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Local coefficients and Euler class groups

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ABSTRACT

In this paper, we establish an isomorphism between the Euler class group $E(\mathbb{R}(X), L)$ for a real smooth affine variety $X = \mathrm{Spec}(A)$ and the 0-th homology group $H_0(M_c; \mathcal{G})$ with local coefficients in a bundle $\mathcal{G}$ of groups constructed from the line bundle $\mathcal{L}$ over M corresponding to the orientation rank-1 projective module L, where M_c is the compact part of the manifold M of real points in X. Then by Steenrod's Poincaré duality between homology and cohomology groups with local coefficients, this isomorphism is identified with the Whitney class homomorphism.

1. Introduction

Obstruction theory in topology is classical, while the advent of obstruction theory in algebra is a more recent phenomenon. In topology, for real smooth manifolds with $\dim(M) = n \geqslant 2$ and line bundles $\mathcal{L}$ over M, there is an obstruction group $\mathcal{H}(M, \mathcal{L})$, and for vector bundles $\mathcal{E}$ of $\mathrm{rank}(\mathcal{E}) = n$ with an orientation $\chi : \mathcal{L} \xrightarrow{\cong} \bigwedge^n \mathcal{E}$, there is an invariant $w(\mathcal{E}, \chi) \in \mathcal{H}(M, \mathcal{L})$ such that $\mathcal{E}$ has a nowhere vanishing section if and only if $w(\mathcal{E}, \chi) = 0$.

In early nineties, Nori outlined a program for an obstruction theory in algebra. The program of Nori mirrors the already existing theory in topology. Accordingly, for smooth affine algebras A over infinite fields with $\dim(A) = n \geqslant 2$ and for projective A-modules L with $\mathrm{rank}(L) = 1$, Nori outlined a definition ([MS], later generalized in [BRS3]) of an obstruction group $E(A, L)$, which contains an invariant $e(P, \chi)$ for any projective A-module P of $\mathrm{rank}(P) = n$ with orientation $\chi : L \xrightarrow{\cong} \bigwedge^n P = \det(P)$, such that conjecturally, $e(P, \chi) = 0$ if and only if $P \cong Q \oplus A$ for some projective A-module Q. Essentially, all the conjectures given at the time when the program was outlined were proved and the program of Nori flourished beyond all expectations. Among the major and important papers on this program

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are [M,MS,MV,BRS1,BRS2,BRS3,BhDaMa]. Readers are referred to [MaSh] for further introductory remarks and history [MkM,Mk,Mu1] of development of obstruction theory in algebra.

While the obstruction theory in algebra was guided by the classical obstruction theory in topology, there has not been a successful attempt to reconcile the theory in algebra and topology. More precisely, let $X = \mathrm{Spec}(A)$ be a real smooth affine variety and let $M = M(X)$ be the manifold of real points of X with $\dim(X) = \dim(M) = n \geqslant 2$. Also, let L be a projective $\mathbb{R}(X)$-module of rank one and let $\mathcal{L}$ be the line bundle over M, whose module of cross sections comes from L, i.e. $\Gamma(\mathcal{L}) = L \otimes_{\mathbb{R}(X)} C(M)$ [Sw], where $\mathbb{R}(X) = S^{-1}A$ for the multiplicative set S of all functions $f \in A$ that do not vanish at any real point of X. The issue in question is whether there is a canonical homomorphism from the algebraic obstruction group $E(A, L)$ to the topological obstruction group $\mathcal{H}(M, \mathcal{L})$. While this fundamental question remained open since the inception of obstruction theory in algebra, it did not draw enough attention.

In the orientable case, the obstruction group $\mathcal{H}(M, M \times \mathbb{R})$ in topology is the cohomology group $H^n(M; \mathbb{Z})$ with integer coefficients. For the general (non-orientable) case, the obstruction group $\mathcal{H}(M, \mathcal{L})$ in topology turns out to be the more sophisticated homology group $H_0(M_c, \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}})$ with local coefficients in a bundle $\mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}$ of groups associated with $\mathcal{K}^* \otimes \mathcal{L}$, where $\mathcal{K}$ is the cotangent determinant bundle of M, and M_c is the union of compact connected components of M. By Steenrod's Poincaré duality, this obstruction group is also naturally isomorphic to the cohomology group $H^n(M, \mathcal{G}_{\mathcal{L}^*})$.

In our earlier paper [MaSh], without using the concept of (co)homology with local coefficients, we addressed our question for the case of oriented real smooth affine varieties and oriented vector bundles (i.e. with $K_X = \bigwedge^n \Omega_{\mathbb{R}(X)/\mathbb{R}} = \mathbb{R}(X)$ and $L = \mathbb{R}(X)$) and defined a canonical isomorphism

$$\zeta : E\big(\mathbb{R}(X), \mathbb{R}(X)\big) \to \mathcal{H}(M, M \times \mathbb{R}) \cong H^n(M; \mathbb{Z}),$$

from the algebraic obstruction group $E(\mathbb{R}(X), \mathbb{R}(X))$ to the topological obstruction group $\mathcal{H}(M, M \times \mathbb{R})$.

In this paper, we consider the general case and establish a canonical isomorphism

$$\zeta : E\big(\mathbb{R}(X), L\big) \to \mathcal{H}(M, \mathcal{L}) \cong H_0(M_c, \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}) \cong H^n(M, \mathcal{G}_{\mathcal{L}^*}).$$

Furthermore, given a projective $\mathbb{R}(X)$-module P of rank n with an L-orientation $\chi : L \xrightarrow{\cong} \bigwedge^n P$, the obstruction classes from algebra and from topology agree, i.e. $\zeta(e(P, \chi)) = w(\mathcal{E}^*, \chi)$, where $\mathcal{E}$ is the vector bundle on M whose sections come from P and the latter χ represents the orientation on $\mathcal{E}^*$ induced by χ. Some applications of our main theorem are given, including an example of an algebraic vector bundle that does not have any algebraic nowhere vanishing section, but with a continuous nowhere vanishing section.

As an interesting consequence, we get a purely algebraic description of such cohomology groups $H^n(M, \mathcal{G}_{\mathcal{L}})$, including the special case of $H^n(M; \mathbb{Z})$. It remains open whether such descriptions can be given for all cohomology groups $H^r(M, \mathcal{G}_{\mathcal{L}})$ for $0 \leqslant r < n$. This obviously relates to the question, whether it is possible to give appropriate definitions for obstruction groups in algebra for projective modules of all ranks and develop an algebraic theory in complete analogy to the existing topological theory.

In this paper, the not so-widely used (co)homology theory with local coefficients, including Poincaré duality theorem of Steenrod [St1], is crucially used. In order to prove our results that relate algebra to topology, we give this theory a relatively modern account, which is of some novelty to our knowledge, making rigorous formal connections between topological, geometrical, and algebraic interpretations of the concept of local orientations. In our presentation, special attention is paid to make the objects and arguments as much coordinate-free as possible.

Here we briefly describe the content of each of the following sections. In Section 2, we recall the theory of (co)homology groups with local coefficients including the computation of the 0-th homology group. In Section 3, we introduce the bundle of groups associated with a bundle of local orientations, and in Section 4, we identify it with the bundle of homotopy groups for a related vector bundle. In

Section 5, the concept of the top Whitney class of a vector bundle is described in terms of cohomology with local coefficients. In Section 6, we recall Steenrod's Poincaré duality theorem, and results about indices of transversal cross sections are deduced from the Euler–Hopf–Poincaré Theorem. Then we obtain canonical isomorphisms from the Euler class group to a 0-th homology group and an n-th cohomology group with local coefficients, respectively in Sections 7 and 8, the latter of which identifies the algebraic Euler class of a projective module and the top Whitney class of an associated vector bundle. Finally in Section 9, we present some applications of our results.

2. Homology with local coefficients

We first recall the notion of homology with local coefficients, which was first formally introduced by Steenrod [St1] in the simplicial homological context, and can be equivalently formulated in the singular homological context [Wh2]. In this section, we work with the singular homological version, and in Section 5, we recall and utilize the simplicial homological version.

For a modern account of the theory of fiber (or vector) bundles that is needed in our discussion, we referred readers to [Hu,St2,Wh2].

Let $\mathfrak{F}$ be the *fundamental groupoid* of a topological space X [Sp]. More explicitly, $\mathfrak{F}$ is the *category* consisting of elements $x \in X$ as objects, and homotopy classes $[\gamma]$ of (continuous) paths $\gamma : [0,1] \to X$ from x_1 to x_2 in X as morphisms from object x_1 to object x_2, with the composition defined as $[\gamma_2] \circ [\gamma_1] := [\gamma_1 * \gamma_2]$ for $[\gamma_1] \in \mathrm{Hom}(x_1, x_2)$ and $[\gamma_2] \in \mathrm{Hom}(x_2, x_3)$, where $\gamma_1 * \gamma_2$ is the standard concatenation of paths γ_1 and γ_2 with γ_2 following γ_1. We use $s(\gamma)$ and $t(\gamma)$ to denote, respectively, the *source* x_1 and the *target* x_2 of the homotopy class $[\gamma] \in \mathrm{Hom}(x_1, x_2)$.

For any (locally trivial) fiber bundle $\mathcal{F} \xrightarrow{p} X$, we denote by $\mathcal{F}|_x := p^{-1}(\{x\})$ the fiber at $x \in X$ and take $\mathcal{F}|_A := p^{-1}(A)$ for $A \subset X$, which also denotes the restricted bundle $\mathcal{F}|_A \xrightarrow{p|_A} A$.

A fiber bundle $\mathcal{G} \xrightarrow{\pi} X$ is called a *bundle of groups* modeled on a group G, if any $x \in X$ has an open neighborhood $U \subset X$ with a homeomorphism $\phi_U : \mathcal{G}|_U \to U \times G$, and the transition maps between such local trivializations are fiberwise *group automorphisms* of G, i.e. the map $g \mapsto (\phi_{U_2} \circ \phi_{U_1}^{-1})(x, g)$ belongs to $\mathrm{Aut}(G)$ for any $x \in U_1 \cap U_2$. Via such local trivializations ϕ_U, each fiber $\mathcal{G}|_x$, $x \in X$, of a bundle $\mathcal{G}$ of groups modeled on G has a well-defined group structure such that $\mathcal{G}|_x \cong G$ as groups, even though in general, there is no canonical choice of an isomorphism between each $\mathcal{G}|_x$ and G.

We remark that a bundle $\mathcal{G}$ of groups modeled on group G is associated with a *principal A-bundle* $\mathfrak{G} \to X$, i.e. $\mathcal{G} = \mathfrak{G} \otimes_A G := \mathfrak{G} \times G/{\sim}$ where $\sim$ is defined by $(z\psi, g) \sim (z, \psi(g))$ for all $z \in \mathfrak{G}$, $g \in G$, and $\psi \in A$, for some subgroup A of $\mathrm{Aut}(G)$. This is an analogue of the property that if the transition maps between local trivializations $\mathcal{E}|_U \to U \times \mathbb{R}^n$ of a vector bundle $\mathcal{E} \to X$ are fiberwise in a matrix subgroup $H \subset GL(n, \mathbb{R})$ then $\mathcal{E} = \mathfrak{H} \otimes_H \mathbb{R}^n$ for some principal H-bundle $\mathfrak{H} \to X$ over X. Note that each bundle $\mathcal{G}$ of groups has a global cross section going through the identity element of each fiber group $\mathcal{G}|_x$, but $\mathcal{G}$ may not be a trivial fiber bundle unless $\mathcal{G}$ is also a G-principal bundle with a well-defined global G-action on $\mathcal{G}$. On the other hand, a G-principal bundle in general is not a bundle of groups. In fact, a bundle $\mathcal{G}$ of groups modeled on G is a trivial fiber bundle if and only if $\mathcal{G}$ is also a G-principal bundle such that $w(z \cdot g) = (wz) \cdot g$ for any $g \in G$ and any z, w in the group $\mathcal{G}|_x$ at any $x \in X$.

Note that for a *discrete* group G, a bundle $\mathcal{G} \xrightarrow{\pi} X$ of groups modeled on G is a covering map, and hence via lifting of paths, the fundamental groupoid $\mathfrak{F}$ acts on $\mathcal{G}$ in a canonical way compatible with its canonical action on X and compatible with the group structure on each fiber of $\mathcal{G}$. More precisely, there is a well-defined continuous map

$$\cdot : \big([\gamma], c\big) \in \mathfrak{F} \times_X \mathcal{G} \mapsto [\gamma] \cdot c \in \mathcal{G}$$

where $\mathfrak{F} \times_X \mathcal{G} := \{([\gamma], c) \mid s(\gamma) = \pi(c)\}$ is the fibered product of $\mathfrak{F}$ and $\mathcal{G}$ over X, such that for any $([\gamma], c) \in \mathfrak{F} \times_X \mathcal{G}$, (i)

$$\pi\big([\gamma] \cdot c\big) = t(\gamma) = [\gamma]\big(s(\gamma)\big),$$

(ii) for any morphism $[\eta]$ composable with $[\gamma]$ in $\mathfrak{F}$,

$$\big([\eta] \circ [\gamma]\big) \cdot c = [\eta] \cdot \big([\gamma] \cdot c\big)$$

and (iii) the action restricted to each fiber

$$[\gamma]\cdot : c \in \mathcal{G}|_{s(\gamma)} \mapsto [\gamma] \cdot c \in \mathcal{G}|_{t(\gamma)}$$

is a group isomorphism. Indeed the path γ in X is lifted against the projection map π to a unique path $\tilde{\gamma}$ in the covering space $\mathcal{G}$ that has $c \in \mathcal{G}|_{s(\gamma)}$ as its start (or source) point, and $[\gamma] \cdot c$ is then exactly the end (or target) point of $\tilde{\gamma}$.

We now recall the construction of the complex of *singular chains with local coefficients* in a bundle $\mathcal{G}$ of groups modeled on a *discrete abelian* group G [Wh2]. Let us view G as an *additive* group with its identity element denoted as 0. First we define the group of k-chains in X with local coefficients in $\mathcal{G}$ as

$$S_k(X;\mathcal{G}) := \left\{\text{finite sum} \sum_{\sigma:\Delta_k \to X \text{ continuous}} c_\sigma \sigma \;\middle|\; c_\sigma \in \mathcal{G}|_{\sigma(e_0)}\right\}$$

consisting of (finite) *formal linear combinations* of *singular k-simplices* $\sigma : \Delta_k \to X$ with coefficients $c_\sigma \in \mathcal{G}|_{\sigma(e_0)}$ where

$$\Delta_k := \left\{\sum_{i=0}^{k} a_i e_i \;\middle|\; a_i \geqslant 0 \text{ with } \sum_{i=0}^{k} a_i = 1\right\} \subset \mathbb{R}^{k+1}$$

is the convex hull of the standard basis vectors $e_0, \ldots, e_k$ of $\mathbb{R}^{k+1}$. In another word,

$$S_k(X;\mathcal{G}) \cong \bigoplus_{\sigma:\Delta_k \to X \text{ continuous}} \mathcal{G}|_{\sigma(e_0)}.$$

The boundary map

$$\partial : S_k(X;\mathcal{G}) \to S_{k-1}(X;\mathcal{G})$$

is defined by

$$\partial\left(\sum_\sigma c_\sigma \sigma\right) = \sum_\sigma \left(\big([\sigma_{01}] \cdot c_\sigma\big)\partial_0\sigma + \sum_{i=1}^{k} (-1)^i c_\sigma \partial_i \sigma\right)$$

where $\sigma_{01} : t \in [0,1] \mapsto \sigma(te_1 + (1-t)e_0)$ is the path in X from $\sigma(e_0)$ to $\sigma(e_1) = (\partial_0\sigma)(e_0)$, and the singular $(k-1)$-simplex $\partial_i\sigma$ is the standard i-th face of the singular k-simplex σ. It can be shown that $\partial \circ \partial = 0$, and hence we get the *chain complex* $(S_\cdot(X;\mathcal{G}), \partial)$ of *singular chains with local coefficients* in $\mathcal{G}$, whose k-th homology group

$$H_k(X;\mathcal{G}) := \frac{\ker(\partial : S_k(X;\mathcal{G}) \to S_{k-1}(X;\mathcal{G}))}{\partial(S_{k+1}(X;\mathcal{G}))}$$

is called the k-th *homology group* of X *with local coefficients* in $\mathcal{G}$, where it is understood that $\partial = 0$ on $S_0(X;\mathcal{G})$ by definition, i.e. every 0-chain is a 0-cycle. We remark that a *relative* version of such homology groups for pairs of spaces (X, A) with $A \subset X$ can be formulated similarly in a standard way and is denoted by $H_k(X, A; \mathcal{G})$.

Furthermore by taking the "dual"

$$S^k(X; \mathcal{G}) \cong \prod_{\sigma:\Delta_k \to X \text{ continuous}} \mathcal{G}|_{\sigma(e_0)}$$

of $S_k(X; \mathcal{G})$, which consists of functions $c \equiv \prod_\sigma c_\sigma \sigma$ sending each singular k-simplex σ to an element $c_\sigma \in \mathcal{G}|_{\sigma(e_0)}$, and the "dual" $\delta : S^k(X; \mathcal{G}) \to S^{k+1}(X; \mathcal{G})$ of ∂ as defined by

$$\delta\Bigg(\prod_{\sigma:\Delta_k \to X \text{ continuous}} c_\sigma \sigma\Bigg) = \prod_{\tau:\Delta_{k+1} \to X \text{ continuous}} \Bigg(\big([\tau_{01}]^{-1} \cdot c_{\partial_0 \tau}\big) + \sum_{i=1}^{k+1} (-1)^i c_{\partial_i \tau}\Bigg)\tau,$$

we get the *singular cochain complex* $(S^\cdot(X; \mathcal{G}), \delta)$ *with local coefficients* in $\mathcal{G}$ whose k-th homology group $H^k(X; \mathcal{G})$ is called the k-th *cohomology group* of X *with local coefficients* in $\mathcal{G}$. Similarly, the relative cohomology group $H^k(X, A; \mathcal{G})$ of (X, A) with local coefficients in $\mathcal{G}$ can be defined.

It is easy to see that $H_k(X; \mathcal{G}) = \bigoplus_j H_k(X_j; \mathcal{G}|_{X_j})$ where X_j's are the path connected components of X. Since our main result will be stated in terms of $H_0(X; \mathcal{G})$ but the homology with local coefficients is not very widely popularized, we include the following known proposition with a proof, which provides some insight to our following discussions.

Proposition 1. *For a path connected topological space X and a bundle $\mathcal{G} \xrightarrow{\pi} X$ of groups modeled on a discrete abelian group G,*

$$H_0(X; \mathcal{G}) \cong \mathcal{G}|_{x_0} / \big\langle \big\{ [\gamma] \cdot c - c \;\big|\; [\gamma] \in \pi_1(X; x_0), \text{ and } c \in \mathcal{G}|_{x_0} \big\} \big\rangle$$

for any fixed $x_0 \in X$, where $\mathcal{G}|_{x_0} \cong G$ and $\langle S \rangle$ denotes the subgroup generated by the set S.

Proof. Note that for any singular 1-simplex γ in X, viewed as a path from $x := \gamma(e_0)$ to $y := \gamma(e_1)$, and for any $c \in \mathcal{G}|_x$, we have

$$\partial(c\gamma) = \big([\gamma] \cdot c\big)y - cx.$$

So the group $\partial(S_1(X; \mathcal{G}))$ of 0-boundaries is generated by $([\gamma] \cdot c)y - cx$ in $S_0(X; \mathcal{G})$, where $x, y \in X$ and γ is a path from x to y, which implies $[([\gamma] \cdot c)y] = [cx]$ in

$$H_0(X; \mathcal{G}) = \frac{S_0(X; \mathcal{G})}{\partial(S_1(X; \mathcal{G}))}.$$

Let $x_0 \in X$ be any point fixed. Since X is path connected, for any $x \in X$, we can fix a path γ_x from x to x_0, with in particular, γ_{x_0} equal to the constant path x_0 so that $[\gamma_{x_0}] \cdot c = c$ for all $c \in \mathcal{G}|_{x_0}$. Since each $[\gamma_x] \cdot$ is a group isomorphism,

$$h : \sum_{x \in X} c_x x \in S_0(X; \mathcal{G}) \mapsto \sum_{x \in X} [\gamma_x] \cdot c_x \in \mathcal{G}|_{x_0}$$

defines a surjective group homomorphism. Note that $h(cx_0) = c$ for all $c \in \mathcal{G}|_{x_0}$.

We note that $\ker(h) \subset \partial(S_1(X;\mathcal{G}))$, because if $h(\sum_{x\in X} c_x x) = 0$, then

$$\sum_{x\in X} c_x x = \sum_{x\in X} c_x x - \left(\sum_{x\in X} [\gamma_x]\cdot c_x\right) x_0 = -\sum_{x\in X}\big(([\gamma_x]\cdot c_x)x_0 - c_x x\big)$$
$$= -\sum_{x\in X} \partial(c_x\gamma_x) = \partial\left(-\sum_{x\in X} c_x\gamma_x\right) \in \partial\big(S_1(X;\mathcal{G})\big).$$

Thus we have

$$H_0(X;\mathcal{G}) = \frac{S_0(X;\mathcal{G})}{\partial(S_1(X;\mathcal{G}))} \cong \frac{S_0(X;\mathcal{G})/\ker(h)}{\partial(S_1(X;\mathcal{G}))/\ker(h)} \cong \frac{\mathcal{G}|_{x_0}}{h(\partial(S_1(X;\mathcal{G})))}.$$

Now it remains to show that $h(\partial(S_1(X;\mathcal{G})))$ equals the subgroup $W \subset \mathcal{G}|_{x_0}$ generated by $[\gamma]\cdot c - c$ with $[\gamma] \in \pi_1(X;x_0)$ and $c \in \mathcal{G}|_{x_0}$. Clearly

$$[\gamma]\cdot c - c = h\big(([\gamma]\cdot c)x_0 - cx_0\big) = h\big(\partial(c\gamma)\big) \in h\big(\partial\big(S_1(X;\mathcal{G})\big)\big)$$

for any $[\gamma] \in \pi_1(X;x_0)$ and $c \in \mathcal{G}|_{x_0}$, and hence $W \subset h(\partial(S_1(X;\mathcal{G})))$. Conversely, for any path γ from x to y and any $c \in \mathcal{G}|_x$,

$$h\big(\partial(c\gamma)\big) = h\big(([\gamma]\cdot c)y - cx\big) = [\gamma_y]\cdot\big([\gamma]\cdot c\big) - [\gamma_x]\cdot c$$
$$= \big([\gamma_y]\circ[\gamma]\circ[\gamma_x]^{-1}\big)\cdot\big([\gamma_x]\cdot c\big) - \big([\gamma_x]\cdot c\big) = \big(\big[\gamma_x^{-1} * \gamma * \gamma_y\big]\big)\cdot\big([\gamma_x]\cdot c\big) - \big([\gamma_x]\cdot c\big)$$

where $[\gamma_x^{-1} * \gamma * \gamma_y] \in \pi_1(X;x_0)$ and $[\gamma_x]\cdot c \in \mathcal{G}|_{x_0}$, which implies $h(\partial(c\gamma)) \in W$. So $h(\partial(S_1(X;\mathcal{G}))) \subset W$. $\square$

If the bundle $\mathcal{G} \xrightarrow{\pi} X$ of groups is modeled on $G = \mathbb{Z}$, then $H_0(X;\mathcal{G})$ can be classified as follows.

Corollary 2. *For a path connected topological space X and a bundle $\mathcal{G} \xrightarrow{\pi} X$ of groups modeled on $\mathbb{Z}$,*

$$H_0(X;\mathcal{G}) \cong \begin{cases} \mathbb{Z} & \text{if } \mathcal{G} \text{ is a trivial bundle}, \\ \mathbb{Z}_2 & \text{if } \mathcal{G} \text{ is a non-trivial bundle}. \end{cases}$$

Proof. First we note that for any $x_0 \in X$, since X is path connected and $\mathcal{G} \xrightarrow{\pi} X$ is a covering map, the group $\pi_1(X;x_0)$ acts trivially on $\mathcal{G}|_{x_0}$ if and only if $\mathcal{G}$ is the trivial bundle $X \times \mathbb{Z}$. Also note that since $\mathrm{Aut}(\mathbb{Z}) = \{\pm\,\mathrm{id}\}$, for any $[\gamma] \in \pi_1(X;x_0)$, either $[\gamma]\cdot c = c$, i.e. $[\gamma]\cdot c - c = 0$, for all $c \in \mathcal{G}|_{x_0}$, or $[\gamma]\cdot c = -c$, i.e. $[\gamma]\cdot c - c = -2c$, for all $c \in \mathcal{G}|_{x_0}$.

If $\mathcal{G}$ is trivial, then $[\gamma]\cdot c - c = 0$ for all $[\gamma] \in \pi_1(X;x_0)$ and $c \in \mathcal{G}|_{x_0}$, and hence $H_0(X;\mathcal{G}) \cong \mathcal{G}|_{x_0}/\{0\} \cong \mathbb{Z}$.

If $\mathcal{G}$ is non-trivial, then there is some $[\gamma] \in \pi_1(X;x_0)$ such that $[\gamma]\cdot c - c = -2c$ for all $c \in \mathcal{G}|_{x_0}$, and hence

$$H_0(X;\mathcal{G}) \cong \mathcal{G}|_{x_0}/(-2\mathcal{G}|_{x_0}) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2. \qquad \square$$

3. Bundle of local orientations

In this paper, we consider only *real* vector bundles of *finite* rank (i.e. of a finite constant fiber dimension over $\mathbb{R}$) and with a *second countable locally compact Hausdorff* base space.

Recall that a rank-n *vector bundle* $\mathcal{E} \to X$ over a topological space X is called *orientable* if the *determinant bundle* $\bigwedge^n \mathcal{E}$ of $\mathcal{E}$ is a trivial line bundle, and a (smooth) manifold M is called orientable if its tangent bundle TM is orientable (or equivalently, its cotangent bundle $T^*M = (TM)^*$ is orientable since $(\bigwedge^n TM)^* = \bigwedge^n T^*M$). Here the rank of a vector bundle $\mathcal{E}$ is referring to the constant dimension of all fibers of $\mathcal{E}$.

In general, we call any line bundle $\mathcal{L}$ over X a *bundle of local orientations* over X, with the determinant bundle $\bigwedge^n \mathcal{E}$ of any rank-n vector bundle $\mathcal{E}$ as an example.

For any vector bundle $\mathcal{E}$ over X, we denote by $\mathcal{E}^\circ$ the bundle $\mathcal{E}$ with the zero vectors in all of its fibers removed. Note that the multiplicative group $\mathbb{R}_+$ of positive real numbers acts canonically on $\mathcal{E}^\circ$ by fiberwise multiplication, and the orbit space $S(\mathcal{E}) := \mathcal{E}^\circ/\mathbb{R}_+$ is a fiber bundle, called the *sphere bundle* associated with $\mathcal{E}$ since each fiber of $S(\mathcal{E})$ is homeomorphic to the unit $(n-1)$-sphere

$$\mathbb{S}^{n-1} \approx \big(\mathbb{R}^n \backslash \{0\}\big)/\mathbb{R}_+$$

if $\mathcal{E}$ is a rank-n vector bundle. Here $\approx$ denotes "being homeomorphic to". Clearly the multiplicative group $\{\pm 1\}$ has a canonical action on the bundle $\mathcal{E}^\circ$ by fiberwise multiplication, which commutes with the $\mathbb{R}_+$-action on $\mathcal{E}^\circ$, and hence $\{\pm 1\}$ has a well-defined canonical action on the sphere bundle $S(\mathcal{E})$.

We remark that for a Riemannian vector bundle $\mathcal{E}$ over X, i.e. a vector bundle $\mathcal{E}$ endowed with a continuous function $\langle\cdot,\cdot\rangle : \mathcal{E} \times_X \mathcal{E} \to \mathbb{R}$ which defines an inner product $\langle\cdot,\cdot\rangle_x$ on the fiber $\mathcal{E}|_x$ for each $x \in X$, the *unit sphere bundle* $\mathbb{S}(\mathcal{E}) := \bigcup_{x\in X} \mathbb{S}(\mathcal{E}|_x)$ of $\mathcal{E}$ is canonically identified with the sphere bundle $S(\mathcal{E})$, where $\mathbb{S}(E)$ denotes the unit sphere in any Euclidean space E. Furthermore, for a Riemannian vector bundle $\mathcal{E}$ over X, we have a vector bundle isomorphism

$$\mathcal{E} \cong \mathcal{E}^*$$

defined by the correspondence $v \in \mathcal{E}|_x \mapsto \langle v, \cdot\rangle \in \mathcal{E}^*|_x$. In this paper, we consider only second countable locally compact Hausdorff spaces X, which are known to be *paracompact* [Ro], and hence using a partition of unity subordinate to a locally finite open covering of X [La], we can construct such a Riemannian structure on any vector bundle $\mathcal{E}$ over X by piecing together local trivial standard Euclidean metric structures.

In the case of a line bundle $\mathcal{L}$ over X, the sphere bundle

$$\mathcal{D}_{\mathcal{L}} := S(\mathcal{L})$$

is a fiber bundle modeled on $\mathbb{S}^0 = \{\pm 1\}$, and hence with the well-defined $\{\pm 1\}$-action, $\mathcal{D}_{\mathcal{L}}$ becomes a principal $\{\pm 1\}$-bundle over X for the multiplicative group $\{\pm 1\}$. But in general, $\mathcal{D}_{\mathcal{L}}$ is *not* a bundle of groups unless it is a trivial principal $\{\pm 1\}$-bundle. Now with ± 1 acting multiplicatively on $\mathbb{Z}$ as group automorphisms, the associated fiber bundle

$$\mathcal{G}_{\mathcal{L}} := \mathcal{D}_{\mathcal{L}} \otimes_{\{\pm 1\}} \mathbb{Z}$$

is a bundle of groups over X modeled on $\mathbb{Z}$. Note that $\mathcal{D}_{\mathcal{L}}$ is a *double covering* of X, and

$$\mathcal{D}_{\mathcal{L}} \subset \mathcal{G}_{\mathcal{L}}$$

by identifying canonically each $z \in \mathcal{D}_{\mathcal{L}}$ with $[(z,1)] \in \mathcal{G}_{\mathcal{L}} = (\mathcal{D}_L \times \mathbb{Z})/\sim$ where $(z_1, m_1) \sim (z_2, m_2)$ in $\mathcal{D}_{\mathcal{L}} \times \mathbb{Z}$ if and only if $(z_2, m_2) = (z_1 g, g^{-1} m_1) = (z_1 g, g m_1)$ for some $g \in \{\pm 1\}$.

It is easy to see that if $\mathcal{D}_{\mathcal{L}}$ is a trivial double covering of X, i.e. $\mathcal{D}_{\mathcal{L}} \cong X \times \{\pm 1\}$, then $\mathcal{G}_{\mathcal{L}}$ is the trivial bundle $X \times \mathbb{Z}$ of groups over X. Conversely, if $\mathcal{G}_{\mathcal{L}}$ is the trivial bundle $X \times \mathbb{Z}$ of groups over X, then $\mathcal{D}_{\mathcal{L}} = X \times \{\pm 1\}$ is a trivial double covering of X since $\mathcal{D}_{\mathcal{L}} \subset \mathcal{G}_{\mathcal{L}}$ consists of the two generators in each fiber of $\mathcal{G}_{\mathcal{L}}$. So $\mathcal{D}_{\mathcal{L}}$ is a trivial double covering of X if and only if $\mathcal{G}_{\mathcal{L}}$ is the trivial bundle $X \times \mathbb{Z}$ of groups over X.

We also note that a line bundle $\mathcal{L}$ is trivial if and only if $\mathcal{L}$ has a nowhere vanishing cross section, or equivalently, the unit sphere bundle $\mathbb{S}(\mathcal{L})$ has a cross section when $\mathcal{L}$ is endowed with a Riemannian structure. So it is clear that $\mathcal{D}_{\mathcal{L}}$ when identified with $\mathbb{S}(\mathcal{L})$ is trivial if and only if $\mathcal{L}$ is trivial. We summarizes the above observations as follows.

Proposition 3. *For a line bundle $\mathcal{L}$ over a second countable locally compact Hausdorff space X, the following are equivalent.* (1) *$\mathcal{L}$ is a trivial bundle.* (2) *$\mathcal{D}_{\mathcal{L}} = S(\mathcal{L})$ is a trivial double covering of X.* (3) *$\mathcal{G}_{\mathcal{L}}$ is a trivial bundle of groups over X modeled on $\mathbb{Z}$.*

Recall that when applying fiberwise the natural functors of vector spaces, like $\otimes$, Hom (the space of $\mathbb{R}$-linear maps), or Isom (the space of $\mathbb{R}$-linear isomorphisms), to vector bundles $\mathcal{E}, \mathcal{F}$ over X, we can construct new fiber bundles like $\mathcal{E} \otimes \mathcal{F}$, $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, or $\mathrm{Isom}(\mathcal{E}, \mathcal{F})$ (for $\mathcal{E}, \mathcal{F}$ of the same rank). Here the group-valued functor Isom applies only to vector spaces of the same fixed dimension. Note that $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is a vector bundle over X whose fiber at each $x \in X$ consists of linear maps from $\mathcal{E}|_x$ to $\mathcal{F}|_x$, and one should *not confuse* $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ with the space of vector bundle homomorphisms from the vector bundle $\mathcal{E}$ to the vector bundle $\mathcal{F}$.

For finite-dimensional vector spaces V and W over $\mathbb{R}$, there is a *natural isomorphism* $\mathrm{Hom}(V, W) \cong V^* \otimes W$, which gives rise to, when applied fiberwise, a vector bundle isomorphism

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F}$$

for vector bundles $\mathcal{E}, \mathcal{F}$ over X. Similarly, the natural homomorphism

$$\bigwedge^n : f \in \mathrm{Hom}(V, W) \mapsto \bigwedge^n f \in \mathrm{Hom}\bigl(\bigwedge^n V, \bigwedge^n W\bigr)$$

gives rise to a vector bundle homomorphism

$$\bigwedge^n : \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}\bigl(\bigwedge^n \mathcal{E}, \bigwedge^n \mathcal{F}\bigr).$$

For a rank-n vector space V and $f \in \mathrm{Hom}(V, V)$, we have $\bigwedge^n V \cong \mathbb{R}$ and the map $\bigwedge^n f : \bigwedge^n V \to \bigwedge^n V$ is the multiplication by the well-defined determinant $\det(f) \in \mathbb{R}$ (independent of the choice of a basis for V), i.e. we have a natural surjective (multiplicative but not linear) map

$$\bigwedge^n : f \in \mathrm{Hom}(V, V) \mapsto \det(f) \in \mathbb{R} \cong \mathrm{Hom}\bigl(\bigwedge^n V, \bigwedge^n V\bigr).$$

So we adopt the notations

$$\det\bigl(\mathrm{Hom}(V, W)\bigr) := \mathrm{Hom}\bigl(\bigwedge^n V, \bigwedge^n W\bigr)$$

and

$$\det\bigl(\mathrm{Isom}(V, W)\bigr) := \mathrm{Isom}\bigl(\bigwedge^n V, \bigwedge^n W\bigr)$$

for vector spaces V, W of the same dimension n. With these notations, we have *surjective* fiber bundle maps

$$\bigwedge^n : \mathrm{Hom}(\mathcal{E},\mathcal{F}) \to \det\bigl(\mathrm{Hom}(\mathcal{E},\mathcal{F})\bigr) := \mathrm{Hom}\bigl(\bigwedge^n\mathcal{E}, \bigwedge^n\mathcal{F}\bigr)$$

and

$$\bigwedge^n : \mathrm{Isom}(\mathcal{E},\mathcal{F}) \to \det\bigl(\mathrm{Isom}(\mathcal{E},\mathcal{F})\bigr) := \mathrm{Isom}\bigl(\bigwedge^n\mathcal{E}, \bigwedge^n\mathcal{F}\bigr)$$

for vector bundles $\mathcal{E}$ and $\mathcal{F}$ of the same dimension n over X. Note that when $\mathcal{E}$ and $\mathcal{F}$ are rank-n vector bundles,

$$\mathrm{Isom}\bigl(\bigwedge^n\mathcal{E}, \bigwedge^n\mathcal{F}\bigr) = \bigl(\mathrm{Hom}\bigl(\bigwedge^n\mathcal{E}, \bigwedge^n\mathcal{F}\bigr)\bigr)^\circ$$

since $\bigwedge^n\mathcal{E}, \bigwedge^n\mathcal{F}$ are line bundles.

We point out that there is a natural isomorphism

$$w \wedge e_1 \wedge \cdots \wedge e_{n-1} \in \bigwedge^n\bigl(W \oplus \mathbb{R}^{n-1}\bigr) \overset{\cong}{\longmapsto} w \in W$$

with respect to 1-dimensional vector spaces W, and hence for any line bundle $\mathcal{L}$ over X, we have a natural isomorphism

$$\bigwedge^n\bigl(\mathcal{L} \oplus \mathcal{R}^{n-1}\bigr) \cong \mathcal{L}$$

with respect to line bundles $\mathcal{L}$ over X, where $\mathcal{R}^{n-1}$ represents the trivial vector bundle $X \times \mathbb{R}^{n-1}$ over X.

Now for a line bundle $\mathcal{L}$ over an n-dimensional manifold M with the cotangent determinant bundle $\mathcal{K} := \bigwedge^n T^*M$, we consider the bundle $\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}$ of groups modeled on $\mathbb{Z}$, that arises from the line bundle

$$\mathcal{K}^* \otimes \mathcal{L} \cong \mathrm{Hom}(\mathcal{K},\mathcal{L}) \cong \det\bigl(\mathrm{Hom}\bigl(T^*X, \mathcal{L} \oplus \mathcal{R}^{n-1}\bigr)\bigr).$$

Proposition 4. *The bundle $\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}$ of groups modeled on $\mathbb{Z}$ is a trivial bundle if and only if $\mathcal{K} \cong \mathcal{L}$.*

Proof. Since $\mathcal{K} \otimes \mathcal{K}^* \cong \mathrm{Hom}(\mathcal{K},\mathcal{K}) = \mathcal{R}$, we have that $\mathcal{K}^* \otimes \mathcal{L} \cong \mathcal{R}$ if and only if $\mathcal{L} \cong \mathcal{K} \otimes \mathcal{K}^* \otimes \mathcal{L} \cong \mathcal{K} \otimes \mathcal{R} \cong \mathcal{K}$. $\square$

We note that $\mathcal{D}_\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}^*}$, and hence $\mathcal{G}_\mathcal{L}$ and $\mathcal{G}_{\mathcal{L}^*}$, are *naturally* isomorphic for line bundles $\mathcal{L}$. Indeed

$$[v] \in \mathcal{L}^\circ|_x/\mathbb{R}_+ = \mathcal{D}_\mathcal{L}|_x \mapsto \bigl[v^*\bigr] \in \bigl(\mathcal{L}^*\bigr)^\circ\big|_x/\mathbb{R}_+ = \mathcal{D}_{\mathcal{L}^*}|_x$$

where for $v \in \mathcal{L}^\circ|_x$, $v^* \in (\mathcal{L}^*)^\circ|_x$ with $v^*(v) \in \mathbb{R}_+$, say, $v^*(v) = 1$, *well defines* a natural isomorphism $\mathcal{D}_\mathcal{L} \to \mathcal{D}_{\mathcal{L}^*}$. So in particular, $\mathcal{D}_\mathcal{K} = \mathcal{D}_{\bigwedge^n T^*M}$ and $\mathcal{D}_{\mathcal{K}^*} = \mathcal{D}_{\bigwedge^n TM}$ are naturally isomorphic for n-dimensional manifolds M, and they can used interchangeably, e.g. in the later discussion of twisted bundle of groups and Poincaré–Steenrod duality. We define for $[v] \in \mathcal{D}_\mathcal{L}|_x$, the dual

$$[v]^* := \bigl[v^*\bigr] \in \mathcal{D}_{\mathcal{L}^*}|_x.$$

4. Bundle of homotopy groups

In this section, we recall the bundle of $(n-1)$-homotopy groups for a rank-n vector bundle $\mathcal{E} \xrightarrow{\pi} M$ over an n-dimensional manifold M with $n > 1$.

We denote by $\mathcal{P}_{n-1}(\mathcal{E}^\circ)$ the disjoint union of the $(n-1)$-homotopy groups $\pi_{n-1}(\mathcal{E}^\circ|_x)$ with $x \in M$, and describe it in the following proposition as a bundle of groups modeled on $\mathbb{Z}$. An open set $U \subset M$ is called *simple* if it is homeomorphic to the open unit n-ball $\mathbb{B}^n$.

Proposition 5. *For an n-dimensional manifold M with n* > 1*, the disjoint union*

$$\mathcal{P}_{n-1}(\mathcal{E}^\circ) = \bigcup_{x \in M} \pi_{n-1}(\mathcal{E}^\circ|_x)$$

of the $(n-1)$*-homotopy groups* $\pi_{n-1}(\mathcal{E}^\circ|_x)$ *with* $x \in M$ *is a bundle of groups modeled on* $\mathbb{Z}$ *with the canonical homotopy group structure on each fiber* $\pi_{n-1}(\mathcal{E}^\circ|_x)$ *and with the topology determined by the local trivializations*

$$\pi_{n-1}(\iota_{U,\cdot}) : \bigcup_{x \in U} \pi_{n-1}(\mathcal{E}^\circ|_x) \to U \times \pi_{n-1}(\mathcal{E}^\circ|_U) \cong U \times \mathbb{Z}$$

over simple open sets $U \subset M$*, defined on each fiber* $\pi_{n-1}(\mathcal{E}^\circ|_x)$ *by*

$$\pi_{n-1}(\iota_{U,\cdot}) : h \in \pi_{n-1}(\mathcal{E}^\circ|_x) \mapsto (x, \pi_{n-1}(\iota_{U,x})(h)) \in \{x\} \times \pi_{n-1}(\mathcal{E}^\circ|_U)$$

where $\pi_{n-1}(\iota_{U,x})$ *is the isomorphism induced by the inclusion map*

$$\iota_{U,x} : \mathcal{E}^\circ|_x \hookrightarrow \mathcal{E}^\circ|_U.$$

Proof. Note that since any simple neighborhood U of a point $x \in X$ is contractible and $\mathcal{E}$ trivializes over it, i.e. $\mathcal{E}^\circ|_U \cong U \times (\mathbb{R}^{n-1} \backslash \{0\})$, the inclusion map $\iota_{U,x}$ induces a group isomorphism

$$\pi_{n-1}(\mathcal{E}^\circ|_x) \xrightarrow{\pi_{n-1}(\iota_{U,x})} \pi_{n-1}(\mathcal{E}^\circ|_U) \cong \pi_{n-1}(\mathbb{R}^{n-1} \backslash \{0\}) \cong \pi_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$$

because $\mathcal{E}^\circ|_x$ is a deformation retract of $\mathcal{E}^\circ|_U$ and is homeomorphic to $\mathbb{R}^{n-1} \backslash \{0\}$. So $\pi_{n-1}(\iota_{U,\cdot})$ is a well-defined bijection which is fiberwise a group isomorphism.

It remains to show that the transition maps between overlapping trivializations are continuous and fiberwise group automorphisms of $\mathbb{Z}$. Let U, V be simple open sets of X with $U \cap V \neq \emptyset$. It is clear that the transition map

$$\pi_{n-1}(\iota_{V,\cdot}) \circ \pi_{n-1}(\iota_{U,\cdot})^{-1}\big|_{(U \cap V) \times \pi_{n-1}(\mathcal{E}^\circ|_U)}$$

is fiberwise a group automorphism since $\pi_{n-1}(\iota_{U,x})$ and $\pi_{n-1}(\iota_{V,x})$ are group isomorphisms for each $x \in U \cap V$. On the other hand, for any contractible open set $W \subset U \cap V$, the inclusions $\iota_{U,x}$ and $\iota_{V,x}$ factor through the same inclusion $\iota_{W,x}$ for all $x \in W$, and the inclusions $\iota_{U,W} : \mathcal{E}^\circ|_W \hookrightarrow \mathcal{E}^\circ|_U$ and $\iota_{V,W} : \mathcal{E}^\circ|_W \hookrightarrow \mathcal{E}^\circ|_V$ induce isomorphisms $\pi_{n-1}(\iota_{U,W})$ and $\pi_{n-1}(\iota_{V,W})$, respectively. So

$$\pi_{n-1}(\iota_{V,\cdot}) \circ \pi_{n-1}(\iota_{U,\cdot})^{-1}\big|_{W \times \pi_{n-1}(\mathcal{E}^\circ|_U)} = \mathrm{id}_W \times \left(\pi_{n-1}(\iota_{V,W}) \circ \pi_{n-1}(\iota_{U,W})^{-1}\right)$$

which is clearly a continuous function from $W \times \pi_{n-1}(\mathcal{E}^\circ|_U) \cong W \times \mathbb{Z}$ to $W \times \pi_{n-1}(\mathcal{E}^\circ|_V) \cong W \times \mathbb{Z}$ for any contractible open set $W \subset U \cap V$. Thus $\pi_{n-1}(\iota_{V,\cdot}) \circ \pi_{n-1}(\iota_{U,\cdot})^{-1}$ restricted to $(U \cap V) \times \pi_{n-1}(\mathcal{E}^\circ|_U)$ is continuous. $\square$

Proposition 6. *For a rank-n vector bundle $\mathcal{E}$ over an n-dimensional manifold M with $n > 1$,*

$$\mathcal{G}_{\bigwedge^n \mathcal{E}} \cong \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big).$$

Proof. Let $\mathcal{L} := \bigwedge^n \mathcal{E}$. First we note that every non-zero element of the line $\mathcal{L}|_x$ is of the form $\bigwedge^n B := v_0 \wedge \cdots \wedge v_{n-1}$ for some ordered basis $B = (v_0, \ldots, v_{n-1})$ of $\mathcal{E}|_x$, which defines $f_B \in \mathrm{Isom}(\mathbb{R}^n, \mathcal{E}|_x)$ by $f_B(e_i) := v_i$ for $0 \leqslant i \leqslant n-1$. Clearly

$$f : B \mapsto f_B \in \mathrm{Isom}\big(\mathbb{R}^n, \mathcal{E}|_x\big)$$

defines a one-to-one correspondence between ordered basis of $\mathcal{E}|_x$ and linear isomorphisms from $\mathbb{R}^n$ to $\mathcal{E}|_x$. We claim that f induces a well-defined *injective* map

$$\phi : \big[\textstyle\bigwedge^n B\big] \in \mathcal{D}_{\mathcal{L}}|_x \mapsto [f_B|_{\mathbb{S}^{n-1}}] \in \pi_{n-1}\big(\mathcal{E}^\circ|_x\big) \cong \mathbb{Z}$$

onto $\{1, -1\} \subset \mathbb{Z}$.

For any ordered bases B, B' of $\mathcal{E}|_x$, it is easy to see that $[\bigwedge^n B] = [\bigwedge^n B']$ in $\mathcal{D}_{\mathcal{L}}|_x = S(\mathcal{L}|_x)$ if and only if $\det(\tau_{B',B}) > 0$, i.e. $\tau_{B',B}$ is path connected to id in $\mathrm{Isom}(\mathcal{E}|_x, \mathcal{E}|_x)$, where $\tau_{B',B} \in \mathrm{Isom}(\mathcal{E}|_x, \mathcal{E}|_x)$ with $\tau_{B',B}(v_i) = v_i'$ for each i. Thus if $[\bigwedge^n B] = [\bigwedge^n B']$ in $\mathcal{D}_{\mathcal{L}}|_x$, then f_B and $f_{B'} = \tau_{B',B} \circ f_B$ are path connected in $\mathrm{Isom}(\mathbb{R}^n, \mathcal{E}|_x)$ and hence $[f_{B'}|_{\mathbb{S}^{n-1}}] = [f_B|_{\mathbb{S}^{n-1}}]$ in $\pi_{n-1}(\mathcal{E}^\circ|_x)$. So ϕ is well defined.

By fixing an ordered basis $C = (u_0, \ldots, u_{n-1})$ of $\mathcal{E}|_x$, we can identify $\mathcal{E}|_x$ with $\mathbb{R}^n$, and get $f_C = \mathrm{id}_{\mathbb{R}^n}$ under this identification. Clearly

$$[f_C|_{\mathbb{S}^{n-1}}] = 1 \in \mathbb{Z} \cong \pi_{n-1}\big(\mathbb{S}^{n-1}\big) \cong \pi_{n-1}\big(\big(\mathbb{R}^n\big)^\circ\big|_x\big) \cong \pi_{n-1}\big(\mathcal{E}^\circ|_x\big).$$

On the other hand, setting $B^- := (v_1, v_0, v_2, \ldots, v_{n-1})$ for any ordered basis $B = (v_0, \ldots, v_{n-1})$ of $\mathcal{E}|_x$, we have

$$\mathcal{D}_{\mathcal{L}}|_x = \big\{\textstyle\bigwedge^n B, \bigwedge^n B^-\big\}$$

since $\bigwedge^n (v_1, v_0, v_2, \ldots, v_{n-1}) = -\bigwedge^n (v_0, \ldots, v_{n-1})$. Since clearly $f_{B^-} = f_B \circ \rho$ for the reflection

$$\rho : (x_0, x_1, x_2, \ldots, x_{n-1}) \mapsto (x_1, x_0, x_2, \ldots, x_{n-1}),$$

we have, as a well-known property of the degree of maps on $\mathbb{S}^{n-1}$, that

$$-[f_B|_{\mathbb{S}^{n-1}}] = [f_B|_{\mathbb{S}^{n-1}} \circ \rho|_{\mathbb{S}^{n-1}}] = [f_{B^-}|_{\mathbb{S}^{n-1}}]$$

in $\pi_{n-1}(\mathcal{E}^\circ|_x)$. Thus $\phi([\bigwedge^n B^-]) = -\phi([\bigwedge^n B])$, and in particular, $\phi([\bigwedge^n C^-]) = -1 \in \mathbb{Z} \cong \pi_{n-1}(\mathbb{S}^{n-1})$. So ϕ is injective and onto $\{1, -1\}$.

Since the above definition of ϕ at each $x \in X$ is canonical, i.e. local trivialization free, it is easy to see that

$$\phi : \mathcal{D}_{\mathcal{L}} \to \bigcup_{x \in M} \pi_{n-1}\big(\mathcal{E}^\circ|_x\big) = \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)$$

is a well-defined injective (continuous) fiber bundle map onto $\mathcal{S} \subset \mathcal{P}_{n-1}(\mathcal{E}^\circ)$, where $\mathcal{S}$ consists of the generators ± 1 in each fiber of $\mathcal{P}_{n-1}(\mathcal{E}^\circ)$. On the other hand, since $\mathcal{G}_{\mathcal{L}} = \mathcal{D}_{\mathcal{L}} \otimes_{\{\pm 1\}} \mathbb{Z}$ and $\mathcal{P}_{n-1}(\mathcal{E}^\circ) = \mathcal{S} \otimes_{\{\pm 1\}} \mathbb{Z}$, it is easy to see that ϕ extends to a well-defined bundle isomorphism from $\mathcal{G}_{\mathcal{L}}$ to $\mathcal{P}_{n-1}(\mathcal{E}^\circ)$. $\square$

Note that for a Riemannian vector bundle $\mathcal{E}$ over X, the unit sphere bundle $\mathbb{S}(\mathcal{E})$ is a deformation retract of $\mathcal{E}^\circ$ via deformations preserving each fiber, and hence $\mathcal{P}_{n-1}(\mathcal{E}^\circ)$ can be canonically identified with the bundle

$$\mathcal{P}_{n-1}\big(\mathbb{S}(\mathcal{E})\big) := \bigcup_{x\in X} \pi_{n-1}\big(\mathbb{S}(\mathcal{E})|_x\big).$$

Corollary 7. *For a rank-n Riemannian vector bundle $\mathcal{E}$ over an n-dimensional manifold M with $n > 1$,*

$$\mathcal{G}_{\bigwedge^n \mathcal{E}} \cong \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big) \cong \mathcal{P}_{n-1}\big(\mathbb{S}(\mathcal{E})\big).$$

Given simple open neighborhoods U, V of x in an n-dimensional manifold M with $U \subset V$, the inclusion map induces an isomorphism $\pi_{n-1}(U\backslash\{x\}) \to \pi_{n-1}(V\backslash\{x\}) \cong \mathbb{Z}$, whose inverse is denoted by

$$r_{U,V} : \pi_{n-1}\big(V\backslash\{x\}\big) \to \pi_{n-1}\big(U\backslash\{x\}\big) \cong \mathbb{Z}.$$

Note that the set of *simple* open neighborhoods U of x in M is a directed set partially ordered by $\supset$ and $r_{U,V} \circ r_{V,W} = r_{U,W}$ for any *simple* open neighborhoods $U \subset V \subset W$ of x. So we have a directed system $\{\pi_{n-1}(U\backslash\{x\}), r_{U,V}\}$, and a well-defined direct limit

$$\mathcal{Q}_x := \varinjlim_{U\searrow\{x\}} \big(\pi_{n-1}\big(U\backslash\{x\}\big)\big) \cong \mathbb{Z}$$

in which every continuous map $\sigma : \mathbb{S}^{n-1} \to U\backslash\{x\} \subset M$ with U a simple open neighborhood of x determines a unique "germ" of $[\sigma] \in \pi_{n-1}(U\backslash\{x\})$, denoted by

$$[[\sigma]] \in \mathcal{Q}_x.$$

Since T_xM is intuitively the linearization of M near x, we can informally view a small open neighborhood U of x as an open neighborhood of 0 in T_xM and hence any continuous map $\sigma : \mathbb{S}^{n-1} \to U\backslash\{x\}$ can be viewed as $\sigma : \mathbb{S}^{n-1} \to (T_xM)^\circ$ which gives rise to an element of $\pi_{n-1}((T_xM)^\circ) = \mathcal{P}_{n-1}((TM)^\circ)|_x$. This informal view indeed gives rise to a canonical identification of $\mathcal{Q}_x$ with $\mathcal{P}_{n-1}((TM)^\circ)|_x$.

Proposition 8. *Let $\mathcal{K} := \bigwedge^n T^*M$ for an n-dimensional manifold M with $n > 1$. There is a canonical group isomorphism defined as*

$$[[\sigma]] \in \mathcal{Q}_x \overset{\cong}{\longmapsto} \big[(d\phi|_x)^{-1} \circ \phi \circ \sigma\big] \in \mathcal{P}_{n-1}\big((TM)^\circ\big)\big|_x \cong \mathcal{G}_{\mathcal{K}^*}|_x$$

where $\phi : U \twoheadrightarrow U' \subset \mathbb{R}^n$ is a chart map of a simple open neighborhood $U \subset M$ of x with $\phi(x) = 0$ and $\mathbb{R}^n$ identified with T_xM via the total derivative $d\phi|_x : T_xM \overset{\cong}{\to} T_0U' = \mathbb{R}^n$, and $\sigma : \mathbb{S}^{n-1} \to U\backslash\{x\} \subset M$ is a continuous map.

Proof. Note that the homeomorphic map $(d\phi|_x)^{-1} \circ \phi$ induces an isomorphism

$$[\sigma] \in \pi_{n-1}\big(U\backslash\{x\}\big) \mapsto \big[(d\phi|_x)^{-1} \circ \phi \circ \sigma\big] \in \pi_{n-1}\big((d\phi|_x)^{-1}\big(U'\backslash\{0\}\big)\big)$$

and $\mathbb{Z} \cong \pi_{n-1}((d\phi|_x)^{-1}(U'\backslash\{0\})) \cong \pi_{n-1}((T_xM)^\circ)$. It is then easy to see that for a fixed chart map $\phi : U \twoheadrightarrow U' \subset \mathbb{R}^n$, we get a well-defined isomorphism

$$[[\sigma]] \in \mathcal{Q}_x \overset{\cong}{\mapsto} \left[(d\phi|_x)^{-1} \circ \phi \circ \sigma\right] \in \pi_{n-1}\left((T_xM)^\circ\right).$$

Next we show that $[(d\phi|_x)^{-1} \circ \phi \circ \sigma] \in \pi_{n-1}((T_xM)^\circ)$ is independent of the choice of the chart map ϕ on U. Without loss of generality, we may assume $(\phi \circ \sigma)(\mathbb{S}^{n-1})$ as close to 0 as we need, by picking a suitable representative of $[[\sigma]] \in \mathcal{Q}_x$.

Let $\psi : U \to U'' \subset \mathbb{R}^n$ be another chart map with $\psi(x) = 0$. Showing that

$$\left[(d\phi|_x)^{-1} \circ \phi \circ \sigma\right] = \left[(d\psi|_x)^{-1} \circ \psi \circ \sigma\right] \in \pi_{n-1}\left((T_xM)^\circ\right)$$

is equivalent to showing that

$$\left[d\left(\psi \circ \phi^{-1}\right)\big|_0 \circ \phi \circ \sigma\right] = \left[\left(\psi \circ \phi^{-1}\right) \circ \phi \circ \sigma\right] \in \pi_{n-1}\left(\left(\mathbb{R}^n\right)^\circ\right).$$

Note that $d(\psi \circ \phi^{-1})|_0$ is a linear approximation to $\psi \circ \phi^{-1}$ near $0 = (\psi \circ \phi^{-1})(0)$, or more precisely,

$$d\left(\psi \circ \phi^{-1}\right)\big|_0(v) - \left(\psi \circ \phi^{-1}\right)(v) = O\left(\|v\|^2\right)$$

for $v \in U' \subset \mathbb{R}^n$. Since $d(\psi \circ \phi^{-1})|_0$ is invertible and hence there is $\delta > 0$ such that $\|d(\psi \circ \phi^{-1})|_0(v)\| \geqslant \delta\|v\|$ for all $v \in \mathbb{R}^n$, we can deform $(d(\psi \circ \phi^{-1})|_0)|_{(\phi\circ\sigma)(\mathbb{S}^{n-1})}$ to $(\psi \circ \phi^{-1})|_{(\phi\circ\sigma)(\mathbb{S}^{n-1})}$ inside $(\mathbb{R}^n)^\circ$ by linear interpolation

$$\begin{aligned} t \in [0,1] &\mapsto \left[(1-t)\left(d\left(\psi \circ \phi^{-1}\right)\big|_0\right) + t\left(\psi \circ \phi^{-1}\right)\right]\big|_{(\phi\circ\sigma)(\mathbb{S}^{n-1})} \\ &= \left\{\left(d\left(\psi \circ \phi^{-1}\right)\big|_0\right) - t\left[d\left(\psi \circ \phi^{-1}\right)\big|_0 - \left(\psi \circ \phi^{-1}\right)\right]\right\}\big|_{(\phi\circ\sigma)(\mathbb{S}^{n-1})} \end{aligned}$$

when $(\phi \circ \sigma)(\mathbb{S}^{n-1}) \subset (\mathbb{R}^n)^\circ$ is sufficiently close to 0. Thus we get $d(\psi \circ \phi^{-1})|_0 \circ \phi \circ \sigma|_{\mathbb{S}^{n-1}}$ homotopic to $(\psi \circ \phi^{-1}) \circ \phi \circ \sigma|_{\mathbb{S}^{n-1}}$ inside $(\mathbb{R}^n)^\circ$.

Finally we note that since two chart maps $U \twoheadrightarrow U'$ and $V \twoheadrightarrow V'$ on simple open neighborhoods U and V of x can restrict to chart maps on a smaller common simple open neighborhood W of x, it is easy to see that any chart map $\phi : U \twoheadrightarrow U' \subset \mathbb{R}^n$ on any simple open neighborhood U of x gives rise to the same isomorphism as defined above. □

From now on, we often view $[[\sigma]] \in \mathcal{Q}_x$ as an element of $\mathcal{G}_{\mathcal{K}^*}|_x \cong \mathcal{P}_{n-1}((TM)^\circ)|_x$ via the canonical isomorphism in Proposition 8.

Proposition 9. *If $\sigma : \overline{\mathbb{B}^n} \to M$ is a continuous injective map with $x \in \sigma(\mathbb{B}^n)$ in an n-dimensional manifold M for $n > 1$, then*

$$[[\sigma|_{\mathbb{S}^{n-1}}]] \in \mathcal{D}_{\mathcal{K}^*}|_x \quad \textit{and} \quad [[\sigma|_{\mathbb{S}^{n-1}}]]^* \in \mathcal{D}_{\mathcal{K}}|_x.$$

Proof. Clearly $[\mathrm{id}_{\mathbb{S}^{n-1}}] = 1$ in $\pi_{n-1}(\overline{\mathbb{B}^n}\backslash\{\sigma^{-1}(x)\}) \cong \mathbb{Z}$, and hence $[\sigma|_{\mathbb{S}^{n-1}}] = \pi_{n-1}(\sigma)([\mathrm{id}_{\mathbb{S}^{n-1}}])$ is a generator of $\pi_{n-1}(\sigma(\overline{\mathbb{B}^n})\backslash\{x\}) \cong \mathbb{Z}$. Thus $[[\sigma|_{\mathbb{S}^{n-1}}]]$ is a *generator* of $\mathbb{Z} \cong \mathcal{Q}_x \equiv \mathcal{G}_{\mathcal{K}^*}|_x$, i.e. $[[\sigma|_{\mathbb{S}^{n-1}}]] \in \mathcal{D}_{\mathcal{K}^*}|_x$. □

5. Whitney class

In this section, we revisit the notion of (co)homology with local coefficients in the context of simplicial complex, which is the framework used in the theory of obstruction involving the Whitney characteristic class. Then we derive some simple consequences of the obstruction theory needed later.

We recall the simplicial homological version of homology with local coefficients that was initially used by Steenrod [St1]. Here we first present it in the more flexible context of CW-complexes instead of simplicial or cell complexes, as done in [Wh2].

Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ be the skeletons of a topological space X with respect to a fixed CW-complex structure on X, and following the convention of [Wh2], we use

$$\sigma : (\Delta_k, \partial\Delta_k) \to (X_k, X_{k-1})$$

to denote a typical k-cell (characteristic map) for this CW-complex structure, instead of $\sigma : (\overline{\mathbb{B}^k}, \mathbb{S}^{k-1}) \to (X_k, X_{k-1})$. Note that the interiors $\sigma(\Delta_k \backslash \partial\Delta_k)$ of all cells σ of all possible dimensions k are pairwise disjoint, and in particular, no part of X is represented as images of two cells with different "orientations". We remark that in the discussion of CW-complexes, Δ_k is often implicitly identified with the unit k-ball $\overline{\mathbb{B}^k}$ in a fixed way that preserves the standard orientation, and identifies $e_0 \in \Delta_k \subset \mathbb{R}^{k+1}$ with $e_0 \in \overline{\mathbb{B}^k} \subset \mathbb{R}^k$ and $\partial\Delta_k$ with $\mathbb{S}^{k-1}$.

Let $\mathcal{G} \xrightarrow{\pi} X$ be a bundle of groups modeled on an abelian group G. Then as in the theory of homology groups with constant coefficient group, we have the relative singular homology group of (X_k, X_{k-1}) with local coefficients in $\mathcal{G}|_{X_k}$ decomposed as

$$H_k(X_k, X_{k-1}; \mathcal{G}|_{X_k}) \cong \bigoplus_{\sigma \text{ a } k\text{-cell}} H_k(\Delta_k, \partial\Delta_k; \mathcal{G}|_{\sigma(e_0)}) \cong \bigoplus_{\sigma \text{ a } k\text{-cell}} \mathcal{G}|_{\sigma(e_0)},$$

and a chain complex

$$\cdots \to C_k(X; \mathcal{G}) \xrightarrow{\partial} C_{k-1}(X; \mathcal{G}) \to \cdots$$

can be constructed by taking

$$C_k(X; \mathcal{G}) := H_k(X_k, X_{k-1}; \mathcal{G}|_{X_k}) \cong \bigoplus_{\sigma \text{ a } k\text{-cell}} \mathcal{G}|_{\sigma(e_0)}$$

and defining ∂, as in the singular case, by

$$\partial\left(\sum_{\sigma \text{ a } k\text{-cell}} c_\sigma \sigma\right) = \left[\sum_{\sigma \text{ a } k\text{-cell}} \left(\left([\sigma_{01}] \cdot c_\sigma\right)\partial_0\sigma + \sum_{i=1}^{k} (-1)^i c_\sigma \partial_i \sigma\right)\right]$$

where $c_\sigma \in \mathcal{G}|_{\sigma(e_0)}$ and $\sum_{\sigma \text{ a } k\text{-cell}} c_\sigma \sigma$ represents the element $\bigoplus_{\sigma \text{ a } k\text{-cell}} c_\sigma \in \bigoplus_{\sigma \text{ a } k\text{-cell}} \mathcal{G}|_{\sigma(e_0)}$. However unless the CW complex structure is actually a simplicial or cell complex structure, we note that on the right-hand side of the equality, the faces $\partial_i\sigma$ themselves may not be $(k-1)$-cells of the CW complex, but the full sum is a $(k-1)$-cycle of $S_\cdot(X_{k-1}, X_{k-2}; \mathcal{G}|_{X_{k-1}})$ and can be rewritten as a homologous sum of $(k-1)$-cells with coefficients in $\mathcal{G}$. It can be shown [Wh2] that the k-th homology group of the chain complex $(C_\cdot(X; \mathcal{G}), \partial)$ is isomorphic to $H_k(X; \mathcal{G})$. So the homology of $(C_\cdot(X; \mathcal{G}), \partial)$ is independent of the choice of the CW-complex decomposition of X.

Similarly the relative singular cohomology group of (X_k, X_{k-1}) with local coefficients in $\mathcal{G}|_{X_k}$ decomposes as

$$H^k(X_k, X_{k-1}; \mathcal{G}|_{X_k}) \cong \prod_{\sigma \text{ a } k\text{-cell}} H^k(\Delta_k, \partial\Delta_k; \mathcal{G}|_{\sigma(e_0)}) \cong \prod_{\sigma \text{ a } k\text{-cell}} \mathcal{G}|_{\sigma(e_0)},$$

and a cochain complex

$$\cdots \to C^k(X; \mathcal{G}) \xrightarrow{\delta} C^{k+1}(X; \mathcal{G}) \to \cdots$$

can be constructed by taking

$$C^k(X; \mathcal{G}) := H^k(X_k, X_{k-1}; \mathcal{G}|_{X_k}) \cong \prod_{\sigma \text{ a } k\text{-cell}} \mathcal{G}|_{\sigma(e_0)}$$

and defining δ, as in the singular case, by

$$\delta\left(\prod_{\sigma \text{ a } k\text{-cell}} c_\sigma \sigma\right) = \left[\prod_{\tau \text{ a } (k+1)\text{-cell}} \left(\left([\tau_{01}]^{-1} \cdot c_{\partial_0 \tau}\right) + \sum_{i=1}^{k+1} (-1)^i c_{\partial_i \tau}\right) \tau\right],$$

where $\prod_{\sigma \text{ a } k\text{-cell}} c_\sigma \sigma$ represents the function sending σ to $c_\sigma \in \mathcal{G}|_{\sigma(e_0)}$. However similar to the homology case, unless the CW complex structure is a simplicial or cell complex structure, $\partial_i \tau$ themselves may not be k-cells of the CW complex for a $(k+1)$-cell τ, and hence the right-hand side has to be suitably interpreted. It can be shown [Wh2] that the k-th homology group of the cochain complex $(C^{\cdot}(X; \mathcal{G}), \delta)$ is isomorphic to $H^k(X; \mathcal{G})$. So the homology of $(C^{\cdot}(X; \mathcal{G}), \delta)$ is also independent of the choice of the CW-complex decomposition of X. We remark that $[\sigma \circ \rho] = -[\sigma]$ in $H_k(X_k, X_{k-1}; \mathbb{Z})$ when ρ is a reflection on $\mathbb{R}^{k+1}$ (preserving Δ_k), and hence if we replace a cell σ in a CW-complex structure by the “same” cell with the *opposite orientation*, then the coefficient c_σ in a homology or cohomology class changes its $\pm$-sign.

Let us now recall the top Whitney class $w_n(\mathcal{E})$ of a vector bundle $\mathcal{E} \to M$ over an n-dimensional manifold M as introduced originally by Whitney in [Wn] and then formalized by Steenrod [St1] in the context of cohomology with local coefficients. For the convenience of discussion, we endow $\mathcal{E}$ continuously with fiberwise inner products and hence the sphere bundle $S(\mathcal{E})$ can be identified with the unit sphere bundle $\mathbb{S}(\mathcal{E})$.

We first recall a statement from [Wn] in the following form and provide a detailed proof.

Proposition 10. *Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ be the skeletons of a CW complex X and let $\mathcal{E} \to X$ be a rank-N Riemannian vector bundle with $N \geqslant n$. For any* (*continuous*) *cross section $s_{n-1} : X_{n-1} \to \mathbb{S}(\mathcal{E})$ which exists, there are* (*continuous*) *cross sections $s_i : X_i \to \mathbb{S}(\mathcal{E})$ over the i-skeleton $X_i \subset X$ for all $0 \leqslant i < n-1$, such that $\{s_i(x)\colon k \leqslant i \leqslant n-1\}$ is an orthonormal set in $\mathbb{S}(\mathcal{E}|_x)$ for all $x \in X_k$ for any $0 \leqslant k \leqslant n-1$.*

Proof. We prove by induction on n. When $n = 1$, the statement is clearly true since X_0 is a discrete set of points in X. As an induction hypothesis, we assume that for any $(N-1)$-dimensional inner product vector bundle $\mathcal{E}' \to X_{n-1}$, such cross sections $s_i : X_i \to \mathbb{S}(\mathcal{E}')$ for $0 \leqslant i \leqslant n-2$ exist.

By Theorem 1.2 of Chapter 8 of [Hu], $\mathcal{E}$ restricted to X_{n-1} contains the trivial 1-dimensional subbundle $\mathcal{R} = X_{n-1} \times \mathbb{R}$ over X_{n-1}, and hence there is a cross section $s_{n-1} : X_{n-1} \to \mathbb{S}(\mathcal{R}) \subset \mathbb{S}(\mathcal{E})$. On the other hand, any such a cross section $s_{n-1} : X_{n-1} \to \mathbb{S}(\mathcal{E})$ determines a copy of the trivial 1-dimensional subbundle $\mathcal{R}$ in $\mathcal{E}$, and hence $\mathcal{E}|_{X_{n-1}} = \mathcal{R} \oplus \mathcal{R}^\perp$ where the orthogonal complement $\mathcal{R}^\perp \to X_{n-1}$ of $\mathcal{R}$ in $\mathcal{E}|_{X_{n-1}}$ is an $(N-1)$-dimensional Riemannian vector bundle over X_{n-1}. So by induction hypothesis, there are cross sections $s_i : X_i \to \mathbb{S}(\mathcal{R}^\perp) \subset \mathbb{S}(\mathcal{E}|_{X_{n-1}})$ over each X_i for all $0 \leqslant i \leqslant n-2$, such that $\{s_i(x)\colon k \leqslant i \leqslant n-2\}$ is an orthonormal set in $\mathbb{S}(\mathcal{R}^\perp|_x)$ for all $x \in X_k$ for all $0 \leqslant k \leqslant n-2$. Since s_{n-1} takes value in $\mathcal{R}$ while s_i takes value in $\mathcal{R}^\perp$ for all $0 \leqslant i \leqslant n-2$, we have $\{s_i(x)\colon k \leqslant i \leqslant n-1\}$ an orthonormal set in $\mathbb{S}((\mathcal{R} \oplus \mathcal{R}^\perp)|_x) = \mathbb{S}(\mathcal{E}|_x)$ for all $x \in X_k$. $\square$

Now we consider the case of an n-dimensional manifold $X = M$ and a rank-n Riemannian vector bundle $\mathcal{E} \to M$, with both having the *same dimension* $n > 1$. For such a case, the above CW-complex structure on $X = M$ can be assumed to be a *simplicial complex* structure as a result of a *triangulation* of the manifold M, which exists [Ca,Wh1,Mkr1].

We define the Whitney class in terms of a cell complex structure on the manifold, which is a structure stronger than the CW complex structure, but weaker than the simplicial complex structure. Roughly speaking, a *cell complex* structure [St2] on M gives rise to a CW decomposition $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = M$ such that each k-cell $\sigma : \Delta_k \to M$ is a homeomorphism onto $\sigma(\Delta_k) \subset X_k$ such that $\sigma(\partial\Delta_k)$ is a disjoint union of the "interiors" $\sigma_i(\Delta_{k_i} \backslash \partial\Delta_{k_i})$ of finitely many k_i-cells σ_i with $k_i < k$.

Applying the above theorem to a cell complex decomposition of M, we see that for each n-cell $\sigma : \Delta_n \to M$, since $\sigma(\Delta_n)$ is homeomorphic to Δ_n and hence contractible, $\mathcal{E}|_{\sigma(\Delta_n)}$ is a trivial vector bundle. So the inclusion map

$$\iota_{\sigma(\Delta_n),x} : \mathbb{S}(\mathcal{E}|_x) \hookrightarrow \mathbb{S}(\mathcal{E}|_{\sigma(\Delta_n)}) \approx \sigma(\Delta_n) \times \mathbb{S}^{n-1}$$

induces a group isomorphism

$$\pi_{n-1}(\iota_{\sigma(\Delta_n),x}) : \pi_{n-1}\big(\mathbb{S}(\mathcal{E}|_x)\big) \to \pi_{n-1}\big(\mathbb{S}(\mathcal{E}|_{\sigma(\Delta_n)})\big) \cong \pi_{n-1}\big(\mathbb{S}^{n-1}\big) \cong \mathbb{Z}$$

for any $x \in \sigma(\Delta_n)$, and we get from $[s_{n-1} \circ \sigma|_{\partial\Delta_n}] \in \pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(\Delta_n)}))$,

$$\big(\pi_{n-1}(\iota_{\sigma(\Delta_n),\sigma(e_0)})\big)^{-1}\big([s_{n-1} \circ \sigma|_{\partial\Delta_n}]\big) \in \pi_{n-1}\big(\mathbb{S}(\mathcal{E}|_{\sigma(e_0)})\big) = \pi_{n-1}\big(\mathcal{E}^\circ|_{\sigma(e_0)}\big)$$

where Δ_n is identified with the standard unit n-ball $\overline{\mathbb{B}^n}$ in a fixed orientation-compatible way with $e_0 \in \mathbb{R}^{n+1}$ identified with $e_0 \in \mathbb{R}^n$. By abuse of notation, we shall denote $(\pi_{n-1}(\iota_{\sigma(\Delta_n),\sigma(e_0)}))^{-1}([s_{n-1} \circ \sigma|_{\partial\Delta_n}])$ also by $[s_{n-1} \circ \sigma|_{\partial\Delta_n}]$ wherever the context is clear about its meaning. The n-cocycle

$$\prod_{\sigma \text{ an } n\text{-cell}} [s_{n-1} \circ \sigma|_{\partial\Delta_n}]\sigma \in C^n\big(M; \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)$$

then defines the *top Whitney class* of $\mathcal{E}$

$$w_n(\mathcal{E}) := \Bigg[\prod_{\sigma \text{ an } n\text{-cell}} [s_{n-1} \circ \sigma|_{\partial\Delta_n}]\sigma\Bigg] \in H^n\big(M; \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big) \cong H^n(M; \mathcal{G}_{\bigwedge^n \mathcal{E}}),$$

which is known to be a well-defined invariant of $\mathcal{E}$, independent of the choice of the cross section s_{n-1} over X_{n-1} and the cell complex decomposition of M [St1,Wh2]. Furthermore since normalizing a cross section s'_{n-1} of $\mathcal{E}^\circ|_{X_{n-1}}$ does not change the homotopy class $[s'_{n-1} \circ \sigma|_{\partial\Delta_n}] \in \pi_{n-1}(\mathcal{E}^\circ|_{\sigma(e_0)})$, the cross section s_{n-1} in the above definition of $w_n(\mathcal{E})$ can be any cross section of $\mathcal{E}^\circ|_{X_{n-1}}$ instead of $\mathbb{S}(\mathcal{E}^\circ)|_{X_{n-1}}$.

We remark that when M is oriented and the vector bundle $\mathcal{E}$ is also *oriented*, say, by an isomorphism $\chi : M \times \mathbb{R} \to \bigwedge^n \mathcal{E}$, the top Whitney class $w_n(\mathcal{E})$ coincides with the Euler class $e(\mathcal{E}, \chi) \in H^n(M; \mathbb{Z})$, when $H^n(M; \mathcal{P}_{n-1}(\mathcal{E}^\circ))$ is identified with $H^n(M; \mathbb{Z})$ via χ. In such a case, the obstruction to $\mathcal{E}$ having a nowhere vanishing cross section is exactly the non-vanishing of the Euler class $e(\mathcal{E}, \chi) \in H^n(M; \mathbb{Z})$ of $\mathcal{E}$, and the top Stiefel–Whitney class $sw_n(\mathcal{E})$ of $\mathcal{E}$ is the image of $e(\mathcal{E}, \chi)$ under the canonical homomorphism $H^n(M; \mathbb{Z}) \to H^n(M; \mathbb{Z}/2\mathbb{Z})$ [MiSf]. When the orientability of $\mathcal{E}$ or M is not assumed, we still have the following known fact.

Theorem 11. *(See [St2].) A rank-n vector bundle $\mathcal{E}$ over an n-dimensional compact manifold M has a nowhere vanishing cross section if and only if $w_n(\mathcal{E}) = 0$ in $H^n(M; \mathcal{P}_{n-1}(\mathcal{E}^\circ)) \cong H^n(M; \mathcal{G}_{\bigwedge^n \mathcal{E}})$.*

To get some basic insight, we note that half of the above theorem can be easily proved as follows. If $\mathcal{E}$ has a nowhere vanishing cross section $s : M \to \mathcal{E}^\circ$, then for the cross section $s_{n-1} := s|_{X_{n-1}}$ over X_{n-1} in a fixed cell complex decomposition $X_0 \subset \cdots \subset X_n = M$, since the function $s_{n-1} \circ \sigma|_{\partial\Delta_n}$ defined on $\partial\Delta_n \approx \partial\mathbb{B}^n$ extends continuously to the function $s_{n-1} \circ \sigma|_{\Delta_n}$ defined on $\Delta_n \approx \overline{\mathbb{B}^n}$, we have $[s_{n-1} \circ \sigma|_{\partial\Delta_n}] = 0$ in $\pi_{n-1}(\mathcal{E}^\circ|_{\sigma(e_0)})$, for any n-cell σ. Thus $w_n(\mathcal{E}) = 0$.

When $(\bigwedge^n \mathcal{E}) \otimes (\bigwedge^n TM)$ is a trivial line bundle for a compact manifold, the above proposition can be derived from the Euler–Hopf–Poincaré Theorem in differential geometry [GrHlVa], which tells us that the sum of well-defined indices of a cross section of $\mathcal{E}$ transversal to its zero section is independent of the choice of the cross section. But when $(\bigwedge^n \mathcal{E}) \otimes (\bigwedge^n TM)$ is a non-trivial line bundle, the above proposition provides some interesting result, to be discussed in the next section, which does not seem to be derivable from Euler–Hopf–Poincaré Theorem.

6. Poincaré–Steenrod duality and indices

In this section, we recall Steenrod's result on Poincaré duality between homology and cohomology groups with local coefficients.

Let $\mathcal{K} := \bigwedge^n T^*M$, called the *canonical bundle* of an n-dimensional manifold M. For any bundle $\mathcal{G}$ of abelian groups over M, we define a companion bundle as the tensor product of bundles of abelian groups

$$\mathcal{G}' := \mathcal{G} \otimes \mathcal{G}_{\mathcal{K}^*}$$

where for all $x \in M$,

$$\mathcal{G}'|_x = \mathcal{G}|_x \otimes \mathcal{G}_{\mathcal{K}^*}|_x \cong \mathcal{G}|_x \otimes \mathbb{Z} \cong \mathcal{G}|_x$$

identifying, in a non-canonical way, $z' \in \mathcal{G}'|_x$ with $z \in \mathcal{G}|_x$ under the relation $z' = z \otimes 1$ in $\mathcal{G}'|_x \cong \mathcal{G}|_x \otimes \mathbb{Z}$.

For $[\gamma] \in \pi(M, x)$, the automorphism $[\gamma]\cdot$ on $\mathcal{G}_{\mathcal{K}^*}|_x$ equals id when γ preserves the orientation of M at x, and equals $-$id when γ reverses the orientation of M at x. From

$$[\gamma] \cdot (z \otimes 1) = \big([\gamma] \cdot z\big) \otimes \big([\gamma] \cdot 1\big) = \begin{cases} ([\gamma] \cdot z) \otimes 1 & \text{if } [\gamma]\cdot = \mathrm{id} \text{ on } \mathcal{G}_{\mathcal{K}^*}|_x, \\ (-[\gamma] \cdot z) \otimes 1 & \text{if } [\gamma]\cdot = -\mathrm{id} \text{ on } \mathcal{G}_{\mathcal{K}^*}|_x, \end{cases}$$

for all $z \otimes 1 \in \mathcal{G}|_x \otimes \mathbb{Z} \cong \mathcal{G}'|_x$, we see that $\mathcal{G}'$ coincides with the *twisted bundle of groups associated with* $\mathcal{G}$ introduced in [St1] when M is not orientable.

Note that when M is orientable, $\mathcal{G}' = \mathcal{G}$. Also we remark that

$$\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)' = \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big) \otimes \mathcal{G}_{\mathcal{K}^*} = \mathcal{G}_{\bigwedge^n \mathcal{E}} \otimes \mathcal{G}_{\mathcal{K}^*} \cong \mathcal{G}_{\mathcal{K}^* \otimes \bigwedge^n \mathcal{E}}.$$

Furthermore, since $\pi(M, x)$ acts trivially on $\mathcal{G}_{\mathcal{K}^*}|_x \otimes \mathcal{G}_{\mathcal{K}^*}|_x$ at any $x \in M$, we have $\mathcal{G}_{\mathcal{K}^*} \otimes \mathcal{G}_{\mathcal{K}^*}$ a trivial bundle of groups modeled on $\mathbb{Z}$, and hence

$$(\mathcal{G}')' = (\mathcal{G} \otimes \mathcal{G}_{\mathcal{K}^*}) \otimes \mathcal{G}_{\mathcal{K}^*} \cong \mathcal{G}.$$

Theorem 12 *(Poincaré–Steenrod duality). (See [St1].)* $H_k(M; \mathcal{G}) \cong H^{n-k}(M; \mathcal{G}')$ *for all* $0 \leqslant k \leqslant n = \dim(M)$ *for any compact manifold* M.

Corollary 13. $H_0(M_c; \mathcal{G}|_{M_c}) \cong H^n(M; \mathcal{G}')$ *and* $H_0(M_c; \mathcal{G}'|_{M_c}) \cong H^n(M; \mathcal{G})$, *for any n-dimensional manifold M with finitely many compact connected components, where M_c is the union of the compact connected components of M.*

Proof. Since $(\mathcal{G}')' \cong \mathcal{G}$, it suffices to prove the first equality.

Recall that any *non-compact* n-dimensional manifold X is homotopy equivalent to an $(n-1)$-dimensional CW complex [NR], and hence $H^n(X;\mathcal{G}) = 0$ for any bundle $\mathcal{G}$ of groups over X. Thus

$$H^n(M;\mathcal{G}') \cong H^n(M_c;\mathcal{G}'|_{M_c}) \cong H_0(M_c;\mathcal{G}|_{M_c})$$

where M_c being the union of finitely many compact components is a compact manifold. $\square$

Now we briefly describe how this duality isomorphism is actually implemented for the case of $k = n$.

First we fix a cell complex decomposition of M. Since $\mathcal{G}_{\mathcal{K}^*}$ restricted to the contractible set $\sigma(\Delta_k)$ is trivial for any k-cell σ, we can have $\mathcal{G}_{\mathcal{K}^*}|_x$'s identified as the same group, called $\mathcal{G}_{\mathcal{K}^*}|_\sigma$, for all $x \in \sigma(\Delta_k)$ when working within the same cell $\sigma(\Delta_k)$. Now there is a *fundamental class*

$$Z = \Bigg[\sum_{\sigma \text{ an } n\text{-cell}} z_\sigma \sigma\Bigg] := \Bigg[\sum_{\sigma \text{ an} n\text{-cell}} [[\sigma|_{\partial\Delta_n}]]\sigma\Bigg] \in H_n(M;\mathcal{G}_{\mathcal{K}^*})$$

where $z_\sigma := [[\sigma|_{\partial\Delta_n}]] \in \mathcal{D}_{\mathcal{K}^*}|_{\sigma(\bar{e})} \cong \mathcal{D}_{\mathcal{K}^*}|_\sigma \subset \mathcal{G}_{\mathcal{K}^*}|_\sigma$ with $\bar{e} = \frac{1}{n}\sum_{i=0}^n e_i$ the barycenter of Δ_n. Note that this definition of Z is independent of the choice of orientation of an n-cell, i.e. if σ is replaced by $\sigma \circ \rho$ then

$$[[\sigma \circ \rho|_{\partial\Delta_n}]](\sigma \circ \rho) = \big(-[[\sigma|_{\partial\Delta_n}]]\big)(-\sigma) = [[\sigma|_{\partial\Delta_n}]]\sigma$$

in $H_n(M;\mathcal{G}_{\mathcal{K}^*})$. For $k = n$, the Poincaré–Steenrod duality isomorphism is defined by

$$\Bigg[\sum_{x \text{ a 0-cell}} c_x x\Bigg] \in H_0(M;\mathcal{G}) \mapsto \Bigg[-\prod_{x \text{ a 0-cell}} \big((\tau_x \cdot c_x) \otimes z_{x^*}\big)x^*\Bigg] \in H^n(M;\mathcal{G}')$$

where x^* is the n-cell corresponding to the 0-cell x in the cell complex structure *dual* (or *reciprocal*) [Mkr2,Le] to the fixed cell complex structure on M, and τ_x is a path from x to $x^*(e_0)$ in $x^*(\Delta_n)$.

Next we recall the Euler–Hopf–Poincaré Theorem (Theorem III in Section 9.9 of [GrHlVa]) and some corollaries, concerning a *smooth* cross section $f : M \to \mathcal{E}$ of a rank-n vector bundle $\mathcal{E}$ over a *compact* n-dimensional manifold M such that f vanishes at *finitely many* $x_i \in M$ and f is *transversal* to the zero section of $\mathcal{E}$ at these x_i's. We call such a smooth cross section a *transversal cross section* of $\mathcal{E}$. When local orientations of M and $\mathcal{E}$ are chosen at a zero x_i, the index $j(f, x_i)$ of f at x_i can be computed as

$$j(f, x_i) = \frac{\det(d(\tilde{\phi} \circ f \circ \psi)|_{x_i})}{|\det(d(\tilde{\phi} \circ f \circ \psi)|_{x_i})|} \in \{\pm 1\}$$

where $\psi : \mathbb{B}^n \twoheadrightarrow U \subset M$ with $\psi(0) = x_i$ is an orientation-compatible chart map of an open neighborhood U of x_i and $\phi : \mathcal{E}|_U \twoheadrightarrow U \times \mathbb{R}^n$ is an orientation-compatible trivialization of $\mathcal{E}|_U$, with $\tilde{\phi} : \mathcal{E}|_U \twoheadrightarrow \mathbb{R}^n$ the second component function of ϕ. Note that the choice of local orientations of M and $\mathcal{E}$ only affects the $\pm$-sign of $j(f, x_i)$, or equivalently, we can say that the index $j(f, x_i)$ is well defined *up to a $\pm$-sign* for a transversal cross section f of $\mathcal{E}$ at a zero x_i, for any manifold M and vector bundle $\mathcal{E}$ orientable or not.

Theorem 14 *(Euler–Hopf–Poincaré Theorem). If $\mathcal{K} \otimes \mathcal{L}$ is a trivial line bundle, where $\mathcal{K} = \bigwedge^n T^*M$ and $\mathcal{L} = \bigwedge^n \mathcal{E}$ for a (smooth) rank-n vector bundle $\mathcal{E}$ over a compact n-dimensional manifold M with $n > 1$, and f is a transversal cross section of $\mathcal{E}$ with (finitely many) zeros x_i, then $|\sum_i j(f, x_i)| = |\omega(\mathcal{E})|$ if M is orientable, and*

$|\sum_i j(f,x_i)| = \frac{1}{2}|\omega(\widetilde{\mathcal{E}})|$ if M is non-orientable and $\widetilde{\mathcal{E}}$ is the pull-back of $\mathcal{E}$ to an orientable double covering $\widetilde{M}$ of M, where $j(f,x_i)$ is the index of f at x_i with respect to a fixed orientation of $\mathcal{K}\otimes\mathcal{L}$, and $\omega(\mathcal{F})$ represents the Euler number of an oriented vector bundle $\mathcal{F}$ over an oriented manifold. Here since $\mathcal{E}$ or $\widetilde{\mathcal{E}}$ is orientable, $|\omega(\mathcal{E})|$ or $|\omega(\widetilde{\mathcal{E}})|$ is well defined.

Since the Euler class $e(\mathcal{F},\chi)=0$ and hence $\omega_n(\mathcal{F})=0$ for any orientable vector bundle $\mathcal{F}$ with a nowhere vanishing cross section over an oriented compact manifold, we have the following corollary.

Corollary 15. *If $\mathcal{K}\otimes\mathcal{L}$ is a trivial line bundle, where $\mathcal{K}=\bigwedge^n T^*M$ and $\mathcal{L}=\bigwedge^n\mathcal{E}$ for a (smooth) rank-n vector bundle $\mathcal{E}$ over a compact n-dimensional manifold M with $n>1$, and either $\mathcal{E}$ when M is orientable or the pull-back $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ to an orientable double covering $\widetilde{M}$ of M when M is not orientable has a nowhere vanishing cross section, then $\sum_i j(f,x_i)=0$ for any transversal cross section f of $\mathcal{E}$ with indices $j(f,x_i)$ at its zeros x_i with respect to any fixed orientation of $\mathcal{K}\otimes\mathcal{L}$.*

When $(\bigwedge^n T^*M)\otimes(\bigwedge^n\mathcal{E})$ is a non-trivial line bundle, the ambiguity of the $\pm$-sign of the indices $j(f,x_i)$ cannot be globally fixed over M, and we can only expect that $\sum_i j(f,x_i)\equiv 0 \bmod 2$. However to prove this, the above results from the Euler–Hopf–Poincaré Theorem do not seem to be useful. Instead, we use Steenrod's Poincaré duality to relate the mod-2 sum of indices to the Whitney class of $\mathcal{E}$ and then prove it.

Theorem 16. *If $\mathcal{K}\otimes\mathcal{L}$ is a non-trivial line bundle, where $\mathcal{K}=\bigwedge^n T^*M$ and $\mathcal{L}=\bigwedge^n\mathcal{E}$ for a (smooth) rank-n vector bundle $\mathcal{E}$ over a compact n-dimensional manifold M with $n>1$, and $\mathcal{E}$ has a nowhere vanishing cross section, then for any transversal cross section f of $\mathcal{E}$ with (finitely many) zeros x_i, we have $\sum_i j(f,x_i)\equiv 0 \bmod 2$, where $j(f,x_i)\in\{\pm1\}$ is the index of f at x_i with respect to any local orientations of $\mathcal{E}$ and T^*M at x_i, i.e. f has an even number of zeros.*

Proof. Since $\mathcal{E}$ has a non-zero cross section, we have $w_n(\mathcal{E})=0$. Next we relate $w_n(\mathcal{E})$ to the indices $j(f,x_i)$.

Let $X_0\subset X_1\subset X_2\subset\cdots\subset X_n=M$ be the skeletons of a *simplicial complex* structure on M such that X_{n-1} does not contain any of the zeros x_i of f and for any n-cell $\sigma:\Delta_n\to X$ in the n-skeleton X_n, at most one zero x_i of f is contained in $\sigma(\Delta_n\backslash\partial\Delta_n)$. Such a simplicial complex decomposition of M can be obtained by refining a triangulation of M and then perturbing the interiors of some $(n-1)$-cells if necessary. We endow $\mathcal{E}$ with a Riemannian structure.

Fixing a homeomorphic orientation-preserving identification of Δ_n with $\overline{\mathbb{B}^n}$ so that $e_0\in\Delta_n$ is identified with $e_0\in\overline{\mathbb{B}^n}$ and $\partial\Delta_n$ is identified with $\mathbb{S}^{n-1}$, we view σ as defined on $\overline{\mathbb{B}^n}$. If $\sigma(\mathbb{B}^n)$ does not contain any zero x_i of f, then

$$\left.\frac{f\circ\sigma}{\|f\circ\sigma\|}\right|_{\mathbb{S}^{n-1}}:\mathbb{S}^{n-1}\to\mathbb{S}(\mathcal{E}|_{\sigma(\overline{\mathbb{B}^n})})\approx\sigma\left(\overline{\mathbb{B}^n}\right)\times\mathbb{S}^{n-1}$$

extends to

$$\frac{f\circ\sigma}{\|f\circ\sigma\|}:\overline{\mathbb{B}^n}\to\mathbb{S}(\mathcal{E}|_{\sigma(\overline{\mathbb{B}^n})})\approx\sigma\left(\overline{\mathbb{B}^n}\right)\times\mathbb{S}^{n-1}$$

and hence $[\frac{f\circ\sigma}{\|f\circ\sigma\|}|_{\mathbb{S}^{n-1}}]=0$ in $\pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(\overline{\mathbb{B}^n})}))\cong\pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(e_0)}))\cong\mathbb{Z}$.

If $\sigma(\mathbb{B}^n)$ contains a zero x_i of f, say, $\sigma(0)=x_i$ without loss of generality, then $[\frac{f\circ\sigma}{\|f\circ\sigma\|}|_{\mathbb{S}^{n-1}}]=[\frac{f\circ\sigma}{\|f\circ\sigma\|}|_{r\mathbb{S}^{n-1}}]$ in $\pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(\overline{\mathbb{B}^n})}))\cong\pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(e_0)}))\cong\mathbb{Z}$ for all $r\in(0,1)$ where $\frac{f\circ\sigma}{\|f\circ\sigma\|}|_{r\mathbb{S}^{n-1}}$ actually represents the map $\frac{(f\circ\sigma)(r\cdot)}{\|(f\circ\sigma)(r\cdot)\|}$ defined on $\mathbb{S}^{n-1}$. For r sufficiently close to 0, we see that

$$\left|\left[\left.\frac{f\circ\sigma}{\|f\circ\sigma\|}\right|_{r\mathbb{S}^{n-1}}\right]\right|=\left|j(f,x_i)\right|=1$$

where $j(f, x_i)$ is the index of f at x_i with respect to any local orientations of $\mathcal{E}$ and M at x_i (cf. [MaSh] for some relevant basic discussion of local index). So $[\frac{f\circ\sigma}{\|f\circ\sigma\|}|_{\mathbb{S}^{n-1}}]$ is a generator of $\pi_{n-1}(\mathbb{S}(\mathcal{E}|_{\sigma(e_0)}))\cong\mathbb{Z}$.

Let σ_i be the n-cell with $\sigma_i(0)=x_i$, and set

$$c_i := \left[\frac{f\circ\sigma_i}{\|f\circ\sigma_i\|}\bigg|_{\mathbb{S}^{n-1}}\right] \in \pi_{n-1}\big(\mathbb{S}(\mathcal{E}|_{\sigma(e_0)})\big) \cong \mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big|_{\sigma(e_0)}$$

a generator of $\mathcal{P}_{n-1}(\mathcal{E}^\circ)|_{\sigma(e_0)}\cong\mathbb{Z}$. Under the Poincaré–Steenrod duality

$$H^n\big(M;\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)\cong H_0\big(M;\big(\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)'\big),$$

we have

$$w_n(\mathcal{E}) = \left[\prod_{\sigma\ \text{an}\ n\text{-cell}}\left[\frac{f\circ\sigma}{\|f\circ\sigma\|}\bigg|_{\mathbb{S}^{n-1}}\right]\sigma\right] = \left[\prod_i c_i\sigma_i\right] \in H^n\big(M;\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)$$

identified with

$$\left[\sum_i g_i\sigma_i(0)\right] = \left[\sum_i g_i x_i\right] \in H_0\big(M;\big(\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)'\big)$$

where

$$g_i := \big(\gamma_i^{-1}\cdot c_i\big)\otimes[[\sigma_i|_{\mathbb{S}^{n-1}}]] \in \big(\mathcal{P}_{n-1}\big(\mathcal{E}^\circ\big)\big)'\big|_{x_i}\cong\mathbb{Z}$$

for $\gamma_i(t) := \sigma_i(te_0)$, $t\in[0,1]$, is a generator of $\mathbb{Z}$, with $[[\sigma_i|_{\mathbb{S}^{n-1}}]]\in\mathcal{D}_{\mathcal{K}^*}$ representing an orientation of $\sigma_i(\mathbb{B}^n)$, and we note that $\sigma(0)$'s are the 0-cells of the cell complex structure dual to the original simplicial complex structure on X.

Thus from $w_n(\mathcal{E})=0$, we get $0=[\sum_i g_i x_i]$ in $H_0(M;(\mathcal{P}_{n-1}(\mathcal{E}^\circ))')$ which is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$. With $(\mathcal{P}_{n-1}(\mathcal{E}^\circ))'$ a bundle of groups modeled on $\mathbb{Z}$, and $g_i\in\{\pm1\}$ in $\mathbb{Z}\cong(\mathcal{P}_{n-1}(\mathcal{E}^\circ))'|_{x_i}$, the condition $0=[\sum_i g_i x_i]$ implies that

$$0 \overset{\text{mod } 2}{\equiv} \sum_i |g_i| = \sum_i 1 = \sum_i \big|j(f,x_i)\big| \overset{\text{mod } 2}{\equiv} \sum_i j(f,x_i). \qquad \square$$

7. Euler map

Before we prove our first main theorem, we point out the following natural isomorphisms for line bundles $\mathcal{L}$ over an n-dimensional manifold M with $\mathcal{K} := \bigwedge^n T^*M$,

$$\begin{tikzcd}
(\mathcal{K}\otimes\mathcal{L}^*)^\circ \arrow[d, equal] & & & \\
\mathrm{Isom}(\mathcal{L},\mathcal{K}) \arrow[r, "\alpha"] & \mathrm{Isom}(\mathcal{K}^*,\mathcal{L}^*) \arrow[r, equal] & (\mathcal{K}\otimes\mathcal{L}^*)^\circ \arrow[dd, equal] & (\mathcal{K}^*\otimes\mathcal{L})^\circ \arrow[dd, equal] \\
h \arrow[r, mapsto] & h^* & & \\
 & & \mathrm{Isom}(\mathcal{L},\mathcal{K}) \arrow[r, "\beta"] & \mathrm{Isom}(\mathcal{K},\mathcal{L}) \\
 & & g \arrow[r, mapsto] & g^{-1}
\end{tikzcd}$$

which induces an isomorphism

$$(\beta \circ \alpha)_* : H_k(M; \mathcal{G}_{\mathcal{K}\otimes\mathcal{L}^*}) \cong H_k(M; \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}).$$

Next, we define the "*differential*" $Ds|_{x_0} : T_{x_0}X \to \mathcal{V}|_{x_0}$ of a smooth cross section s of a smooth vector bundle $\mathcal{V}$ over a manifold X at a point (*a zero*) $x_0 \in X$ with $s(x_0) = 0$. Indeed using $\mathcal{V}|_{x_0}$ as a prototype of the fibers of $\mathcal{V}$ around x_0, any (smooth) local trivialization of $\mathcal{V}$ at x_0 is of the form

$$\phi : \mathcal{V}|_U \to U \times (\mathcal{V}|_{x_0})$$

for some open neighborhood U of x_0 with $\phi|_{\mathcal{V}|_{x_0}} = \mathrm{id}$ on $\mathcal{V}|_{x_0}$. (Here we use the vector space $\mathcal{V}|_{x_0}$ and an open subset $U \subset X$ instead of, respectively, a concrete Euclidean space and an open set in a Euclidean space as more commonly used, because in our following definition of $Ds|_{x_0}$, the concept used is well known to be independent of the choice of such explicit Euclidean objects.) We define

$$Ds|_{x_0} \equiv d(\tilde{\phi} \circ s)|_{x_0} : T_{x_0}X \to \mathcal{V}|_{x_0}$$

as the total derivative of the function $(\tilde{\phi} \circ s)|_U : U \to \mathcal{V}|_{x_0}$ at x_0 where $\tilde{\phi} : \mathcal{V}|_U \to \mathcal{V}|_{x_0}$ is the second component function of ϕ. The well-definedness of $Ds|_{x_0}$ is given by the following lemma.

Lemma 17. *Let $\mathcal{V}$ be a smooth vector bundle over a manifold X. If $s : X \to \mathcal{V}$ is a smooth cross section of $\mathcal{V}$ with $s(x_0) = 0$ at a point $x_0 \in X$ and $\phi_i : \mathcal{V}|_{U_i} \to U_i \times \mathcal{V}|_{x_0}$ with $i = 1, 2$ are smooth local trivializations of $\mathcal{V}$ over some open neighborhoods U_i of x_0 with $\phi_i|_{\mathcal{V}|_{x_0}} = \mathrm{id}$ on $\mathcal{V}|_{x_0}$, then $d(\tilde{\phi}_1 \circ s)|_{x_0} = d(\tilde{\phi}_2 \circ s)|_{x_0}$.*

Proof. Replacing U_i by $U_1 \cap U_2$, we may assume $U := U_1 = U_2$. Then $\phi_1 \circ \phi_2^{-1} : U \times \mathcal{V}|_{x_0} \to U \times \mathcal{V}|_{x_0}$ is of the form

$$\left(\phi_1 \circ \phi_2^{-1}\right)(x, v) = \left(x, \psi(x)(v)\right)$$

for some smooth function $\psi : U \to \mathrm{End}(\mathcal{V}|_{x_0})$ such that

$$\psi(x) \circ \tilde{\phi}_2|_{\mathcal{V}|_x} = \tilde{\phi}_1|_{\mathcal{V}|_x} : \mathcal{V}|_x \to \mathcal{V}|_{x_0}$$

for any $x \in U$ with $\psi(x_0) = \mathrm{id}$ on $\mathcal{V}|_{x_0}$ and hence

$$\left(\psi(x)\right)\left(\tilde{\phi}_2\left(s(x)\right)\right) = \tilde{\phi}_1\left(s(x)\right)$$

for all $x \in U$. Thus by the product rule for differentiation, we get

$$\begin{aligned} d(\tilde{\phi}_1 \circ s)|_{x_0} &= \left((d\psi)|_{x_0}\right)\left(\tilde{\phi}_2\left(s(x_0)\right)\right) + \left(\psi(x_0)\right)\left(d(\tilde{\phi}_2 \circ s)|_{x_0}\right) \\ &= \left(\psi(x_0)\right)\left(d(\tilde{\phi}_2 \circ s)|_{x_0}\right) = d(\tilde{\phi}_2 \circ s)|_{x_0} \end{aligned}$$

since $s(x_0) = 0$ and hence $\tilde{\phi}_2(s(x_0)) = 0$. □

Notation. Now we fix some notations that we use subsequently. Let $X = \mathrm{Spec}(A)$ be a real smooth affine variety and let $M = M(X)$ be the manifold of real points in X with $\dim(M) = n > 1$. Furthermore we set $\mathbb{R}(X) := S^{-1}A$ for the multiplicative set S of all functions $f \in A$ that do not vanish at any real point of X. We note that M has *finitely many* compact connected components [Mi] and hence the union M_c of all compact connected components of M is a compact submanifold of M.

We recall that for a finitely generated projective module P over $\mathbb{R}(X)$, there is a unique vector bundle $\mathcal{V}(P)$ over M such that $\Gamma(\mathcal{V}(P)) \cong P \otimes_{\mathbb{R}(X)} C(M)$ [Sw], where $\Gamma(\mathcal{E})$ is the space of continuous cross sections of $\mathcal{E}$ and $C(M)$ is the space of all real-valued continuous functions on M.

Lemma 18. *Let $J = \bigcap_{i=1}^N m_i$ for finitely many maximal ideals m_i of $\mathbb{R}(X)$ corresponding to $x_i \in M$, and let P be a projective $\mathbb{R}(X)$-module of rank $n = \dim(M)$ which determines a vector bundle $\mathcal{E} \to M$ by $\Gamma(\mathcal{E}) \cong P \otimes_{\mathbb{R}(X)} C(M)$. Any surjective $\mathbb{R}(X)$-homomorphism $\omega : P \twoheadrightarrow J$ determines an element $g_i \in \operatorname{Isom}(\mathcal{E}|_{x_i}, T^*_{x_i}M)$ for each i, satisfying*

$$g_i\big(f(x_i)\big) = d\big(\omega(f)\big)\big|_{x_i}$$

for all $f \in P \subset \Gamma(\mathcal{E})$, and a smooth cross section s of the dual bundle of $\mathcal{E}^* \to M$ with $s \circ f = \omega(f)$ for all $f \in P \subset \Gamma(\mathcal{E})$, which vanishes exactly at the x_i's and intersects the zero section transversally at the zeros x_i, such that

$$Ds|_{x_i} = g_i^* \in \operatorname{Isom}\big(T_{x_i}M, \mathcal{E}^*|_{x_i}\big).$$

Proof. First we recall that under the well-known relation $\Gamma(\mathcal{E}) \cong P \otimes_{\mathbb{R}(X)} C(M)$ between projective modules P and vector bundles $\mathcal{E}$ [Sw], we have an isomorphism

$$\eta \equiv \bigoplus_{i=1}^N [\eta_i] : \frac{P}{JP} \cong \bigoplus_{i=1}^N \frac{P}{m_i P} \cong \bigoplus_{i=1}^N \frac{\Gamma(\mathcal{E})}{m_i P \otimes_{\mathbb{R}(X)} C(M)} \xrightarrow{\cong} \bigoplus_{i=1}^N \mathcal{E}|_{x_i}$$

implemented by the evaluation maps

$$\eta_i : f \in \Gamma(\mathcal{E}) \mapsto f(x_i) \in \mathcal{E}|_{x_i}$$

that appear in the short exact sequence

$$0 \to m_i P \otimes_{\mathbb{R}(X)} C(M) \to P \otimes_{\mathbb{R}(X)} C(M) \cong \Gamma(\mathcal{E}) \xrightarrow{\eta_i} \mathcal{E}|_{x_i} \cong \mathbb{R}^n \to 0$$

which with P an $\mathbb{R}(X)$-projective module, is derived from the short exact sequence

$$0 \to m_i P \to P \to P/(m_i P) \cong \mathbb{R}^n \to 0.$$

On the other hand, there is also a well-known isomorphism

$$d_i : [f] \in J/(m_i J) \overset{\cong}{\longmapsto} df|_{x_i} \in T^*_{x_i}M$$

for each i, which gives rise to an isomorphism

$$\bigoplus_{i=1}^N d_i : [f] \in J/J^2 \cong \bigoplus_{i=1}^N \frac{J}{m_i J} \overset{\cong}{\longmapsto} \bigoplus_{i=1}^N df|_{x_i} \in \bigoplus_{i=1}^N T^*_{x_i}M.$$

So a surjective $\mathbb{R}(X)$-homomorphism $\omega : P \twoheadrightarrow J$ induces the commuting diagram

$$\begin{array}{rcc}
 & \bigoplus_{i=1}^N \mathcal{E}|_{x_i} & \bigoplus_{i=1}^N T^*_{x_i}M \\
 & \| & \| \\
\omega/J \equiv \omega \bigotimes_{\mathbb{R}(X)} (\mathbb{R}(X)/J): & P/(JP) & \longrightarrow\!\!\!\!\!\rightarrow \; J/J^2 \\
 & \downarrow & \downarrow \\
\omega/m_i \equiv \omega \bigotimes_{\mathbb{R}(X)} (\mathbb{R}(X)/m_i): & P/(m_i P) & \longrightarrow\!\!\!\!\!\rightarrow \; J/(m_i J) \\
 & \| \, \eta_i & \| \, d_i \\
 & \mathcal{E}|_{x_i} & T^*_{x_i}M
\end{array}$$

where the horizontal map ω/m_i for each i, and hence ω/J, is a linear isomorphism. Now each

$$g_i := d_i \circ (\omega/m_i) \circ \eta_i^{-1} \in \operatorname{Isom}\bigl(\mathcal{E}|_{x_i}, T^*_{x_i}M\bigr)$$

determined by ω clearly satisfies

$$g_i\bigl(f(x_i)\bigr) = \bigl(d_i \circ (\omega/m_i)\bigr)(f) = d_i\bigl([f]\bigr) = df|_{x_i}.$$

The surjection $\omega : P \twoheadrightarrow J$ also induces a linear functional

$$s(x) : \mathcal{E}|_x \cong P/(m_x P) \twoheadrightarrow J/(m_x \cap J) \subset \mathbb{R}(X)/m_x \cong \mathbb{R}$$

at each $x \in M$, where m_x is the maximal ideal of $\mathbb{R}(X)$ corresponding to x. Since $J \subset m_x$ if and only if $m_x = m_i = m_{x_i}$, i.e. $x = x_i$, for some i, we get $s(x) = 0$ if and only if $x = x_i$ for some i. Note that $s(f(x)) = \omega(f)(x)$ with $\omega(f) \in J \subset C^\infty(M)$ for all $f \in P$ which contains all coordinate cross sections of $\mathcal{E}$ in a smooth local trivialization at any $x \in X$, and hence s is a smooth cross section of $\mathcal{E}^*$.

It remains to show that $Ds|_{x_i} = g_i^*$ for each i, which then clearly implies that $Ds|_{x_i} \in \operatorname{Isom}(T_{x_i}M, \mathcal{E}^*|_{x_i})$ and hence s is transversal to the zero cross section at each x_i.

To show that $Ds|_{x_i} = g_i^*$, it is helpful to first fix some basis for each $\mathcal{E}|_{x_i}$ constructed from suitably chosen cross sections $f_j \in P \subset \Gamma(\mathcal{E})$ as follows.

Fixing a set of generators $[\omega(f_1)], \dots, [\omega(f_n)]$ of J/J^2 with $f_i \in P$, we have already shown that

$$g_i : f_j(x_i) \in \mathcal{E}|_{x_i} \mapsto d\bigl(\omega(f_j)\bigr)\big|_{x_i} \in T^*_{x_i}M$$

for each $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant N$.

Note that with $d(\omega(f_j))|_{x_i}$ linearly independent at each i, these f_j's give rise to a chart map

$$\bigl(\omega(f_1), \dots, \omega(f_n)\bigr) : U_i \to \mathbb{R}^n$$

locally at each x_i on some open neighborhood U_i of x_i. Furthermore, $f_j(x_i) = g_i^{-1}(d(\omega(f_j))|_{x_i})$, $1 \leqslant j \leqslant n$, are linearly independent at each x_i. So by continuity, we may assume that $f_1(x), \dots, f_n(x) \in \bigoplus_{i=1}^N \mathcal{E}|_x$ are linearly independent at each $x \in U_i$ by taking U_i sufficiently small, and hence we get a trivialization

$$\psi_i : \mathcal{E}|_{U_i} \to U_i \times (\mathcal{E}|_{x_i})$$

of $\mathcal{E}$, defined by the well-defined inverse isomorphism

$$\psi_i^{-1} : \left(x, \sum_{j=1}^{n} a_j f_j(x_i)\right) \in U_i \times (\mathcal{E}|_{x_i}) \overset{\cong}{\mapsto} \sum_{j=1}^{n} a_j f_j(x) \in \mathcal{E}|_x \subset \mathcal{E}|_{U_i}.$$

Clearly ψ_i induces a trivialization

$$\phi_i : \mathcal{E}^*|_{U_i} \to U_i \times (\mathcal{E}|_{x_i})^*$$

of $\mathcal{E}^*$ satisfying that if $\xi \in \mathcal{E}^*$ and $\xi(f_j(x)) = a_j$ for all $1 \leqslant j \leqslant n$, then $\phi_i(\xi) = (x, \tilde{\phi}_i(\xi))$ where the second component function of $\tilde{\phi}_i$ satisfies

$$\big(\tilde{\phi}_i(\xi)\big)\big(f_j(x_i)\big) = a_j$$

for all $1 \leqslant j \leqslant n$.

In particular, since $s(x)(f_j(x)) = \omega(f_j)(x)$ for all j, the function $(\tilde{\phi}_i \circ s)|_{U_i} : U_i \to (\mathcal{E}|_{x_i})^*$ satisfies

$$\big((\tilde{\phi}_i \circ s)(x)\big)\big(f_j(x_i)\big) = \big(\tilde{\phi}_i\big(s(x)\big)\big)\big(f_j(x_i)\big) = \omega(f_j)(x)$$

for all $x \in U_i$, which implies that the linear map

$$Ds|_{x_i} = d(\tilde{\phi}_i \circ s)|_{x_i} : T_{x_i} M \to (\mathcal{E}|_{x_i})^*$$

satisfies

$$\big((Ds|_{x_i})(\cdot)\big)\big(f_j(x_i)\big) = \big(d(\tilde{\phi}_i \circ s)|_{x_i}(\cdot)\big)\big(f_j(x_i)\big) = d\big(\omega(f_j)\big)\big|_{x_i}(\cdot).$$

Thus for all $1 \leqslant j \leqslant n$ and $v \in T_{x_i} M$,

$$\big(g_i\big(f_j(x_i)\big)\big)(v) = d\big(\omega(f_j)\big)\big|_{x_i}(v) = \big((Ds|_{x_i})(v)\big)\big(f_j(x_i)\big),$$

i.e. for all $w \in \mathcal{E}|_{x_i}$ and $v \in T_{x_i} M$,

$$\big(g_i(w)\big)(v) = \big((Ds|_{x_i})(v)\big)(w),$$

or equivalently, $Ds|_{x_i} = g_i^*$. $\square$

Now we briefly recall the definition of the *Euler class group* $E(\mathbb{R}(X), L)$. There are mainly two ways to define $E(\mathbb{R}(X), L)$, and here we take the first approach that was initiated by Nori ([MS], also see Remark 4.7 of [BRS3]). Let F be the *free abelian* group generated by (m, ω) where m is a maximal ideal of $\mathbb{R}(X)$ and

$$\omega : L/(mL) \xrightarrow{\cong} \bigwedge^n \big(m/m^2\big)$$

is an isomorphism, called a *local orientation* of m. An isomorphism $\omega_J : L/(JL) \xrightarrow{\cong} \bigwedge^n(J/J^2)$ called a *local orientation* of J, where $J = \bigcap_{i=1}^N m_i$ for distinct maximal ideals m_i, induces $\omega_i : L/(m_i L) \xrightarrow{\cong} \bigwedge^n(m_i/m_i^2)$ for each i, and we denote

$$(J, \omega_J) := \sum_{i=1}^{N} (m_i, \omega_i) \in F.$$

With the canonical isomorphisms $\bigwedge^n(L \oplus \mathbb{R}(X)^{n-1}) \cong L$ and hence

$$\bigwedge^n\left(\frac{L \oplus \mathbb{R}(X)^{n-1}}{J(L \oplus \mathbb{R}(X)^{n-1})}\right) \cong \frac{L}{JL},$$

a local orientation ω_J determines an equivalence class of isomorphisms

$$\tilde{\omega}_J : \frac{L \oplus \mathbb{R}(X)^{n-1}}{J(L \oplus \mathbb{R}(X)^{n-1})} \to J/J^2$$

with $\omega_J = \bigwedge^n \tilde{\omega}_J$, where $\tilde{\omega}_{J,1} \sim \tilde{\omega}_{J,2}$ when $\tilde{\omega}_{J,1} = \tilde{\omega}_{J,2} \circ \eta$ for some

$$\eta \in SL_{\mathbb{R}(X)/J}\left(\frac{L \oplus \mathbb{R}(X)^{n-1}}{J(L \oplus \mathbb{R}(X)^{n-1})}\right).$$

Let H be the subgroup of F generated by $(J, \omega_J) \in F$ with ω_J a *global orientation*, i.e. ω_J liftable to a surjective homomorphism

$$\omega : L \oplus \mathbb{R}(X)^{n-1} \twoheadrightarrow J$$

in the sense that $\omega_J = \bigwedge^n(\omega/J)$ where

$$\omega/J \equiv \omega \bigotimes_{\mathbb{R}(X)} \big(\mathbb{R}(X)/J\big) : \frac{L \oplus \mathbb{R}(X)^{n-1}}{J(L \oplus \mathbb{R}(X)^{n-1})} \twoheadrightarrow J/J^2.$$

Then we define $E(\mathbb{R}(X), L) := F/H$.

Theorem 19. *Let $X = \operatorname{Spec}(A)$ be a smooth real affine variety with $\dim(X) = n > 1$ and let M be the n-dimensional manifold of real points of X. For the line bundle $\mathcal{L}$ (with $\Gamma(\mathcal{L}) = L \otimes_{\mathbb{R}(X)} C(M)$) associated with a projective $\mathbb{R}(X)$-module L of rank one and $\mathcal{K} := \bigwedge^n T^*M$, there is a canonical group isomorphism*

$$\varepsilon : E\big(\mathbb{R}(X), L\big) \to H_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})$$

where M_c is the union of all compact connected components of M.

Proof. (i) First we construct a group homomorphism $\varepsilon' : F \to S_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})$. Let M_c be the union of all compact connected components of M. Note that

$$\big(\mathcal{K}^* \otimes \mathcal{L}\big)^\circ \cong \operatorname{Isom}\big(\bigwedge^n T^*M, \bigwedge^n\big(\mathcal{L} \oplus \mathcal{R}^{n-1}\big)\big) \cong \det\big(\operatorname{Isom}\big(T^*M, \mathcal{L} \oplus \mathcal{R}^{n-1}\big)\big)$$

for the trivial line bundle $\mathcal{R}$ over M, and hence

$$\mathcal{D}_{\mathcal{K}^* \otimes \mathcal{L}} \cong \det\big(\operatorname{Isom}\big(T^*M, \mathcal{L} \oplus \mathcal{R}^{n-1}\big)\big)/\mathbb{R}_+.$$

For any ideal $J = \bigcap_{i=1}^N m_i$ where m_i is the maximal ideal corresponding to a distinct point $x_i \in M$, any local L-orientation $\omega_J : L/(JL) \xrightarrow{\cong} \bigwedge^n(J/J^2)$ determines an equivalence class of isomorphisms

$$\tilde{\omega}_J : \frac{L \oplus \mathbb{R}(X)^{n-1}}{J(L \oplus \mathbb{R}(X)^{n-1})} \twoheadrightarrow J/J^2$$

with $\omega_J = \bigwedge^n \tilde{\omega}_J$, which can be decomposed into a direct sum of

$$\tilde{\omega}_i : \frac{L \oplus \mathbb{R}(X)^{n-1}}{m_i(L \oplus \mathbb{R}(X)^{n-1})} \twoheadrightarrow J/(m_i J)$$

with $\omega_i = \bigwedge^n \tilde{\omega}_i$.

By Lemma 18, $\tilde{\omega}_J$ determines

$$g_i \in \operatorname{Isom}\bigl(\mathcal{L}|_{x_i} \oplus \mathbb{R}^{n-1}, T^*_{x_i} M\bigr)$$

at each x_i. In fact, from its proof, we actually have $g_i = \tilde{\omega}_i$ after the canonical identifications of their domains and range spaces. Furthermore $\bigwedge^n g_i \in \det(\operatorname{Isom}(\mathcal{L}|_{x_i} \oplus \mathbb{R}^{n-1}, T^*_{x_i} M))$ depends only on the equivalence class of $\tilde{\omega}_J$, i.e. on ω_J.

Now with

$$\bigl[\bigwedge^n g_i^{-1}\bigr] \in (\mathcal{D}_{\mathcal{K}^* \otimes \mathcal{L}})|_{x_i} \subset (\mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}})|_{x_i}$$

for each i, we get a singular 0-cycle

$$\varepsilon'(J, \omega_J) := \sum_{i \text{ with } x_i \in M_c} \bigl[\bigwedge^n g_i^{-1}\bigr] x_i \in S_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})$$

with local coefficients in $\mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}$. We note that $\varepsilon'(m_x, \omega) = 0$ for the maximal ideal m_x determined by any $x \notin M_c$, and for $(J, \omega_J) = \sum_{i=1}^N (m_i, \omega_i)$ in F, it is straightforward to check that

$$\varepsilon'(J, \omega_J) = \sum_{i=1}^N \varepsilon'(m_i, \omega_i).$$

So ε' well defines a group homomorphism

$$\varepsilon' : F \to S_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c}).$$

(ii) If ω_J is actually a *global* L-orientation, i.e. $\omega_J = \bigwedge^n(\omega/J)$ for some $\mathbb{R}(X)$-epimorphism $\omega : L \oplus \mathbb{R}(X)^{n-1} \twoheadrightarrow J$, then we claim, and prove below, that $\varepsilon'(J, \omega_J)$ is a singular 0-*boundary* with local coefficients in $\mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}$, i.e. $[\varepsilon'(J, \omega_J)] = 0$ in $H_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})$, which then implies that ε' induces a well-defined group homomorphism

$$\varepsilon : E\bigl(\mathbb{R}(X), L\bigr) = F/H \to \frac{S_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})}{\partial(S_1(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c}))} = H_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c}).$$

By Lemma 18, the epimorphism $\omega : L \oplus \mathbb{R}(X)^{n-1} \twoheadrightarrow J$ determines a transversal cross section s of the bundle $\mathcal{L}^* \oplus \mathcal{R}^{n-1} = (\mathcal{L}|_{x_i} \oplus \mathcal{R}^{n-1})^*$ vanishing only at x_i's such that

$$h_i := Ds|_{x_i} = g_i^* \in \operatorname{Isom}\bigl(T_{x_i} M, \mathcal{L}^*|_{x_i} \oplus \mathbb{R}^{n-1}\bigr).$$

So $[\bigwedge^n h_i] \in (\mathcal{D}_{\mathcal{K} \otimes \mathcal{L}^*})|_{x_i} \subset (\mathcal{G}_{\mathcal{K} \otimes \mathcal{L}^*})|_{x_i} \cong \mathbb{Z}$ satisfies

$$\bigl|\bigl[\bigwedge^n h_i\bigr]\bigr| = \bigl|\operatorname{Ind}_{x_i}(s)\bigr| = \bigl|j(s, x_i)\bigr| = 1$$

[MaSh] where the index $j(s,x_i)$ is globally well defined if $\mathcal{K}\otimes\mathcal{L}^*$ is trivial, and is only defined up to a $\pm$-sign if $\mathcal{K}\otimes\mathcal{L}^*$ is non-trivial.

With $n>1$, the vector bundle $(\mathcal{L}\oplus\mathcal{R}^{n-1})^*\cong\mathcal{L}^*\oplus\mathcal{R}^{n-1}$ has a nowhere vanishing cross section over *each compact* component M' of M, and hence by Corollary 15 and Theorem 16, we have

$$\left|\sum_{i\text{ with }x_i\in M'} j(s,x_i)\right|\begin{cases}=0 & \text{if } (\mathcal{K}\otimes\mathcal{L}^*)|_{M'} \text{ is trivial,}\\ \equiv 0 \bmod 2 & \text{if } (\mathcal{K}\otimes\mathcal{L}^*)|_{M'} \text{ is non-trivial,}\end{cases}$$

which then implies

$$\left[\sum_{i\text{ with }x_i\in M'}\left[\bigwedge^n h_i\right]x_i\right]=0\in H_0\big(M';\mathcal{G}_{\mathcal{K}\otimes\mathcal{L}^*}|_{M'}\big)$$
$$\begin{cases}=\mathbb{Z} & \text{if } (\mathcal{K}\otimes\mathcal{L}^*)|_{M'} \text{ is trivial,}\\ \equiv\mathbb{Z}/2\mathbb{Z} & \text{if } (\mathcal{K}\otimes\mathcal{L}^*)|_{M'} \text{ is non-trivial.}\end{cases}$$

Now under the natural isomorphism

$$\beta\circ\alpha: h\in\mathcal{K}\otimes\mathcal{L}^*\mapsto\big(h^*\big)^{-1}\in\mathcal{K}^*\otimes\mathcal{L}$$

we get

$$(\beta\circ\alpha)_*: 0=\left[\sum_{i\text{ with }x_i\in M_c}\left[\bigwedge^n h_i\right]x_i\right]\in H_0(M_c;\mathcal{G}_{\mathcal{K}\otimes\mathcal{L}^*}|_{M_c})$$
$$\mapsto\left[\sum_{i\text{ with }x_i\in M_c}\left[\bigwedge^n g_i^{-1}\right]x_i\right]\in H_0(M_c;\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M_c})$$

since $H_0(M_c,\mathcal{G})\cong\bigoplus_{M'}H_0(M';\mathcal{G}|_{M'})$ for any $\mathcal{G}$, and hence

$$\big[\varepsilon'(J,\omega_J)\big]=\left[\sum_{i\text{ with }x_i\in M_c}\left[\bigwedge^n g_i^{-1}\right]x_i\right]=0\quad\text{in } H_0(M_c;\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M_c}).$$

So we conclude that

$$\varepsilon:(J,\omega_J)\in E\big(\mathbb{R}(X),L\big)\mapsto\left[\sum_{x_i\in M_c}\left[\bigwedge^n g_i^{-1}\right]x_i\right]\in H_0(M_c;\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M_c})$$

well defines a group homomorphism.

(iii) Let M' be any *compact* connected component of M. Fix a maximal ideal m corresponding to a point $x\in M'$ and a local L-orientation

$$\omega_m: L/(mL)\to\bigwedge^n\big(m/m^2\big)$$

which determines an element $g\in\operatorname{Isom}(\mathcal{L}|_x\oplus\mathcal{R}^{n-1},T_x^*M)$, we have

$$\varepsilon(m,\omega_m)=\big[\big[\bigwedge^n g^{-1}\big]x\big]\neq 0\quad\text{in } H_0(M';\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'}).$$

Indeed since M' is connected, $H_0(M', \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'})$ is isomorphic to $(\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}})_x \cong \mathbb{Z}$ modulo the action of $\pi_1(M'; x)$ by automorphisms of $\mathbb{Z}$, which is either $\mathbb{Z}$ if $(\mathcal{K}^* \otimes \mathcal{L})|_{M'}$ is a trivial line bundle, or $\mathbb{Z}/2\mathbb{Z}$ if $(\mathcal{K}^* \otimes \mathcal{L})|_{M'}$ is a non-trivial line bundle. Since

$$\left[\bigwedge^n g^{-1}\right] \in \mathcal{D}_{\mathcal{K}^*\otimes\mathcal{L}}|_x \cong \{\pm 1\} \subset \mathbb{Z} \cong \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_x$$

is a generator of the group $\mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_x \cong \mathbb{Z}$, the class $[[\bigwedge^n g^{-1}]x] \neq 0$ in the quotient group $H_0(M', \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'})$ and is in fact a generator of the group $H_0(M', \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'})$. Thus we have proved that

$$\varepsilon : E\big(\mathbb{R}(X), L\big) \to H_0(M_c; \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M_c}) \cong \bigoplus_{M'} H_0(M', \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'})$$

is a non-zero surjective homomorphism.

Let M_i be the compact connected components of M with $\mathcal{K}|_{M_i} \cong \mathcal{L}|_{M_i}$ and let M_j be the compact connected components of M with $\mathcal{K}|_{M_j} \ncong \mathcal{L}|_{M_j}$. We recall that by [BhDaMa], we have

$$E\big(\mathbb{R}(X), L\big) \cong \left(\bigoplus_i \mathbb{Z}\right) \oplus \left(\bigoplus_j \mathbb{Z}_2\right).$$

On the other hand, we also have $H_0(M_i, \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}) \cong \mathbb{Z}$ and $H_0(M_j, \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}) \cong \mathbb{Z}_2$. Hence

$$E\big(\mathbb{R}(X), L\big) \cong H_0(M_c; \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M_c}) \cong \bigoplus_{M'} H_0(M'; \mathcal{G}_{\mathcal{K}^*\otimes\mathcal{L}}|_{M'})$$

over all compact connected components M' of M.

Now we see that ε is a group homomorphism between groups of the same structure $(\bigoplus_i \mathbb{Z}) \oplus (\bigoplus_j \mathbb{Z}_2)$. So it is easy to verify that this non-zero surjective homomorphism ε is indeed an isomorphism. □

Remark. Since $(\mathcal{K}^* \otimes \mathcal{L})^\circ \cong (\mathcal{K} \otimes \mathcal{L}^*)^\circ$ naturally, we can instead work with the map

$$(J, \omega_J) \mapsto \left[\sum_i \left[\bigwedge^n g_i\right] x_i\right] \in H_0(M_c; \mathcal{G}_{\mathcal{K}\otimes\mathcal{L}^*}|_{M_c})$$

which also well defines a canonical isomorphism from $E(\mathbb{R}(X), \mathcal{L})$ to $H_0(M_c, \mathcal{G}_{\mathcal{K}\otimes\mathcal{L}^*}|_{M_c})$.

8. Whitney map

In this section, we show that under the Poincaré–Steenrod duality between homology and cohomology groups with local coefficients, the isomorphism ε in Theorem 19 coincides with the Whitney class map for vector bundles.

Proposition 20. *Let s be a smooth cross section of a smooth rank-n vector bundle $\mathcal{E}$ over an n-dimensional manifold M with $\mathcal{K} := \bigwedge^n T^*M$ for $n > 1$, and let $\sigma : \overline{\mathbb{B}^n} \to M$ be a continuous injective map with $x = \sigma(0)$. If s has an isolated zero at x and is transversal to the zero section at x, then*

$$\left[\bigwedge^n (Ds|_x)\right] \otimes [[\sigma|_{\mathbb{S}^{n-1}}]] = [s \circ \sigma|_{\mathbb{S}^{n-1}}] \quad \text{in } \mathcal{G}_{\mathcal{K}\otimes\bigwedge^n\mathcal{E}}|_x \otimes \mathcal{G}_{\mathcal{K}^*}|_x \cong \mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$$

where $\mathcal{K}\otimes\mathcal{K}^*\otimes\bigwedge^n\mathcal{E}$ is naturally identified with $\bigwedge^n\mathcal{E}$ via the canonical pairing $a\otimes b\in\mathcal{K}\otimes\mathcal{K}^*\mapsto b(a)\in\mathbb{Z}$, and $[s\circ\sigma|_{\mathbb{S}^{n-1}}]\in\pi_{n-1}(\mathcal{E}^\circ|_{\sigma(\overline{\mathbb{B}^n})})$ which is identified with $\pi_{n-1}(\mathcal{E}^\circ|_x)\equiv\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$.

Proof. Clearly $[s\circ\sigma|_{\mathbb{S}^{n-1}}]=[s\circ\sigma|_{r\mathbb{S}^{n-1}}]$ in $\pi_{n-1}(\mathcal{E}^\circ|_x)\equiv\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$ and $[[\sigma|_{\mathbb{S}^{n-1}}]]=[[\sigma|_{r\mathbb{S}^{n-1}}]]$ in $\mathcal{G}_{\mathcal{K}^*}|_x\equiv\mathcal{Q}_x$ for all $r\in(0,1)$, where $\sigma|_{r\mathbb{S}^{n-1}}$ represents the function $\sigma(r\cdot)|_{\mathbb{S}^{n-1}}$. So without loss of generality, we may assume that $\sigma(\overline{\mathbb{B}^n})\subset U$ for some simple open neighborhood U of x, in which s has x as the only zero, with a fixed smooth chart map $\phi:U\to\mathbb{B}^n$ such that $\phi(x)=0$, and that $\sigma(\overline{\mathbb{B}^n})$ is as close to x as we need.

By Proposition 8, $[[\sigma|_{\mathbb{S}^{n-1}}]]\equiv[(d\phi|_x)^{-1}\circ\phi\circ\sigma|_{\mathbb{S}^{n-1}}]$ in $\mathcal{Q}_x\equiv\mathcal{P}_{n-1}((TM)^\circ)|_x\equiv\mathcal{G}_{\mathcal{K}^*}|_x$. On the other hand, let $\psi:\mathcal{E}|_U\to U\times\mathcal{E}|_x$ be a smooth trivialization with $\psi(v)=(x,v)$ for all $v\in\mathcal{E}|_x$. Then $Ds|_x=d(\tilde\psi\circ s)|_x$ where $\tilde\psi$ is the second component function of ψ. Note that since s is transversal to the zero section, $d(\tilde\psi\circ s)|_x$ is invertible.

Since the concepts involved are coordinate-free, i.e. well defined, independent of chart maps, we can identify U with $\mathbb{B}^n$ (and x with 0) by the chart map ϕ in order to simplify the presentation. So in the following, we keep in mind that $U=\mathbb{B}^n$, $x=0$, and $T_xU=\mathbb{R}^n\supset U$. In particular, we have $[[\sigma|_{\mathbb{S}^{n-1}}]]\equiv[\sigma|_{\mathbb{S}^{n-1}}]$ in $\pi_{n-1}((T_xU)^\circ)|\equiv\mathcal{G}_{\mathcal{K}^*}|_x$.

Note that $[s\circ\sigma|_{\mathbb{S}^{n-1}}]=[\tilde\psi\circ s\circ\sigma|_{\mathbb{S}^{n-1}}]$ in $\pi_{n-1}(\mathcal{E}^\circ|_x)$. On the other hand, since the *invertible* linear map

$$Ds|_x=d(\tilde\psi\circ s)|_x:T_xU=\mathbb{R}^n\to\mathcal{E}|_x\cong\mathbb{R}^n$$

is the linear approximation to $\tilde\psi\circ s$ at $x\in U\subset T_xU=\mathbb{R}^n$ and $\sigma(\overline{\mathbb{B}^n})$ can be assumed to be as close to x as needed, by an argument used for a similar situation in the proof of Proposition 8, we get

$$[Ds|_x\circ\sigma|_{\mathbb{S}^{n-1}}]=[\tilde\psi\circ s\circ\sigma|_{\mathbb{S}^{n-1}}]=[s\circ\sigma|_{\mathbb{S}^{n-1}}]$$

in $\pi_{n-1}(\mathcal{E}^\circ|_x)\equiv\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$.

It remains to show that $[Ds|_x\circ\sigma|_{\mathbb{S}^{n-1}}]\equiv[\bigwedge^n(Ds|_x)]\otimes[\sigma|_{\mathbb{S}^{n-1}}]$. Note that the canonical pairing

$$\mathcal{G}_{\mathcal{K}\otimes\bigwedge^n\mathcal{E}}|_x\otimes\mathcal{G}_{\mathcal{K}^*}|_x=\mathcal{G}_{\mathrm{Hom}(\mathcal{K}^*,\bigwedge^n\mathcal{E})}|_x\otimes\mathcal{G}_{\mathcal{K}^*}|_x\xrightarrow{\cong}\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$$

identifies $[h]\otimes[v]\in\mathcal{D}_{\mathrm{Hom}(\mathcal{K}^*,\bigwedge^n\mathcal{E})}|_x\otimes\mathcal{D}_{\mathcal{K}^*}|_x$ with $[h(v)]\in\mathcal{D}_{\bigwedge^n\mathcal{E}}|_x$ where $h\in\mathrm{Isom}(\mathcal{K}|_x,\bigwedge^n\mathcal{E}|_x)$ and $v\in(\mathcal{K}^*)^\circ|_x$. From the proof of Proposition 6, we have $\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x\equiv\pi_{n-1}(\mathcal{E}^\circ|_x)$ by the canonical identification

$$F_{\mathcal{E}}:\left[\bigwedge^nB\right]\underset{\{\pm1\}}{\otimes}k\in\mathcal{D}_{\bigwedge^n\mathcal{E}}|_x\underset{\{\pm1\}}{\otimes}\mathbb{Z}=\mathcal{G}_{\bigwedge^n\mathcal{E}}|_x\overset{\cong}{\longmapsto}k[f_B|_{\mathbb{S}^{n-1}}]\in\pi_{n-1}\left(\mathcal{E}^\circ|_x\right)$$

where B is an ordered basis of $\mathcal{E}|_x$ and $f_B\in\mathrm{Isom}(\mathbb{R}^n,\mathcal{E}|_x)$ sends the standard basis of $\mathbb{R}^n$ to B. Similarly, we have $\mathcal{G}_{\mathcal{K}^*}|_x=\mathcal{G}_{\bigwedge^nTM}|_x\equiv\pi_{n-1}((T_xM)^\circ)$ by the canonical identification $F_{TM}([\bigwedge^nC])=[f_C|_{\mathbb{S}^{n-1}}]$ for ordered basis C of T_xM. Since $\bigwedge^n(Ds|_x)\in\mathrm{Isom}(\bigwedge^nT_xM,\bigwedge^n\mathcal{E}|_x)$ sends $\bigwedge^nC$ to $\bigwedge^n(Ds|_x(C))$ for any ordered basis C of $TM|_x$ while $\pi_{n-1}(Ds|_x)$ sends $[f_C|_{\mathbb{S}^{n-1}}]$ to

$$[Ds|_x\circ f_C|_{\mathbb{S}^{n-1}}]=[f_{(Ds|_x(C))}|_{\mathbb{S}^{n-1}}],$$

we have the commuting diagram

$$\begin{array}{ccc} [\bigwedge^n(Ds|_x)]: & \mathcal{G}_{\mathcal{K}^*}|_x & \longrightarrow & \mathcal{G}_{\bigwedge^n\mathcal{E}}|_x \\ & {\scriptstyle F_{TM}}\big\downarrow\equiv & & {\scriptstyle F_{\mathcal{E}}}\big\downarrow\equiv \\ \pi_{n-1}(Ds|_x) & \pi_{n-1}((T_xM)^\circ) & \longrightarrow & \pi_{n-1}(\mathcal{E}^\circ|_x) \end{array}$$

where $[\bigwedge^n(Ds|_x)] \in \mathcal{D}_{\mathcal{K}\otimes\bigwedge^n\mathcal{E}}|_x$ is viewed as a homomorphism $\mathcal{G}_{\mathcal{K}^*}|_x \to \mathcal{G}_{\bigwedge^n\mathcal{E}}|_x$ via the above canonical pairing. This commuting diagram shows that $[\bigwedge^n(Ds|_x)] \equiv \pi_{n-1}(Ds|_x)$ under canonical identifications. Thus under the above identification by pairing, we get

$$\left[\bigwedge\nolimits^n(Ds|_x)\right] \otimes [\sigma|_{\mathbb{S}^{n-1}}] \equiv \pi_{n-1}(Ds|_x)\big([\sigma|_{\mathbb{S}^{n-1}}]\big) = [Ds|_x \circ \sigma|_{\mathbb{S}^{n-1}}]. \qquad \square$$

Let P be a projective $\mathbb{R}(X)$-module of rank n with an orientation $\chi : L \xrightarrow{\cong} \bigwedge^n P$, and let $\mathcal{E}$ be the corresponding rank-n vector bundle over the n-dimensional manifold M of real points in X, i.e. with $P \otimes_{\mathbb{R}(X)} C(M) = \Gamma(\mathcal{E})$. Then induced by χ, there is a vector bundle isomorphism

$$\chi : \mathcal{L} \xrightarrow{\cong} \bigwedge\nolimits^n \mathcal{E}$$

still denoted by χ, the meaning of which should be clear from the context. For $\mathcal{K} := \bigwedge^n T^*M$, we have

$$\mathcal{K}^* \otimes \mathcal{L} = \operatorname{Hom}(\mathcal{K}, \mathcal{L}) \overset{\operatorname{Hom}(\mathrm{id},\chi)}{\cong} \operatorname{Hom}\big(\bigwedge\nolimits^n T^*M, \bigwedge\nolimits^n \mathcal{E}\big) \cong \det\big(\operatorname{Hom}\big(T^*M, \mathcal{E}\big)\big).$$

The element $e(P,\chi)$ in $E(\mathbb{R}(X), L)$ determined by P and χ can be described as follows [BRS3]. There is a surjective homomorphism $\phi : P \twoheadrightarrow J$, where $J = \bigcap_{i=1}^N m_i$ for some distinct maximal ideals m_i corresponding to x_i, and $e(P,\chi) = (J, \omega_J)$ with $\omega_J = \bigwedge^n \tilde{\omega}_J$ for some isomorphism

$$\tilde{\omega}_J : (L/JL) \oplus \big(\mathbb{R}(X)/J\big)^{n-1} \to J/J^2$$

defined by the commuting diagram

$$\begin{tikzcd}
(L/JL) \oplus (\mathbb{R}(X)/J)^{n-1} \arrow[rr, dashed, "\tilde{\omega}_J"] \arrow[dr, "\gamma"'] & & J/J^2 \\
& P/JP \arrow[ur, "\phi/J"'] &
\end{tikzcd}$$

where $\gamma : (L/JL) \oplus (\mathbb{R}(X)/J)^{n-1} \to P/JP$ is an isomorphism with

$$\bigwedge\nolimits^n \gamma = \chi/J : L/JL \cong \bigwedge\nolimits^n\big((L/JL) \oplus \big(\mathbb{R}(X)/J\big)^{n-1}\big) \to \bigwedge\nolimits^n(P/JP),$$

and for all $f \in P \subset \Gamma(\mathcal{E})$,

$$(\phi/J)\big(f(x_i)\big) = d\big(\phi(f)\big)\big|_{x_i} \in T^*_{x_i}M \subset \bigoplus_i T^*_{x_i}M \cong J/J^2.$$

By Lemma 18, $\phi : P \twoheadrightarrow J$ determines $g_i \in \operatorname{Isom}(\mathcal{E}|_{x_i}, T^*_{x_i}M)$ and a transversal cross section $s : M \to \mathcal{E}^*$ such that $g_i(f(x_i)) = d(\phi(f))|_{x_i}$ for all $f \in P \subset \Gamma(\mathcal{E})$ and $g_i^* = Ds|_{x_i} \in \operatorname{Isom}(T_{x_i}M, \mathcal{E}^*|_{x_i})$, which implies that for all $v \in T_{x_i}M$ and all $f \in P \subset \Gamma(\mathcal{E})$,

$$\big((Ds|_{x_i})^*\big(f(x_i)\big)\big)(v) = \big(g_i\big(f(x_i)\big)\big)(v) = d\big(\phi(f)\big)\big|_{x_i}(v) = \big[(\phi/J)\big(f(x_i)\big)\big](v),$$

i.e. $(Ds|_{x_i})^* = (\phi/J)|_{\mathcal{E}|_{x_i}}$.

Thus the commuting diagram

$$\begin{tikzcd} L/JL \arrow[rr, dashed, "\bigwedge^n \tilde{\omega}_J"] \arrow[dr, "\bigwedge^n \gamma = \chi/J"'] & & \bigwedge^n (J/J^2) \\ & \bigwedge^n (P/JP) \arrow[ur, "\bigwedge^n (\phi/J)"'] & \end{tikzcd}$$

implies

$$\bigwedge^n \tilde{\omega}_J = \bigwedge^n \big((Ds)^*/J\big) \circ (\chi/J) : L/JL \to \bigwedge^n \big(J/J^2\big) = \bigoplus_i \bigwedge^n \big(T^*_{x_i} M\big)$$

where

$$(Ds)^*/J = \bigoplus_i \big((Ds)|_{x_i}\big)^* : \bigoplus_i \mathcal{E}|_{x_i} \to \bigoplus_i T^*_{x_i} M.$$

From this description of $\bigwedge^n \tilde{\omega}_J$, we get

$$\begin{aligned} \varepsilon\big[e(P,\chi)\big] = \varepsilon\big[(J,\omega_J)\big] &= \Bigg[\sum_{i \text{ with } x_i \in M_c} \big[(\chi/J)^{-1} \circ \bigwedge^n \big(\big(\big((Ds)|_{x_i}\big)^*\big)^{-1}\big)\big] x_i\Bigg] \\ &\in H_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c}) \end{aligned}$$

with $[(\chi/J)^{-1} \circ \bigwedge^n ((((Ds)|_{x_i})^*)^{-1})] \in \mathcal{D}_{\mathcal{K}^* \otimes \mathcal{L}} \subset \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}$.

Applying the inverse of the isomorphism

$$(\beta \circ \alpha)_* : H_0(M_c; \mathcal{G}_{\mathcal{L}^* \otimes \mathcal{K}}|_{M_c}) \xrightarrow{\cong} H_0(M_c; \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}|_{M_c})$$

induced by $\mathcal{G}_{\mathcal{L}^* \otimes \mathcal{K}} \overset{\mathcal{G}_{\beta \circ \alpha}}{\cong} \mathcal{G}_{\mathcal{K}^* \otimes \mathcal{L}}$ to $\varepsilon[e(P,\chi)]$, we get

$$(\beta \circ \alpha)_*^{-1}\big(\varepsilon\big[e(P,\chi)\big]\big) = \Bigg[\sum_{i \text{ with } x_i \in M_c} \big[\big(\chi^*/J\big) \circ \bigwedge^n (Ds|_{x_i})\big] x_i\Bigg] \in H_0(M_c; \mathcal{G}_{\mathcal{L}^* \otimes \mathcal{K}}|_{M_c}).$$

There is a simplicial complex structure on M such that x_i's are the barycenters of different n-cells σ_i's, i.e. $x_i = \sigma_i(\bar{e})$ where $\bar{e} := \frac{1}{n+1} \sum_{k=0}^n e_k$. Thus applying the inverse of the isomorphism

$$\Phi : H_0\big(M_c; (\mathcal{G}_{\mathcal{L}^*})'|_{M_c}\big) = H_0(M_c; \mathcal{G}_{\mathcal{L}^* \otimes \mathcal{K}}|_{M_c}) \xrightarrow{\cong} H^n(M; \mathcal{G}_{\mathcal{L}^*})$$

induced by the Poincaré–Steenrod duality as in Corollary 13, we get

$$\begin{aligned} \Phi\big((\beta \circ \alpha)_*^{-1}\big(\varepsilon\big[e(P,\chi)\big]\big)\big) &= \Bigg[\sum_{i \text{ with } x_i \in M_c} \big(\big[\big(\chi^*/J\big) \circ \bigwedge^n (Ds|_{x_i})\big] \otimes [[\sigma_i|_{\partial \Delta_n}]]\big)\sigma_i\Bigg] \\ &= \Bigg[\sum_{i \text{ with } x_i \in M_c} \big(\chi^*/J\big)_*\big([s \circ \sigma_i|_{\partial \Delta_n}]\big)\sigma_i\Bigg] \in H^n(M_c; \mathcal{G}_{\mathcal{L}^*}|_{M_c}). \end{aligned}$$

Note that for any n-cell σ in this simplicial complex, we have

$$[s \circ \sigma|_{\partial \Delta_n}] = 0 \in \mathcal{P}_{n-1}\big((\mathcal{E}^*)^\circ\big)\big|_{\sigma(e_0)} \cong \mathbb{Z}$$

if σ is different from all σ_i's since s then extends to a cross section of $(\mathcal{E}^*)^\circ$ over $\sigma(\Delta_n)$. So we have

$$\begin{aligned} w_n(\mathcal{E}^*) &= \Big[\prod_{\sigma \text{ an } n\text{-cell in } M_c} [s \circ \sigma|_{\partial \Delta_n}]\sigma\Big] = \Big[\prod_{i \text{ with } x_i \in M_c} [s \circ \sigma_i|_{\partial \Delta_n}]\sigma_i\Big] \\ &\in H^n\big(M_c; \mathcal{P}_{n-1}\big((\mathcal{E}^*)^\circ\big)\big|_{M_c}\big). \end{aligned}$$

Hence under the isomorphism

$$(\chi *)_* : H^n\big(M; \mathcal{P}_{n-1}\big((\mathcal{E}^*)^\circ\big)\big) \xrightarrow{\cong} H^n(M; \mathcal{G}_{\mathcal{L}^*})$$

induced by $\mathcal{P}_{n-1}((\mathcal{E}^*)^\circ) = \mathcal{G}_{\bigwedge^n \mathcal{E}^*} \overset{\mathcal{G}_{\chi *}}{\cong} \mathcal{G}_{\mathcal{L}^*}$, we get

$$(\chi *)_*\big(w_n(\mathcal{E}^*)\big) = \Big[\prod_{i \text{ with } x_i \in M_c} \big(\chi^*/J\big)_* [s \circ \sigma_i|_{\partial \Delta_n}]\sigma_i\Big] = \Phi\big((\beta \circ \alpha)_*^{-1}\big(\varepsilon\big[e(P, \chi)\big]\big)\big).$$

We now summarize the above result. First we define the isomorphism

$$\zeta = \big(\Phi \circ (\beta \circ \alpha)_*^{-1}\big) \circ \varepsilon : E\big(\mathbb{R}(X), L\big) \xrightarrow{\cong} H^n(M; \mathcal{G}_{\mathcal{L}^*})$$

called the *Whitney map*, and adopt the notation $w(\mathcal{E}^*, \chi) := \zeta(e(P, \chi))$.

Theorem 21. *Let $X = \operatorname{Spec}(A)$ be a smooth real affine variety with $\dim(X) = n > 1$ and let M be the n-dimensional manifold of real points of X. For a projective $\mathbb{R}(X)$-module P of rank n with an orientation $\chi : L \xrightarrow{\cong} \bigwedge^n P$, if $\mathcal{E}$ and $\mathcal{L}$ are the vector bundles over the manifold M of real points in X corresponding to P and L respectively, then the Euler class $e(P, \chi) \in E(\mathbb{R}(X), L)$ coincides with the Whitney class of the vector bundle $\mathcal{E}^*$ under the isomorphism $(\chi *)_*^{-1} \circ \zeta$, i.e.*

$$(\chi *)_*^{-1}\big(\zeta\big(e(P, \chi)\big)\big) = (\chi *)_*^{-1}\big(w\big(\mathcal{E}^*, \chi\big)\big) = w_n\big(\mathcal{E}^*\big) \in H^n(M; \mathcal{G}_{\bigwedge^n \mathcal{E}^*}),$$

where $\chi * : \bigwedge^n \mathcal{E}^* \xrightarrow{\cong} \mathcal{L}^*$ is the isomorphism canonically induced by χ.

Corollary 22. *With notations as in Theorem 21, P has a unimodular element if and only if $\mathcal{E}$ has a nowhere vanishing continuous cross section.*

Proof. Note that $w_n(\mathcal{E}^*) = 0$ if and only if $\mathcal{E}^*$, or equivalently, $\mathcal{E}$ has a nowhere vanishing continuous cross section, and that $e(P, \chi) = 0$ if and only if P has a unimodular element. We get clearly the corollary. □

Remark. Note that the Whitney class $w_n(\mathcal{E}^*) \in H^n(M; \mathcal{P}_{n-1}((\mathcal{E}^*)^\circ))$ where both $w_n(\mathcal{E}^*)$ and $H^n(M; \mathcal{P}_{n-1}((\mathcal{E}^*)^\circ))$ are intrinsic to the bundle $\mathcal{E}$ in the sense that they depend on the bundle $\mathcal{E}$ but not an external "orientation" $\chi : \mathcal{L} \xrightarrow{\cong} \bigwedge^n \mathcal{E}$ prescribed by a line bundle $\mathcal{L}$. On the other hand, $\zeta : E(\mathbb{R}(X), L) \to H^n(M; \mathcal{G}_{\mathcal{L}^*})$ depends only on a given "orientation" L (or $\mathcal{L}$) and can be applied to $e(P, \chi)$ corresponding to any vector bundle $\mathcal{E}$ over M with any prescribed "orientation"

$\chi : \mathcal{L} \xrightarrow{\cong} \bigwedge^n \mathcal{E}$. So to get $w_n(\mathcal{E}^*)$ from $e(P,\chi)$, we have to include the isomorphism $(\chi *)_*^{-1}$ which involves explicitly $\mathcal{E}$.

Remark. Also note that when $\mathcal{E}$ is endowed with a Riemannian vector bundle structure, we have $\mathcal{E}^* \cong \mathcal{E}$ and hence $w_n(\mathcal{E}^*) = w_n(\mathcal{E})$.

9. Applications

Subsequently, for a regular ring A and $X = \mathrm{Spec}(A)$, the Chow group of zero cycles modulo rational equivalence will be denoted by $CH_0(A)$ or $CH_0(X)$.

Example 23. Suppose $X = \mathrm{Spec}(A)$ is a smooth real affine variety with $\dim(X) = n \geqslant 2$ and M is the manifold of real points of X. We assume M is non-empty. Let $CH^{\mathbb{C}}(X)$ be the subgroup of $CH_0(X)$ generated by the complex points of X. Assume $CH^{\mathbb{C}}(X) \neq 0$. Let L be a projective A-module with $\mathrm{rank}(L) = 1$. Then, there is a projective A-module P with $\mathrm{rank}(P) = n$ and $\det(P) \cong L$ such that P does not have a unimodular element, but the corresponding vector bundle $\mathcal{E} = \mathcal{V}(P)$, whose module of sections $\Gamma(\mathcal{E}) = P \otimes_A C(M)$ has a nowhere vanishing section.

Proof. Let x be a complex point in X such the cycle $[x] \in CH_0(X)$ is non-zero. Let m be a maximal ideal corresponding to x. By the proof of [BRS2, Lemma 5.1], there is a local complete intersection ideal I of A with height n such that (1) $I = (f_1, f_2, \dots, f_n) + I^2$, (2) $m + I = A$, (3) I is contained in only complex maximal ideals, and (4) with

$$I' = (f_1, f_2, \dots, f_{n-1}) + I^{(n-1)!},$$

we have $m \cap I'$ a complete intersection.

By [DM], there is a projective A-module of rank n and $\det(P) = L^*$ and an orientation $\chi : L \xrightarrow{\sim} \bigwedge^n P$ such that there is a surjective map $P \twoheadrightarrow I'$ and the Euler class $e(P,\chi) = (I',\omega)$ for some local orientation ω. Since

$$\mathrm{cycle}(A/I') = -\mathrm{cycle}(A/m) = -x \neq 0 \in CH_0(X),$$

we have $e(P,\chi) \neq 0$ and so P does not have a nowhere vanishing section. Since I' is supported in complex points, $e(P \otimes \mathbb{R}(X), \chi \otimes \mathbb{R}(X)) = 0$. Let $\mathcal{E}$ be the vector bundle corresponding to P. Then, by our theorem,

$$w_n(\mathcal{E}^*) = (\chi *)_*\bigl(\zeta\bigl(e\bigl(P \otimes \mathbb{R}(X), \chi \otimes \mathbb{R}(X)\bigr)\bigr)\bigr) = 0.$$

Hence $\mathcal{E}^*$ has a nowhere vanishing section and so does $\mathcal{E}$. $\square$

Example 24. M.P. Murthy communicated an explicit example of a real smooth affine variety $X = Spec(A)$, as follows, for which Example 23 is applicable. Let

$$A = \frac{\mathbb{R}[X_0, X_1, \dots, X_n]}{(\sum X_i^d - 1)} \quad \text{where } d > n+1.$$

Then $CH^{\mathbb{C}}(A) \neq 0$.

Proof. Let

$$B = \frac{\mathbb{Q}[T_0, T_1, \dots, T_n]}{(\sum T_i^d - 1)} \quad \text{and} \quad C = \frac{k[X_0, X_1, \dots, X_n]}{(\sum X_i^d - 1)}$$

where k is the fraction field of B. Let $Y \subseteq \mathbb{P}_k^{n+1}$ be the projective subvariety defined by the equation $\sum_{i=0}^n X_i^d - Z^d = 0$. Then, the geometric genus $p_g(Y) = \dim H^0(Y, \Omega_{Y/k}^n) > 0$. It follows from the results of Mumford ([Mu], also see Bloch [Bl]) that the cycle $x \in CH_0(C)$ corresponding to the maximal ideal $(X_0 - T_0, \ldots, X_n - T_n)C \subseteq C$ has infinite order.

We can find $t_0, t_1, \ldots, t_{n-1} \in \mathbb{R}$ transcendental over $\mathbb{Q}$, and small enough, so that there is $t_n \in \mathbb{R}$ with $\sum_{i=0}^n t_i^d = 1$. Using this, we can assume that $k = \mathbb{Q}(t_0, t_1, \ldots, t_n) \subseteq \mathbb{R}$ and $C \subseteq A$ is a subring.

Now, we claim that the image of x in $CH_0(A)$ has infinite order. If not, then there is a field extension $k \subseteq K \subseteq \mathbb{R}$ such that K is finitely generated over k and the image of x in $CH_0(K \otimes_k C)$ is a torsion element. This is impossible.

Since, $CH_0(A)/CH^{\mathbb{C}}(A)$ is finite (see [BRS2, Theorem 4.10]), it follows that $CH^{\mathbb{C}}(A) \neq 0$. This completes the proof. □

The following is a generalized version of a result in [MaSh].

Theorem 25. *Let $X = \operatorname{Spec}(A)$, $\mathbb{R}(X)$, and $M \neq \phi$ be as in Theorem* 21. *Then, the following diagram*

$$\begin{array}{ccc} E(\mathbb{R}(X), L) & \xrightarrow{\zeta} & H^n(M; \mathcal{G}_{\mathcal{L}^*}) \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \mu} \\ CH_0(\mathbb{R}(X)) & \xrightarrow{\zeta_0} & H^n(M; \mathbb{Z}_2) \end{array}$$

commutes, where Θ and μ are the natural homomorphisms and ζ_0 is an isomorphism.

Proof. For any bundle $\mathcal{G}$ of groups modeled on $\mathbb{Z}$ over a path connected space M, it is easy to see that there is a canonical surjective bundle homomorphism η from $\mathcal{G}$ to the trivial bundle $\mathcal{Z} = M \times \mathbb{Z}_2$ of groups modeled on $\mathbb{Z}_2$ over M, sending any $c \in \mathcal{G}_x \cong \mathbb{Z}$ to c modulo 2 in $\mathbb{Z}_2$ for all $x \in M$, independent of the choice of the isomorphism $\mathcal{G}|_x \cong \mathbb{Z}$. Furthermore f induces a canonical surjective homomorphism

$$H_0(f) : H_0(M; \mathcal{G}) \twoheadrightarrow \mathcal{Z}|_x \cong H_0(M; \mathcal{Z}) \cong H_0(M; \mathbb{Z}_2)$$

for any fixed $x \in M$, since $H_0(M; \mathcal{G})$ is isomorphic to $\mathcal{G}|_x$ modulo the $\pi_1(M, x)$-action, where $\pi_1(M, x)$ acts on $\mathcal{G}|_x \cong \mathbb{Z}$ as multiplication by ± 1 as discussed in Proposition 1. Thus if M is a connected n-dimensional manifold, we have a surjective homomorphism

$$H^n(f) : H^n(M; \mathcal{G}) \twoheadrightarrow H^n(M; \mathcal{Z}) \cong H^n(M; \mathbb{Z}),$$

as either gotten from the Poincaré–Steenrod duality when M is compact, or from $H^n(M; \mathcal{G}) = 0 = H^n(M; \mathcal{Z})$ when M is non-compact and hence homotopy equivalent to an $(n-1)$-dimensional CW complex [NR].

Applying the above result to each connected component M' of M including those r compact ones with $\mathcal{K}|_{M'} \cong \mathcal{L}|_{M'}$ and those s compact ones with $\mathcal{K}|_{M'} \ncong \mathcal{L}|_{M'}$, we get a surjective homomorphism μ in the following commuting diagram

$$\begin{array}{ccccccc} \mathbb{Z}^r \oplus (\mathbb{Z}_2)^s & \cong & E(\mathbb{R}(X), L) & \xrightarrow{\zeta} & H^n(M; \mathcal{G}_{\mathcal{L}^*}) & \cong & \mathbb{Z}^r \oplus (\mathbb{Z}_2)^s \\ & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \text{mod } 2} \\ (\mathbb{Z}_2)^{r+s} & \cong & CH_0(\mathbb{R}(X)) & & H^n(M; \mathbb{Z}_2) & \cong & (\mathbb{Z}_2)^{r+s} \end{array}$$

where Θ is the well-known surjective homomorphism [BRS1]. Thus the isomorphism ζ clearly factors through an isomorphism $\zeta_0 : CH_0(\mathbb{R}(X)) \to H^n(M;\mathbb{Z}_2)$. $\square$

We remark that the Whitney class $w_n(\mathcal{E})$ modulo 2, i.e. $sw_n(\mathcal{E}) := \mu(\zeta(e(P^*, (\chi^*)^{-1})))$, is the so-called Stiefel–Whitney class of the vector bundle $\mathcal{E}$ [Wh2]. The following extends a result obtained in [MaSh].

Theorem 26. *We use the notations as in Theorem* 25. *Then, the diagram*

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(\mathbb{R}(X)) & \longrightarrow & KO(M) \\ \downarrow{\scriptstyle C_0} & & \downarrow{\scriptstyle C_0} & & \downarrow{\scriptstyle sw_n} \\ CH_0(A) & \longrightarrow & CH_0(\mathbb{R}(X)) & \xrightarrow{\zeta_0} & H^n(M;\mathbb{Z}_2) \end{array}$$

commutes, where $K_0(A)$, $K_0(\mathbb{R}(X))$ (respectively, $KO(M)$) denote the Grothendieck group of finitely generated projective modules over the corresponding ring (respectively, of real vector bundles over M), C_0 denotes the top Chern class homomorphism, and sw_n denotes the top Stiefel–Whitney class homomorphism.

Proof. We only need to prove that the second rectangle commutes. Let $\tau = [P] - [\mathbb{R}(X)^n] \in K_0(\mathbb{R}(X))$ be any element, where P is a projective $\mathbb{R}(X)$-module, with $\mathrm{rank}(P) = n$. Write $\bigwedge^n P = L$ and fix an orientation $\chi : L \xrightarrow{\cong} \bigwedge^n P$. Let $\mathcal{E} = \mathcal{V}(P)$ and $\mathcal{L} = \mathcal{V}(L)$ be the vector bundles on M, respectively, corresponding to P and L. Let the orientation of $\mathcal{E}$ induced by χ be still denoted by χ as before. By Theorem 21, $\zeta(e(P^*, (\chi^*)^{-1})) = w(\mathcal{E}, \chi)$.

Recall, that the top Chern class of P is given by a generic section $f : P^* \twoheadrightarrow J$ (see [Mu1, Remark 3.6]). Now, by Theorem 25 (also see [MiSf, Property 9.5]), we have

$$\begin{aligned}\zeta_0 C_0(\tau) &= \zeta_0\big(C_0(P)\big) = \zeta_0\big(\Theta\big(e\big(P^*, \big(\chi^*\big)^{-1}\big)\big)\big) \\ &= \mu\big(\zeta\big(e\big(P^*, \big(\chi^*\big)^{-1}\big)\big)\big) = sw_n(\mathcal{E}). \qquad \square\end{aligned}$$

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