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Bases of ideals and Rees valuation rings

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ABSTRACT

Let I be a regular proper ideal in a Noetherian ring R. We prove that there exists a simple free integral extension ring A of R such that the ideal IA has a Rees-good basis; that is, a basis $c_1, \dots, c_g$ such that $c_i W = IW$ for $i = 1, \dots, g$ and for all Rees valuation rings W of IA. Moreover, A may be constructed so that: (i) IA and I have the same Rees integers (with possibly different cardinalities), and (ii) A_P is unramified over $R_{P \cap R}$ for each asymptotic prime divisor P of IA. Indeed, if H is a regular ideal in R such that each asymptotic prime divisor of H is contained in an asymptotic prime divisor of I, then (ii) holds for HA. If $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$, we prove that (i) also holds for HA and H. If $I = (b_1, \dots, b_g)R$ and $b_1, \dots, b_g$ is an asymptotic sequence, we prove that $b_1, \dots, b_g$ is a Rees-good basis of I.

1. Introduction

All rings in this paper are commutative with a unit $1 \neq 0$. Let I be a regular proper ideal of the Noetherian ring R, that is, I contains a regular element of R and $I \neq R$. The set $\operatorname{Rees} I$ of Rees valuation rings of I is a finite set of rank one discrete valuation rings (DVRs) that determine the integral closure $(I^k)_a$ of I^k for every positive integer k and is the unique minimal set of DVRs having this property. Recall that $(I^k)_a = \{x \in R \mid \text{there exists a positive integer } h \text{ and elements } i_j \in I^{kj}, \text{ for } j = 1, \dots, h, \text{ such that } x^h + i_1 x^{h-1} + \dots + i_h = 0\}$. If $(V_1, N_1), \dots, (V_n, N_n)$ are the Rees valuation rings of I, then the integers $(e_1, \dots, e_n)$, where $IV_i = N_i^{e_i}$, are the **Rees integers** of I.

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We introduce the following terminology.

Definition 1.1. Let I be a regular proper ideal in a Noetherian ring R. An element $b \in I$ is said to be **Rees-good** for I in case $bV = IV$ for all Rees valuation rings V of I. A basis $b_1, \ldots, b_g$ of I is said to be **Rees-good** in case b_i is Rees-good for I for $i = 1, \ldots, g$.

If R is a Noetherian integral domain, the existence of an element $b \in I$ that is Rees good for I implies that all the Rees valuation rings of I are obtained as localizations of the integral closure of $R[I/b]$ at height-one primes containing b. Thus the existence of a Rees good element b for I allows one to focus on the one affine piece $R[I/b]$ of the blowup $\operatorname{Proj} R[It]$ of R along I, cf. [20, pp. 194–195]. Concerning Rees-good bases, H.T. Muhly and M. Sakuma prove in [11, Lemma 3.1] that some power I^k of I contains an element b such that $bV = I^kV$ for all Rees valuation rings V of I, or equivalently of I^k. It then follows that b^h has the analogous property for I^{kh} for all positive integers h. It is shown in [4, (3.19) and (3.20)] that if either (i) R contains an infinite field, or (ii) R is a local ring with an infinite residue field, then every ideal I in R has a Rees-good basis. On the other hand, it is asked in [4, (3.9)] if there always exists a power I^k of I that has a Rees-good basis. Theorem 3.7 of [4] shows that if I has a Rees-good basis and if the least common multiple of the Rees integers of I is a unit of R, then there exists a finite free integral extension ring A of R such that $J := \operatorname{Rad}(IA)$ is projectively equivalent to IA and all the Rees integers of J are equal to one. This motivates our interest in the question of the existence of Rees-good bases. In this paper we examine this question and consider properties of the Rees integers of I and of IA, where A is a finite free integral extension ring of R.

In Section 2 we give several sufficient conditions for I to have a Rees-good basis, and demonstrate in Example 2.3 the existence of a Gorenstein local ring (R, M) of altitude one such that no power of M has a Rees-good basis. Recall that an ideal J is **projectively equivalent** to I if $(I^m)_a = (J^n)_a$ for some positive integers m and n. In addition to the papers [2–4] and the references listed there, further interesting results concerning projective equivalence can be found in [6, Proposition 2.1], [7–9]. We also use the following definition, see [10] and [20, p. 111].

Definition 1.2. (1.2.1) Let $\overline{A}^*(I) = \{P \in \operatorname{Spec}(R) \mid P \in \operatorname{Ass}(R/(I^i)_a)$ for some positive integer $i\}$. Then $\overline{A}^*(I)$ is the set of **asymptotic prime divisors** of I.

(1.2.2) The sequence of elements $b_1, \ldots, b_g$ in R is an **asymptotic sequence** provided that $(b_1, \ldots, b_g)R \neq R$ and for each $i \in \{1, \ldots, g\}$ the element b_i is not in any asymptotic prime divisor of $(b_1, \ldots, b_{i-1})R$. (In particular, b_1 is not in any minimal prime ideal in R.)

Theorem 2.14 shows that if for each asymptotic prime divisor p of I that is a maximal ideal of R one has $\operatorname{Card}(R/p) \geqslant \operatorname{Card}(\operatorname{Rees} I)$, then every ideal H projectively equivalent to I has a Rees-good basis. In the final part of Section 2 it is shown that a similar approach can be used to prove that if a commutative ring R contains a set of $n-1$, $n > 2$, units $u_1, \ldots, u_{n-1}$ such that $u_i - u_j$ is a unit in R for all $i \neq j$ in $\{1, \ldots, n-1\}$, then no finitely generated R-module is the union of any $k \leqslant n$ proper submodules.

Recall that a ring (R, M) is quasi-local if M is the unique maximal ideal of R and R is not necessarily Noetherian. We use Definition 1.3 in Section 3.

Definition 1.3. (1.3.1) A quasi-local ring (R', M') is **unramified** over a quasi-local ring (R, M) in case R is a subring of R', $M' = MR'$, and R'/M' is separable over R/M. A prime ideal p' of R' is **unramified** over $p' \cap R$ in case $R'_{p'}$ is unramified over $R_{p' \cap R}$.

(1.3.2) $\mathbf{R}(R, I)$ denotes the **Rees ring of** R **with respect to** I, so $\mathbf{R}(R, I)$ is the graded subring $R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = \frac{1}{t}$.

(1.3.3) Let $z_1, \ldots, z_r$ be the minimal prime ideals z in R such that $z + I \neq R$, for $i = 1, \ldots, r$ let $R_i = R/z_i$, let F_i be the quotient field of R_i, let $\mathbf{R}'_i$ be the integral closure in $F_i(u)$ of $\mathbf{R}_i = \mathbf{R}(R_i, (I + z_i)/z_i)$ (see (1.3.2)), let $p_{i,1}, \ldots, p_{i,h_i}$ be the (height-one) prime divisors of $u\mathbf{R}'_i$, let $w_{i,j}$ be the valuation of the discrete valuation ring $W_{i,j} = \mathbf{R}'_{i\,p_{i,j}}$, let $e_{i,j} = w_{i,j}(u)$, let $V_{i,j} = W_{i,j} \cap F_i$, and define $v_{i,j}$ on R by $v_{i,j}(x) = w_{i,j}(x + z_i)$. Then the **Rees valuations** of I are the valuations $v_{1,1}, \ldots, v_{r,h_r}$, and the **Rees valuation rings** of I are the rings $V_{1,1}, \ldots, V_{r,h_r}$. We use **Rees** I to denote the set $\{V_{i,j} \mid i = 1, \ldots, r \text{ and } j = 1, \ldots, h_r\}$ of all the Rees valuation rings of I.

(1.3.4) If $\operatorname{Rees} I = \{(V_1, N_1), \ldots, (V_n, N_n)\}$ and $IV_i = N_i^{e_i}$, then e_i is the **Rees integer of** I **with respect to** V_i.

We prove in Theorem 3.7 that there always exists a simple free integral extension ring A of R such that IA has a Rees-good basis. Moreover, A may be constructed so that, for each regular ideal H in R whose asymptotic prime divisors are contained in the union of the asymptotic prime divisors of I and for which $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$, A_P is unramified over $R_{P \cap R}$ for each asymptotic prime divisor P of HA and HA has a Rees-good basis and the same Rees integers as H (with possibly different cardinalities). It follows from Theorem 3.7 that the assumption that I has a Rees-good basis in [3] and [4] is, in fact, superfluous. In this connection, see also [5, Theorem 4.1].

If $I = (b_1, \ldots, b_g)R$ and $b_1, \ldots, b_g$ is an asymptotic sequence, we prove in Theorem 4.2 that $b_1, \ldots, b_g$ is a Rees-good basis of I.

Our notation is as in [12] and [21]. Thus, for example, a **basis** of a module or ideal is a set of elements that generate the module or ideal, and the **altitude** of a ring is the maximal length of a chain of prime ideals in the ring.

2. Rees-good bases for ideals

Recall that if I is a regular ideal of a Noetherian ring R, then

$$\operatorname{Rees} I = \bigcup \{\operatorname{Rees} IR/z \mid z \text{ is a minimal prime ideal of } R \text{ such that } I + z \neq R\}.$$

See for example [2, (2.2)(c) and (2.4)]. Thus if $(V, N) \in \operatorname{Rees} I$, then V is a valuation ring of the quotient field of R/z for some minimal prime ideal z of R and the **center of** V in R is $\phi^{-1}(N) = \phi^{-1}(N \cap (R/z))$, where $\phi : R \to R/z$ is the canonical map. If $H \subseteq V$ is an ideal of V, we sometimes write $\phi^{-1}(H)$ as $H \cap R$.

We fix the following notation.

Notation 2.1. Let I be a regular proper ideal in a Noetherian ring R and let $\{(V_i, N_i)\}_{i=1}^n$ be the set of Rees valuation rings of I. For $j \in \{1, \ldots, n\}$ let $H_j = \{x \in I \mid xV_j \subsetneq IV_j\}$.

Lemma 2.2 describes the relation between Rees-good elements $b \in R$ and the sets H_i of Notation 2.1, and also gives two cases when I has a Rees-good basis.

Lemma 2.2. *With the notation of* (2.1)*, the following hold*:

(2.2.1) *$H_i = H_i V_i \cap I$ is an ideal in R that is properly contained in I for $i = 1, \ldots, n$.*
(2.2.2) *An element $b \in I$ is Rees-good for I if and only if $b \notin H_1 \cup \cdots \cup H_n$.*
(2.2.3) *If either I is principal or $n = 1$, then I has a Rees-good basis.*

Proof. Using Notation 2.1 and Definition 1.1, (2.2.1) and (2.2.2) follow from basic properties of valuations.

For (2.2.3), if $I = bR$ is principal, then it is clear that b is a Rees-good basis of I. On the other hand, if I has only one Rees valuation ring V and if $c_1, \ldots, c_g$ is an arbitrary basis of I, then $c_i V = IV$ for some $i \in \{1, \ldots, g\}$. We may assume by renumbering that $c_i V = IV$ for $i = 1, \ldots, k$, where $1 \leqslant k \leqslant g$. It is then readily checked that $c_1, \ldots, c_k, c_{k+1} + c_1, \ldots, c_g + c_1$ is a Rees-good basis for I. $\square$

Example 2.3 exhibits a Gorenstein local ring (R, M) of altitude one such that M has no Rees-good elements and such that no power of M has a Rees-good basis.

Example 2.3. Let F be the field with two elements, let X, Y be independent indeterminates over F, let $R = F[\![X,Y]\!]/(XY(X+Y))$, and let x, y denote the images in R of X, Y, respectively. Then $M = (x, y)R$ has three Rees valuation rings

$$V_1 := F[\![X,Y]\!]/(X), \qquad V_2 := F[\![X,Y]\!]/(Y), \quad \text{and} \quad V_3 := F[\![X,Y]\!]/(X+Y).$$

With notation as in (2.1), notice that

$$H_1 = xR + M^2, \qquad H_2 = yR + M^2, \quad \text{and} \quad H_3 = (x+y)R + M^2.$$

Therefore $M = H_1 \cup H_2 \cup H_3$, so M does not have any Rees-good elements, by Lemma 2.2.2. Since $xy(x+y) = 0$ and F is of characteristic two, one has $x^2y = xy^2$, and for $n \geqslant 3$

$$x^{n-1}y = x^{n-2}y^2 = \cdots = xy^{n-1}.$$

Thus $\{x^n, x^{n-1}y, y^n\}$ is a minimal basis of M^n for every $n \geqslant 2$. It follows that the only Rees-good element for M^n, up to congruence mod M^{n+1}, is $x^n + x^{n-1}y + y^n$, for every $n \geqslant 2$. For $g \in M^n$ can be written $g = ax^n + bx^{n-1}y + cy^n + h$ with $a, b, c \in F$ and $h \in M^{n+1}$, and g is a Rees-good element for M^n if and only if $a = b = c = 1$.

Remark 2.4. For M as in Example 2.3, there exist principal ideals that are projectively equivalent to M. Therefore there exist ideals projectively equivalent to M that have a Rees-good basis.

Our main interest in this paper is to determine when I has a Rees-good basis. In view of (2.2.3) we assume throughout the remainder of this section that $n > 1$; that is, that I has more than one Rees valuation ring.

Lemma 2.5. *With the notation of* (2.1)*, let $u, v \in R$ and $x, y \in I$. Then the following hold*:

(2.5.1) *If $x + uy, x + vy \in H_1$ and if $(u - v)V_1 = V_1$, then $x, y \in H_1$.*
(2.5.2) *If $x \in H_1$ and $y \notin H_1$, then $rx + wy \notin H_1$ for all elements $r, w \in R$ with $wV_1 = V_1$.*

Proof. For (2.5.1), if $x + uy, x + vy \in H_1$, then $(u - v)y \in H_1$. Since $(u - v)V_1 = V_1$, it follows that $y \in H_1$. Therefore $x = (x + uy) - uy \in H_1$.

For (2.5.2), if $x \in H_1$ and $rx + wy \in H_1$, then $wy \in H_1$. Since $wV_1 = V_1$, it follows that $y \in H_1$. $\square$

Lemma 2.5.1 applied to each V_i suggests the following definition.

Definition 2.6. With the notation of (2.1), let $U = \{u_1, \ldots, u_n\} \subseteq R$, $n \geqslant 2$. Assume first that R is an integral domain with quotient field F and let $D = \bigcap_{i=1}^n V_i$. Then U is said to be a set of D-**units with** D-**unit-differences** in case $u_1, \ldots, u_n$ and the $u_i - u_j$ are units in D for all $i \neq j$ in $\{1, \ldots, n\}$. If R is not an integral domain, then by abuse of terminology we continue to say that U is a set of D-units with D-unit-differences in case the images in V_k $(k = 1, \ldots, n)$ of $u_1, \ldots, u_n$ and the $u_i - u_j$ are units in V_k for all $i \neq j$ in $\{1, \ldots, n\}$. If $u_1, \ldots, u_n$ and the $u_i - u_j$ are, in fact, units in R, then it is said that U is a set of R-**units with** R-**unit-differences**.

Using the notation of Definitions 1.2.1 and 2.6, we have

Lemma 2.7.

(2.7.1) $\operatorname{Ass}(R/(I^i)_a) \subseteq \operatorname{Ass}(R/(I^{i+1})_a)$ *for every positive integer i, and $\overline{A}^*(I)$ is a finite set.*
(2.7.2) *The prime ideals in $\overline{A}^*(I)$ are precisely the prime ideals that are the center in R of some Rees valuation ring of I.*

(2.7.3) *Let $u_1, \dots, u_n \in R$. Then $u_1, \dots, u_n$ are D-units with D-unit-differences if and only if $\{u_1, \dots, u_n\} \cup \{u_i - u_j \mid i \neq j \in \{1, \dots, n\}\} \subseteq R \setminus \bigcup\{P \mid P \in \bar{A}^*(I)\}$.*

Proof. (2.7.1) is proved in [17, (2.4) and (2.7)].

(2.7.3) follows immediately from (2.7.2), and (2.7.2) follows from the construction/definition of Rees valuation rings given in [2, Definition 2.3]. □

Lemma 2.8. *With the notation of* (2.1)*, if there exists a set $U = \{u_1, \dots, u_{n-1}\} \subseteq R$, $n \geqslant 3$, of D-units with D-unit-differences, then $I \neq \bigcup_{i=1}^{n} H_i$.*

Proof. By possibly removing some of the H_i, we may assume that no H_i is contained in H_j with $i \neq j$. First observe that $I \neq H_1 \cup H_2$. Indeed, if $x_1 \in H_1 \setminus H_2$ and $x_2 \in H_2 \setminus H_1$, then by (2.5.2), $x_1 + x_2 \in I \setminus (H_1 \cup H_2)$.

Now assume that $2 \leqslant h < n$ and $I \neq H_{i_1} \cup \dots \cup H_{i_h}$ for any subset $\{H_{i_1}, \dots, H_{i_h}\} \subseteq \{H_1, \dots, H_n\}$. Suppose $I = H_1 \cup \dots \cup H_{h+1}$. Then there exist $x_1 \in H_1 \setminus \bigcup_{i=2}^{h+1} H_i$ and $x_2 \in \bigcup_{i=2}^{h+1} H_i \setminus H_1$. So $x_1 + u_i x_2 \notin H_1$ for $i = 1, \dots, n-1$, by (2.5.2).

Also, $x_2 \in H_m$ for some $m \in \{2, \dots, h+1\}$, so $x_1 + u_i x_2 \notin H_m$ for $i = 1, \dots, n-1$, by (2.5.2). Therefore, since $h + 1 \leqslant n$, at least one of the $h - 1$ ($\leqslant n - 2$) submodules H_l (with $l \in \{2, \dots, m-1, m+1, \dots, h+1\}$) must contain $x_1 + u_i x_2$ and $x_1 + u_j x_2$ for some $i \neq j \in \{1, \dots, n-1\}$. But this, together with (2.5.1), implies that $x_1 \in H_l$, contradicting the choice of x_1. Thus $I \neq H_1 \cup \dots \cup H_{h+1}$. □

Lemma 2.9. *With the notation of* (2.1)*, assume that $n \geqslant 2$ and that there exists a set $U = \{u_1, \dots, u_n\} \subseteq R$ of D-units with D-unit-differences. Let $c_1, \dots, c_g$ be a* (*not necessarily minimal*) *basis of I, and assume that there exists an integer k such that*: (a) $1 \leqslant k < g$; *and,* (b) *$c_i \notin H_1 \cup \dots \cup H_n$ if and only if $i = 1, \dots, k$. Then there exists $x \in I$ such that $x \notin H_1 \cup \dots \cup H_n$ and $c_1, \dots, c_k, x, c_{k+2}, \dots, c_g$ is a basis of I.*

Proof. For $h = 1, \dots, n$ let $x_h = c_{k+1} + u_h c_1$, so $c_1, \dots, c_k, x_h, c_{k+2}, \dots, c_g$ is a basis of I for $h = 1, \dots, n$. Also, by (b) there exists at least one $j \in \{1, \dots, n\}$ such that $c_{k+1} \in H_j$. Since $u_h c_1 \notin H_j$ for $h, j \in \{1, \dots, n\}$, by (b) (since u_h is a D-unit in R), it follows from (2.5.2) that $x_h = c_{k+1} + u_h c_1 \notin \bigcup\{H_j \mid c_{k+1} \in H_j\}$ for $h = 1, \dots, n$. Also, by the hypotheses on c_1 and the u_h, it follows from (2.5.1) that each H_j can contain at most one of the n elements x_h, so it follows that there exists at least one $h \in \{1, \dots, n\}$ such that $x_h \notin H_1 \cup \dots \cup H_n$. Therefore the conclusion follows by letting $x = x_h$. □

Remark 2.10. It follows from Lemma 2.9 and its proof that if $c_1, \dots, c_g$ is a basis of I such that (a) and (b) hold, and if $U = \{u_1, \dots, u_n\}$ is as in (2.9), then there exist not necessarily distinct D-units $v_1, \dots, v_{g-k}$ in U such that $xV_j = IV_j$ for each $x \in \{c_1, \dots, c_k, c_{k+1} + v_1 c_1, \dots, c_g + v_{g-k} c_1\}$, that is, by Definition 1.1, such that $c_1, \dots, c_k, c_{k+1} + v_1 c_1, \dots, c_g + v_{g-k} c_1$ is a Rees-good basis for I.

Proposition 2.11. *With the notation of* (2.1)*, assume that $n \geqslant 2$ and that there exists a set $U = \{u_1, \dots, u_n\} \subseteq R$ of D-units with D-unit-differences. Then I has a Rees-good basis.*

Proof. By (2.2.3) it may be assumed that each basis $b_1, \dots, b_g$ of I has $g > 1$. Let $c_1, \dots, c_g$ be an arbitrary basis of I and let $H_1, \dots, H_n$ be as in (2.1). If $c_1, \dots, c_g \notin H_1 \cup \dots \cup H_n$, then the conclusion holds with $b_i = c_i$ for $i = 1, \dots, g$ by (1.1) and (2.2.2). On the other hand, if $c_i \in H_1 \cup \dots \cup H_n$ and $c_j \notin H_1 \cup \dots \cup H_n$ for some $i, j \in \{1, \dots, g\}$, then it may be assumed that $c_1, \dots, c_k \notin H_1 \cup \dots \cup H_n$ and that $c_{k+1}, \dots, c_g \in H_1 \cup \dots \cup H_n$, so the conclusion follows from Remark 2.10. Therefore it may be assumed that $c_1, \dots, c_g \in H_1 \cup \dots \cup H_n$. Then it follows from Lemma 2.8 that there exists $b_1 \in I \setminus (H_1 \cup \dots \cup H_n)$. Then $b_1, c_1, \dots, c_g$ is a basis of I with $b_1 \notin H_1 \cup \dots \cup H_n$ and $c_1, \dots, c_g \in H_1 \cup \dots \cup H_n$, so the conclusion follows from Remark 2.10. □

Remark 2.12. Let $c_1, \dots, c_g$ be an arbitrary basis of I and let $U = \{u_1, \dots, u_n\}$, $n \geqslant 2$, be a set of D-units with D-unit-differences. The proof of Proposition 2.11 shows*:*

(i) If $c_1, \dots, c_g \notin H_1 \cup \dots \cup H_n$, then $c_1, \dots, c_g$ is a Rees-good basis of I.
(ii) If $c_1, \dots, c_k \notin H_1 \cup \dots \cup H_n$ and $c_{k+1}, \dots, c_g \in H_1 \cup \dots \cup H_n$, then there exist not necessarily distinct D-units $v_1, \dots, v_{g-k}$ in U such that $c_1, \dots, c_k, c_{k+1} + v_1 c_1, \dots, c_g + v_{g-k} c_1$ is a Rees-good basis for I.
(iii) If $c_1, \dots, c_g \in H_1 \cup \dots \cup H_n$, then there exists $b_1 \in I \setminus (H_1 \cup \dots \cup H_n)$ and not necessarily distinct D-units $v_1, \dots, v_g$ in U such that $b_1, c_1 + v_1 b_1, \dots, c_g + v_g b_1$ is a Rees-good basis for I.

Remark 2.13. Let R be a Noetherian ring. With the terminology as in Definition 2.6, assume that there exists in R a set $U = \{u_1, \dots, u_n\}$, $n \geqslant 2$, of R-units with R-unit-differences. Then Proposition 2.11 implies that every regular proper ideal I in R such that $\operatorname{Card}(\operatorname{Rees} I) \leqslant n$ has a Rees-good basis. Moreover, the proof of Proposition 2.11 shows how to obtain such a basis.

Theorem 2.14. *With the notation of* (2.1) *and* (1.2)*, assume that $n \geqslant 2$. The following properties are equivalent.*

(1) $\operatorname{Card}(R/p) > n$ *for each $p \in \overline{A}^*(I)$ that is a maximal ideal of R.*
(2) *There exists a set $U = \{u_1, \dots, u_n\} \subseteq R$ of D-units with D-unit-differences.*

If these equivalent properties hold, then each regular ideal H in R such that $\bigcup\{q \mid q \in \overline{A}^*(H)\} \subseteq \bigcup\{p \mid p \in \overline{A}^*(I)\}$ and $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$ *has a Rees-good basis. In particular, each ideal H of R that is projectively equivalent to I has a Rees-good basis.*

Proof. (1) $\Rightarrow$ (2) The ideals $p_i = N_i \cap R$, $i = 1, \dots, n$, are the (not necessarily distinct) elements of $\overline{A}^*(I)$, by Lemma 2.7.2, so let $p_1, \dots, p_k$ be the (distinct) maximal members of $\overline{A}^*(I)$, and let $T = R \setminus \bigcup\{p \mid p \in \overline{A}^*(I)\}$. Then R_T is semi-local with maximal ideals $p_1 R_T, \dots, p_k R_T$, IR_T is contained in the Jacobson radical of R_T, and $\operatorname{Rees} IR_T = \operatorname{Rees} I$, by [17, (6.5) and (6.8)] and the definition of T. For $i = 1, \dots, k$ let $u_{i,1}, \dots, u_{i,n} \in R_T$ be such that their images in $R_T/p_i R_T$ are nonzero and distinct. These exist by hypothesis if p_i is a maximal ideal of R. Otherwise, R/p_i is infinite. For $j = 1, \dots, n$ there exists $u_j \in R_T$ such that $u_j - u_{i,j} \in p_i R_T$ for $i = 1, \dots, k$, by comaximality (see [21, Theorem 31, p. 177]). Let $t \in T$ be such that $tu_j \in R$ for $j = 1, \dots, n$. We claim that $\{tu_1, \dots, tu_n\}$ is a set of D-units in R with D-unit-differences. To see this it suffices to show for each Rees valuation ring (V_i, N_i) of I that the images of $tu_1, \dots, tu_n$ in V/N_i are nonzero and distinct. If $p_i = N_i \cap R$ is maximal in $\overline{A}^*(I)$, they are nonzero and distinct in R/p_i by construction, and thus nonzero and distinct in V_i/N_i. If $p_i = N_i \cap R$ is not maximal in $\overline{A}^*(I)$, we have a homomorphism $R/p_i \to R/p_j$ for some prime ideal p_j that is maximal in $\overline{A}^*(I)$. Since the images of $tu_1, \dots, tu_n$ are nonzero and distinct in R/p_j, their images in R/p_i must be nonzero and distinct, and thus nonzero and distinct in V_i/N_i.

(2) $\Rightarrow$ (1) If $p \in \overline{A}^*(I)$ is a maximal ideal of R, then we have homomorphisms $R \to R/p \to V_i/N_i$ and the elements of U map to n distinct nonzero elements of V_i/N_i by hypothesis. Thus the elements of U map to n distinct elements of R/p. So $\operatorname{Card}(R/p) > n$.

It follows from Proposition 2.11 that the condition (2) implies that I has a Rees-good basis. But conditions (1) and (2) depend only on the set of maximal asymptotic prime divisors of I, so it follows that each regular ideal H in R such that $\bigcup\{q \mid q \in \overline{A}^*(H)\} \subseteq \bigcup\{p \mid p \in \overline{A}^*(I)\}$ and $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$ also has a Rees-good basis. The last statement follows from this, since projectively equivalent ideals have the same asymptotic prime divisors, by [2, Theorem 3.4]. □

The next two corollaries follow immediately from Theorem 2.14.

Corollary 2.15. *Let I be a regular proper ideal in a Noetherian ring R and assume that no member of $\overline{A}^*(I)$ is a maximal ideal of R. Then I has a Rees-good basis.*

Corollary 2.16. *Let R be a Noetherian ring and assume that R/M is infinite for all maximal ideals M in R. Then every regular proper ideal in R has a Rees-good basis.*

In the final three results in this section we show that, with a slight change of perspective, there is a useful result analogous to Lemma 2.8 concerning modules equal to a finite union of proper submodules.

The proof of Lemma 2.17 is similar to the proof of Lemma 2.5, so it is omitted.

Lemma 2.17. *Let R be a commutative ring, let M be a finitely generated R-module, let N be a submodule of M, and let $u, v \in R$ and $x, y \in M$. Then the following hold:*

(2.17.1) *If $x + uy, x + vy \in N$ and if $u - v$ is a unit in R, then $x, y \in N$.*
(2.17.2) *If $x \in N$ and $y \notin N$, then, for all elements r and units w in R, $rx + wy \notin N$.*

Theorem 2.18. *Let R be a commutative ring, let $U = \{u_1, \dots, u_{n-1}\}$ ($n \geqslant 3$) be a set of R-units with R-unit-differences, let M be a finitely generated R-module, and let $N_1, \dots, N_n$ be proper submodules of M. Then $M \neq \bigcup_{i=1}^n N_i$.*

Proof. The proof is similar to the proof of Lemma 2.8, using Lemma 2.17 in place of Lemma 2.5, so we omit the details. □

Proposition 2.19 is a variation of Theorem 2.18.

Proposition 2.19. *Let R be a commutative ring, let $v_1, \dots, v_{n+1}$ ($n \geqslant 1$) be elements in R such that $v_i - v_j$ is a unit in R for $i \neq j$ in $\{1, \dots, n+1\}$, let M be a finitely generated R-module, and let $N_1, \dots, N_n$ be proper submodules of M. Then $M \neq \bigcup_{i=1}^n N_i$.*

The proof is similar to the proof of Theorem 2.18, but since it is not assumed that $v_1, \dots, v_{n+1}$ are units in R, (2.17.2) cannot be used to assume that no $x_1 + v_i x_2$ is in N_1, and (2.17.2) cannot be used to assume that no $x_1 + v_i x_2$ is in N_m.

Results which are related to (2.18) and (2.19) for the case that R contains an uncountable set $\{u_\lambda \mid \lambda \in \Lambda\}$ with unit differences are given in [19, (2.5) and (2.6)].

3. Rees-good bases in finite integral extensions

We use Lemma 3.1 in Proposition 3.2.

Lemma 3.1. *Let $I = (b_1, \dots, b_g)R$ be a proper ideal in a Noetherian ring R with each b_i regular and let A be a finite integral extension ring of R.*

(1) *Let $(V, N) \in \operatorname{Rees} I$ and let z be the unique minimal prime ideal of R such that $(V, N) \in \operatorname{Rees} IR/z$. Let w be a (necessarily minimal) prime ideal of A lying over z and let (W, Q) be a DVR overring of A/w such that $W \cap K = V$, where K is the quotient field of R/z. Then $(W, Q) \in \operatorname{Rees} IA$.*
(2) *Let $(W, Q) \in \operatorname{Rees} IA$ and let w be the unique minimal prime ideal of A such that $(W, Q) \in \operatorname{Rees} IA/w$, and assume that $w \cap R = z$ is minimal in $\operatorname{Ass}(R)$. Then $W \cap K = V \in \operatorname{Rees} I$, where K is the quotient field of R/z.*

Proof. (1) After exchanging $R \subseteq A$ for $R/z \subseteq A/w$, we may assume that R and A are integral domains with quotient fields K and F respectively. For some $i \in \{1, \dots, g\}$, we have $V = (R[I/b_i]')_P$ for some height-one prime ideal P of the integral closure $R[I/b_i]'$ of $R[I/b_i]$. Then $R[I/b_i]' \subseteq A[I/b_i]'$ is an integral extension of domains, and since $W \cap K = V = (R[I/b_i]')_P$, we have $A[I/b_i]' \subseteq W$ and $Q \cap R[I/b_i]' = P$. Thus $P' = Q \cap A[I/b_i]'$ is a prime ideal of $A[I/b_i]'$ lying over P, so P' is a height-one prime ideal of $A[I/b_i]'$ and $(A[I/b_i]')_{P'} \subseteq W$. Since $(A[I/b_i]')_{P'}$ is a DVR, we have $W = (A[I/b_i]')_{P'} \in \operatorname{Rees} IA$.

(2) Again, after exchanging $R \subseteq A$ for $R/z \subseteq A/w$, we may assume that R and A are integral domains with quotient fields K and F respectively. Then $W = (A[I/b_i]')_{P'}$ for some $i \in \{1, \dots, g\}$

and some height-one prime ideal P' of $A[I/b_i]'$. Since $R[I/b_i]' \subseteq A[I/b_i]'$ satisfies going-down, [12, (10.13)], it follows that $P = P' \cap R[I/b_i]'$ also has height-one. Therefore $W \cap K \neq K$, $(R[I/b_i]')_P \subseteq W \cap K$, and $(R[I/b_i]')_P \in \operatorname{Rees} I$, hence $W \cap K = (R[I/b_i]')_P \in \operatorname{Rees} I$. $\square$

Proposition 3.2. *Let I be a regular proper ideal in a Noetherian ring R and let A be a finite integral extension ring of R. Assume that some prime ideal P in A that contains I is unramified over $P \cap R$. There exists a Rees valuation ring (W, Q) of IA whose center in A is contained in P, and for each such W there exists a Rees valuation ring (V, N) of I such that W is an extension of V. Moreover, W is unramified over V, so the Rees integer of IA with respect to W is the Rees integer of I with respect to V.*

Proof. Since $IA \subseteq P$, the prime ideal P contains some minimal associated prime ideal p of IA, and it is clear p (and every minimal prime divisor of IA) is in $\overline{A}^*(IA)$. Also, $\overline{A}^*(IA) = \{Q' \cap A \mid Q'$ is the maximal ideal of some Rees valuation ring W' of $IA\}$, by Lemma 2.7.2, so there exists at least one Rees valuation ring (W, Q) of IA such that $Q \cap A \subseteq P$. Further, since by hypothesis A_P is unramified over $R_{P\cap R}$, it follows from [12, (38.6)] that minimal prime ideals in A_P contract to minimal prime ideals in $R_{P\cap R}$. Therefore it follows from Lemma 3.1 that the Rees valuation rings of IA whose centers in A are contained in P are among the extensions of the Rees valuation rings of I whose centers in R are contained in $P \cap R$. Let (V, N) be the Rees valuation ring of I such that W is an extension of V.

To see that W is unramified over V, let $q = Q \cap A$ and let $p = N \cap R$, so $q \cap R = p$. Since, by hypothesis, $q \subseteq P$ and A_P is unramified over $R_{P\cap R}$, it follows from [12, (38.8)] that A_q is unramified over R_p. Let w be the minimal prime ideal in A such that W is a Rees valuation ring of IA/w. Then $w \subset Q \cap A = q$, so A_q/wA_q is unramified over $R_p/(wA_q \cap R_p)$, by [12, (38.7)], and $z = (wA_q \cap R_p) \cap R$ is a minimal prime ideal in R that is the unique minimal prime ideal z' in R such that V is a Rees valuation ring of IR/z'. Therefore it may be assumed that A_q and R_p are integral domains.

It follows from [12, (38.6)] that A_q is unramified over R_p if and only if there exists an element $u \in A_q$ for which A_q is a ring of quotients of $R_p[u]$ and for which there is a polynomial $f(X) \in R_p[X]$ with derivative $f'(X)$ such that $f(u) \in qA_q$ and $f'(u) \notin qA_q$. Since A_q is unramified over R_p, such an element u and polynomial $f(X)$ exist for A_q with respect to R_p. We have $A_q \subseteq W$ since $R_p \subseteq V$. Also $Q \cap A_q = qA_q$ implies that $f'(u)$ maps to a nonzero element in the subfield A_q/qA_q of W/Q. Therefore u and $f(X)$ imply by [12, (38.6)] that $V[u]_{Q\cap V[u]}$ is unramified over V. It follows that $(Q \cap V[u])V[u]_{Q\cap V[u]} = NV[u]_{Q\cap V[u]}$ is principal, so $V[u]_{Q\cap V[u]}$ is a DVR contained in W and with the same quotient field as W, so $V[u]_{Q\cap V[u]} = W$. Hence W is unramified over V. $\square$

Corollary 3.3. *Let I be a regular proper ideal in a Noetherian ring R and let A be a finite integral extension ring of R. Assume that each maximal ideal P in A that contains I is unramified over $P \cap R$. Then each Rees valuation ring of IA is an extension of a Rees valuation ring of I over which it is unramified, so the Rees integers of I and IA are the same (with possibly different cardinalities).*

Proof. Lemma 3.1 implies that the Rees valuation rings of IA are the extensions of the Rees valuation rings of I, so Corollary 3.3 follows from Proposition 3.2. $\square$

The following remark also follows immediately from Proposition 3.2.

Remark 3.4. With the notation and assumptions of Corollary 3.3, let H be a regular ideal in R such that $\bigcup\{q \mid q \in \overline{A}^*(H)\} \subseteq \bigcup\{p \mid p \in \overline{A}^*(I)\}$ and $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$. Then each Rees valuation ring of HA is an extension of a Rees valuation ring of H over which it is unramified, so the Rees integers of H and HA are the same (with possibly different cardinalities).

Proposition 3.5. *Let $(S, M_1, \dots, M_m)$ be a semi-local ring. Then the following hold:*

(3.5.1) *For each positive integer k there exists a simple free integral extension ring $S_k = S[x_k]$ of S that contains a set U_k of $2^k - 1$ S_k-units with S_k-unit-differences. Moreover, S_k may be chosen so that:*

(a) *S_k has exactly m maximal ideals*; *and* (b) *for $i = 1, \dots, m$, the unique maximal ideal in S_k that lies over M_i is, depending on i, either $M_i S_k$ or $(M_i, x_k)S_k$.*

(3.5.2) *For each positive integer k the ring S has a simple free integral extension ring $S_k = S[x_k]$ that contains a set U_k of $2^k - 1$ S_k-units with S_k-unit-differences and each prime ideal P in S_k is unramified over $P \cap S$, so for each regular proper ideal I in S, the ideals I and IS_k have the same Rees integers* (*with possibly different cardinalities*).

Proof. For (3.5.1), if $\mathrm{Card}(S/M_i) \geqslant 2^k$ for each $i \in \{1, \dots, m\}$, let $U_i = \{u_{i,1}, \dots, u_{i,2^k-1}\} \subseteq S$ be such that the images of the $u_{i,j}$ in S/M_i are distinct and nonzero. For each i and j, we may choose w_j such that, for $i = 1, \dots, m$, $w_j \equiv u_{i,j} \bmod M_i$, by comaximality (see [21, Theorem 31, p. 177]). Then $U = \{w_1, \dots, u_{2^k-1}\}$ is a set of $2^k - 1$ S-units with S-unit-differences.

Therefore assume that $m' \in \{1, \dots, m\}$ is such that $\mathrm{Card}(S/M_i) < 2^k$ for $i = 1, \dots, m'$ and $\mathrm{Card}(S/M_i) \geqslant 2^k$ for $i = m' + 1, \dots, m$. For $i = 1, \dots, m$ let $F_i = S/M_i$, for $i = 1, \dots, m'$ let $f_i(X)$ be a monic irreducible polynomial of degree k in $F_i[X]$, and for $i = m' + 1, \dots, m$ let $f_i(X) = X^k$. By comaximality we may choose a monic $f(X) \in S[X]$ of degree k such that, for $i = 1, \dots, m$, its image in $F_i[X]$ is $f_i(X)$. Then $f(X)$ is irreducible of degree k.

For $i = 1, \dots, m'$ let $N_i = (M_i, f(X))S[X]$, and for $i = m' + 1, \dots, n$ let $N_i = (M_i, X)S[X]$. Then for $i = 1, \dots, n$, N_i is a maximal ideal, since, for $i = 1, \dots, m'$, $S[X]/N_i \cong F_i[X]/(f_i(X))$, while for $i = m' + 1, \dots, n$, $S[X]/N_i \cong S/M_i$. Therefore $S_k = S[x_k] = S[X]/(f(X))$ is a simple free integral extension ring of S and, for $i = 1, \dots, m'$, $P_i = M_i S_k = N_i/(f(X)S[X])$ (resp., for $i = m' + 1, \dots, m$, $P_i = (M_i, x_k)S_k = N_i/(f(X)S[X])$) is the only maximal ideal in S_k that lies over M_i. It therefore follows that S_k is a semi-local ring such that (a) and (b) hold and for each maximal ideal P of S_k we have $\mathrm{Card}(S_k/P) \geqslant 2^k$. It now follows, as in the first paragraph of this proof that S_k contains a subset U of $2^k - 1$ S_k-units with S_k-unit-differences.

For (3.5.2), if all S/M_i are infinite, then it follows from the first paragraph of the proof of (3.5.1) that (3.5.2) holds with $S_k = S[1] = S$ for all positive integers k, so it may be assumed that $S/M_1, \dots, S/M_d$ are finite and $S/M_{d+1}, \dots, S/M_m$ are infinite. Since S/M_i is finite for $i = 1, \dots, d$, there exists for each positive integer k a monic irreducible and separable polynomial $f_i(X) \in (S/M_i)[X]$ of degree k. Fix k and for $i = d + 1, \dots, m$ let $f_i(X) = (X - s_{i,1}) \cdots (X - s_{i,k})$, where $s_{i,1}, \dots, s_{i,k}$ are distinct nonzero elements in S/M_i (this is possible, since S/M_i is infinite). By the Chinese Remainder Theorem there exists a monic polynomial $f(X) \in S[X]$ of degree k such that $f(X)$ modulo $M_i S[X]$ is equal to $f_i(X)$ for $i = 1, \dots, m$. Let $S_k = S[x_k] = S[X]/(f(X))$.

Then, for $i = 1, \dots, d$, it follows as in the second preceding paragraph that $Q_i = M_i S_k$ is a maximal ideal, and $S[x_k]_{Q_i}$ is integral over S_{M_i} since Q_i is the only maximal ideal in S_k that lies over M_i. Also S_k/Q_i is separable over S/M_i by the choice of $f_i(X)$, so $S[x_k]_{Q_i}$ is unramified over S_{P_i}. For the remaining i the field S/M_i is infinite and the ideal $M_i S_k$ factors into a product of k distinct maximal ideals $Q_{i,1}, \dots, Q_{i,k}$ such that $Q_{i,j} S[x_k]_{Q_{i,j}} = M_i S[x_k]_{Q_{i,j}}$ and $S_k/Q_{i,j} \cong S/P_i$, so $S[x_k]_{Q_{i,j}}$ is unramified over S_{P_i}. Therefore it follows from [12, (38.8)] that each prime ideal P in S_k is unramified over $P \cap S$, and it follows from (3.2) that the Rees integers of I and IS_k are the same (with possibly different cardinalities) for each regular proper ideal I in S.

Finally, it follows from the last paragraph of the proof of (3.5.1) that S_k contains a set U_k of $2^k - 1$ S_k-units with S_k-unit-differences. □

Remark 3.6. (3.6.1) In (3.5.1), assume[1] that, for $i = m' + 1, \dots, n$, there exists an irreducible polynomial $g_i(X)$ of degree k in $(S/M_i)[X]$. Then it follows immediately from the third paragraph of the proof of (3.5.1) that, by choosing $g_i(X)$ in place of $f_i(X) = X^k$, the maximal ideals in S_k are the ideals $M_i S_k$, $i = 1, \dots, n$.

[1] S/M_i may have no extension field K with $[K : (S/M_i)] = k$; for example, it is shown in [18, Example 3] that if $\{p_1, \dots, p_h\}$ is a finite set of distinct prime integers, then there exist fields F of characteristic zero having the property that F admits an extension field of degree k if and only if none of the p_i divides k.

(3.6.2) If each S/M_i has a separable extension field of degree k (for example, if each S/M_i is finite), then it follows from the proof of (3.5.2) that S_k may be chosen so that S and S_k have the same number of maximal ideals in (3.5.2).

(3.6.3) In (3.5), assume that I is a regular proper ideal in a Noetherian ring R, that $T = R \setminus \bigcup\{p \mid p \in \overline{A}^*(I)\}$, and that $S = R_T$. Then IS is contained in the Jacobson radical of S, $\overline{A}^*(IS) = \{pS \mid p \in \overline{A}^*(I)\}$, by [17, (6.5) and (6.8)], and it follows from the definition of Rees valuation rings (for example [2, (2.2)(c) and (2.4)]) that $\operatorname{Rees} I = \operatorname{Rees} IS$ and that I and IS have the same Rees integers. Also, by Proposition 3.5, the Rees integers of IS and of $IS[x_k]$ are the same (with possibly different cardinalities). Further, since x_k is integral over S and $S[x_k]$ is a simple free integral extension ring of S, there exists $t_1 \in T$ such that t_1x_k is integral over R and $R[t_1x_k]$ is a simple free integral extension ring of R such that $S[x_k] = (R[t_1x_k])_T$. Moreover, the centers in $R[t_1x_k]$ of the Rees valuation rings of $IR[t_1x_k]$ are disjoint from T (by integral dependence, the definition of T, and [10, Proposition 3.22]), so it follows as just above that $\operatorname{Rees} IS[x_k] = \operatorname{Rees} IR[t_1x_k]$ and that $IS[x_k]$ and $IR[t_1x_k]$ have the same Rees integers, hence I and $IR[t_1x_k]$ have the same Rees integers (with possibly different cardinalities). Finally, if $U = \{u_1, \dots, u_n\}$ is a set of D^*-units with D^*-unit-differences in $S[x_k]$ (here, D^* is, in the domain case, the intersection of the Rees valuation rings of $IS[x_k]$), then there exists t_2 in T such that $\{t_2u_1, \dots, t_2u_n\}$ is a set of D^*-units with D^*-unit-differences in $R[t_2x_k]$, so if we let $t = t_1t_2$, then $t \in T$ is such that $R[tx_k]$ is a simple free integral extension ring of R, $\{tu_1, \dots, tu_n\}$ is a set of D^*-units with D^*-unit-differences in $R[tx_k]$, and the Rees integers of I and $IR[tx_k]$ are the same (with possibly different cardinalities).

Theorem 3.7 is the main result in this paper.

Theorem 3.7. *Let I be a regular proper ideal in a Noetherian ring R. There exists a simple free integral extension ring A of R such that*:

(1) *For each regular ideal H in R whose asymptotic prime divisors are contained in the union of the asymptotic prime divisors of I and for which* $\operatorname{Card}(\operatorname{Rees} H) \leqslant \operatorname{Card}(\operatorname{Rees} I)$, *the ring A_P is unramified over $R_{P \cap R}$ for each asymptotic prime divisor P of HA*;
(2) *Each Rees valuation ring of HA is unramified over its contraction to a Rees valuation ring of H*; *and*
(3) *The ideal HA has a Rees-good basis and the same Rees integers as H* (*with possibly different cardinalities*).

In particular, these properties hold for the ideal $H = I$.

Proof. Let $T = R \setminus \bigcup\{p \mid p \in \overline{A}^*(I)\}$ and $S = R_T$. Then IS is contained in the Jacobson radical of S, $\overline{A}^*(IS) = \{pS \mid p \in \overline{A}^*(I)\}$, and $\operatorname{Rees} IS = \operatorname{Rees} I$, by the first part of (3.6.3). Also, it follows from (3.6.3) that if there exists a finite free integral extension ring A^* of S that contains a set of $\operatorname{Card}(\operatorname{Rees} IA^*)$ D^*-units with D^*-unit-differences, then there exists a finite free integral extension ring A of R that contains a set of $\operatorname{Card}(\operatorname{Rees} IA)$ D-units with D-unit-differences. Further, since $\overline{A}^*(IS) = \{pS \mid p \in \overline{A}^*(I)\}$, it follows from the definitions that if the asymptotic prime divisors (resp., the Rees valuation rings) of HA^* are unramified over their contractions to S (resp., the total quotient ring of S), then the asymptotic prime divisors (resp., the Rees valuation rings) of HA are unramified over their contractions to R (resp., the total quotient ring of R), hence the Rees integers of HA and H are the same. Therefore it may be assumed to begin with that R is semi-local with I contained in the Jacobson radical of R, and it suffices (by Remark 3.6.3, Corollary 3.3, and Remark 3.4) to construct a finite free integral extension ring A of R such that: (a) A contains a set of $\operatorname{Card}(\operatorname{Rees} IA)$ D-units with D-unit-differences; and, (b) each maximal ideal M in A is unramified over $M \cap R$.

To construct such a ring A such that (a) holds, let h_0 be the number of minimal prime ideals in R, so $h_0 \geqslant 1$, and let $n = \operatorname{Card}(\operatorname{Rees} I)$ (so $n \geqslant 1$). If $n = 1$, then the conclusion follows with $A = R$, by (2.2.3), so it may be assumed that $n \geqslant 2$. Since the integers h_0 and n are fixed, for all large integers k it holds that

$$2^k > h_0nk^2. \tag{3.7.1}$$

Fix such an integer k, and let $A = R_k$, where $R_k = R[X]/f(X)R[X]$ is as in (3.5), so A has a set U of $2^k - 1$ A-units with A-unit-differences. Therefore, to show that (a) holds for A, it remains to show that $\mathrm{Card}(\mathrm{Rees}\, IA) \leqslant 2^k - 1$.

For this, let $z_1, \ldots, z_{h_0}$ be the minimal prime ideals in R. Then the degree of the image $g_j(X)$ of $f(X)$ in $(R/z_j)[X]$ is k and $g_j(X)$ is monic, so $g_j(X)$ has at most k minimal prime divisors in $(R/z_j)[X]$, so $f(X)$ has at most kh_0 minimal prime divisors in $R[X]$. Therefore A has at most kh_0 minimal prime ideals and for each minimal prime ideal z_j in R there are at most k minimal prime ideals w in A that lie over z_j. Let $w_1, \ldots, w_h$ be the minimal prime ideals in A, so

$$h \leqslant kh_0 \tag{3.7.2}$$

and

$$\left[(A/w_j)_{(0)} : \left(R/(w_j \cap R)\right)_{(0)}\right] \leqslant k \quad \text{for } j = 1, \ldots, h. \tag{3.7.3}$$

The Rees valuation rings of $(IA + w_j)/w_j$ are the extensions of the Rees valuation rings of $(I + (w_j \cap R))/(w_j \cap R)$ to the quotient field $(A/w_j)_{(0)}$ of A/w_j. By [22, Theorem 19, p. 55] each Rees valuation ring of $(I + (w_j \cap R))/(w_j \cap R)$ has at most $[(A/w_j)_{(0)} : (R/(w_j \cap R))_{(0)}]$ extensions to $(A/w_j)_{(0)}$. Using (3.7.3), it follows that

$$r_j \leqslant kq_j, \tag{3.7.4}$$

where r_j is the number of Rees valuation rings of $(IA + w_j)/w_j$, and q_j is the number of Rees valuation rings of $(I + (w_j \cap R))/(w_j \cap R)$. It is clear that $r_1 + \cdots + r_h = \mathrm{Card}(\mathrm{Rees}\, IA)$, so it follows from (3.7.4) that $\mathrm{Card}(\mathrm{Rees}\, IA) = r_1 + \cdots + r_h \leqslant kq_1 + \cdots + kq_h \leqslant k(hn)$ (since $q_j \leqslant n$ for $j = 1, \ldots, h$). Since $h \leqslant kh_0$, by (3.7.2), it follows that $\mathrm{Card}(\mathrm{Rees}\, IA) \leqslant khn \leqslant k(kh_0)n = h_0nk^2 \leqslant 2^k - 1$, by (3.7.1), as desired.

Finally, to see that (b) holds for A, since R is semi-local, (3.5.2) implies that A may be constructed so that each prime ideal P in A is unramified over $P \cap R$. Hence by Corollary 3.3 and Remark 3.4 each Rees valuation ring of HA is unramified over its contraction to a Rees valuation ring of H, so it follows that the ideals HA and H have the same Rees integers (with possibly different cardinalities). □

Remark 3.8. (3.8.1) Concerning the conclusion of Theorem 3.7, if H is an ideal in R that is projectively equivalent to I, then H and I have the same Rees valuation rings, by [2, Theorem 3.4], hence it follows from (3.7) that HA has a Rees-good basis and the same Rees integers as H (with possibly different cardinalities).

(3.8.2) It follows from (3.7) and its proof that if $M_1, \ldots, M_k$ are finitely many regular maximal ideals in a Noetherian ring R and if n is a given positive integer, then there exists a simple free integral extension ring A_n of R such that, for all ideals I in R with $\bigcup\{p \mid p \in \overline{A}^*(I)\} \subseteq M_1 \cup \cdots \cup M_k$ and $\mathrm{Card}(\mathrm{Rees}\, I) \leqslant n$, the ideal IA_n has a Rees-good basis and the same Rees integers as I (with possibly different cardinalities).

4. An asymptotic sequence is a Rees-good basis

Let $I = (b_1, \ldots, b_g)R$ be a regular ideal in the Noetherian ring R. An interesting result of Swanson and Huneke [20, Proposition 10.2.6] asserts that if (R, M) is a quasi-unmixed local ring of altitude g and $b_1, \ldots, b_g$ are analytically independent modulo each minimal prime of R, then $b_1, \ldots, b_g$ is a Rees-good basis of I. We prove in this section a related result. Theorem 4.2 asserts that if $b_1, \ldots, b_g$ is an asymptotic sequence, then $b_1, \ldots, b_g$ is a Rees-good basis of I. We collect in Remark 4.1 facts used to prove this.

Remark 4.1. Let $b_1, \ldots, b_g$ be an asymptotic sequence in a Noetherian ring R and let $I = (b_1, \ldots, b_g)R$. Then:

(4.1.1) [15, (2.3.3)]: For each $i \in \{1, \dots, g\}$, we have $\mathrm{ht}((b_1, \dots, b_i)R) = i$.

(4.1.2) [15, (6.1)]: For each minimal prime ideal z in R, the z-residue classes of $b_1, \dots, b_g$ is an asymptotic sequence in R/z.

(4.1.3) [15, (2.9.1)]: If S is a multiplicatively closed subset of R such that $IR_S \neq R_S$, then the images of $b_1, \dots, b_g$ in R_S is an asymptotic sequence in R_S.

(4.1.4) Assume that R is local with maximal ideal M. If $M \in \overline{A}^*(I)$, then $b_1, \dots, b_g$ is a maximal asymptotic sequence in R (by either Definition 1.2.2 or [15, (2.9.3)]).

(4.1.5) [15, (2.11)]: Assume that R is local and let $\widehat{R}$ denote the completion of R. If $b_1, \dots, b_g$ is a maximal asymptotic sequence in R, then

$$\min\bigl\{\mathrm{altitude}(\widehat{R}/z) \bigm| z \text{ is a minimal prime ideal in } \widehat{R}\bigr\} = g.$$

(4.1.6) Assume that R is an integral domain. It is shown in [16, (2.5.3) and (2.5.4)] that there exists a prime ideal H in $R_{g-1} = R[X_1, \dots, X_{g-1}]$, where the X_i are independent indeterminates, such that $R[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}] \cong R_{g-1}/H$ and $H \subset pR_{g-1}$ for each minimal prime divisor p of I. Therefore, for each prime ideal P in R such that $I \subseteq P$ it follows from the factor-of-a-factor isomorphism theorem that

$$R\left[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}\right] \Big/ \left(PR\left[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}\right]\right) \cong R_{g-1}/(PR_{g-1}) \cong (R/P)[X_1, \dots, X_{g-1}].$$

(4.1.7) Assume that R is local with maximal ideal M, let z be a minimal prime ideal in R, and let $\widehat{R}$ be the M-adic completion of R. If R/z is quasi-unmixed, then since $\widehat{R/z}$ is isomorphic to $\widehat{R}/z\widehat{R}$ it follows from the definition of quasi-unmixed that for every minimal prime divisor z^* of $z\widehat{R}$ one has $\mathrm{altitude}(\widehat{R}/z^*) = \mathrm{altitude}(R/z)$.

(4.1.8) Assume that (R, M) is a local domain and let $(\widehat{R}, \widehat{M})$ be the M-adic completion of R. Let $C = R[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}]$, and let $C^* = \widehat{R}[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}]$. It is shown in [14, Lemma 3.2] that there exists a one-to-one correspondence between the prime ideals Q in C such that $Q \cap R = M$ and the prime ideals Q^* in C^* such that $Q^* \cap \widehat{R} = \widehat{M}$, and then $Q^* = QC^*$, $Q = Q^* \cap C$, and C_Q is a dense subspace of ${C^*}_{Q^*}$.

(4.1.9) [13, Proposition 3.5]: Assume that (R, M) is a local domain and let $\widehat{R}$ be the M-adic completion of R. There exists a height-one maximal ideal in the integral closure of R if and only if there exists a minimal prime z of $\widehat{R}$ such that $\mathrm{altitude}(\widehat{R}/z) = 1$.

(4.1.10) [13, Corollary 2.14 and Theorem 3.1]: Assume that R is a Noetherian integral domain and that A is a finitely generated extension domain of R. If R is locally quasi-unmixed, then A is locally quasi-unmixed.

Theorem 4.2. *Let $I = (b_1, \dots, b_g)R$ be a regular ideal in a Noetherian ring R. If $b_1, \dots, b_g$ is an asymptotic sequence, then it is a Rees-good basis for I.*

Proof. Let (V, N) be a Rees valuation ring of I. It suffices to show that $b_iV = IV$ for $i = 1, \dots, g$. There exists a minimal prime ideal z in R such that $R/z \subseteq V \subseteq F$, where F is the quotient field of R/z. By Remark 4.1.2, the z-residue classes of $b_1, \dots, b_g$ is an asymptotic sequence in R/z, and V is a Rees valuation ring of $(I + z)/z$, by construction/definition (see [20, Section 10.1]), so it may be assumed to begin with that R is an integral domain.

Let $(L, M) = (R_{N \cap R}, (N \cap R)R_{N \cap R})$. By Remark 4.1.3, $b_1, \dots, b_g$ is an asymptotic sequence in L, and V is a Rees valuation ring of IL, by construction/definition (see [20, Section 10.1]), so it may also be assumed to begin with that R is a local domain such that its maximal ideal $M = N \cap R$. Thus $M \in \overline{A}^*(I)$, by Lemma 2.7.2. Therefore $b_1, \dots, b_g$ is a maximal asymptotic sequence in R, by Remark 4.1.4. Let $\widehat{R}$ denote the M-adic completion of R. Remark 4.1.5 implies that

$$\min\bigl\{\mathrm{altitude}(\widehat{R}/z) \bigm| z \text{ is a minimal prime of } \widehat{R}\bigr\} = g. \tag{4.2.1}$$

Since V is a valuation ring and $I = (b_1, \dots, b_g)R$, there exists $i \in \{1, \dots, g\}$ such that $IV = b_i V$, so by possibly relabeling it may be assumed that $b_1 V = IV$. Then $V = C'_{p'}$, where C' is the integral closure of $C = R[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}]$ in F and p' is a height-one prime divisor of $b_1 C'$. Now $p' \cap R = M$, by the start of this paragraph, so Remark 4.1.6 shows that $C/MC \cong R_{g-1}/(MR_{g-1}) \cong (R/M)[X_1, \dots, X_{g-1}]$, where the X_i are independent indeterminates.

Assume it is known that $MC = N \cap C$. Then $R \subseteq C \subseteq V$ and $N \cap C = MC$ imply that $C/MC \subseteq V/N$ and the MC-residue classes of $\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}$ are algebraically independent over R/M. In particular, the N-residue classes of the $g-1$ elements $\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}$ are nonzero, so $\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}$ are units in V; that is, $\frac{b_i}{b_1} V = V$ for $i = 1, \dots, g$, so $b_i V = b_1 V = IV$ (by the preceding paragraph) for $i = 1, \dots, g$, hence $b_1, \dots, b_g$ is a Rees-good basis of I. Therefore it remains to show that $MC = N \cap C$.

For this, let $P = N \cap C$. Then $MC \subseteq P$, and it remains to show that $P = MC$. Suppose, by way of contradiction, that $MC \subsetneq P$. Since $C/(MC) \cong (R/M)[X_1, \dots, X_{g-1}]$, there exists a nonzero polynomial

$$f(X_1, \dots, X_{g-1}) \in R[X_1, \dots, X_{g-1}]$$

such that $f(\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}) \in P \setminus MC$, so the P-residue classes of $\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}$ are not algebraically independent over $R/(P \cap R) = R/M$. Hence

$$\text{trans. deg.}\big((C/P)/(R/M)\big) < g - 1. \tag{4.2.2}$$

Since $(\widehat{R}, \widehat{M})$ is the M-adic completion of (R, M), let $C^* = \widehat{R}[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}]$, and let $P^* = PC^*$. By Remark 4.1.8, P^* is a prime ideal such that $P^* \cap C = P$ and C_P is a dense subspace of $C^*_{P^*}$. Let $D = C_P$ and let $D^* = C^*_{P^*}$. Remark 4.1.8 implies that the PD-adic completion $\widehat{D}$ of D is also the P^*D^*-adic completion of D^*. Notice that $C'_{(C \setminus P)}$ is the integral closure D' of D in its quotient field and $p'D'$ is a height-one maximal ideal in D', since $p'D' \cap D = PD$, the maximal ideal of D. By Remark 4.1.9, there exists a minimal prime $\widehat{w}$ in $\widehat{D}$ such that altitude$(\widehat{D}/\widehat{w}) = 1$.

Let $w^* = \widehat{w} \cap D^*$, $z = \widehat{w} \cap C^*$, and $w = \widehat{w} \cap \widehat{R}$. Using that $\widehat{D}$ is the P^*D^*-adic completion of D^*, w^* is a minimal prime ideal in D^*, so z is a minimal prime ideal in C^* and w is a minimal prime ideal in $\widehat{R}$ (since C^* and $\widehat{R}$ have the same total quotient ring), hence it follows from (4.2.1) that

$$\text{altitude}(\widehat{R}/w) \geqslant g. \tag{4.2.3}$$

Since $\widehat{R}/w$ is a complete local domain and therefore unmixed and quasi-unmixed, Remark 4.1.10 implies that C^*/z is locally quasi-unmixed, so D^*/w^* is quasi-unmixed. Since $\widehat{w}$ is a minimal prime divisor of $w^*\widehat{D^*}$ and altitude$(\widehat{D^*}/\widehat{w}) = 1$, it follows from Remark 4.1.7 that altitude$(D^*/w^*) = 1$, so ht$(P^*D^*/w^*) = 1$, hence

$$\text{ht}(P^*/z) = 1. \tag{4.2.4}$$

Since $\widehat{R}/w$ is quasi-unmixed, [13, Theorem 3.1] implies that $\widehat{R}/w$ satisfies the altitude formula. Hence

$$\text{ht}(P^*/z) + \text{trans. deg.}\big((C^*/P^*)/(\widehat{R}/\widehat{M})\big) = \text{ht}(\widehat{M}/w) + \text{trans. deg.}\big((C^*/z)/(\widehat{R}/w)\big);$$

that is

$$1 + t = \text{altitude}(\widehat{R}/w) + 0, \tag{4.2.5}$$

by (4.2.4), where $t = \text{trans. deg.}((C^*/P^*)/(\widehat{R}/\widehat{M}))$. Since C_P is a dense subspace of $C^*_{P^*}$, it follows that $C^*/P^* = C/P$ (and $\widehat{R}/\widehat{M} = R/M$), so $t < g - 1$, by (4.2.2), and altitude$(\widehat{R}/w) \geqslant g$, by (4.2.3), and this contradicts (4.2.5). Therefore the supposition in the preceding paragraph is false, so $P = N \cap C = MC$, hence $b_1, \dots, b_g$ is a Rees-good basis of I. $\square$

Remark 4.3. Let $b_1, \dots, b_g$ be an asymptotic sequence in a Noetherian ring R, let $C = R[\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}]$, and let P be a prime ideal in C such that $b_1 \in P$. Then it follows from the proof of Theorem 4.2 that if $P \cap R \in \overline{A}^*((b_1, \dots, b_g)R)$ (equivalently, $P \in \overline{A}^*(b_1 C)$), then: $P = (P \cap R)C$; the P-residue classes of $\frac{b_2}{b_1}, \dots, \frac{b_g}{b_1}$ are algebraically independent over $R/(P \cap R)$; and, there exists a height-one prime ideal p' in the integral closure C' of C such that $p' \cap C = P$.

We obtain as a corollary the following result of Swanson and Huneke [20, Proposition 10.2.8].

Corollary 4.4. *Let $b_1, \dots, b_g$ be an R-sequence in a locally quasi-unmixed Noetherian ring R and let $I = (b_1, \dots, b_g)R$. Then I has a Rees-good basis.*

Proof. This follows immediately from Theorem 4.2, since an R-sequence is a strong version of an asymptotic sequence, by [15, (2.3.5)]. □

In the next corollary, an ideal I is of the **principal class** in case I has a basis consisting of $h = \mathrm{ht}(I)$ elements.

Corollary 4.5. *Let I be a regular proper ideal of the principal class in a locally quasi-unmixed Noetherian ring R. Then each basis $\{b_1, \dots, b_h\}$ for I with $h = \mathrm{ht}(I)$ is a Rees-good basis for I.*

Proof. It is shown in [15, (2.3.6)] that if $I = (b_1, \dots, b_h)R$ is an ideal with $h = \mathrm{ht}(I)$ in a locally quasi-unmixed Noetherian ring then $b_1, \dots, b_h$ is an asymptotic sequence. So this follows immediately from Theorem 4.2. □

Corollaries 4.6 and 4.7 are somewhat analogous to [1, Theorem 1] and also to [16, (2.13)], in that all four results concern chains of radical ideals. These corollaries are also sharpened versions of [4, (3.18)]. In these two corollaries, an ideal I is **projectively full** in case the only integrally closed ideals J that are projectively equivalent to I are the ideals $(I^i)_a$, where i is an arbitrary positive integer. We use the following notation in Corollaries 4.6 and 4.7. If m is a positive integer and $b_1, \dots, b_g$ are regular elements of the Noetherian ring R, we let $A_m = R[X_1, \dots, X_g]/(X_1^m - b_1, \dots, X_g^m - b_g) = R[x_1, \dots, x_g]$ where x_i is the residue class of X_i in A_m for each i.

Corollary 4.6. *Let $b_1, \dots, b_g$ be an asymptotic sequence in a Noetherian ring R, for $i = 1, \dots, g$ let $I_i = (b_1, \dots, b_i)R$, let e_i^* be the least common multiple of the Rees integers of I_i, and let m be a common multiple of $e_1^*, \dots, e_g^*$. Assume that m is a unit in R and for $i = 1, \dots, g$, let $B_i = (x_1, \dots, x_i)A_m$. Then A_m is a finite free integral extension ring of R and for $i = 1, \dots, g$, $(B_i)_a$ is a projectively full radical ideal that is projectively equivalent $I_i A_m$ and the Rees integers of B_i are all equal to one.*

Proof. By Theorem 4.2, $b_1, \dots, b_i$ is a Rees-good basis of I_i, since $b_1, \dots, b_i$ is an asymptotic sequence in R. Also, m is a unit in R that is a multiple of e_i^*, so it follows from [4, Theorem 3.7] that the subring $C_i = R[x_1, \dots, x_i]$ of A_m is a finite free integral extension ring of R and $((x_1, \dots, x_i)C_i)_a$ is a projectively full radical ideal that is projectively equivalent to $I_i C_i$ and the Rees integers of $(x_1, \dots, x_i)C_i$ are all equal to one. Also, the $g - i$ elements $b_{i+1}, b_{i+2}, \dots, b_g$ are not in the centers in R of the Rees valuation rings of I_i, so there is no ramification in the extension of the Rees valuation rings of $(x_1, \dots, x_i)C_i$ to the Rees valuation rings of B_i, by [4, Corollary 3.2], and the conclusion readily follows from this. □

Corollary 4.7. *Let I be an ideal contained in the Jacobson radical of a semi-local ring $(R; M_1, \dots, M_h)$, assume that I is generated by an asymptotic sequence $b_1, \dots, b_g$ of regular elements in R, and for each of the $2^g - 1$ ideals $G \in \mathbf{G} = \{(b_{\pi(1)}, \dots, b_{\pi(i)})R \mid \pi$ is an arbitrary permutation of $\{1, \dots, g\}$ and $i = 1, \dots, g\}$ let e_G be the least common multiple of the Rees integers of G. Let m be a common multiple of the integers in $\{e_G \mid G \in \mathbf{G}\}$, and assume that m is a unit in R. For $i = 1, \dots, g$ let $A_m = R[x_1, \dots, x_g]$ as above, and for each*

$G = (b_{\pi(1)}, \ldots, b_{\pi(i)})R \in \mathbf{G}$ let $B_G = (x_{\pi(1)}, \ldots, x_{\pi(i)})A_m$. Then A_m is a finite free integral extension ring of R and for each $G \in \mathbf{G}$ it holds that $(B_G)_a$ is a projectively full radical ideal that is projectively equivalent GA_m and the Rees integers of B_G are all equal to one.

Proof. Since an asymptotic sequence contained in the Jacobson radical of a Noetherian ring is a permutable asymptotic sequence, by [15, (2.10)], the proof is similar to the proof of Corollary 4.6. □

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