Journal of Algebra 322 (2009) 1830–1858

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the center and semi-center of enveloping algebras in prime characteristic

Amiram Braun *, Gil Vernik

Department of Mathematics, University of Haifa, Haifa 31905, Israel

ARTICLE INFO

Article history:
Received 22 November 2008
Communicated by Michel Van den Bergh

Keywords:
Enveloping algebra
Lie algebra
Semi-center
Unique factorization
Divisor class group

ABSTRACT

Theorems of J. Dixmier and C. Moeglin on the semi-center of enveloping algebras in zero characteristic and of R.P. Stanley and H. Nakajima on polynomial relative invariants of finite groups, are shown to have analogs for enveloping algebras and for Lie algebra polynomial invariants, in the prime characteristic case. We shall illustrate by examples the extent of these analogies. A key result is the realization of the semi-center as a fixed ring of the action of a finite set of nilpotent derivations on the center of an enveloping algebra of a related Lie algebra.

1. Introduction

Let L be a finite dimensional Lie algebra over a field F and $U(L)$ be its enveloping algebra. For each $\lambda \in Hom_F(L,F)$, let $U(L)_\lambda \equiv \{a \in U(L) \mid [x,a] = \lambda(x)a,\ \ x \in L\}$. If $U(L)_\lambda \neq \{0\}$ we call λ, a weight of L (on $U(L)$) and set:

$$Sz\big(U(L)\big) = \bigoplus_{\lambda} U(L)_\lambda,$$

the semi-center of $U(L)$. Clearly $U(L)_0 = Z(U(L))$, the center of $U(L)$. By [11] $Sz(U(L))$ is a commutative ring. The notion of $Sz(U(L))$ is proven to be very useful in case $char\, F = 0$. This is evident from many results in [11, 2]. It is shown by C. Moeglin [23], if F is also algebraically closed that $Sz(U(L))$ is a factorial domain, thus extending an earlier result of J. Dixmier, in the nilpotent case [12].

In a totally different strand R. Stanley [29] and H. Nakajima [24] considered for a finite group G, the factorial and complete intersection properties of $S(V)^G$, where V is a finite dimensional G-module

* Corresponding author.
E-mail addresses: abraun@math.haifa.ac.il (A. Braun), gvernik@math.haifa.ac.il (G. Vernik).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.06.012

and $S(V)$ its symmetric algebra. The semi-invariants ring (or the ring of relative invariants, see e.g. [7, p. 279]):

$$Sz\big(S(V)\big) = \bigoplus_{\lambda} S(V)_{\lambda}, \quad \text{for } \lambda \in Hom\big(G, F \setminus \{0\}\big),$$

where $S(V)_{\lambda} = \{a \in S(V) \mid g \cdot a = \lambda(g)a, g \in G\}$, had an important role in their papers (though somewhat implicit in [29]). In particular $Cl(S(V)^G)$ is completely described (see also [4]) and it is shown in [24] that $S(V)^G$ is factorial if and only if $Sz(S(V))$ is free (finitely generated) over $S(V)^G$.

Let V be a finite dimensional L-module and $S(V)$ the symmetric algebra of V. Clearly the linear action of L on V extends to an action by derivations on $S(V)$. We denote by $S(V)^L = \{a \in S(V) \mid x \cdot a = 0, x \in L\}$, the fixed ring of L. Similarly $Sz(S(V)) = \bigoplus_{\lambda} S(V)_{\lambda}$, is the ring of L-semi-invariants, where

$$S(V)_{\lambda} = \big\{a \in S(V) \mid x \cdot a = \lambda(x)a, \ x \in L\big\}, \quad \lambda \in Hom_F(L, F).$$

The present paper will be mainly concerned with similar questions for a finite dimensional L, having $char\, F = p > 0$. The questions of when are $Z(U(L))$, $Sz(U(L))$, $S(V)^L$ factorial domains, are dealt with as well as the structure of the divisor class groups of $Z(U(L))$ and $S(V)^L$, $Cl(Z(U(L)))$ and $Cl(S(V)^L)$ respectively. We shall also consider the complete intersection property of $Sz(U(L))$. It will become clear that both strands do have common analogs in our case and we shall also explore the extent of validity of this analogy.

We shall now give a more detailed account of our results.

The following theorem is our key result in the answering the previous questions for $Z(U(L))$ and $Sz(U(L))$.

Theorem A. *Let L be a finite dimensional arbitrary Lie algebra over an algebraically closed field F with $char\, F = p > 0$. Then*

$$Sz\big(U(L)\big) = Z\big(U(L_0)\big)^{a\,da_1,\ldots,a\,da_r},$$

where L_0 is a Lie sub-algebra of $U(L)$, sharing a common ideal with L, which contains $[L, L]$, and $\dim_F L_0 = \dim_F L$. Here $a_1, \ldots, a_r$ are p-polynomials in some elements of L, acting nilpotently as derivations on $Z(U(L_0))$.

The following is an immediate consequence of Theorem A, and of a classical result of Zassenhaus [32], which grants the normality of $Z(U(L))$.

Theorem B. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with $char\, F = p > 0$. Then $Sz(U(L))$ is a normal affine domain.*

The next result is also a consequence of Theorem A. It clearly establishes an analog to Moeglin's result [23], in the prime characteristic case.

Theorem C. *Let L be a finite dimensional Lie algebra over an algebraically closed field F, with $char\, F = p > 0$. Suppose that $[L, L]$ is nilpotent. Then $Sz(U(L))$ is a factorial domain.*

Recall that by Lie's theorem $[L, L]$ is nilpotent in case L is solvable and $char\, F = 0$. However the analogy to the $char\, F = 0$ case, is not complete and we exhibit for each $p > 0$ a finite dimensional solvable Lie algebra L, over an algebraically closed field F, with a non-factorial semi-center.

Still inspite of the previous counter example, one can say more in the general solvable case. Recall that a Noetherian domain is factorial iff each of its height one prime ideals is principal. The next result shows that $Z(U(L))$ and $Sz(U(L))$ complement each other in this regard.

Theorem D. *Let L be a solvable finite dimensional Lie algebra over an algebraically closed field F with char $F = p \geqslant 3$. Let q be a height one prime ideal in $Z(U(L))$ and v the unique (height one) prime ideal in $Sz(U(L))$ with $v \cap Z(U(L)) = q$. Then at least one of the following holds*:

(i) *v is principal,*
(ii) *$q = (d)$ and $v^{(p)} = dSz(U(L))$, where $v^{(p)}$ is the p-th symbolic power of v.*

Theorem D leads to a complete description of the divisor class group $Cl(Z(U(L)))$ as follows.

Theorem E. *Let L be a finite dimensional solvable Lie algebra over an algebraically closed field F with char $F = p \geqslant 3$. Then there exists an exact sequence*:

$$0 \longrightarrow Cl\big(Z\big(U(L)\big)\big) \xrightarrow{\varphi} \{\lambda \mid \lambda \text{ is a weight of } L\} \xrightarrow{\psi} span_{\mathbb{Z}/p\mathbb{Z}}\{\lambda_1, \dots, \lambda_n\} \longrightarrow 0,$$

where λ_i is a weight corresponding to a prime non-central weight element which ramifies over $Z(U(L))$, for $1 \leqslant i \leqslant n$.

The analogy to the description of $Cl(S(V)^G)$, in the finite group case, is evident and can be seen, e.g. in [4]. However the interpretation of each λ_i as coming from a “pseudo-reflection” is missing here.

The next result is a corollary of Theorem E.

Proposition F. *Let L be a finite dimensional solvable Lie algebra over an algebraically closed field F with char $F = p > 0$. Then $Cl(Z(U(L)))$ and $Cl(Sz(U(L)))$ are finite elementary abelian p-groups.*

The next result assembles together several theorems determining when $Z(U(L))$ is a U.F.D.

Theorem G. *Let L be a finite dimensional solvable Lie algebra over a field F with char $F = p > 0$. Consider the following statements*:

(i) *$Z(U(L))$ is a U.F.D.,*
(ii) *$Sz(U(L))$ is a free (or merely a projective) $Z(U(L))$-module,*
(iii) *$Sz(U(L))$ has exactly $\log_p[Q(Sz(U(L))) : Q(Z(U(L)))]$ non-central prime weight elements,*
(iv) *The extension $Sz(U(L))/Z(U(L))$ has a finite p-basis (in the terminology of* [20, *p.* 76]*),*
(v) *The extension $Sz(U(L))/Z(U(L))$ is a global complete intersection (in the terminology of* [20, *p.* 317]*).*

Then (i) is equivalent to (ii). Moreover if in addition, F is algebraically closed and $p \geqslant 3$, then all statements are equivalent.

In the next theorem we group together all the results about Lie algebra polynomial invariants.

Theorem H. *Let L be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Let V be an F-finite dimensional L-module. Then all the results described in Theorems* D, E, F, G *hold for $S(V)^L$ (replacing $Z(U(L))$) and $Sz(S(V))$ (replacing $Sz(U(L))$).*

Observe that L need not be solvable here. As a sample consequence we mention the following result about torus invariants.

Proposition I. *Let $L \subseteq \mathfrak{gl}_F(V)$ be a commutative Lie subalgebra consisting of semi-simple elements, where $\dim_F V$ is finite and F is algebraically closed field with char $F = p > 0$. Then $S(V)^L$ is factorial if and only if it is a polynomial ring.*

The results presented so far suggest the plausibility of the following conjectures.

Conjecture J. *Theorems* E, F *and* G *are valid for every F-finite dimensional Lie algebra L with* $\operatorname{char} F = p > 0$.

In fact, this will be the case if one proves the following conjecture, which extends [6, Conjecture E].

Conjecture K. *Let L be a finite dimensional Lie algebra over a field F with $\operatorname{char} F = p > 0$. Suppose $[L, L] = L$. Then $Z(U(L))$ is factorial and every height one prime ideal in $U(L)$ is generated by a central element.*

Special cases of Conjecture K (and hence of [6, Conjecture E]) were already verified in [2,25]. Moreover in a very recent preprint [30], R. Tange confirmed Conjecture K for L which is a Lie algebra of a connected reductive algebraic group G (with some extra mild assumptions). In particular it holds for all the classical (that is, of non-Cartan type) semi-simple Lie algebras.

2. Preliminaries

The next result appears in [11, Proposition 4.3.5]. We reproduce it here in order to show that it is characteristic free. In fact Dixmier proves a more general result about the semi-center of prime quotients of $U(L)$ and this however seems to require the $\operatorname{char} F = 0$ assumption.

Proposition 2.1. *Let L be a finite dimensional Lie algebra over a field F of arbitrary characteristic. Then $Sz(U(L))$ is commutative.*

Proof. We prove the result by induction on $\dim L$. If $[L, L] = L$ then $Z(U(L)) = Sz(U(L))$ and there is nothing to prove. So assume that $[L, L] \subset L$. Therefore there exists an ideal H, $H \supseteq [L, L]$ with $H + Fx = L$. So, by the P-B-W theorem, each $u \in U(L)$ has a unique expression in the form: $u = x^n u_n + x^{n-1} u_{n-1} + \cdots + u_0$, where $u_i \in U(H)$, for $i = 1, \ldots, n$. Suppose $u \in U(L)_\lambda$, then for each $y \in L$ we have $[y, u] = \lambda(y)u = x^n \lambda(y) u_n + x^{n-1}\lambda(y)u_{n-1} + \cdots + \lambda(y)u_0$. But also $[y, u] = [y, x^n]u_n + [y, x^{n-1}]u_{n-1} + \cdots + [y, x]u_1 + x^n[y, u_n] + \cdots + x[y, u_1] + [y, u_0]$. Consequently, since $[y, x^i] \in \sum_{j \leqslant i-1} x^j U(H)$ for each i, then we get by comparing highest terms, that $[y, u_n] = \lambda(y)u_n$ for all $y \in L$. That is $u_n \in U(L)_\lambda \cap U(H)$ and therefore $uu_n^{-1} \in Z(Q(U(L)))$. Let $a \in U(L)_\lambda$, $b \in U(L)_\mu$ then the above grants $a = a_1 z$, $b = b_1 c$, where $a_1, b_1 \in Sz(U(H))$ and $z, c \in Z(Q(U(L)))$. By induction $[a_1, b_1] = 0$ and since z, c are central this clearly shows that $ab = ba$. $\square$

The next lemma is valid due to the $\operatorname{char} F = p > 0$ assumption.

Lemma 2.2. *Let L be a finite dimensional Lie algebra over a field F with $\operatorname{char} F = p > 0$. Then $[Q(U(L))_\lambda : Q(Z(U(L)))] = 1$ for each L-weight λ.*

Proof. Let $u, v \in Q(U(L))_\lambda$, then $uv^{-1} \in Z(Q(U(L))) = Q(Z(U(L)))$, where the last equality holds for every prime PI ring. $\square$

The next result is a consequence of [9]. Observe that L is solvable here.

Proposition 2.3. *Let L be a finite dimensional solvable Lie algebra over a field F with $\operatorname{char} F = p > 0$. If $Sz(U(L)) = Z(U(L))$ then $Z(U(L))$ is factorial.*

Proof. Let q be a height one prime ideal in $Z(U(L))$. By "Going up" between $Z(U(L))$ and $U(L)$, there exists a (unique) height one prime ideal P in $U(L)$ such that $P \cap Z(U(L)) = q$. Now by [9, Theorem 5.4] $P = cU(L)$. Consequently $c \in U(L)_\lambda$ for some weight λ and by assumption $\lambda = 0$, that is $c \in Z(U(L))$. Thus $q = P \cap Z(U(L)) = cU(L) \cap Z(U(L)) = cZ(U(L))$. $\square$

We now recall some basic definitions. Let I be an ideal in a prime PI Notherian ring R which is integral over its center. We set $I^* \equiv \{y \in Q(R) \mid yI \subseteq R\}$ and $I^{**} \equiv \{y \in Q(R) \mid I^* y \subseteq R\}$. Then I^{**} is a two-sided ideal in R with $I \subseteq I^{**}$. I is said to be reflexive if $I = I^{**}$.

The next result appears implicitly in [8].

Proposition 2.4. *Let L be a solvable finite dimensional Lie algebra over a field F with $\operatorname{char} F = p > 0$. Let I be a reflexive ideal in $U(L)$. Then $I = dU(L)$.*

Proof. Let P be a height one prime in $U(L)$ and $q = P \cap Z(U(L))$. Since $U(L)_q$ is a maximal order with $\text{K.dim}\, U(L)_q = 1$, then P_q is its unique maximal ideal implying that $\bigcap_i P_q^i = \{0\}$ and consequently $\bigcap_i P^i = \{0\}$. Let P_1 be a height one prime ideal in $U(L)$ such that $I \subseteq P_1 = d_1 U(L)$. Choose n_1 maximal so that $I \subseteq P_1^{n_1}$. Then $d_1^{-n_1} I$ is a two-sided ideal in $U(L)$ and $d_1^{-n_1} I \nsubseteq P_1$. Choose $P_2 = d_2 U(L)$ a height one prime ideal so that $d_1^{-n_1} I \subseteq P_2^{n_2}$, where n_2 is a maximal to satisfy it. Hence $d_2^{-n_2} d_1^{-n_1} I$ is a reflexive two-sided ideal in $U(L)$ which is not contained in P_1 and P_2. Iterations of this process must stop after finitely many steps since I is contained in only finitely many height one prime ideals. Thus $d_k^{-n_k} \cdots d_1^{-n_1} I = U(L)$, for some k, equivalently $I = d_1^{n_1} \cdots d_k^{n_k} U(L)$. □

Our next result is slight generalization of [5, Lemma 1].

Lemma 2.5. *Let L be a finite dimensional Lie algebra over a field F with $\operatorname{char} F = p > 0$. Let H be a proper ideal in L with $L = Fx + H$. Let $m(t) \in F[t]$ be a p-polynomial of degree p^s. Then $\{1, x, x^2, \ldots, x^{p^s-1}\}$ is a free basis of $U(L)$ as a $U(H)\{m(x)\}$-module (from both sides).*

Proof. Set $C \equiv U(H)\{m(x)\}$, the subring of $U(L)$ generated by $U(H)$ and $m(x)$. If $j \geqslant p^s$ find k so that $j - kp^s < p^s$. Then $x^j - m(x)^k$ has a smaller degree than j and we argue by induction that $x^j - m(x)^k \in \sum_{i<p^s} Cx^i$, showing that $\{1, x, \ldots, x^{p^s-1}\}$ generates $U(L)$ as a C-module from both sides. Suppose $\sum_{i<p^s} f_i x^i = 0$, with $f_i \in C$ and we shall only consider $f_i \neq 0$ (for $i \neq 0$). Using the fact that $m(x)$ is a p-polynomial we have that $f_i = \sum a_{ij} m(x)^j$, with $a_{ij} \in U(H)$ for each j. Therefore $\sum_i \sum_j a_{ij} m(x)^j x^i = 0$. Writing $m(x)^j$ as a sum of monomials in x shows that the maximal exponent of x will have the form $x^{p^s j}$ and therefore the maximal exponent coming from $m(x)^j x^i$ is $x^{p^s j + i}$, (with $0 < i < p^s$). Since $\{p^s j + i \mid 0 < i < p^s\}$ consists of different numbers, the corresponding terms do not cancel each other and we reach a contradiction since their sum is 0. □

Remark 2.6. By a modification of the proof the last result holds for any polynomial $m(t)$.

Lemma 2.7. *Let $L = H + Fx_1 + \cdots + Fx_r$ be a finite dimensional Lie algebra over a field F with $\operatorname{char} F = p > 0$, where $H \supseteq [L, L]$ is an ideal in L with $\dim_F H = \dim_F L - r$. Then*

1. *$L_s \equiv H + Fx_{s+1} + \cdots Fx_r + Fm_1(x_1) + \cdots + Fm_s(x_s)$, is a Lie algebra, H is an ideal in L_s, and $\dim_F L_s = \dim_F(L)$, where $m_s(t)$ is an arbitrary p-polynomial, of degree p^{n_s}, for $s = 1, \ldots, r$.*
2. *$U(L_s) \cong U(H + Fx_{s+1} + \cdots + Fx_r)\{m_1(x_1), \ldots, m_s(x_s)\} \equiv C$, for $s = 1, \ldots, r$.*

Proof. By assumption one can complete $\{x_1, \ldots, x_r\}$ to a basis of L by adding any basis of H. Consequently by the P-B-W theorem $\{x_{s+1}, \ldots, x_r, m_1(x_1), \ldots, m_s(x_s)\}$ are algebraically independent over $U(H)$ and consequently $\dim_F L_s = \dim_F L$ for each s. The fact that L_s is a Lie algebra follows since $m_i(t)$ is a p-polynomial and consequently $[H, m_i(x_i)] \subseteq [L, L] \subseteq H$ as well as $[m_i(x_i), m_j(x_j)] \subseteq [L, L] \subseteq H$. This also shows that H is an ideal in L_s and settles item (1). Now as in Lemma 2.5, it is easy to see that $\{x_1^{i_1} \cdots x_s^{j_s} \mid 0 \leqslant i_j < p^{n_j},\ j = 1, \ldots, s\}$ is a finite generating set of $U(L)$ as a C-module (from both sides). Consequently $\text{K.dim}\, C = \text{K.dim}\, U(L) = \dim_F L = \dim_F L_s = \text{K.dim}\, U(L_s)$. Now by the universal property of $U(L_s)$ there is an onto homomorphism φ from $U(L_s)$ onto C, thus the last equality forces $\ker \varphi = \{0\}$. This settles item (2). □

The next result will be used in the proof of Theorem A.

Lemma 2.8. *Let $L = H + Fx_1 + \cdots + Fx_r$, be a finite dimensional Lie algebra over a field F with char $F = p > 0$, where $H \supseteq [L, L]$, is an ideal in L with $\dim_F H = \dim_F L - r$. Set $H_i \equiv H + Fx_1 + \cdots + \widehat{Fx_i} + \cdots + Fx_r$, and let $m_i(t)$ be a p-polynomial in $F[t]$ with $\deg m_i(t) = p^{n_i}$, for $i = 1, \ldots, r$. Then*

$$\bigcap_{i=1}^{r} U(H_i)\{m_i(x_i)\} = U(H)\{m_1(x_1), \ldots, m_r(x_r)\}.$$

Proof. By assumption $\{x_1, \ldots, x_r\}$ can be completed to a basis of L by adding any basis of H. Consequently $\dim_F H_i = \dim_F L - 1$ and $H_1 \cap \cdots \cap \hat{H}_i \cap \cdots \cap H_r \nsubseteq H_i$ for each i, which shows (by induction) that $\dim_F H_1 \cap \cdots \cap H_s = \dim_F L - s$, for each $1 \leqslant s \leqslant r$ and therefore $H_1 \cap \cdots \cap H_s = H + Fx_{s+1} + \cdots + Fx_r$. We shall prove by induction on s ($\leqslant r$) that

$$\bigcap_{i=1}^{s} U(H_i)\{m_i(x_i)\} = U\left(\bigcap_{i=1}^{s} H_i\right)\{m_1(x_1), \ldots, m_s(x_s)\}. \tag{1}$$

Therefore, by induction, the l.h.s. of (1) can be written in the following form

$$U\left(\bigcap_{i=1}^{s-1} H_i\right)\{m_1(x_1), \ldots, m_{s-1}(x_{s-1})\} \cap U(H_s)\{m_s(x_s)\}. \tag{2}$$

Now by Lemma 2.5 and Lemma 2.7 $U(\bigcap_{i=1}^{s-1} H_i)\{m_1(x_1), \ldots, m_{s-1}(x_{s-1})\} \cong U(L_{s-1})$ and has the set $\{1, x_s, \ldots, x_s^{p^{n_s}-1}\}$, as a free basis over the r.h.s. of (1). Therefore if $a \in (2)$, but $a \notin$ r.h.s. of (1), then $a = \sum_{i<p^{n_t}} a_i x_s^i$, where $a_i \in$ r.h.s. of (1). We shall only consider terms with $a_i \neq 0$ (for $i \neq 0$). A maximal occurrence of x_s in a_i after straightening (that is, writing it as a sum of monomials with powers of x_s appearing on the right), will have the form $x_s^{p^{ns} k_i}$. Consequently the maximal power of x in $a_i x_s^i$ will be $x_s^{p^{ns} k_i + i}$ (where $i \neq 0$). Now the members of $\{x_s^{p^{ns} k_i + i} \mid a_i \neq 0,\ 0 < i < p^{n_s}\}$ are distinct and therefore the corresponding monomial do not cancel. Thus the maximal power of x_s appearing in a after straightening will be $x_s^{p^{ns} k_m + m}$, for some $0 < m < p^{n_s}$. However, by (2) $a \in U(H_s)\{m(x_s)\}$, and so $a = \sum_{i=0}^{k} b_i m(x_s)^i$, with $b_i \in U(H_s)$, $b_k \neq 0$. Therefore after straightening the maximal power of x_s appearing in these monomials is $x_s^{p^{ns} k}$. This is in contradiction to the previous discussion. Therefore we have that l.h.s. of (1) $\subseteq$ r.h.s. of (1). Since the reverse inclusion is obvious, (1) is established. □

3. A reduction theorem

Let L be a finite dimensional Lie algebra over an algebraically closed field F with *char* $F = p > 0$. We shall show here that $Z(U(L)) \subseteq Z(U(L_0))$ and $Sz(U(L)) \subseteq Sz(U(L_0))$, where L_0 is a Lie subalgebra of $U(L)$, with $\dim_F(L) = \dim_F(L_0)$. Moreover L and L_0 share in common a co-dimension one ideal. The point is that $Z(U(L_0))$ and $Sz(U(L_0))$ are better behaved then $Z(U(L))$, $Sz(U(L))$ (respectively); thus enabling us to get in Section 4, after a finite number of iterations, our key result Theorem A.

Lemma 3.1. *Let L be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Let $a \in U(L)_\lambda$, $y \in L$ and $i \geqslant 0$. Then*

(1) *$[y^i, a] = aw$, for some $w \in U(L)$,*
(2) *$[m(y), a] = m(\lambda(y))a$, for each p-polynomial $m(t)$.*

Proof. By [15, p. 116], $[y^i, a] = \sum_{k=1}^{i} \binom{i}{k} (a\,dy)^k(a) y^{i-k} = a(\sum_{k=1}^{i} \lambda(y)^k y^{i-k})$. This settles (1). Now (2) follows from the fact that $[y^p, a] = (a\,dy)^p(a) = \lambda(y)^p(a)$. □

Corollary 3.2. *Let $L = H + Fx$, be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Suppose that H is a proper ideal in L. Set $B = U(H)\{m(x)\}$, for some p-polynomial $m(t)$. Let $a \in U(L)_\lambda$ and $w \in B$. Then $[w, a] = va$ for some $v \in B$.*

Proof. By Lemma 2.7 B is the enveloping algebra of the Lie algebra $L_1 = H + Fm(x)$. Let $h_1, \ldots, h_t$ be a basis of H. Then w can be written, by the P-B-W theorem, as a sum of monomials of the form $h_1^{i_1} \cdots h_t^{i_t} m(x)^j$. So we only need to consider w of this form. Iterations of Lemma 3.1 items (1) and (2), yield the result. □

The following is our first reduction result.

Lemma 3.3. *Let $L = H + Fx$, be a finite dimensional Lie algebra over a field F with char $F = p > 0$ and H a proper ideal in L. Let $B = U(H)\{x^p - \eta x\}$ where $\eta \in F$. Suppose $[x, Z(B)] \neq 0$. Then*

(1) $Z(U(L)) \subseteq Z(B)$,
(2) $Sz(U(L)) \subseteq Sz(B)$.

Proof. Let $a \in Z(U(L))$. By assumption and the $a\,dx$ stability of $Z(B)$, there exists $u \in Z(B)$ with $0 \neq [x, u] \equiv w \in Z(B)$. Now by Lemma 2.5 $\{1, \ldots, x^{p-1}\}$ is a free basis of $U(L)$ over B. Suppose $a = \gamma_m x^m + \sum_{i<m} \gamma_i x^i$, where $\gamma_i \in B$, $i = 1, \ldots, m$ and $\gamma_m \neq 0$, where $m < p$. Then

$$0 = [a, u] = \gamma_m \big[x^m, u\big] + \sum_{i<m} \gamma_i \big[x^i, u\big]. \tag{3}$$

Now $[x^i, u] \in \sum_{j \leqslant i-1} Bx^j$ and $[x^m, u] = mwx^{m-1} + \sum_{j<m-1} b_j x^j$ where $b_j \in B$ for each j. Therefore the coefficients of the highest term in x in (3) is $mw\gamma_m$ which must be 0. This leads to a contradiction. Thus $a \in B$ and, since $B \subseteq U(L)$, it implies that $a \in Z(B)$. Suppose now that $a \in U(L)_\lambda$, and assume that $a = \gamma_m x^m + \sum_{i<m} \gamma_i x^i$, with $m < p$, $\gamma_m \neq 0$ and $\gamma_i \in B$, for $i = 1, \ldots, m$. Retaining $u, w \in Z(B)$ satisfying $[x, u] = w \neq 0$, as above, we have by Corollary (3.2), that there exists $v \in B$ such that: $v(\gamma_m x^m + \sum_{i<m} \gamma_i x^i) = va = [u, a] = \gamma_m [u, x^m] + \sum_{i<m} \gamma_i [u, x^i]$. Now if $v \neq 0$ then in the r.h.s. of the last equality all the appearing powers of x are smaller then m, while the coefficient of x^m in the l.h.s. is $v\gamma_m$. Therefore $v\gamma_m = 0$ and consequently $\gamma_m = 0$, that is $a \in B$. If $v = 0$ then $0 = \gamma_m [x^m, u] + \sum_{i>m} \gamma_i [x^i, u]$ and as in the early part with $a \in Z(U(L))$ we get $\gamma_m = 0$, that is $a \in B$. □

The next two results deal with some elementary properties of p-polynomials. A different proof which yields more on η was kindly communicated to us by the referee.

Lemma 3.4. *Let F be an algebraically closed field with char $F = p > 0$. Let $f(t) \in F[t]$, be a monic p-polynomial (which is by definition with no constant term). Then exactly one of the following holds:*

(i) $f(t) = t^{p^n}$,
(ii) $f(t) = g(t^p - \eta t)$, *for some $\eta \neq 0$ in F and a monic p-polynomial $g(t) \in F[t]$.*

Proof. Suppose that $f(t) \neq t^{p^n}$ for all n. So $f(t) = t^{p^n} + \sum_{i \geqslant 1} \alpha_i t^{p^{n-i}}$, where $\alpha_j \neq 0$ for some j. Consider $g(t) = t^{p^{n-1}} + \sum_{i \geqslant 1}^{n-1} \beta_i t^{p^{n-1-i}}$, where $\beta_i \in F$. The equality $f(t) = g(t^p - \eta t)$, with $\eta \in F$, is equivalent to the following equations:

$$\beta_i - \beta_{i-1} \eta^{p^{n-i}} = \alpha_i \quad \text{for } i = 1, \ldots, n-1,\ \beta_0 = 0 \text{ and } \beta_{n-1}\eta = \alpha_n. \tag{4}$$

Starting with $\beta_1 = \eta^{p^{n-1}} + \alpha_1$ and using (4) we get for $j = 1, \dots, n-1$, $\beta_j = \eta^{(p^{n-1}+p^{n-2}+\dots+p^{n-j})} + \sum_{i=1}^{j-1} \alpha_i \eta^{(p^{n-1-i}+\dots+p^{n-j})}$. Thus all but the last equality of (4) hold (for an arbitrary η in F). To satisfy the $\beta_{n-1}\eta = \alpha_n$, we plug

$$\beta_{n-1} = \eta^{(p^{n-1}+\dots+p)} + \sum \alpha_i \eta^{(p^{n-1-i}+\dots+p)}$$

into it and we get a monic polynomial equation in η with some of the α_i's being non-zero. Now use the algebraically closed assumption to find a suitable non-zero solution. $\square$

The next result already appears in [26, Lemma 1].

Lemma 3.5. *Let $E \supset D$ be two fields of positive characteristic p. Suppose that $\delta \in \operatorname{Der}_D E$ satisfies $m(\delta) = 0$, where $m(t)$ is a minimal monic p-polynomial with coefficients in D. Then every p-polynomial $f(t)$, with coefficients in D, such that $f(\delta) = 0$ satisfies $f(t) = a_0 m(t)^{p^n} + a_1 m(t)^{p^{n-1}} + \dots + a_{n-1} m(t)$, where $a_i \in D$, for $i = 0, \dots, n-1$.*

Proof. This is merely the p-division algorithm. The proof is by induction on $\deg f(t)$. We clearly may assume that $\deg f(t) = p^r > \deg m(t) = p^s$. Let a_0 be the leading term coefficient of $f(t)$. Then $f(t) - a_0(m(t))^{p^{r-s}} = g(t)$ satisfies $g(\delta) = 0$ and is a p-polynomial with $\deg g(t) < \deg f(t)$, we can now use induction to finish. $\square$

Corollary 3.6. *Let $L = H + Fx$, be a Lie algebra over an algebraically closed field F, with $\operatorname{char} F = p > 0$. Suppose H is a proper ideal in L. Set $E = Q(Z(U(H)))$ and $D = Q(Z(U(H)))^{a\,dx}$. Let $m(t)$ denote the minimal monic p-polynomial that $a\,dx|_E$ satisfies as an element of $\operatorname{Der}_D E$. Then all the coefficients of $m(t)$ are in F.*

Proof. $a\,dx \in \operatorname{Der}_F L$ and therefore satisfies a (monic) p-polynomial $f(t)$ who's coefficients are in F. Consequently $a\,dx \in \operatorname{Der}_F U(L)$ satisfies $f(t)$ as well and so is $a\,dx|_E$. Now by Lemma 3.5 $f(t) = a_0 m(t)^{p^n} + a_1 m(t)^{p^{n-1}} + \dots + a_{n-1} m(t)$. Therefore all the roots of $m(t)$ (in the algebraic closure of D) are roots of $f(t)$ and are therefore in F. This shows that all the coefficients of $m(t)$ are in F. $\square$

Remark 3.7. Let $D = Q(Z(U(H)))^{a\,dx} = Q(Z(U(H))^{a\,dx})$ and $E = Q(Z(U(H)))$. It is a consequence of Corollary 3.6 that the minimal monic p-polynomial that $a\,dx|_E \in End_D E$ satisfies is the same as the minimal one that $a\,dx|_E \in End_F E$ satisfies, and is also the same minimal p-polynomial $m(t) \in F[t]$ such that $[m(x), Z(U(H))] = 0$.

The next result is crucial.

Corollary 3.8. *Let $L = Fx + H$ be a finite dimensional Lie algebra over an algebraically closed field F, with $\operatorname{char} F = p > 0$. Suppose H is a proper ideal in L and let $m(t)$ be chosen as in Corollary* 3.6. *Let $\eta \in F$ satisfy $m(t) = g(t^p - \eta t)$, for some p-polynomial $g(t) \in F[t]$ (as in Lemma* 3.4*). We set $B = U(H)\{x^p - \eta x\}$. Then $[x, Z(U(H))] \neq 0$ implies that $[x, Z(B)] \neq 0$.*

Proof. Suppose that $[x, Z(B)] = 0$, then since $[H, Z(B)] = 0$ we have $Z(B) \subseteq Z(U(L))$. Let $u \in Z(U(H))^{a\,d(x^p - \eta x)}$, then $[u, x^p - \eta x] = 0$ and since $[u, H] = 0$, we get $u \in Z(B) \subseteq Z(U(L))$, thus $[x, u] = 0$ and $u \in Z(U(H))^{a\,dx}$. The reverse inclusion $Z(U(H))^{a\,dx} \subseteq Z(U(H))^{a\,d(x^p - \eta x)}$, now shows that $Z(U(H))^{a\,dx} = Z(U(H))^{a\,d(x^p - \eta x)}$. Consequently $Q(Z(U(H)))^{a\,dx} = Q(Z(U(H)))^{a\,d(x^p - \eta x)}$. Let $E = Q(Z(U(H)))$ and $D = Q(Z(U(H))))^{a\,dx}$ as in Remark 3.7. Then by [18, p. 536] $[E : D] = \deg m(t)$. Now $D = E^{a\,d(x^p - \eta x)}$ and $g(a\,d(x^p - \eta x)|_E) = m(a\,dx|_E) = 0$, show by a second application of [18, p. 536] that $\deg(m(t)) = [E : D] \leqslant \deg g(t)$, a contradiction. $\square$

We finally arrive at our main reduction result.

Theorem 3.9. *Let $L = H + Fx$ be a finite dimensional Lie algebra over an algebraically closed field with char $F = p > 0$. Suppose H is a proper ideal in L and let $m(t)$ be the minimal monic p-polynomial in $F[t]$ such that $[m(x), Z(U(H))] = 0$. Set $A \equiv U(H)\{m(x)\}$ and $L_0 = H + Fm(x)$. Suppose that $[x, Z(U(H))] \neq 0$. Then*

(1) $Z(U(L)) \subseteq Z(A)$,
(2) $Sz(U(L)) \subseteq Sz(A)$,
(3) *For each weight λ on $U(L)$, there exists an L_0-weight $\widetilde{\lambda}$ on A such that*

$$U(L)_\lambda \subseteq A_{\widetilde{\lambda}}.$$

Proof. Recall that $A \cong U(L_0)$ by Lemma 2.7. The proof is by induction on the p-degree of $m(t)$. Let $\eta \in F$ be chosen, as in Lemma 3.4, to satisfy $m(t) = g(t^p - \eta t)$. Observe that if $m(t) = t^{p^n}$, then $g(t) = t^{p^{n-1}}$ and $\eta = 0$. If p-degree $(m(t)) = 1$, then $m(t) = t^p - \eta t$ and $A = U(H)\{x^p - \eta x\}$. Corollary 3.8 and $[x, Z(U(H))] \neq 0$, imply by Lemma 3.3 that items (1) and (2) are true. Let $x_1 \equiv x^p - \eta x$. If $[x_1, Z(U(H))] = 0$, then $m(t) = t^p - \eta t$ and we are back to the previous case. So assume that $[x_1, Z(U(H))] \neq 0$. Moreover the minimality of $m(t)$ implies that $g(t)$ is the minimal monic p-polynomial such that $[g(x_1), Z(U(H))] = 0$. Let $L_1 \equiv H + Fx_1$. Then one easily verifies that L_1 is a Lie algebra and H is an ideal of a co-dimension one in L_1. Set $A' \equiv U(H)\{g(x_1)\}$. Consequently by induction $Z(U(L_1)) \subseteq Z(A')$ and $Sz(U(L_1)) \subseteq Sz(A')$. Since $m(x) = g(x_1)$ then $A' = A$ and the last inclusions translate into $Z(U(H)\{x_1\}) \subseteq Z(A)$ and $Sz(U(H)\{x_1\}) \subseteq Sz(A)$. Now $U(H)\{x_1\} = U(H)\{x^p - \eta x\} \equiv B$ and by Corollary 3.8 $[x, Z(U(H))] \neq 0$ implies $[x, Z(B)] \neq 0$. Therefore by Lemma 3.3 we have that $Z(U(L)) \subseteq Z(B)$ and $Sz(U(L)) \subseteq Sz(B)$. This combined with the previous inclusions complete the proof of items (1) and (2). Finally let $a \in U(L)_\lambda$, then by Lemma 3.1 item (2), a is a weight vector with respect to the adjoint action of $L_0 \equiv H + Fm(x)$. Now by (2), $a \in Sz(A)$ and since $A \cong U(L_0)$ this settles item (3), where $\widetilde{\lambda}$ is defined via $\widetilde{\lambda}(h) = \lambda(h)$, for $h \in H$ and $\widetilde{\lambda}(m(x)) = m(\lambda(x))$. □

4. Theorem A and some of its consequences

The introduction of $Sz(U(L))$ in case char $F = 0$, is due to Dixmier, and its importance is manifested in [11]. Moeglin showed in [23] that $Sz(U(L))$ is factorial if in addition F is algebraically closed. Later work, for an arbitrary field F with char $F = 0$, is done in [10,21]. It is proved in [27] that $Sz(U(L)) \cong S(L)^{[L,L]}$, in case char $F = 0$, and F algebraically closed. This is an extension of Duflo's theorem [11, Theorem 10.4.5] stating that $Z(U(L)) \cong S(L)^L$.

Theorem A, whose proof is the main result of the present section shows, in case char $F = p > 0$ and F algebraically closed, that $Sz(U(L)) \cong Z(U(L_0))^{a\,da_1,\ldots,a\,da_r}$ where L_0 is a finite dimensional Lie algebra related to L. Moreover L and L_0 share a common ideal which contains $[L, L]$. This has some artificial resemblance to the mentioned above result of Rentschler–Vergne. However it is more similar in nature (but not in proof) to [3, Satz 6.1].

Lemma 4.1. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with char $F = p > 0$. Let α be a non-zero L-weight on $U(L)$. Set $K = \ker\alpha$ and let $x \in L \setminus K$. Then $[x, Z(U(K))] \neq 0$.*

Proof. Clearly $[L, L] \subseteq K$, follows from the fact that α is a weight. Consequently K is a co-dimension one ideal in L with $L = K + Fx$. Suppose by negation that $[x, Z(U(K))] = 0$, then $Z(U(K)) \subseteq Z(U(L))$ and in particular $Q(U(K)) \subseteq Q(U(L))$. Therefore by the Skolem–Noether theorem [14, p. 100] there exists $t \in Q(U(K))$ with $a\,dx|_{Q(U(K))} = a\,d(-t)|_{Q(U(K))}$. Thus $[x + t, y] = 0$ for each $y \in Q(U(K))$, and in particular $[x + t, t] = 0$, as well as $[x, t] = 0$. Consequently $[x + t, x] = 0$ showing together with $[x + t, y] = 0$ where $y \in Q(U(K))$, that $x + t \in Z(Q(U(L)))$. Let $0 \neq a \in U(L)_\alpha$. Then $\alpha(K) = 0$ implies $[k, a] = 0$ for each $k \in K$ and consequently $[y, a] = 0$ for each $y \in Q(U(K))$. Now, $[x + t, a] = 0$, since $x + t$ is central. However, $[x + t, a] = [x, a] + [t, a] = [x, a] = \alpha(x)a$ and therefore $\alpha(x) = 0$, a contradiction. □

Lemma 4.2. *Let L, α, K, x, a be as in Lemma* 4.1. *Let $m(t) \in F[t]$ be the minimal p-polynomial that $a\,dx$ satisfies as a derivation on $Z(U(K))$. Set $L_1 = K + Fm(x)$. Then $U(L)_\alpha \subseteq Z(U(L_1))$.*

Proof. Clearly K is a co-dimension one ideal in the Lie algebra L_1. By Lemma 2.7 $U(L_1) \cong U(K)\{m(x)\}$, where the later stands for the subring in $U(L)$, generated by $U(K)$ and $m(x)$. We have by definition that $[m(x), Z(U(K))] = 0$. Now the argument in Lemma 4.1 (with x being replaced with $m(x)$) shows that $m(x) + t_1 \in Z(Q(U(L_1)))$, where $t_1 \in Q(U(K))$ is chosen to satisfy $a\,dm(x)|_{Q(U(K))} = a\,d(-t_1)|_{Q(U(K))}$. Recall that by Theorem 3.9 $Sz(U(L)) \subseteq Sz(U(L_1))$ and in particular $a \in U(L_1)$. So if $a \in U(L)_\alpha$ then $0 = [m(x) + t_1, a] = [m(x), a]$, where the 2nd equality holds since a commutes with K. Thus a commutes with $m(x)$ and with K implying that $a \in Z(U(L_1))$. □

Remark 4.3. Let $\Delta = \{\alpha \mid \alpha$ is a non-zero L-weight on $U(L)\}$. Now the finite generation of $Sz(U(L))$ as a $Z(U(L))$-module, implies that Δ is finite. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a maximal F-linearly independent subset in Δ. Set $H_i \equiv \operatorname{Ker}\alpha_i = \alpha_i^\perp$, for $i = 1, \ldots, r$, and $H \equiv \bigcap_{i=1}^r H_i$.

The following lemma is proved by using linear algebra (using $g \notin span\{g_1, \ldots, g_t\}$ iff $g^\perp \not\supseteq span\{g_1, \ldots, g_t\}^\perp$).

Lemma 4.4. *Let H, α_i, H_i, for $i = 1, \ldots, r$, be as in the Remark* 4.3. *Then the following properties hold*:

(1) $\dim_F(H_i) = \dim_F(L) - 1$, *for $i = 1, \ldots, r$,*
(2) $\dim_F(H) = \dim_F(L) - r$,
(3) *$[L, L] \subseteq H_i$, $[L, L] \subseteq H$, and H, H_i are ideals in L, for $i = 1, \ldots, r$,*
(4) *$\alpha(h) = 0$ for each $\alpha \in \Delta$ and $h \in H$,*
(5) *One can choose $x_i \in L \setminus H_i$, for $i = 1, \ldots, r$ satisfying $\alpha_i(x_j) = \delta_{ij}$,*
(6) *$H_i = H + \sum_{j \neq i} Fx_j$, for $i = 1, \ldots, r$ and $L = H + Fx_1 + \cdots + Fx_r$.*

Proof. (1) is a consequence of $H_i = \ker\alpha_i$. Next, α_i is an L-weight and consequently $[L, L] \subseteq \ker\alpha_i = H_i$, for $i = 1, \ldots, r$, which establishes (3). Now $H \subseteq H_i$ and consequently $\alpha_i(h) = 0$ for each $h \in H$ and $i = 1, \ldots, r$. Thus since $\alpha \in span_F(\alpha_1, \ldots, \alpha_r\}$ for each $\alpha \in \Delta$ this settles (4). Next, the linear independence of $\{\alpha_1, \ldots, \alpha_r\}$ and the duality of $\perp$ show that $H_i = \alpha_i^\perp \not\supseteq (span\{\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_r\})^\perp = H_1 \cap \cdots \cap \hat{H}_i \cap \cdots \cap H_r$. Therefore one can prove (by induction) on k, using (1), that $\dim_F H_1 \cap \cdots \cap H_k = \dim_F L - k$, and (2) is established by taking $k = r$. Now $H_i \not\supseteq H_1 \cap \cdots \cap \hat{H}_i \cap \cdots \cap H_r$ permits the choice of $x_i \notin H_i$, so that $x_i \in H_1 \cap \cdots \cap \hat{H}_i \cap \cdots \cap H_r$, $\alpha_i(x_i) = 1$ and $\alpha_j(x_i) = 0$ for $j \neq i$. Thus (5) is established. By choice $\{x_1, \ldots, x_r, h_1, \ldots, h_{\dim L - r}\}$ is a linearly independent set for each basis $\{h_1, \ldots, h_{dimL-r}\}$ of H. So (2) implies that $L = H + Fx_1 + \cdots + Fx_r$ and since $H_i \supseteq H + \sum_{i \neq j} Fx_j$, the last equality also shows that $H_i = H + \sum_{i \neq j} Fx_j$. □

Lemma 4.5. *Retaining all the notations of Remark* 4.3 *and let $m_i(t)$ be the minimal p-polynomial that $a\,dx_i$ satisfies on $Z(U(H_i))$ for $i = 1, \ldots, r$. Let $L_r = H + Fm_1(x_1) + Fm_2(x_2) + \cdots + Fm_r(x_r)$. Then the following hold*:

(1) *L_r is a Lie subalgebra of $U(L)$ with $\dim_F(L_r) = \dim_F L$,*
(2) *$[L, L_r] \subseteq [L, L]$, $[L_r, L_r] \subseteq [L, L]$ and consequently H is an ideal in L_r,*
(3) *$U(L)_{\alpha_i} \subseteq Z(U(L_r))$ for $i = 1, \ldots, r$,*
(4) *$U(H)\{m_1(x_1), \ldots, m_r(x_r)\} \cong U(L_r)$,*
(5) *Let $\beta \in \Delta$, then $U(L)_\beta \subseteq U(L_r)_{\widetilde{\beta}}$ for some weight $\widetilde{\beta}$ on $U(L_r)$,*
(6) *$Sz(U(L)) \subseteq Sz(U(L_r))$ and $Z(U(L)) \subseteq Z(U(L_r))$.*

Proof. Using the fact that $m_i(x_i)$ is a p-polynomial in x_i we get that $[m_i(x_i), m_j(x_j)] \in [L, L]$ for each i, j and $[H, m_i(x_i)] \subseteq [L, m_i(x_i)] \subseteq [L, L] \subseteq H$. This clearly validates most of items (1) and (2). Also $\dim_F L_r = \dim_F L$ is true by Lemma 4.4 item (2) and Lemma 2.7 (with $s = r$). Now (4) also follows from 2.7. Therefore by Lemma 2.8 $\bigcap_{i=1}^{r} U(H_i)\{m_i(x_i)\} = U(H)\{m_1(x_1), \dots, m_r(x_r)\} = U(L_r)$. Let $0 \neq a \in U(L)_\beta$. Then by Theorem 3.9 item (3) (or Lemma 3.1) a is a weight vector with respect to H and $m_1(x), \dots, m_r(x_r)$. Now by Theorem 3.9 item (2), we have that $Sz(U(L)) \subseteq \bigcap_{i=1}^{r} Sz(U(H_i)\{m_i(x_i)) \subseteq \bigcap_{i=1}^{r} U(H_i)\{m_i(x_i) = U(L_r)$, so by the previous discussion a is a L_r-weight vector in $U(L_r)$, that is $a \in U(L_r)_{\widetilde{\beta}}$ and consequently $U(L)_\beta \subseteq U(L_r)_{\widetilde{\beta}}$, for some $\widetilde{\beta}$. This also settles the 1st part of (6). For the second part of (6), recall by Theorem 3.9 item (1), $Z(U(L)) \subseteq \bigcap_{i=1}^{r} Z(U(H_i))\{m_i(x_i)\} \subseteq \bigcap_{i=1}^{r} U(H_i)\{m_i(x_i)\} = U(L_r) \subseteq U(L)$ and consequently $Z(U(L)) \subseteq Z(U(L_r))$. Finally to settle (3) one observes that by Lemma 4.2 and (6) $U(L)_{\alpha_i} \subseteq Z(U(H_i)\{m_i(x_i)\}) \cap U(L_r)$. Now since $U(L_r) \subseteq U(H_i)\{m_i(x_i)\}$, this shows that $U(L)_{\alpha_i} \subseteq Z(U(L_r))$, for $i = 1, \dots, r$. □

Proof of Theorem A. We retain all notations appearing in Lemma 4.2, Remark 4.3 and Lemma 4.5. Let $\beta_1, \dots, \beta_s \in \Delta \setminus \{\alpha_1, \dots, \alpha_r\}$ be chosen via Lemma 4.5 item (6) so that $\{\widetilde{\beta}_1, \dots, \widetilde{\beta}_s\}$ is a maximal F-linearly independent set in $\{\widetilde{\beta} \mid \beta \in \Delta \setminus \{\alpha_1, \dots, \alpha_r\}\}$. We set $\widetilde{H}_i \equiv \ker \widetilde{\beta}_i$, $i = 1, \dots, s$ and $\widetilde{H} = \bigcap_{i=1}^{s} \widetilde{H}_i$. As in Lemma 4.4, we choose $\{y_1, \dots, y_s\} \subseteq L_r$ with $\widetilde{\beta}_i(y_j) = \delta_{ij}$, $i = 1, \dots, s$ (in fact $y_1, \dots, y_s$ can be chosen in $Fm_1(x_1) + \cdots + Fm_r(x_r)$). Moreover if $n_i(t)$ denotes the minimal p-polynomial that $a\,dy_i$ satisfies over $Z(U(\widetilde{H}_i))$, then as in Lemma 4.5 we define $L_{r+1} \equiv \widetilde{H} + Fn_1(y_1) + \cdots + Fn_s(y_s)$. By applying Lemma 4.5 we have that $U(L)_{\beta_i} \subseteq U(L)_{\widetilde{\beta}_i} \subseteq Z(U(L_{r+1}))$ for $i = 1, \dots, s$, as well as $Sz(U(L)) \subseteq Sz(U(L_r)) \subseteq Sz(U(L_{r+1}))$. Now $[L_{r+1}, L_{r+1}] \subseteq [L_r, L_r] \subseteq [L, L]$ and since $[L, n_i(y_i)] \subseteq [L, L_r] \subseteq [L, L]$ we get that $[L, L_{r+1}] = [L, \widetilde{H} + \sum_{i=1}^{s} Fn_i(y_i)] \subseteq [L, L_r] + [L, L] \subseteq [L, L]$. Also $H \subseteq \widetilde{H} \subseteq L_{r+1}$ and $[L_{r+1}, L_{r+1}] \subseteq [L, L] \subseteq H$ show that H is an ideal in L_{r+1}. Iterating this process finitely many times we arrive at a Lie algebra L_0 having the following properties:

$$\begin{gathered} \dim_F L_0 = \dim_F L, \quad U(L_0) \subseteq U(L), \quad H \text{ is an ideal in } L_0 \quad \text{and} \\ U(L)_\alpha \subseteq Z\big(U(L_0)\big), \quad \text{for each } \alpha \in \Delta. \end{gathered} \tag{5}$$

Moreover $[L, L_0] \subseteq [L, L]$ shows that $U(L_0)$ and (therefore) $Z(U(L_0))$ are $a\,dL$-stable. Also since, $[L, L] \subseteq H$ and $H \subseteq L_0$ then $[L, L]$ commute with $Z(U(L_0))$. Consequently $[a\,dx|_{Z(U(L_0))}, a\,dy|_{Z(U(L_0))}] = 0$ for each $x, y \in L$. Let $p_i(t)$ be the minimal p-polynomial that $a\,dx_i$ satisfies on L (and therefore on $U(L)$) for $i = 1, \dots, r$. We can always write $p_i(t) = (q_i(t))^{p^{n_i}}$, with $q_i(t)$ being a semi-simple p-polynomial, for $i = 1, \dots, r$. Then $q_i(x_i) \equiv a_i \in U(L)$ and $a\,da_i = q_i(a\,dx_i)$ is a nilpotent derivation on $U(L)$ for each $i = 1, \dots, r$. We shall prove that

$$Sz\big(U(L)\big) = Z\big(U(L_0)\big)^{a\,da_1, \dots, a\,da_r}. \tag{6}$$

In one direction, let $a \in U(L)_\lambda$. Then by Lemma 3.1 $0 = [p_i(x_i), a] = [q_i(x_i)^{p^{n_i}}, a] = q_i(\lambda(x_i))^{p^{n_i}} a$ and therefore $[a_i, a] = q_i(\lambda(x_i))a = 0$, for $i = 1, \dots, r$, namely $a \in Z(U(L_0))^{a\,da_1, \dots, a\,da_r}$. For the converse direction, one observes that $a\,da_i = q_i(a\,dx_i)$ acts as the zero map on $Z(U(L_0))^{a\,da_1, \dots, a\,da_r}$, for $i = 1, \dots, r$, and consequently $a\,dx_i$, for $i = 1, \dots, r$, acts semi-simply on $Z(U(L_0))^{a\,da_1, \dots, a\,da_r}$. Also $\{a\,dx_i|_{Z(U(L_0))^{a\,da_1, \dots, a\,da_r}}\}$ consists of commuting elements. Now $[H, Z(U(L_0))] = 0$ implies that $a\,dh|_{Z(U(L_0))^{a\,da_1, \dots, a\,da_r}} = 0$ for each $h \in H$. Therefore we can decompose $Z(U(L_0))^{a\,da_1, \dots, a\,da_r}$ into a direct sum of weight spaces with respect to $a\,dx_1, \dots, a\,dx_r$ and $a\,dH$. This is a decomposition of $Z(U(L_0))^{a\,da_1, \dots, a\,da_r}$ into L-weight spaces implying that $Z(U(L_0))^{a\,da_1, \dots, a\,da_r} \subseteq Sz(U(L))$. □

Proposition 4.6. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p > 0$. Let L_0 be as in the proof of Theorem A. Suppose that one of the following holds:*

(i) $[L_0, L_0] = [L, L]$,
(ii) $[L, L]$ *is nilpotent.*

Then $Z(U(L_0)) = Sz(U(L_0))$.

Proof. We pick by negation $0 \neq a \in U(L_0)_\gamma$, with $\gamma \neq 0$. We shall firstly show that either one of (i) or (ii) implies that $U(L_0)_\gamma$ is adL-stable. Let $x \in L$, $y \in L_0$. Then

$$\bigl[y, [x,a]\bigr] = \bigl[[y,x],a\bigr] + \bigl[x,[y,a]\bigr]. \tag{7}$$

If (i) holds then $[y,x] \in [L_0, L] \subseteq [L,L] = [L_0, L_0]$ showing that $[[y,x],a] = 0$ and (7) takes the form $[y,[x,a]] = \gamma(y)[x,a]$. Since $[x,a] \in U(L_0)$ by the adL-stability of $U(L_0)$ this shows that $[x,a] \in U(L_0)_\gamma$ for each $x \in L$. In case (ii) $ad[y,x]$ acts nilpotently on $U(L)$ and since $U(L_0) \subseteq U(L)$ it is also nilpotent on $U(L_0)$. Now $[y,x] \in [L, L_0] \subseteq [L,L] \subseteq H \subset L_0$ and hence $[[y,x],a] = \gamma([y,x])a$. The nilpotency of $ad[y,x]$ and iterations show that $\gamma([y,x]) = 0$. Therefore, as before, (7) takes the form $[y,[x,a]] = \gamma(y)[x,a]$ and again $U(L_0)_\gamma$ is adL-stable. In fact by exactly the same arguments, if $x, y \in L$ then $[x,y] \in [L,L]$ and in case (i) this shows that $[[x,y],a] = 0$. In case (ii) $[x,y] \in [L,L] \subseteq H$ is *ad* nilpotent on $U(L)$ and again $[[x,y],a] = 0$. This shows, in both cases, that $[adx|_{U(L_0)_\gamma}, ady|_{U(L_0)_\gamma}] = 0$ for each $x, y \in L$. Also recall that $H \subset L_0$ and consequently $adh(a) = \gamma(h)a$ for each $h \in H$. Let $p_i(t)$ be the minimal p-polynomial that adx_i satisfies on L, where x_i is as in the previous theorems, for each $i = 1, \dots, r$. Consider

$$W \equiv span_F\bigl\{\bigl(adx_1^{t_1}\bigr)\bigl(adx_2^{t_2}\bigr)\cdots\bigl(adx_r^{t_r}\bigr)(adh)^j(a)\bigm| t_i < \deg\bigl(p_i(t)\bigr) \text{ for } i = 1, \dots, r\bigr\}.$$

Then $W \subset U(L_0)_\gamma$ is adL-stable and $\dim_F(W) < \infty$. The previous considerations show that $adL = F adx_1 + \cdots + F adx_r + adH$, acts as a commutative Lie algebra of linear transformations on W. Thus by standard linear algebra (using that F is algebraically closed), there exists a non-zero common adL eigenvector $b \in U(L_0)_\gamma$. Now since $b \in U(L_0) \subseteq U(L)$ this shows in particular that $b \in Sz(U(L)) = Z(U(L_0))^{ada_1,\dots,ada_r}$. Therefore $b \in Z(U(L_0))$, in contradiction to the assumption $\gamma \neq 0$. □

The next result confirms, in particular, the validity of Theorem C.

Theorem 4.7. *Let L be a solvable finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p > 0$. Suppose that one of the following holds*:

(i) $[L_0, L_0] = [L, L]$, *where L_0 is as in Theorem* A,
(ii) $[L, L]$ *is nilpotent.*

Then $Sz(U(L))$ is a factorial domain.

Proof. In both cases we have $[L_0, L_0] \subseteq [L,L]$ and since L is solvable the same holds for L_0. Therefore by Proposition 2.3 and Proposition 4.6, $Z(U(L_0))$ is factorial. Now, by Theorem A, $Sz(U(L)) = Z(U(L_0))^{ada_1,\dots,ada_r}$, with $ada_1, \dots, ada_r$ being nilpotent derivations on $Z(U(L_0))$ (and on $U(L)$). Thus by [16, Corollary 17.3] this implies that $Sz(U(L))$ is factorial. □

Remark 4.8. A more direct proof of Theorem 4.7 item (ii) can be found in [31].

Corollary 4.9. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p > 0$. Then $Sz(U(L))$ is factorial in the following cases*:

(i) $L \subseteq \mathfrak{b}_n$, *the standard Borel subalgebra of* $\mathfrak{gl}_n$,
(ii) $L =$ *the Lie algebra of a connected solvable algebraic group.*

The following is another consequence of Theorem A.

Theorem 4.10. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p > 0$. Then $Cl(Sz(U(L)))$ is a d-torsion group, where $d = \mathrm{PI.deg}\, U(L)$.*

Proof. By Theorem A, using the fact that $ada_1, \ldots, ada_r$ are nilpotent derivations, we conclude by [16, Corollary 17.3] that $Cl(Sz(U(L))) \subseteq Cl(Z(U(L_0)))$. Now by [6, Theorem A] the later is a d_0-torsion group where $d_0 = \mathrm{PI.deg}\, U(L_0)$. Now the result will follow once we show that d_0 divides d. To this end recall that we have the following inclusions of division rings:

$$Q\bigl(Z\bigl(U(L)\bigr)\bigr) \subset Q\bigl(Z\bigl(U(L_0)\bigr)\bigr) \subset Q\bigl(U(L_0)\bigr) \subset Q\bigl(U(L)\bigr).$$

Consequently $d^2 = [Q(U(L)) : Q(Z(U(L)))] = [Q(U(L)) : Q(U(L_0))] \cdot [Q(U(L_0)) : Q(Z(U(L_0)))] \cdot [Q(Z(U(L_0))) : Q(Z(U(L)))] = d_0^2 [Q(U(L)) : Q(U(L_0))] \cdot [Q(Z(U(L_0))) : Q(Z(U(L)))]$. This clearly shows that d_0 divides d. □

The following conjecture is suggested by Theorem 4.10.

Conjecture 4.11. *Let L be a finite dimensional Lie algebra over an algebraically closed field of prime characteristic p. Then $Cl(Sz(U(L)))$ is a finite elementary abelian p-group.*

Remark 4.12. The previous argument confirms the validity of this conjecture, in case L is solvable, as a consequence of Corollary 6.3.

5. The proof of Theorem D

Recall that a Noetherian domain is factorial if and only if all its height one prime ideals are principal. It is fairly easy to produce a solvable finite dimensional Lie algebra L over a prime characteristic field F with $Z(U(L))$ being non-factorial.

Consider for example $Fx + Fy + Fz = L$, subject to the Lie products $[x, y] = y$, $[x, z] = z$ and $[y, z] = 0$. Then $Z(U(L)) = F[x^p - x, y^i z^{p-i}, i = 0, \ldots, p]$, is clearly a non-factorial domain. If L is nilpotent this cannot happen by [5]. Given by [23] that $Sz(U(L))$ is factorial if $\mathit{char}\, F = 0$, it might suggest that $Sz(U(L))$ is better behaved in case $\mathit{char}\, F = p > 0$, as well. This is only partially true as follows from Theorem 4.7 item (ii) and Example 9.1.

The purpose of the present section is to report, in case L is solvable, on the nature of relation between height one primes of $Z(U(L))$ and those of $Sz(U(L))$. The main result here is as follows:

Theorem D. *Let L be a finite dimensional solvable Lie algebra over an algebraically closed field F with $\mathit{char}\, F = p \geqslant 3$. Let q be a height one prime ideal in $Z(U(L))$ and v the unique height one prime ideal in $Sz(U(L))$ with $v \cap Z(U(L)) = q$. Then at least one of the following holds*:

(i) *v is principal,*
(ii) *$q = (d)$ and $v^{(p)} = dSz(U(L))$.*

This theorem is crucial in the determination of $Cl(Z(U(L)))$ in Section 6, as well as in finding, in Section 7, criteria for the factoriality of $Z(U(L))$.

Recall that if $[L, L]$ is nilpotent then one can do better, that is $Sz(U(L))$ is factorial. However, in Section 9, we exhibit an example (in case $p = 5$) where neither $Z(U(L))$ nor $Sz(U(L))$ are factorial.

The next result is probably well known and is included for lack of a suitable reference. See the discussion before Proposition 2.4 for the definition of I^{**}.

Lemma 5.1. *Let R be a prime Noetherian PI maximal order and $A \subseteq Z(R)$, a normal Noetherian domain with R being integral over A. Let I be an ideal in R. Then $I^{**} = \bigcap_q I_q$, where q runs on all height one prime ideals of A.*

Proof. R being a maximal order implies that $\bigcap_h R_h = R$, where h runs on all height one prime ideals of $Z(R)$. Since the extension $Z(R)/A$ satisfies "Going down" we have that $R = \bigcap_q R_q$, where q is a height one prime ideal in A. Moreover R_q being an hereditary maximal order implies that $I_q^{**} = I_q$ and therefore we may replace I by I^{**} and assume that $I = I^{**}$. Now let $y \in \bigcap_q I_q$, then $y \in \bigcap_q R_q = R$. Consider $\tau = \{z \in A \mid zy \in I\}$. Clearly τ is a two-sided ideal in A which is not contained in any height one prime ideal of A. Consequently by "Going down" between A and R [28, Theorem 4.4.24] τR is not contained in any height one prime ideal of R. Now $R/I = \overline{R}$ has an Artinian quotient ring [8, Proposition 1.3], [13, Theorem 3.3] and the minimal primes are all images of height one primes in R (which contain I). Therefore τR contains a regular element (mod I) and therefore the inclusion $(\tau R)y \subseteq I$ shows that $y \in I$, as needed □

Given a prime ideal P in R one denotes by $P^{(e)}$ the e-th symbolic power of P meaning $P^{(e)} = P_q^e \cap R$, where $q = P \cap Z(R)$.

The following connects two of the previous notions:

Lemma 5.2. *Let R be a prime Noetherian PI ring which is a maximal order and P a height one prime ideal in R. Then $P^{(e)} = (P^e)^{**}$*

Proof. By Lemma 5.1 $(P^e)^{**} = \bigcap_{ht(p')=1} (P^e)_{p'} = P_q^e \cap (\bigcap_{p' \neq q} P_{p'}^e) = P_q^e \cap (\bigcap_{p' \neq q} R_{p'})$. Intersection of both ends of the previous equality with R_q, yields $(P^e)^{**} = (P^e)^{**} \cap R_q = P_q^e \cap (\bigcap_{p' \neq q} R_{p'} \cap R_q) = P_q^e \cap R = P^{(e)}$. □

Lemma 5.3. *Let L be an F-finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p > 0$. Let v be a height one, adL stable, reflexive ideal in $Sz(U(L))$. Then $(vU(L)))^{**} \cap Sz(U(L)) = v$.*

Proof. By the adL-stability of v we have that $vU(L)$ is a two-sided ideal in $U(L)$ and so is $(vU(L))^{**}$. Let $rad(v) = v_1 \cap \cdots \cap v_k$ where $v_1, \ldots, v_k$ are all the height one primes in $Sz(U(L))$ which contain v. Let $p_i \equiv v_i \cap Z(U(L))$ for $i = 1, \ldots, k$. Then v_i is the unique prime in $Sz(U(L))$ "lying over" p_i and by the "Going down" theorem height $p_i = 1$ for $i = 1, \ldots, k$. Consequently $Sz(U(L))_{p_i}$ is a D.V.R., implying that $U(L)_{p_i}$ is a free (left, right) $Sz(U(L))_{p_i}$-module. Therefore $Sz(U(L))_{p_i}$ is a direct summand of $U(L)_{p_i}$ and therefore $v_{p_i} U(L)_{p_i} \cap Sz(U(L))_{p_i} = v_{p_i}$, for $i = 1, \ldots, k$. Let q denote an arbitrary height one prime ideal in $Z(U(L))$ satisfying $q \neq p_1, \ldots, p_k$. Then clearly $(vU(L))_q^{**} = (vU(L))_q = U(L)_q$ as well as $v_q = Sz(U(L))_q$. Finally $(vU(L))^{**} \cap Sz(U(L)) \subseteq [\bigcap_{i=1}^k (v_{p_i} U(L)_{p_i} \cap Sz(U(L))_{p_i})] \cap [\bigcap_q (vU(L))_q \cap Sz(U(L))_q] = [\bigcap_{i=1}^k v_{p_i}] \cap [\bigcap_q v_q] = v$, where the last equality is due to Lemma 5.1 taking $R = Sz(U(L))$. The reverse inclusion, being obvious, implies the required equality. □

Lemma 5.4. *Let L be a finite dimensional solvable Lie algebra over an algebraically closed field F of prime characteristic. Let v be an adL-stable height one prime ideal in $Sz(U(L))$. Then v is principal.*

Proof. Let $q = v \cap Z(U(L))$. By "Going down" between $Z(U(L))$ and $Sz(U(L))$ we have height $q = 1$. Moreover since v is the unique prime above q then $v^n \subseteq qSz(U(L))$ for some n. Let P be the unique height one in $U(L)$ which contracts to q. Thus $vU(L)$ is a two-sided ideal satisfying $(vU(L))^n \subseteq qU(L) \subseteq P$. That is $vU(L) \subseteq P$ and consequently $(vU(L))^{**} \subseteq P$ is a proper reflexive ideal in $U(L)$. Now, by Proposition 2.4 $(vU(L))^{**} = dU(L)$ for some $d \in U(L)_\lambda$. Therefore by Lemma 5.3, applied twice, we have $v = (vU(L))^{**} \cap Sz(U(L)) = dU(L) \cap Sz(U(L)) = dSz(U(L))$. □

Corollary 5.5. *Say L, v are as in Lemma 5.4 and let $q = v \cap Z(U(L))$. Then $(qSz(U(L))))^{**} = v$ implies that v is principal.*

Proof. Clearly $qSz(U(L))$ is adL-stable and consequently $(qSz(U(L)))^{**} = v$ is adL-stable. Therefore by Lemma 5.4 v is principal. □

Lemma 5.6. *Let L be a finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p$. Let q be a height one prime ideal in $Z(U(L))$ and v the unique height one prime ideal in $Sz(U(L))$ satisfying $q = v \cap Z(U(L))$. Suppose that q is not principal and v is principal. Then $v = (qSz(U(L)))^{**}$.*

Proof. Equivalently, we have to show that $v_q = q_q Sz(U(L))_q$. By assumption $v = bSz(U(L))$ and therefore $v_q = bSz(U(L))_q$. Now $b^p \in Z(U(L))$ shows that $e \leqslant p$, where e is the ramification degree of the D.V.R. extension $Sz(U(L))_q/Z(U(L))_q$. It is standard that $[Q(Sz(U(L))) : Q(Z(U(L)))]$ divides $[Q(U(L)) : Q(Z(U(L)))]$ and the later is a power of p. Thus $e = 1$ or $e = p$. Suppose by negation that $q_q Sz(U(L))_q \subset v_q$. Hence $q_q Sz(U(L)) = v_q^p$ and therefore $(qSz(U(L))))^{**} = v^{(p)} = v^p = b^p Sz(U(L))$. Intersection of both ends with $Z(U(L))$ yield: $q = (qSz(U(L)))^{**} \cap Z(U(L)) = b^p Sz(U(L)) \cap Z(U(L)) = b^p Z(U(L))$, a contradiction. $\square$

Remark 5.7. The assumption that F is algebraically closed is needed to ensure that $Sz(U(L))_q$ is a discrete valuation ring (D.V.R.), and it is a consequence of the normality of $Sz(U(L))$ which holds by Theorem B.

The next result explains our present need for the assumption *char* $F = p \geqslant 3$.

Lemma 5.8. *Let L and L_0 be as in Theorem* A *and char* $F = p \geqslant 3$. *Let $a \in Sz(U(L_0))$. Then $a^p \in Z(U(L))$.*

Proof. Suppose firstly that $a \in U(L_0)_\lambda$. Let $x \in L$, $y \in L_0$. Then $[x, y] \in L_0$ implying that $[y, [x, a]] = [[y, x], a] + [x, [y, a]] = \lambda([y, x])a + \lambda(y)[x, a]$. Therefore $[[y, [x, a]], a] = \lambda(y)[[x, a], a]$. Consequently $[y, [[x, a], a]] = [[y, [x, a]], a] + [[x, a], [y, a]] = \lambda(y)[[x, a], a] + \lambda(y)[[x, a], a]$. That is $[[x, a], a] \in U(L_0)_{2\lambda}$. Now since $a \in U(L_0)_\lambda$, this implies, by the commutativity of $Sz(U(L_0))$, that $[[[x, a], a], a] = 0$. Thus, since $p \geqslant 3$ we have that $[x, a^p] = 0$ for each $x \in L$; equivalently $a^p \in Z(U(L))$. For an arbitrary $a \in Sz(U(L))$ we write a as a sum of weight vectors and use the previous result. $\square$

We shall now prove a theorem which implies (and in fact is equivalent to) Theorem D. This crucially depends on Theorem A.

Theorem 5.9. *Let L be an F-finite dimensional solvable Lie algebra with F algebraically closed and char $F = p \geqslant 3$. Let q be a height one prime ideal in $Z(U(L))$ and v the unique height one prime ideal in $Sz(U(L))$ satisfying $v \cap Z(U(L)) = q$. Then at least one of the following holds*:

(i) *q is principal,*
(ii) *v is principal and $(qSz(U(L)))^{**} = v$.*

Proof. The proof is by induction on $\dim_F[L, L]$, the case $[L, L] = 0$ being obvious. By Theorem A, we have $Sz(U(L)) = Z(U(L_0))^{a\,da_1,\dots,a\,da_r}$ (retaining all our previous notations). Assume by negation that q is not principal. Let u be the unique prime ideal in $Z(U(L_0))$ with $u \cap Sz(U(L)) = v$ and w the unique prime ideal in $Sz(U(L_0))$ satisfying $w \cap Z(U(L_0)) = u$. If $[L_0, L_0] = [L, L]$ then by Proposition 4.6 $Sz(U(L_0)) = Z(U(L_0))$ implying by Theorem 4.7 that $Z(U(L_0))$ is factorial. Now since $a\,da_1, \dots, a\,da_r$ are nilpotent derivations then by [16, 17.3] $Sz(U(L))$ is factorial. In particular v is principal and the rest follows from Lemma 5.6. We may therefore assume that $[L_0, L_0] \subset [L, L]$. Thus by the inductive assumption either u is principal or if not, then w is principal and $(uSz(U(L_0)))^{**} = w$. Recall that $Z(U(L_0))$ being $a\,dL$-stable implies that $c^p \in Z(U(L))$, for each $c \in Z(U(L_0))$ and consequently the ramification degree e of the extension $Z(U(L_0))_q/Z(U(L))_q$ is either 1 or p. Suppose firstly that $e = 1$, that is $q_q Z(U(L_0))_q = u_q$. Consider $q_q Sz(U(L))_q \subseteq v_q$. If the last inclusion is proper, then since $Sz(U(L))_q$ is a direct summand of $Z(U(L_0))_q$ we get that $q_q Sz(U(L))_q \cdot Z(U(L_0))_q \subset v_q(Z(U(L_0)))_q$ and therefore $q_q Z(U(L_0))_q \subset v_q Z(U(L_0))_q \subseteq u_q$, in contradiction to our assumption. Thus $q_q Z(U(L_0))_q = u_q$ implies $q_q Sz(U(L))_p = v_q$. Equivalently $(qSz(U(L)))^{**} = v$ and by Corollary 5.5 v is principal as needed. We may therefore assume that $e = p$, that is $q_q Z(U(L_0))_q = u_q^{(p)}$ or equivalently $(qZ(U(L_0)))^{**} = u^{(p)}$.

If u is principal, $u = dZ(U(L_0))$, then $d^p \in Z(U(L))$ and $(qZ(U(L_0)))^{**} = u^{(p)} = u^p = d^p Z(U(L_0))$. Intersecting both sides with $Z(U(L))$ yield: $q = (qZ(U(L_0)))^{**} \cap Z(U(L)) = d^p Z(U(L_0)) \cap Z(U(L)) = d^p Z(U(L))$ and q is principal, contradicting the assumption. We may therefore assume that $w =$

$bSz(U(L_0))$ is principal and $(uSz(U(L_0)))^{**} = w$. By Lemma 5.8 we have that $b^p \in Z(U(L))$. Therefore combining it with $q_q Z(U(L_0))_q = u_q^p$ we get

$$q_q Z\big(U(L_0)\big)_q Sz\big(U(L_0)\big)_q = u_q^p Sz\big(U(L_0)\big)_q = \big(uSz\big(U(L_0)\big)_q\big)^p = w_q^p = b^p Sz\big(U(L_0)\big)_q.$$

Therefore $q_q Sz(U(L_0))_q = b^p Sz(U(L_0))_q$. Equivalently $(qSz(U(L_0)))^{**} = b^p Sz(U(L_0))$. Finally intersecting both sides with $Z(U(L))$ and using $b^p \in Z(U(L))$ we get $q = (qSz(U(L_0)))^{**} \cap Z(U(L)) = b^p Sz(U(L_0)) \cap Z(U(L)) = b^p Z(U(L))$, in contradiction to our assumption. □

Proof of Theorem D. Suppose that v is not principal, then by Theorem 5.9 q is principal, $q = dZ(U(L))$. We have by Corollary 5.5, that $(qSz(U(L)))^{**} \subset v$. Considerations about the ramification degree (as in the proof of Theorem 5.9) now show that $(qSz(U(L)))^{**} = v^{(p)}$. Finally since $qSz(U(L)) = dSz(U(L))$ then $(qSz(U(L)))^{**} = dSz(U(L))$, as claimed. □

The next result is a consequence.

Corollary 5.10. *Let L be a solvable finite dimensional Lie algebra over an algebraically closed field F with $\operatorname{char} F = p \geqslant 3$. Then the following map i^*, induced by inclusion, is trivial*:

$$i^* : Cl\big(Z\big(U(L)\big)\big) \to Cl\big(Sz\big(U(L)\big)\big).$$

Proof. Let q be a non-principal height one prime ideal in $Z(U(L))$. It is well known that $Cl(Z(U(L)))$ is generated by all $[q]$. Now by Theorem 5.9 $(qSz(U(L)))^{**} = v$ and v is principal, that is $[v] = 0$ in $Cl(Z(U(L)))$. Since $i^*([q]) = [qSz(U(L)))^{**}] = [v] = 0$, we reach the desired result. □

6. The divisor class group of $Z(U(L))$

Our main concern here is to determine $Cl(Z(U(L)))$, the divisor class group of $Z(U(L))$. More precisely we shall show, in case L is solvable, how to embed $Cl(Z(U(L)))$ into the additive group of L-weights and then we shall identify the co-kernel of this embedding with a distinguished subgroup. The resulting exact sequence has a striking resemblance to analogues results of Nakajima [24] in case of polynomial invariants of finite group (see [4, Section 3] for a detailed account). The embedding part is motivated by Samuel's radical descent theory and, in the polynomial invariants of Lie algebra case, is not really new. For an authoritative account on this topic one is referred to [16, Section 17]. The embedding part does not require, in our case, any additional assumption on F appart from $\operatorname{char} F = p > 0$. One immediate consequence is that $Cl(Z(U(L)))$ is a finite elementary abelian p-group.

Let $G \equiv \{\alpha \mid \alpha \text{ is an } L\text{-weight on } U(L)\}$. Alternatively, G consists of all α appearing in the decomposition $\bigoplus_\alpha U(L)_\alpha = Sz(U(L))$. Clearly G is an elementary abelian p-group with respect to addition. Also, since $Sz(U(L))$ is a finitely generated $Z(U(L))$-module, then G is a finite group as well. We next consider the following subgroups of G.

Let $H \equiv span_{\mathbb{Z}/p\mathbb{Z}}\{\alpha \mid \exists \text{ a non-central prime element } a \in U(L)_\alpha \text{ with } (qSz(U(L)))^{**} = (a) \text{ where } q = (a) \cap Z(U(L))\}$.

Set $K \equiv span_{\mathbb{Z}/p\mathbb{Z}}\{\alpha \mid \exists \text{ a non-central prime element } a \in U(L)_\alpha \text{ with } q = (a^p) \text{ where } q = (a) \cap Z(U(L))\}$.

We now define $\varphi : Cl(Z(U(L))) \longrightarrow G$ as follows. Let q be a reflexive ideal in $Z(U(L))$, then $(qU(L))^{**}$ is a reflexive ideal in $U(L)$. Since L is solvable we have by Proposition 2.4 that $(qU(L))^{**} = aU(L)$. Clearly a is a weight vector that is $a \in U(L)_\alpha$ for some α. We set

$$\varphi\big([q]\big) = \alpha. \tag{8}$$

The next lemma is very similar to Lemma 5.3 and its proof is therefore omitted.

Lemma 6.1. *Let q be a reflexive ideal in $Z(U(L))$. Then*

$$\big(qU(L)\big)^{**} \cap Z\big(U(L)\big) = q.$$

Proposition 6.2. *Let L be a finite dimensional solvable Lie algebra over a field F with char $F = p > 0$. Then*

$$\varphi : Cl\big(Z\big(U(L)\big)\big) \longrightarrow G \equiv \big\{\alpha \bigm| \alpha \text{ is a weight on } U(L)\big\},$$

is an injective homomorphism. Moreover if F is algebraically closed and $p \geqslant 3$, then $Im(\varphi) = H$.

Proof. We firstly need to show that φ is a well-defined map. If $[\tau] = [\tau_1]$ in $Cl(Z(U(L)))$ with τ, τ_1 reflexive ideals, then $c\tau = c_1\tau_1$ for some $c, c_1 \in Z(U(L))$ implying that $ca = c_1a_1f$ with $f \in F$ and therefore $\alpha(x)ca = [x, ca] = [x, c_1a_1f] = \alpha_1(x)c_1a_1f$, that is $\alpha(x) = \alpha_1(x)$ for each $x \in L$. Next, let τ, σ be two reflexive ideals in $Z(U(L))$. Then $(\tau U(L))^{**} = aU(L)$, $a \in U(L)_\alpha$ and $(\sigma U(L))^{**} = bU(L)$, with $b \in U(L)_\beta$. Now $[\tau][\sigma] = [(\tau\sigma)^{**}]$ in $Cl(Z(U(L)))$ and therefore $\varphi([\tau][\sigma]) = \varphi([(\tau\sigma)^{**}])$. Now by Lemma 5.1 we have $((\tau\sigma)^{**}U(L))^{**} = ((\tau\sigma)U(L))^{**} = (\tau U(L)\sigma U(L))^{**} = ((\tau U(L))^{**}(\sigma U(L))^{**})^{**} = (aU(L)bU(L))^{**} = (abU(L))^{**} = abU(L)$. Now $ab \in U(L)_{\alpha+\beta}$ and we conclude that $\varphi([\tau][\sigma]) = \alpha + \beta = \varphi(\tau) + \varphi(\sigma)$, and φ is a homomorphism. Suppose now that $\varphi([\tau]) = 0$, that is $(\tau U(L))^{**} = aU(L)$ with $a \in Z(U(L))$, then by Lemma 6.1 we have $\tau = (\tau U(L))^{**} \cap Z(U(L)) = aU(L) \cap Z(U(L)) = (a)$, implying that τ is principal and $[\tau] = 0$ in $Cl(Z(U(L)))$. This shows that φ is injective. Finally recall that $Cl(Z(U(L)))$ is generated by all $[q]$ where q is a non-principal height one prime ideal in $Z(U(L))$. Then by Theorem 5.9 $(qSz(U(L)))^{**} = v$ and $v = (a)$, with $a \in U(L)_\alpha$. Thus $(qU(L))^{**} = (qSz(U(L))U(L))^{**} = [(qSz(U(L)))^{**}U(L)]^{**} = (aU(L))^{**} = aU(L)$, showing that $\varphi([q]) = \alpha$ and α is by definition a generator of H. $\square$

Corollary 6.3. *Let L be a finite dimensional solvable Lie algebra over a field F with char $F = p > 0$. Then $Cl(Z(U(L)))$ is a finite elementary abelian p-group.*

Proof. Clearly $G = \{\alpha \mid \alpha$ is a weight on $U(L)\}$ is a finite elementary abelian p-group, and by the Proposition 6.2, $Cl(Z(U(L)))$ can be realized as a subgroup of G. $\square$

Retaining the notations of the beginning of the section we have:

Lemma 6.4. $H \cap K = \{0\}$.

Proof. Suppose by negation that $0 \neq \gamma = \sum_{k=1}^{r} i_k\alpha_k = \sum_{n=1}^{s} j_n\beta_n$, $1 \leqslant i_k, j_n \leqslant p-1$, where α_k is a generator of H for $1 \leqslant k \leqslant r$ and β_n is a generator of K for $1 \leqslant n \leqslant s$. We choose $0 \neq a_k \in U(L)_{\alpha_k}$ and $0 \neq b_n \in U(L)_{\beta_n}$, prime elements, for $1 \leqslant n \leqslant s$, $1 \leqslant k \leqslant r$. Therefore $\Pi_1^r a_k^{i_k}$ and $\Pi_1^s b_n^{j_n}$ are both in $U(L)_\gamma$. Consequently, by Lemma 2.2 there exist $z_1, z_2 \in Z(U(L))$ satisfying

$$z_1\big(\Pi_1^r a_k^{i_k}\big) = z_2\big(\Pi_1^s b_n^{j_n}\big). \tag{9}$$

Suppose $z_1 \in (b_1)$, then $z_1 \in (b_1) \cap Z(U(L)) = (b_1^p)$ and so $z_1 = c_1b_1^p$ with c_1 in $Z(U(L))$. Therefore $c_1b_1(\Pi_1^r a_k^{i_k}) = z_2(\Pi_2^s b_n^{j_n})$. Now the prime ideals $(a_1), \dots, (a_k), (b_1), \dots, (b_s)$ are mutually different implying that $z_2 \in (b_1)$ and therefore $z_2 = c_2b_1^p$ with $c_2 \in Z(U(L))$. We may therefore cancel b_1^p in both sides of (9), thus replacing z_i by c_i, $i = 1, 2$. This process terminates after finitely many steps, so we may assume to start with that $z_1 \notin (b_1)$. Following the same reasoning we may also assume that $z_1 \notin (b_2), \dots, (b_s)$. Therefore (9) shows for some n and k, that $a_k \in (b_n)$, equivalently $(a_k) = (b_n)$. This is clearly a contradiction. $\square$

Theorem 6.5. $G = H \oplus K$.

Proof. Let $\alpha \in G$ and $0 \neq a \in U(L)_\alpha$. By Proposition 2.4, $aU(L) = P_1^{e_1} \cdots P_k^{e_k}$ where P_i is a height one prime ideal in $U(L)$, $P_i = a_i U(L)$ and consequently $a_i \in U(L)_{\alpha_i}$, for $i = 1, \ldots, k$. Therefore $a = a_1^{e_1} \cdots a_k^{e_k} \delta'$, with δ' in F. Consequently $\alpha = e_1\alpha_1 + \cdots + e_k\alpha_k$. We shall show that if $\alpha_i \neq 0$ then either $\alpha_i \in H$ or $\alpha_i \in K$. This will prove the theorem. Let $p_i = P_i \cap Z(U(L))$ and ν_i be the unique height one prime ideal in $Sz(U(L))$ with $\nu_i \cap Z(U(L)) = p_i$. We already saw in Section 5 that either $(p_i Sz(U(L)))^{**} = \nu_i$ or $(p_i Sz(U(L)))^{**} = \nu_i^{(p)}$. Now $p_i \subseteq P \cap Sz(U(L)) = a_i Sz(U(L))$ implies that $(p_i Sz(U(L)))^{**} \subseteq a_i Sz(U(L)) \subseteq \nu_i$ for each $i = 1, \ldots, k$. If $(p_i Sz(U(L)))^{**} = \nu_i$ then $a_i Sz(U(L)) = \nu_i$ and consequently α_i is a generator of H. So we may assume that $(p_i Sz(U(L)))^{**} = \nu_i^{(p)}$. If $a_i Sz(U(L)) = \nu_i$ then by its very definition, α_i is a generator of K. So we may assume that $(p_i Sz(U(L)))^{**} \subseteq a_i Sz(U(L)) \subset \nu_i$. Consequently $(p_i Sz(U(L)))_{p_i} \subseteq a_i Sz(U(L))_{p_i} \subset \nu_{i_{p_i}}$. Let $b_i \in \nu_i$ satisfying $b_i Sz(U(L))_{p_i} = \nu_{i_{p_i}}$. Then $a_i Sz(U(L))_{p_i} = b_i^{j_i} Sz(U(L))_{p_i}$ with $1 < j_i \leqslant p$. Suppose that $j_i < p$. Therefore for each $x \in L$ we have $j_i[x, b_i] b_i^{j_i - 1} \in b_i^{j_i} Sz(U(L))_{p_i}$; equivalently $[x, b_i] \in b_i Sz(U(L))_{p_i} = \nu_{i_{p_i}}$. Therefore $\nu_{i_{p_i}}$ is adL-stable and consequently ν_i is adL-stable. This shows by Lemma 5.4 that $\nu_i = d_i Sz(U(L))$ with $d_i \in U(L)_{\beta_i}$. Now $\nu_i = (d_i)$ being adL-stable and $\nu_i^{(p)} = (p_i Sz(U(L)))^{**}$ imply that $\beta_i \in K$. Also $a_i Sz(U(L))_{p_i} = d_i^{j_i} Sz(U(L))_{p_i}$ implies that $a_i Sz(U(L)) = \nu_i^{(j_i)} = d_i^{j_i} Sz(U(L))$ and therefore $a_i = d_i^{j_i} \delta_i$, $\delta_i \in F$. Hence $\alpha_i = j_i \beta_i$ implying that $\alpha_i \in K$. Finally suppose that $j_i = p$, then $a_i Sz(U(L))_{p_i} = b_i^p Sz(U(L))_{p_i} = \nu_i^p Sz(U(L))_{p_i}$ and since ν_i is the unique height one prime above a_i we get that $a_i Sz(U(L)) = (a_i Sz(U(L)))^{**} = \nu_i^{(p)}$. If $\nu_i = (d_i)$ is principal then $\nu_i^{(p)} = \nu_i^p = d_i^p Sz(U(L))$ implying that $a_i = d_i^p \epsilon_i$, with ϵ_i in F. Therefore $a_i \in Z(U(L))$, equivalently $\alpha_i = 0$. If ν_i is not principal then by Theorem D, $\nu_i^{(p)} = d_i Sz(U(L))$ where $p_i = (d_i)$. Again $a_i = d_i \delta_i$, with δ_i in F implying that $a_i \in Z(U(L))$ and therefore $\alpha_i = 0$. $\square$

As a consequence of Proposition 6.2, Lemma 6.4 and Theorem 6.5 we have now arrived at the main result of the present section.

Theorem 6.6. *Let L be a solvable F-finite dimensional Lie algebra with F an algebraically closed field and $\operatorname{char} F = p \geqslant 3$. Then the following is an exact sequence*:

$$0 \longrightarrow Cl\big(Z\big(U(L)\big)\big) \xrightarrow{\varphi} \big\{\alpha \bigm| \alpha \text{ is a weight on } U(L)\big\} \xrightarrow{\psi} K \longrightarrow 0,$$

where ψ is the projection in Theorem 6.5, *on the second component.*

Remark 6.7. (1) The analogy to the exact sequence for polynomial invariants of finite groups is evident (e.g. [4, Section 3.9]). Indeed in this case, the middle term is replaced by $Hom(G, F^*)$, the set of all group homomorphisms from G to $F^* \equiv F \setminus \{0\}$ and K similarly, is replaced by the subgroup generated by all group homomorphisms corresponding to ramified height one primes.

(2) This analogy however, has its limitations. In fact, in the case of polynomials invariants of a finite group Γ, the subgroup K can be characterized using the subgroup of Γ which is generated by pseudo-reflections. A particular nice consequence of this is the following result of Nakajima [24, Proposition 3.6]: "Suppose Γ is solvable, $(|\Gamma|, F) = 1$ and F is algebraically closed. Then $S(V)^\Gamma$ is factorial iff it is a polynomial ring". Nothing of this nature holds for the center of the enveloping algebra, as the following example shows: Let $L = Fx + Fy + Fz + Ft$, subject to the Lie products $[x, y] = y$, $[x, z] = z$, $[y, z] = t$ and t central. Then one checks that $Z(U(L)) = Sz(U(L))$ and by Proposition 2.3 it is a factorial domain. It can be shown by direct computations, that the hypersurface $Z(U(L))$ has singularities and in particular it is not a polynomial ring.

(3) Still, see Theorem 8.2 for a positive result in the polynomial invariant case.

The next result shows that the canonical generators of K are $\mathbb{Z}/p\mathbb{Z}$-linearly independent. This will be used in the next section. Observe that here L need not be solvable.

Lemma 6.8. *Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of all different weights corresponding to prime weight elements whose p-powers are prime element in $Z(U(L))$. Then $\{\lambda_1, \ldots, \lambda_n\}$ is $\mathbb{Z}/p\mathbb{Z}$-independent. Equivalently $\{\lambda_1, \ldots, \lambda_n\}$ is a $\mathbb{Z}/p\mathbb{Z}$-basis of K.*

Proof. Let b_i be a prime element in $Sz(U(L))$ corresponding to λ_i, $i = 1, \ldots, n$. Suppose that $m_1\lambda_1 + \cdots + m_n\lambda_n = 0$, with $m_j \leqslant p-1$ for $j = 1, \ldots, n$. Assume by negation that $m_i \geqslant 1$, for $1 \leqslant i \leqslant r \leqslant n$ and $m_i = 0$, for $i > r$. Therefore $z \equiv b_1^{m_1} \cdots b_r^{m_r} \in Z(U(L))$. Now $z \in (b_1) \cap Z(U(L)) = (b_1^p)$ implies that $z = b_1^{pk_1} c_1$ with $c_1 \in Z(U(L))$ and $c_1 \notin (b_1)$. Then $b_1^{pk_1 - m_1} c_1 = b_2^{m_2} \cdots b_r^{m_r}$ if $r > 1$, implying that $b_j \in (b_1)$ or $(b_j) = (b_1)$, for some $j \geqslant 2$, a contradiction. If $r = 1$ then $b_1^{pk_1 - m_1} c_1 = 1$, implies that $1 \in (b_1)$, another absurd. $\square$

7. When is $Z(U(L))$ factorial

Throughout this chapter L will denote a finite dimensional solvable Lie algebra over a field F with *char* $F = p > 0$. We shall prove here several theorems, providing necessary and sufficient conditions for $Z(U(L))$ to be factorial. These theorems are grouped together in the introduction, under the header of "Theorem G". The main results here are as follows.

Theorem 7.1. *Let L be a solvable finite dimensional Lie algebra over a field of prime characteristic. Then the following are equivalent*:

(i) *$Z(U(L))$ is a U.F.D.,*
(ii) *$Sz(U(L))$ is a finitely generated free (or a projective) $Z(U(L))$-module.*

The proof makes use of a result, analogous to a theorem of Kang [19], asserting that $Pic(Z(U(L))) = 0$. A theorem of a different nature is the following:

Theorem 7.2. *Let L be a solvable F-finite dimensional Lie algebra and F is algebraically closed with char $F = p \geqslant 3$. Then the following are equivalent*:

(i) *$Z(U(L))$ is a U.F.D.,*
(ii) *$Sz(U(L))$ has exactly $\log_p[Q(Sz(U(L))) : Q(Z(U(L)))]$ non-central different prime weight elements.*

The theorem makes use of Theorem B, thus explaining the algebraically closed assumption on the field F. It is applied in Section 9, showing for the relevant example (with $p = 5$) that $Z(U(L))$ is not a U.F.D. Another result concerning factoriality is the following:

Theorem 7.3. *Let L be as in Theorem* 7.2. *Then the following are equivalent*:

(i) *$Z(U(L))$ is a factorial,*
(ii) *the extension $Sz(U(L))/Z(U(L))$ is a global (relative) complete intersection (see e.g.* [20, *p.* 317]),
(iii) *the extension $Sz(U(L))/Z(U(L))$ has a finite p-basis (see e.g.* [20, *p.* 76].)

The next Lemma appears in [21, Corollary 3]. The proof however uses a result of [23] and therefore seems to depend on the *char* $F = 0$ assumption. Our proof in contrast is PI dependent and therefore relies on the *char* $F = p > 0$ assumption. Observe that there is no need for the solvability assumption, in the next 3 results.

Lemma 7.4. *$U(L) \cap Q(Sz(U(L))) = Sz(U(L))$.*

Proof. Using the fact that $U(L)$ is a PI ring, it is easy to verify that $Q(Sz(U(L))) = Sz(Q(U(L))) = Sz(U(L))_{Z(U(L))\setminus\{0\}}$, where the last term stands for the localization of $Sz(U(L))$ with respect to

$Z(U(L)) \setminus \{0\}$. It is also a consequence of PI theory that $U(L) \cap Z(Q(U(L))) = Z(U(L))$. Therefore each $r \in U(L) \cap Q(Sz(U(L)))$ can be written in the form $r = q_1 + \cdots + q_k$, where $q_i \in Q(U(L))_{\lambda_i}$, for $i = 1, \ldots, k$, and by grouping together elements of equal weights we may assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Observe that if $q_i \in U(L)$ for $i = 1, \ldots, k$ then $q_i \in U(L) \cap Q(U(L))_{\lambda_i} = U(L)_{\lambda_i}$ and thus $r = q_1 + \cdots + q_k \in Sz(U(L))$. We choose by negation $r \in U(L) \cap Q(Sz(U(L)))$ having the minimal number of q_i's which are not in $U(L)$. By moving the q_i's which are in $U(L)$ to the left hand side we may assume that $q_i \notin U(L)$, for each $i = 1, \ldots, k$. If $k = 1$ then $r \in U(L) \cap Q(U(L))_{\lambda_1} = U(L)_{\lambda_1}$ and we are done. So we may assume that $k > 1$. Let $x \in L$ be chosen so that $\lambda_1(x) \neq \lambda_2(x)$. Then

$$[x, r] - \lambda_1(x) r = \bigl(\lambda_2(x) - \lambda_1(x)\bigr) q_2 + \cdots + \bigl(\lambda_k(x) - \lambda_1(x)\bigr) q_k. \tag{10}$$

Thus by the minimal choice of r, $q_2 \in U(L)$, $r - q_2 \in U(L) \cap Q(Sz(U(L)))$ and by the minimality of $r, q_1, q_3, \ldots, q_k$ are all in $U(L)$. $\square$

Lemma 7.5. *$Sz(U(L)) = \bigcap_q Sz(U(L))_q$, where the intersection runs on all height one prime ideals q in $Z(U(L))$.*

Proof. $U(L)$ being a maximal order satisfies the intersection property that is $U(L) = \bigcap_q U(L)_q$, where q runs on all height one prime ideals q in $Z(U(L))$. Consequently $\bigcap_q Sz(U(L))_q \subseteq U(L) \cap Q(Sz(U(L))) = Sz(U(L))$, where the last equality is due to Lemma 7.4. The reverse inclusion is obvious. $\square$

Remark 7.6. The previous result is an immediate consequence of the normality of $Sz(U(L))$, which is granted by Theorem B, in case F algebraically closed.

Corollary 7.7. *Let λ be an L-weight on $U(L)$. Then $U(L)_\lambda$ is a reflexive rank one $Z(U(L))$-module.*

Proof. That $\operatorname{rank}_{Z(U(L))} U(L)_\lambda = 1$ follows from Lemma 2.2. We may therefore consider $U(L)_\lambda$ as an ideal of $Z(U(L))$. Also $U(L)_\lambda$ is a $Z(U(L))$-direct summand of $Sz(U(L))$. Therefore by Lemma 7.5 $U(L)_\lambda = \bigcap_q (U(L)_\lambda)_q$, where q runs on all height one prime ideals of $Z(U(L))$. This implies by Lemma 5.1 that $U(L)_\lambda$ is a reflexive $Z(U(L))$-module. $\square$

The following result provides an enveloping algebra analog of a theorem of Kang [19]. The original theorem deals with a finite group G acting on a polynomial ring $S(V)$ and states that $Pic(S(V)^G) = \{0\}$. The result will be used in the proof of Theorem 7.1. Our proof follows Kang's argument as presented in [4, Theorem 3.6.1].

Proposition 7.8. *Let L be a finite dimensional solvable Lie algebra over a field F with $\operatorname{char} F = p > 0$. Then $Pic(Z(U(L))) = \{0\}$. Equivalently each projective ideal I in $Z(U(L))$ is principal.*

Proof. Recall that I being projective amounts in this case to $I^*I = II^* = Z(U(L))$, where $I^* = \{y \in Q(Z(U(L))) \mid yI \subseteq Z(U(L))\}$. Clearly $IU(L)$ is a two-sided ideal in $U(L)$ and $I^*U(L) = U(L)I^*$ satisfies $(I^*U(L))(IU(L)) = (IU(L))(I^*U(L)) = U(L)$, implying that $IU(L)$ is an invertible ideal in $U(L)$. In particular $IU(L)$ is projective and obviously reflexive. Consequently by Proposition 2.4, $IU(L) = aU(L)$ for some $a \in U(L)$. Moreover since $IU(L)$ is adL-stale we conclude that $a \in U(L)_\lambda$, for some weight λ. Let $I = (a_1, \ldots, a_s)$ then $I^*I = Z(U(L))$ grants the existence of $b_1, \ldots, b_s \in I^*$ satisfying $\sum_{i=1}^s a_i b_i = 1$. Now $a_i \in IU(L) = aU(L)$ implies that $c_i \equiv (1/a)a_i \in U(L)$ for each $i = 1, \ldots, s$. Similarly $d_i = b_i a \in I^*IU(L) = U(L)$. Consequently $\sum_{i=1}^s c_i d_i = 1$, where $c_i, d_i \in U(L)$ for each $i = 1, \ldots, s$. Now $LU(L)$ is a two-sided ideal in $U(L)$, satisfying by [11, 2.12, p. 66], $U(L) = F \oplus LU(L)$. Therefore by the previous equality we may assume that $c_1 \notin LU(L)$, that is $c_1 = f + g$, with $0 \neq f \in F$, $g \in LU(L)$. Since $[y, (1/a)] = -\lambda(y)(1/a)$ for each $y \in L$, we deduce, using the centrality of a_1, that $(1/a)a_1 = c_1 \in U(L)_{-\lambda}$. However $c_1 = f + g$, $f \neq 0$, yields: $-\lambda(y)(f + g) = -\lambda(y)c_1 = [y, c_1] = [y, f] + [y, g] = [y, g]$. Thus $-\lambda(y)f = \lambda(y)g + [y, g]$ is in $LU(L)$ for each $y \in L$, an obvious contradiction unless $\lambda(y) = 0$, for each $y \in L$. Namely $a \in Z(U(L))$. Therefore $I = IU(L) \cap Z(U(L)) = aU(L) \cap Z(U(L)) = aZ(U(L))$, where the first equality is due to Lemma 6.1. $\square$

Proof of Theorem 7.1. Suppose firstly that $Z(U(L))$ is a U.F.D. This implies by Corollary 7.7 that, for each weight λ, $U(L)_\lambda$ is a free rank one $Z(U(L))$-module. Consequently $Sz(U(L))$ is a finitely generated free $Z(U(L))$-module. Conversely assume that $Sz(U(L))$ is a finitely generated projective $Z(U(L))$-module. Then $U(L)_\lambda$ is a rank one projective $Z(U(L))$-module. Thus by Proposition 7.8 $U(L)_\lambda$ is a free $Z(U(L))$-module, that is $U(L)_\lambda = f_\lambda Z(U(L))$. Let q be a height one prime ideal in $Z(U(L))$, then $q = P \cap Z(U(L))$ for some height one prime ideal in $U(L)$ and since $P^{**} = P$ we get that $(qU(L))^{**} \subseteq P$ and in particular it is a proper reflexive two-sided ideal in $U(L)$. Thus by Proposition 2.4, $(qU(L))^{**} = dU(L)$, for some $d \in U(L)$. Moreover since $q \subseteq Z(U(L))$ we get that $(q(U(L))^{**}$ is an adL-stable ideal. Consequently by [5, Lemma 5] $d \in U(L)_\mu$, for some μ. We next observe that $dU(L) \cap Sz(U(L)) = dSz(U(L))$. Indeed if $dy \in Sz(U(L))$ with $y \in U(L)$, then $y \in Q(Sz(U(L))) \cap U(L) = Sz(U(L))$, where the last equality is due to Lemma 7.4. This equality could be also deduced from Lemma 5.3 if F is in addition algebraically closed. Thus $q = (qU(L))^{**} \cap Z(U(L)) = (dU(L) \cap Sz(U(L))) \cap Z(U(L)) = dSz(U(L)) \cap Z(U(L)) = dU(L)_{-\mu}$. Therefore $q = dU(L)_{-\mu} = df_{-\mu}Z(U(L))$ and $q = (df_{-\mu})$. $\square$

Remark 7.9. The analogy to Nakajima's polynomial invariants of finite group result [24, Theorem 2.11] is evident. We believe that the theorem is valid without the solvability assumption on L. In fact this is the case if L is acting as derivations on $S(V)$, as can be seen in Theorem 8.1.

The next two results are needed in the proof of Theorem 7.2. The first one is a consequence of Nagata's theorem [16, Section 7].

Lemma 7.10. *Let A be a Krull domain where all but possibly finitely many height one prime ideals, are principal. Then A is factorial.*

Proof. Let $\{p_1, \dots, p_k\}$ be the set of height one primes which may not be principal. Set $S = A \setminus p_1 \cup \dots \cup p_k$. Then by [16, Corollary 7.2] we have the following exact sequence

$$0 \longrightarrow K \longrightarrow Cl(A) \longrightarrow Cl(A_S) \longrightarrow 0,$$

where K is generated by all height one primes q in A satisfying $q \cap S \neq \emptyset$. But $q \cap S \neq \emptyset$ means that $q \neq p_i$, for $i = 1, \dots, k$, that is q is principal. Hence $K = \{0\}$ and $Cl(A) \cong Cl(A_S)$. Now A_S is a Krull domain with K.dim A_S=1, therefore A_S is a Dedekind domain with finitely many maximal ideals. Hence A_S is a principal ideal ring [22, Theorem 12.2] and in particular A_S is factorial. That is $Cl(A) \cong Cl(A_S) = \{0\}$. $\square$

Lemma 7.11. *Let L be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Let $a \in Sz(U(L)) \setminus Z(U(L))$ be a prime element. Then a^p is an irreducible element in $Z(U(L))$.*

Proof. Clearly $a^p \in Z(U(L))$. Suppose by negation that $a^p = uv$, where $u, v \in Z(U(L))$ are non-invertible elements. By considering u, v as elements in $Sz(U(L))$ and using the primeness of a we have that $u = a^i c$, $v = a^j d$, where c, d are not in (a) and $0 \leqslant i, j \leqslant p$. Hence $a^p = a^{i+j}cd$. If $i + j > p$ then a is invertible, a contradiction. If $i + j < p$ then c or d are in (a) which is another contradiction. Thus $i + j = p$. If $i = 0$ this leads to $a^p = ua^p d$ and u is invertible, a contradiction. A similar contradiction is achieved if $j = p$. So we may assume that $1 \leqslant i, j \leqslant p - 1$. Now $a^p = a^p cd$ implies that c and d are invertible and consequently $c, d \in F$. Let $x \in L$. Then $u \in Z(U(L))$ implies that $0 = [x, u] = [x, a^i c] = i[x, a]a^{i-1}c$. Consequently $[x, a] = 0$ for each $x \in L$ and $a \in Z(U(L))$, a contradiction. $\square$

Proof of Theorem 7.2. Suppose firstly that $Z(U(L))$ is factorial. Recall that by Lemma 6.8 $[Q(Sz(U(L))) : Q(Z(U(L)))] = p^n$, where $n =$ the number of different weights corresponding to prime weight elements in $Sz(U(L))$ whose p-powers are prime elements in $Z(U(L))$. Assume by negation that $Sz(U(L)) \setminus Z(U(L))$ has more then $\log_p[Q(Sz(U(L))) : Q(Z(U(L)))] = n$, prime weight elements. Then there exists a prime weight element $b \in Sz(U(L)) \setminus Z(U(L))$ such that b^p is not a prime element in $Z(U(L))$, but this is impossible since by Lemma 7.11 b^p is irreducible and $Z(U(L))$ is factorial. This proves the implication (i) $\Rightarrow$ (ii). Conversely suppose that there are exactly $\log_p[Q(Sz(U(L))) : Q(Z(U(L)))]$ different non-central prime elements, which are also weight elements. Let q be a height one prime ideal in $Z(U(L))$ which is not principal, and let v be the unique height one prime ideal in $Sz(U(L))$ with $v \cap Z(U(L)) = q$. By Theorem 5.9 we have $(qSz(U(L)))^{**} = v$ and v is principal, say $v = (a)$. Now q being in $Z(U(L))$ implies that $(qSz(U(L)))^{**}$ is adL-stable, and therefore a is a weight element which is also prime. Therefore, by the uniqueness of the correspondence between the q's and the v's, there are at most $\log_p[Q(Sz(U(L))) : Q(Z(U(L)))]$ height one primes q in $Z(U(L))$ which are not principal. The result is therefore established by applying Lemma 7.10. $\square$

We shall now prove the implication (i) $\Rightarrow$ (iii) of Theorem 7.3.

Proposition 7.12. *Let L be a finite dimensional solvable Lie algebra over an algebraically closed field F with $\operatorname{char} F = p \geqslant 3$. Suppose that $Z(U(L))$ is a U.F.D. Then $Sz(U(L))$ has a finite p-basis over $Z(U(L))$.*

Proof. Retaining the notations of Lemma 6.8 and let α be a non-zero weight. Then by Theorem 6.6 the U.F.D. property of $Z(U(L))$ and $H = \{0\}$ we have $\alpha = m_1\lambda_1 + \cdots + m_n\lambda_n$, with $m_j \leqslant p-1$ for $j = 1, \ldots, n$. We may assume that $1 \leqslant m_j \leqslant p-1$ for $1 \leqslant j \leqslant r \leqslant n$ and $m_j = 0$ for $j > r$. Therefore $b_1^{m_1} \cdots b_r^{m_r} \in U(L)_\alpha$. Now by Theorem 7.1 $U(L)_\alpha = Z(U(L))x_\alpha$. Let v be a height one prime in $Sz(U(L))$ which is minimal over $x_\alpha Sz(U(L))$. Then $b_1^{m_1} \cdots b_r^{m_r} \in U(L)_\alpha \subseteq x_\alpha Sz(U(L))$ implies that $b_j \in v$ for some j and consequently $(b_j) = v$. Therefore we may assume after renumbering, that $(b_1) = v_1, \ldots, (b_k) = v_k$, $k \leqslant r$ are all the height one prime ideals containing $x_\alpha Sz(U(L))$. By the normality of $Sz(U(L))$ and the primary decomposition in $Sz(U(L))$ we have $x_\alpha Sz(U(L)) = v_1^{(e_1)} \cap \cdots \cap v_k^{(e_k)} = v_1^{e_1} \cdots v_k^{e_k} = b_1^{e_1} \cdots b_k^{e_k} Sz(U(L))$, where the second equality is due to Lemmas 5.1 and 5.2 once equality is verified after localizing at each height one prime. Therefore $x_\alpha = b_1^{e_1} \cdots b_k^{e_k}\delta$, with $\delta \in F$, since the only units in $U(L)$ are in F. Moreover $b_1^{m_1} \cdots b_r^{m_r} \in x_\alpha Sz(U(L))$ shows, with the aid of the last equality that $e_i \leqslant m_i \leqslant p-1$, for $i = 1, \ldots, k$. Also $x_\alpha = b_1^{e_1} \cdots b_k^{e_k}\delta$ shows that $\alpha = e_1\lambda_1 + \cdots + e_k\lambda_k$. This together with $\alpha = m_1\lambda_1 + \cdots + m_r\lambda_r$ and the $\mathbb{Z}/p\mathbb{Z}$ independence of $\{\lambda_1, \ldots, \lambda_n\}$ yield $k = r$ and $m_i = e_i$, that is $x_\alpha = b_1^{m_1} \cdots b_r^{m_r}\delta$. Conversely consider $b \equiv b_1^{m_1} \cdots b_n^{m_n}$ with $m_i \leqslant p-1$, for $i = 1, \ldots, n$. Then $b \in U(L)_\alpha$, where $\alpha \equiv m_1\lambda_1 + \cdots + m_n\lambda_n$ and the previous argument shows that $U(L)_\alpha = Z(U(L))b$. This shows that $\{b_1^{m_1} \cdots b_n^{m_n} \mid 0 \leqslant m_i \leqslant p-1\}$ is a p-basis of $Sz(U(L))$ over $Z(U(L))$. $\square$

Remark 7.13. The reverse implication (iii) $\Rightarrow$ (i) of Theorem 7.3, follows from Theorem 7.1, since a p-basis is also a free basis of $Sz(U(L))$ over $Z(U(L))$.

Lemma 7.14. *Let L be as in Proposition 7.12. Let $y_1, \ldots, y_n$ be variables and $b_1, \ldots, b_n$ the prime elements corresponding to $\lambda_1, \ldots, \lambda_n$ as in Lemma 6.8. Then*

$$Z\big(U(L)\big)[y_1, \ldots, y_k]/\big\langle y_1^p - b_1^p, \ldots, y_k^p - b_k^p\big\rangle \cong Z\big(U(L)\big)[b_1, \ldots, b_k] \tag{11}$$

for each $1 \leqslant k \leqslant n$.

Proof. Let ψ be defined on the l.h.s. of (11) by $\psi(z) = z$ for $z \in Z(U(L))$ and $\psi(\overline{y_i}) = b_i$, for each class $\overline{y_i}$, $i = 1, \ldots, k$. Clearly ψ extends to an onto ring homomorphism. Recall that $\{\overline{y_i}^{m_1} \cdots \overline{y_k}^{m_k} \mid 0 \leqslant m_i \leqslant p-1,\ i = 1, \ldots, k\}$ is a generating set of the l.h.s. of (11) over $Z(U(L))$. Any non-trivial element in $\ker\psi$ will give a non-trivial dependence in $\{b_1^{m_1} \cdots b_k^{m_k} \mid 0 \leqslant m_i \leqslant p-1\}$ over $Z(U(L))$, in contradiction to their p-basis property (by Proposition 7.12) $\square$

Proposition 7.15. *Let L be an F-finite dimensional solvable Lie algebra over an algebraically closed field F with char $F = p \geqslant 3$. Suppose $Z(U(L))$ is a U.F.D. Then*

$$Sz\big(U(L)\big) \cong Z\big(U(L)\big)[y_1, \dots, y_n]/\big\langle y_1^p - b_1^p, \dots, y_n^p - b_n^p\big\rangle,$$

and consequently $Sz(U(L))$ is a global (relative) complete intersection over $Z(U(L))$.

Proof. Let $y_1, \dots, y_n$ be variables and $b_1, \dots, b_n$ the prime weight elements which correspond to $\lambda_1, \dots, \lambda_n$, as in Lemma 6.8. Then the isomorphism follows from Lemma 7.14 and Proposition 7.12. To show that $y_1^p - b_1^p, \dots, y_n^p - b_n^p$ is a regular sequence, one observes that $\{y_1^p - b_1^p, \dots, y_k^p - b_k^p\}$ generates by Lemma 7.14 a prime ideal in $Z(U(L))[y_1, \dots, y_k]$. Consequently $\langle y_1^p - b_1^p, \dots, y_k^p - b_k^p\rangle$, its extension to $Z(U(L))[y_1, \dots, y_n]$, is a prime ideal in $Z(U(L))[y_1, \dots, y_n]$, implying that $y_1^p - b_1^p, \dots, y_k^p - b_k^p$ is a regular sequence in $Z(U(L))[y_1, \dots, y_n]$ for each $k \leqslant n$. □

Proof of Theorem 7.3. The equivalence of (i) and (iii) follows from Proposition 7.12 and Remark 7.13. Now the implication (i) $\Rightarrow$ (ii), follows from Proposition 7.15. For the opposite direction observe that $Sz(U(L))$ being a global (relative) complete intersection over $Z(U(L))$ implies that $Sz(U(L))$ is flat over $Z(U(L))$. So being finitely generated as a $Z(U(L))$-module makes it a projective $Z(U(L))$-module [22, p. 53, Corollary]. Therefore by Theorem 7.1, $Z(U(L))$ is a U.F.D. □

The next result is an immediate consequence.

Corollary 7.16. *Let L be an F-finite dimensional solvable Lie algebra over an algebraically closed field F, with char $F = p \geqslant 3$. Suppose $Z(U(L))$ is a polynomial ring. Then $Sz(U(L))$ is a "standard" complete intersection.*

Proof. Clearly $Z(U(L))[y_1, \dots, y_n]$ is a polynomial ring and $y_1^p - b_1^p, \dots, y_n^p - b_n^p$ is a regular sequence in it, one now applies Proposition 7.15 □

8. Polynomial invariants of Lie algebras

In this section L will always be a finite dimensional Lie algebra over a field F with *char* $F = p > 0$. Let V be a finite dimensional L-module. We consider here $S(V)^L$ (respectively $Sz(S(V))$) the invariant (respectively semi-invariant) ring. As a rule all the previous theorems for $Z(U(L))$ and $Sz(U(L))$ holds for $S(V)^L$ and $Sz(S(V))$ with no further restriction on L or F. The proofs are either similar or even easier.

Starting with the analog of Theorem 7.1 we have in the present set up:

Theorem 8.1. *Let L be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Let V be a finite dimensional L-module. Then the following are equivalent:*

(i) *$S(V)^L$ is a factorial domain,*
(ii) *$Sz(S(V))$ is a finitely generated free (projective) $S(V)^L$-module.*

The proof is essentially the same as in Theorem 7.1 where $U(L)$ is replaced by $S(V)$ thus, avoiding the need for the solvability of L since $S(V)$ is factorial and therefore $(q(S(V))^{**}$ is principal. The analog of Kang's theorem in this setting is valid as well, with exactly the same proof.

The following result on torus invariants is in the spirit of [24, Proposition 3.6].

Theorem 8.2. *Let V be a finite dimensional vector space over an algebraically closed field F with char $F = p > 0$. Let $L \subseteq \mathfrak{gl}_F(V)$ be a commutative subalgebra consisting of semi-simple elements. Then the following are equivalent:*

(i) *$S(V)^L$ is a factorial domain,*
(ii) *$S(V)^L$ is a polynomial ring.*

Proof. The semi-simplicity of the elements in L and the algebraic closed property of F imply that $Sz(S(V)) = S(V)$. Therefore if $S(V)^L$ is factorial, then by Theorem 8.1, $S(V)$ is a free $S(V)^L$-module. Now $S(V)^L$ is a graded subalgebra of $S(V)$, the grading being induced by the one on $S(V)$, and by [4, Corollary 6.2.3] $S(V)^L$ is therefore a polynomial ring. The implication (ii) $\Rightarrow$ (i) is standard. □

Remark 8.3. See [1] for other results on the ring of invariants of a single derivation in the prime characteristic case.

To get analogs of the theorems appearing in Section 7 one needs the following analog of Theorem D. The proof is markedly easier and no need is required of a version of Theorem A.

Theorem 8.4. *Let L, V, $S(V)$ and $S(V)^L$ be as before. Let q be a height one prime ideal in $S(V)^L$ and v the unique height one prime ideal in $Sz(S(V))$ satisfying $v \cap S(V)^L = q$. Then at least one of the following holds*:

(1) *v is principal,*
(2) *$q = (d)$ and $v^{(p)} = dSz(V)$.*

Proof. Let w be the unique height one prime ideal in $S(V)$ satisfying $w \cap Sz(S(V)) = v$. Suppose by negation that v is not principal. Since $b^p \in S(V)^L$ for every $b \in S(V)$ then the ramification degree e of the extension $S(V)_q/S(V)^L_q$ is either 1 or p. That is either $qS(V)_q = w_q$ or $qS(V)_q = w^p_q$. Consequently either $(qS(V))^{**} = w$ or $(qS(V))^{**} = w^p$. Now $S(V)$ is a U.F.D. and so $w = (b)$, implying that $w^{(p)} = w^p$. If $(qS(V))^{**} = w = bS(V)$ then since $qS(V)$ is adL-stable we get that $bS(V)$ is adL-stable. This shows that $b \in S(V)_\lambda$ for some λ and so $b \in w \cap S(V) = v$. Consequently $v = bS(V) \cap Sz(S(V)) = bSz(V)$ in contradiction to our assumption. Therefore we may assume that $(qS(V))^{**} = w^{(p)} = w^p = b^pS(V)$. Now $b^p \in S(V)^L$ and intersecting both ends of the last equality with $S(V)^L$, yields: $q = (qS(V))^{**} \cap S(V)^L = b^pS(V) \cap S(V)^L = b^pS(V)^L = (b^p)$. Finally let e_1 be the ramification degree of the extension $S(V)_q/Sz(S(V))_q$, and e_2 the ramification degree of the extension $Sz(S(V))_q/S(V)^L_q$. Since $p = e_1e_2$ then either $e_1 = p$ or $e_2 = p$. If $e_1 = p$ then $w^{(p)} = (vS(V))^{**}$ and therefore $b^pS(V) = (vS(V))^{**}$ so by contracting with $Sz(S(V))$ we get $b^pSz(S(V)) = v$, so v is principal, which was excluded. Therefore $e_2 = p$, that is $v^p_q = qSz(S(V))_q = b^pSz(S(V))_q$ and therefore since v is the unique prime ideal above $q = (b^p)$, $v^{(p)} = b^pSz(S(V))$, as needed. □

Theorem 8.5. *Let L be a finite dimensional Lie algebra over a field F with char $F = p > 0$. Then the analog of Theorems* 7.2, 7.3, 6.6 *holds for $S(V)^L$* (*replacing $Z(U(L))$*) *and $Sz(S(V))$* (*instead of $Sz(U(L))$*).

Proof. The proofs are essentially the same, as in the cited theorems, making use of Theorem 8.4 and replacing $U(L)$ by $S(V)$ throughout. We have no need here for the restriction $p \geqslant 3$ since $b^p \in S(V)^L$ for every $b \in S(V)$. Moreover there is no need to assume that F is algebraically closed, since no version of Theorem A is used. □

In fact the following more practical version of Theorem 7.2 is true, where $Sz(S(V))$ is replaced by $S(V)$.

Theorem 8.6. *Let L, V and $S(V)$ be as in Theorem* 8.5. *Then the following are equivalent*:

(i) *$S(V)^L$ is a U.F.D.,*
(ii) *$S(V) \setminus S(V)^L$ has exactly $\log_p[Q(Sz(S(V))) : Q(S(V)^L)]$ different prime weight elements.*

Proof. We only need to show that a is a prime weight element in $Sz(S(V))$ if and only if it is a prime weight element in $S(V)$. Clearly if it is a prime weight element in $S(V)$, then being in $Sz(S(V))$ it is also a prime (weight) element in $Sz(S(V))$. Conversely let $a \in Sz(S(V)) \setminus S(V)^L$ be a prime weight element and let w be the unique height one prime ideal in $S(V)$ satisfying $w \cap Sz(S(V)) = (a)$. So by the previous ramification reasonings either $aS(V) = w^{(p)}$ or $aS(V) = w$. In the second case a is also a prime element in $S(V)$ and we are done. In the first case, let $w = (b)$ then $aS(V) = w^{(p)} = w^p = b^p S(V)$, showing that $a = b^p\delta$, for some δ in F. This implies that $a \in S(V)^L$, a contradiction. $\square$

9. Jacobson's example revisited

We shall exhibit here a family of examples, showing that for a finite dimensional solvable Lie algebra over an algebraically closed field of prime characteristic, $Sz(U(L))$ and $Z(U(L))$ need not be a U.F.D. Thus the assumption that $[L, L]$ is nilpotent, in Theorem C, is really essential.

Let $Fx + Fy = S$ be the solvable two-dimensional Lie algebra over an algebraically closed field F of prime characteristic p, subject to the relation $[x, y] = x$. A representation for this Lie algebra is given by the $p \times p$ matrices

$$x \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad y \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & p-1 \end{pmatrix}$$

acting faithfully (and irreducibly) on the right, on $V \equiv F^p$, the p-dimensional F-vector space with respect to the standard basis $e_i = (0, \dots, 1, \dots, 0)$, $i = 1, \dots, p$.

This example is used in [17, p. 53] to show that S does not satisfy Lie's theorem and by considering the semi-direct product $L = V \oplus S$ one observes (e.g. [17]) that $[L, L]$ is not nilpotent.

The main results of the present section are as follows:

Example 9.1. Let L be as above. Then

(1) $Sz(U(L))$ is not factorial, for $p \geqslant 3$,
(2) $Z(U(L))$ is not factorial for $p = 5$,
(3) $Z(U(L))$ is a polynomial ring if $p = 3$.

For $p = 2$ we "double" the previous example to get:

Example 9.2. Let $L_0 = S \oplus (V \oplus V)$, where V, S are as above. Then $Sz(U(L_0))$ is not factorial for $p \geqslant 2$.

The proof of Example 9.1 is achieved via a series of steps.

Let $H = Fx + V$. Clearly V is a codimension one ideal in H and H is a codimension one ideal in L. One easily verifies that $m(t) = t^{p^2} - t^p$ is the minimal p-polynomial that $a\,dx$ satisfies on V, H and L.

Claim 9.3. *$Z(U(H)) = U(V)^{a\,dx}[x^{p^2} - x^p]$ and $Sz(U(H)) \subseteq U(V)[x^{p^2} - x^p]$.*

Proof. The above shows that $m(x) = x^{p^2} - x^p$ is central in $U(L)$ and therefore $B = U(V)\{x^{p^2} - x^p\} = U(V)[x^{p^2} - x^p]$. This combined with $[x, U(V)] \neq 0$ implies by Theorem 3.9 that $Z(U(H)) = U(V)^{a\,dx}[x^{p^2} - x^p]$ and $Sz(U(H)) \subseteq U(V)[x^{p^2} - x^p]$. $\square$

Claim 9.4. *Assuming $p \geqslant 3$, then $[y, Z(U(H))] \neq 0$.*

Proof. Let $v \equiv e_1 e_2 \cdots e_p \in U(V)$. Then $x^p = I_V$ implies $(a\,dx)^p(v) = 0$. We also have $a\,dx(v) \neq 0$. Let i be the maximal integer such that $w \equiv (a\,dx)^i(v) \neq 0$. So $0 < i < p$, and $a\,dx(w) = 0$, imply by Claim 9.3 that $w \in Z(U(H))$. Also $[y, v] = (1 + 2 + \cdots + (p-1))v = 0$ (this uses $p \geqslant 3$). Consequently $[y, w] =$

$[y, (adx)^i(v)] = [y, [x, [x, \dots [x, v]]]] = -i[x, [x, [\dots [x, v]]]] + [x, [x, [\dots [y, v]]]] = -i(adx)^i(v) = -iw$. Thus $[y, Z(U(H))] \neq 0$. $\square$

Claim 9.5. $Z(U(L)) = U(V)^{adx,ady}[y^p - y, x^{p^2} - x^p]$ *and* $Sz(U(L)) = U(V)^{adx}[y^p - y, x^{p^2} - x^p] = Z(U(H))[y^p - y]$.

Proof. From Claims (9.4) and (9.3) we get that $Z(U(L)) = Z(U(H))^{ady}[y^p - y]$ and $Sz(U(L)) \subseteq Sz(U(H))[y^p - y] \subseteq U(V)[x^{p^2} - x^p, y^p - y]$. This combined with Claim 9.3 yield $Z(U(L)) = U(V)^{adx,ady}[x^{p^2} - x^p, y^p - y]$. Now let $U(L)_\lambda \neq 0$ be a weight space. Then $[x, y] = x$ implies $\lambda(x) = 0$, that is $U(L)_\lambda \subseteq U(L)^{adx}$. Consequently $Sz(U(L)) \subseteq U(V)^{adx}[x^{p^2} - x^p, y^p - y]$. To show the reverse inclusion one needs to show that $U(V)^{adx} \subseteq Sz(U(L))$. To show this observe that $U(V)^{adx}$ is ady stable. Now ady acts semi-simply on L (with weights $0, 1, \dots, p-1$). Therefore $U(V)^{adx}$ can be decomposed into direct sum of ady-weight spaces. On each one of these weight spaces, adx acts trivially. Thus each one of this summands is an adL weight space, that is $U(V)^{adx} \subseteq Sz(U(L))$. $\square$

As a consequence of the previous claim it suffices to show that $Z(U(H))$ is not a U.F.D., for $p \geqslant 3$.

Claim 9.6. *The minimal polynomial of* x *(on* V*) is* $t^p - 1 = (t-1)^p$.

Proof. If not then $(x-1)^{p-1} = 0_V$. Consequently since $(x-1)^{p-1} = I_V + \dots + x^{p-1}$ we get by applying it on e_1, the following contradiction $0 = 0_V(e_1) = e_1 + e_2 + \dots + e_p$. $\square$

Corollary 9.7. *The Jordan canonical form of* x *with respect to a new basis of* V, $u_p, u_{p-1}, \dots, u_1$ *is given by the* $p \times p$ *matrix*

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now considering the right lower 3×3 *block and using* $p \geqslant 3$ *we have in* L:

$$[x, u_1] = -u_1, \qquad [x, u_2] = -(u_2 + u_1), \qquad [x, u_3] = -(u_3 + u_2).$$

Claim 9.8. *Suppose* $p \geqslant 3$. *Then* $\zeta \equiv -2u_3u_1^{p-1} + u_2^2u_1^{p-2} \in Z(U(H))$.

Proof. $[x, \zeta] = -2[x, u_3]u_1^{p-1} - 2u_3[x, u_1^{p-1}] + [x, u_2^2]u_1^{p-2} + u_2^2[x, u_1^{p-2}] = 2(u_2 + u_3)u_1^{p-1} + 2(p-1)u_3u_1^{p-1} - 2(u_2 + u_1)u_2u_1^{p-2} - (p-2)u_2^2u_1^{p-2} = 0$. The claim now easily follows since $\zeta \in U(V)$ clearly commutes with V. $\square$

Claim 9.9. *Suppose* $p \geqslant 3$ *and let* $P \equiv u_1U(H)$, $q = P \cap Z(U(H))$. *Then* q *is a height one prime in* $Z(U(H))$ *which is not principal.*

Proof. u_1 being a normal element in H implies that P is a two-sided ideal in $U(H)$. Moreover since u_1 is part of a basis of H, $u_1U(H) \equiv P$ is in fact a prime ideal in $U(H)$ and by the principal ideal theorem in $U(H)$ it is of height one. Consequently by the “Going down” between $U(H)$ and $Z(U(H))$, height $q = 1$. Now u_1^p is irreducible in $U(V)^{adx}$ since its proper factors in $U(V)$, u^i, for $1 < i < p$, are not in $U(V)^{adx}$. Consequently by Claim 9.3, u_1^p is an irreducible element in $Z(U(H))$. Moreover, $u_1^p \in q$. Now if q is principal then $q \equiv (u_1^p)$ and since $\zeta \in P \cap Z(U(H)) = q$ we get that $-2u_3u_1^{p-1} +$

$u_2^2 u_1^{p-2} = \alpha u_1^p$, with $\alpha \in Z(U(H))$. By canceling we get $u_2^2 = 2u_3u_1 + \alpha u_1^2 \in P = u_1 U(H)$. This is clearly in contradiction with the P-B-W theorem. $\square$

This settles item (1) of Example 9.1, by showing that $Sz(U(L)) = Z(U(H)[y^p - y]$ is not factorial. We next need the following:

Lemma 9.10. $[Q(S(V)) : Q(S(V)^L)] = p^3$.

Proof. Let $\mathfrak{h} = Fx + Fx^p + Fy$, considered as a Lie subalgebra of $U(L)$. Now $(a\,dx)^{p^2} = (a\,dx)^p$, $(a\,dy)^p = a\,dy$ shows that $a\,d\mathfrak{h}$ is a restricted Lie algebra with the $[p]$-operation being the regular associative p-power. Moreover $a\,d\mathfrak{h}$ can be considered as a Lie subalgebra of $\mathrm{Der}(S(V))$ and consequently also of $\mathrm{Der}(Q(S(V)))$. Now one easily observes that $Q(S(V)^L) = Q(S(V))^L = Q(S(V))^{a\,d\mathfrak{h}}$. Next by using the Hochschild–Serre formula (e.g. [22, p. 197]) and Jacobson's formulas for associative p-powers of a sum of two elements, one gets that $Q(S(V))a\,d\mathfrak{h}$ is a p-$Q(S(V))$-Lie subalgebra of $\mathrm{Der}(Q(S(V)))$ (in the terminology if [18, p. 533]). Consequently by [18, Theorem 8.43], $[Q(S(V)) : Q(S(V))^L] = p^{\dim Q(S(V))a\,d\mathfrak{h}}$. The result will therefore follow once we show that $\dim_{Q(S(V))} Q(S(V))a\,d\mathfrak{h} = 3$. Now clearly $\dim_{Q(S(V))} Q(S(V))a\,d\mathfrak{h} \leqslant 3$. Suppose that $a \cdot a\,dx + b \cdot a\,dx^p = a\,dy$, with $a, b \in Q(S(V))$. Now by applying to $e_1 \cdots e_p$ one gets that $a\,dx(e_1 \cdots e_p) = x(e_1 \cdots e_p) \neq 0$, but $a\,dx^p(e_1 \cdots e_p) = x^p(e_1 \cdots e_p) = 0$ and $a\,dy(e_1 \cdots e_p) = (0 + 1 + \cdots + p - 1)e_1 \cdots e_p = 0$. Consequently $ax(e_1 \cdots e_p) = 0$ and therefore $a = 0$. Thus $ba\,dx^p = a\,dy$. Now since $a\,dy(e_1) = y(e_1) = 0$ and $a\,dx^p(e_1) = x^p(e_1) = e_1$, we get $be_1 = 0$ and so $b = 0$. The only other possible linear dependence between $\{a\,dx, a\,dx^p, a\,dy\}$ over $Q(S(V))$ is of the form $c \cdot a\,dx = a\,dx^p$, $c \in Q(S(V))$. Now since $a\,dx(e_i) = e_{i+1}$, for $i = 1, \dots, p-1$, $a\,dx(e_p) = e_1$, and $a\,dx^p(e_i) = e_i$ for $i = 1, \dots, p$, this implies that $ce_{i+1} = e_i$, for $i = 1, \dots, p-1$ and $ce_p = e_1$. Consequently $(ce_2) \cdots (ce_p)(ce_1) = e_1 \cdots e_p$ and $c^p e_1 \cdots e_p = e_1 \cdots e_p$, that is $c^p = 1$ and $c = 1$. This cannot hold since it implies that $e_2 = ce_2 = e_1$, an obvious contradiction. $\square$

We shall now proceed to prove the other parts of the example.

The next result verifies item (3) of Example 9.1.

Lemma 9.11. *$Z(U(L))$ is a polynomial ring if $p = 3$.*

Proof. For any p, $U(L)$ is a free module of rank p^3 over $S(V)[x^{p^2} - x^p, y^p - y]$. Also by Lemma 9.10 $[Q(S(V)) : Q(S(V))^L] = p^3$, which implies since $Z(U(L)) = S(V)^L[x^{p^2} - x^p, y^p - y]$ that $[Q(U(L)) : Q(Z(U(L)))] = p^6$. Now if $p = 3$ then clearly $U(L)$ is free of rank 3^6 over the central subring $F[e_1^3, e_2^3, e_3^3, x^9 - x^3, y^3 - y] \equiv C$. Therefore for $p = 3$, we have $Q(Z(U(L))) = Q(C)$. Now $Z(U(L))$ is a finite C-module, implying by the normality of C that $C = Z(U(L))$. $\square$

We shall now consider item (2) of Example 9.1. To this end recall that if $U(L)_\lambda \neq 0$ for some non-zero weight λ, then $\lambda(u) = 0$ for each $u \in V$ and $[x, y] = x$ implies $\lambda(x) = 0$. This shows that λ solely depends on y and hence $\lambda(y) \in \{0, 1, 2, \dots, p-1\}$. Thus $Sz(U(L)) = \bigoplus_{i=0}^{p-1} U(L)_{i\lambda}$. Therefore

$$\big[Q\big(Sz\big(U(L)\big)\big) : Q\big(Z\big(U(L)\big)\big)\big] = p \quad \text{and} \quad \log_p\big[Q\big(Sz\big(U(L)\big)\big) : Q\big(Z\big(U(L)\big)\big)\big] = 1.$$

Consequently by Theorem 7.2 (or Theorem 8.6), in order to verify the non-factorial property of $Z(U(L))$, for $p = 5$, we only need to exhibit two weight elements which generate different prime ideals in $Sz(U(L))$. This is done next with the aid of a computer. Presumably a similar result holds for $p > 5$.

Consider the element:

$$w = 4e_2^4e_4 + 4e_1e_2^2e_3e_4 + e_1^2e_3^2e_4 + 3e_1^2e_2e_4^2 + 4e_1^2e_2e_3e_5 + e_1e_2^3e_5 + 3e_1^3e_2^2 = ae_5 + b,$$

where $a = e_1e_2(4e_1e_3 + e_2^2)$ is a product of 3 prime elements, and $b = 4e_2^4e_4 + 4e_1e_2^2e_3e_4 + e_1^2e_3^2e_4 + 3e_1^2e_2e_4^2 + 3e_1^3e_2^2$. One checks, with the aid of a computer, that $w = (a\,dx)^4(e_1^2e_2^3)$ and consequently $a\,dx(w) = 0$. Moreover using $[y, e_1^2e_2^3] = 2e_1 0 + 3e_1^2e_2^2([y, e_2]) = -3e_1^2e_2^3 = 2e_1^2e_2^3$, we get that $[y, w] = [y, (a\,dx)^4(e_1^2e_2^3)] = -4(a\,dx)^4(e_1^2e_2^3) + (a\,dx)^4([y, e_1^2e_2^3]) = -4w + 2w = 3w$. Therefore w is a weight 3 element with respect to $a\,dy$. Now regarding $w = ae_5 + b$ as a degree one element in e_5, the only possible decomposition of w will be of the form $w = a_1(a_2e_5 + b_2)$, that is $a_1a_2 = a$, $a_1b_2 = b$, where a_1, a_2, b_2 are in $F[e_1, e_2, e_3, e_4]$. Now the possibility $a_1 = e_1$ can't work since e_1 is not a divisor of $4e_2^2e_4$ but it is a divisor of all the other terms appearing in b. Similarly the possibility $a_1 = e_2$ does not work since e_2 is not a divisor of $e_1^2e_3^2e_4$ but divides all the other terms in b. Now if $a_1 = 4e_1e_3 + e_2^2$, then a monomial in any multiple of a_1 will be divisible by either e_1e_3 or e_2^2. However $3e_1^2e_2e_4^2$, which is a monomial in b is not divisible by neither one of them. Thus w is an irreducible element in $F[e_1, e_2, e_3, e_4, e_5]$ and by its factorial property it is also a prime weight element with $w \in S(V) \setminus S(V)^L$. Next consider the element $v = (a\,dx)^4(e_3^2e_4^3)$. One shows, with the aid of a computer, that $v = 4e_1e_4^4 + 4e_1e_3e_4^2e_5 + e_1e_3^2e_5^2 + 4e_2e_3^2e_4e_5 + 3e_1^2e_3^2e_4 + 3e_3^3e_4^2 + e_2e_3e_4^3 = ae_2 + b$, where $a = e_3e_4(4e_3e_5 + e_4^2)$, is a product of 3 prime elements and $b = 4e_1e_4^4 + 4e_1e_3e_4^2e_5 + e_1e_3^2e_5^2 + 3e_1^2e_3^2e_4 + 3e_3^3e_4^2$. As before if v is reducible, then $v = a_1(a_2e_2 + b_2)$ which forces us to check the following possibilities: $a_1 = e_3$, $a_1 = e_4$ and $a_1 = 4e_2e_5 + e_4^2$. Now $a_1 = e_3$ is impossible since e_3 does not divide $4e_1e_4^4$. Next $a_1 = e_4$ is excluded since e_4 does not divide $e_1e_3^2e_5^2$. Finally $a_1 = (4e_3e_5 + e_4^2)$, can not happen since $3e_1^2e_3^2e_4$ is not divisible by either $4e_3e_5$ or e_4^2, and any monomial in a multiple of a_1, and in particular b, must have this property. Consequently v is prime element in $S(V) = F[e_1, e_2, e_3, e_4, e_5]$. Also $a\,dy(v) = a\,dy((a\,dx)^4(e_3^2e_4^3)) = -4(a\,dx)^4(e_3^2e_4^3) + (a\,dx)^4([y, e_3^2e_4^3]) = -4v + 2v = 3v$. Since $a\,dx(v) = 0$, we see that v is also a prime weight element and $v \notin S(V)^L$. Finally, clearly v is not a scalar multiple of w, showing that they generate two different height one prime ideals in $S(V)$.

Next we consider the details of Example 9.2. The case $p > 2$ is carried out as in Example 9.1 and so is omitted. Suppose therefore that $p = 2$. Let $\{e_1, e_2\}$ be the standard basis of $(V, 0)$ and $\{f_1, f_2\}$ the standard basis of $(0, V)$, where $V = F^2$. Then x and y have the following presentation as 4×4 matrices:

$$x \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad y \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with multiplication table:

$$[x, e_1] = e_2, \qquad [x, e_2] = e_1, \qquad [x, f_1] = f_2, \qquad [x, f_2] = f_1,$$
$$[y, e_1] = [y, f_1] = 0, \qquad [y, e_2] = e_2, \qquad [y, f_2] = f_2.$$

Then $v \equiv [x, e_1e_2f_1f_2] = (e_2^2 + e_1^2)f_1f_2 + (e_1e_2)(f_1^2 + f_2^2)$. It is easily checked that $[x, v] = 0$ and $[y, v] = v \neq 0$. Set $H_0 \equiv Fx + (V \oplus V)$. Then $v \in Z(U(H_0))$ and consequently $[y, Z(U(H_0))] \neq 0$. Therefore as in Claim 9.5 we have $Z(U(L_0)) = U(V \oplus V)^{a\,dx, a\,dy}[y^2 - y, x^4 - x^2]$ and $Sz(U(L_0)) = U(V \oplus V)^{a\,dx}[y^2 - y, x^4 - x^2] = Z(U(H_0))[y^2 - y]$. Hence to show that $Sz(U(L_0))$ is not factorial we merely need showing that $Z(U(H_0))$ is not factorial. To this end we exhibit two distinct prime weight 1 elements in $Sz(U(H_0))$ implying that $U(H_0)_1$ is not a free $Z(U(H_0))$ module and then use Theorem 7.1. Now $[x, e_1 + e_2] = e_1 + e_2$, $[x, f_1 + f_2] = f_1 + f_2$ so $e_1 + e_2, f_1 + f_2$ are both in $U(H_0)_1$ and both are clearly prime weight elements in $Sz(U(H_0))$, being such in $U(V \oplus V)$. This verifies the properties of Example 9.2.

Remark 9.12. (1) If one wants to have that both $Z(U(L))$ and $Sz(U(L))$ are non-factorial in case $p = 2$, the easiest is to consider $L = S \oplus (V \oplus V \oplus V)$, and follow the previous discussion.

(2) By using different methods one can verify that $Z(U(L))$ is factorial in Example 9.1, for $p = 2$. Also $Z(U(L)) = Sz(U(L))$ in this case.

References

[1] A.G. Aramova, L.L. Avramov, Singularities of quotients by vector fields in characteristic p, Math. Ann. 273 (1986) 629–645.
[2] J.-M. Bois, Gelfand–Kirillov conjecture in positive characteristics, J. Algebra 305 (2006) 820–844.
[3] W. Borho, P. Gabriel, R. Rentschler, Prime ideals in Einhüllenden auflösbarer Lie-Algebren, Lecture Notes in Math, vol. 357, Springer-Verlag, New York, 1973.
[4] D.J. Benson, Polynomial Invariants of Finite Groups, London Math. Soc. Lecture Notes, vol. 190, Cambridge Univ. Press, 1993.
[5] A. Braun, Factorial properties of the universal enveloping algebra of a nilpotent Lie algebra in prime characteristic, J. Algebra 308 (2007) 1–11.
[6] A. Braun, C.R. Hajarnavis, Smooth PI rings with almost factorial center, J. Algebra 299 (2006) 124–150.
[7] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, 1996.
[8] M. Chamarie, Anneaux de Krull non commutatifs, J. Algebra 72 (1981) 210–222.
[9] A.W. Chatters, D.A. Jordan, Non-commutative unique factorization rings, J. London Math. Soc (2) (1986) 22–32.
[10] I. Delvaux, E. Nauwelarts, A.I. Ooms, On the semi-center of a universal enveloping algebra, J. Algebra 94 (1985) 324–346.
[11] J. Dixmier, Enveloping Algebras, Grad. Stud. Math., vol. II, Amer. Math. Soc., 1996.
[12] J. Dixmier, Sur l'algebra enveloppante d'une algebra de Lie nilpotente, Arch. Math. 10 (1959) 321–326.
[13] C.R. Hajarnavis, S.S. Williams, Maximal orders in Artinian rings, J. Algebra 90 (1984) 375–384.
[14] I.N. Herstein, Noncommutative Rings, Carus Math. Monogr., 1968.
[15] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., Springer-Verlag, Heidelberg–Berlin–New York, 1972.
[16] R.M. Fossum, The Divisor Class Group of a Krull Domain, Springer-Verlag, Berlin–Heidelberg–New York, 1973.
[17] N. Jacobson, Lie Algebras, Interscience, 1962.
[18] N. Jacobson, Basic Algebra II, W.H. Freeman and Company, 1970.
[19] M.-G. Kang, Picard groups of some rings of invariants, J. Algebra 58 (1979) 455–461.
[20] E. Kunz, Kähler Differentials, Friedr. Vieweg and Sohn, 1986.
[21] L. Le Bruyn, A.I. Ooms, The semicenter of an enveloping algebra is factorial, Proc. Amer. Math. Soc. 93 (3) (1985) 397–400.
[22] H. Matsumura, Commutative Ring Theory, Cambridge Univ. Press, 2002.
[23] C. Moeglin, Factorialite dans les algebres enveloppantes, C. R. Acad. Sci. Paris A 282 (1976) 1269–1272.
[24] H. Nakajima, Relative invariants of finite groups, J. Algebra 79 (1982) 218–234.
[25] A. Premet, R. Tange, Zassenhaus varietes of general Linear Lie algebras, J. Algebra 294 (2005) 177–195.
[26] A. Premet, Regular Cartan subalgebras and nilpotent elements in restricted Lie algebras, Math. USSR Sb. 66 (2) (1990) 555–570.
[27] R. Rentschler, M. Vergne, Sur le semi-centre du corps enveloppant d'une algebre de Lie, Ann. Sci. École Norm. Sup. 6 (3) (1973) 389–405.
[28] L.W. Rowen, Polynomial Identities in Ring Theory, Academic Press, 1980.
[29] R.P. Stanley, Relative invariants of finite groups generated by pseudoreflections, J. Algebra 49 (1977) 134–148.
[30] R. Tange, The Zassenhaus variety of a reductive Lie algebra in positive characteristic, arXiv:0811.4568v2.
[31] G. Vernik, On the center and the semi-center of the enveloping algebra of finite dimensional solvable Lie algebra over a field of prime characteristic, PhD thesis, University of Haifa, August 2008.
[32] H. Zassenhaus, The representations of Lie algebras of prime characteristic, Proc. Glasg. Math. Ass. 2 (1954) 1–36.