} \sum_{j=0}^{n} f(i + j) = \dim\bigl(B[It]\bigr)_{(i,j)\leqslant(n,n)}.$$

Since $f(m)$ is a numerical polynomial, $\deg g = \deg f + 2$. So $\operatorname{GKdim} B[It] = \deg f + 2 = \operatorname{GKdim} B + 1$. □

For general k-algebras R, S, we have $\operatorname{GKdim}(R \otimes_k S) \leqslant \operatorname{GKdim} R + \operatorname{GKdim} S$ [11, Lemma 3.10]. However, for the tensor product of a twisted homogeneous coordinate ring and a general k-algebra, we have equality, as in the commutative case.

Proposition 4.8. *Let $\mathcal{L}_\sigma$ be σ-ample on a projective scheme X, and let $B = B(X; \mathcal{L}_\sigma)$. Let S be any k-algebra. Then*

$$\operatorname{GKdim}(B \otimes S) = \operatorname{GKdim} B + \operatorname{GKdim} S.$$

Proof. There exists a Veronese subalgebra $B^{(n)}$ of B such that $f(m) = \dim B_m^{(n)}$ is a polynomial for $m > 0$ and $\operatorname{GKdim} B = \operatorname{GKdim} B^{(n)}$ [7, (6.3)–(6.4)]. We may also assume that $B^{(n)}$ is generated in degree one [8, Theorem 7.17]. Let $V = B_0 \oplus B_1^{(n)}$. Then $\dim V^m$ is a polynomial in m, so $\operatorname{GKdim}(B \otimes S) = \operatorname{GKdim} B + \operatorname{GKdim} S$ [11, Proposition 3.11]. □

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