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Graph components of prime spectra

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ABSTRACT

The prime spectrum, Spec(A), of a ring A is a T_0-space and it is partially ordered by inclusion between prime ideals. The partial order makes Spec(A) into a graph – the vertices are the prime ideals, and there is an edge between two vertices if there is a containment relation between them. The graph and the topological space Spec(A) both have connected components, which are called graph components and topological components. Every topological component is a union of graph components. The paper is devoted to a study of the graph components. The main question is how properties of the graph components of the prime spectrum correspond to arithmetic properties of a ring. Given a property $\mathcal{P}$ that graph components may or may not have, let $\mathbf{R}(\mathcal{P})$ be the class of rings A such that every graph component of Spec(A) has property $\mathcal{P}$. For which properties $\mathcal{P}$ is it true that $\mathbf{R}(\mathcal{P})$ is an elementary class of rings in the sense of model theory?

0. Introduction

Every topological space carries a relation that is called *specialization*: Let x, y be two points of a topological space X. Then x is a *specialization* of y (and y is a *generalization* of x) if $x \in \overline{\{y\}}$; one writes $y \rightsquigarrow x$. The specialization relation is a partial order if and only if X is a T_0-space. Specialization

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provides every topological space with the structure of a directed graph. The vertices are the points of the space; the edges are the arrows of the specialization relation.

Viewed as a topological space, or as a graph, X has connected components, which are called *topological components*, or *graph components*. The graph component of a point x is the set

$$\Gamma[x] = \{y \in X \mid \exists k \in \mathbb{N}\ \exists x_0, \ldots, x_k\colon\ x = x_0 \wedge y = x_k \wedge \forall i\colon\ x_{i-1} \rightsquigarrow x_i \vee x_i \rightsquigarrow x_{i-1}\}.$$

The graph components are the minimal nonempty subsets that contain every specialization and every generalization of each of its points. The topological component of x is denoted by $\mathrm{T}[x]$. The topological components and the graph components form partitions of X. The corresponding quotient spaces are denoted by X/T for the topological components, X/G for the graph components; the canonical maps from X to the quotient spaces are denoted by $p_\mathrm{T} = p_\mathrm{T}^X : X \to X/\mathrm{T}$ and $p_\mathrm{G} = p_\mathrm{G}^X : X \to X/\mathrm{G}$. Every topological component is a union of graph components. Hence there is a canonical continuous map $p_{\mathrm{T},\mathrm{G}} = p_{\mathrm{T},\mathrm{G}}^X : X/\mathrm{G} \to X/\mathrm{T}$. The space X is *graph connected* if there is only one graph component.

We study rings that are commutative, have a multiplicative unit and are reduced, i.e., have no nilpotent elements. The prime ideals of a ring A, equipped with the Zariski topology, form a topological space, which is denoted by $\mathrm{Spec}(A)$ and is called the *prime spectrum*. The class of prime spectra of rings is exactly the class of *spectral spaces*, [10]. (Section 1 contains a short introduction to spectral spaces.) Specialization in $\mathrm{Spec}(A)$ coincides with inclusion of prime ideals, i.e., $p \subseteq q$ if and only if $p \rightsquigarrow q$. We study the graph components of the prime spectrum of a ring.

The topological components of $\mathrm{Spec}(A)$ are determined by the idempotents of A. Every idempotent defines a partition of $\mathrm{Spec}(A)$ into two closed and open subsets. The Boolean algebra of idempotents is denoted by $E(A)$. If $p \in \mathrm{Spec}(A)$ then $p \cap E(A)$ is a maximal ideal of $E(A)$. This defines a surjective spectral map $p_E : \mathrm{Spec}(A) \to \mathrm{Spec}(E(A))$. Its codomain is a Boolean space and is canonically homeomorphic to the space $\mathrm{Spec}(A)/\mathrm{T}$ of topological components. The spectrum is connected if and only if $E(A) = \{0, 1\}$, i.e., there are no nontrivial idempotents – the ring is indecomposable.

The space of graph components of $\mathrm{Spec}(A)$ is a finer invariant than the space of topological components. There are indecomposable rings with only one graph component – trivial examples are integral domains and local rings. There are also indecomposable rings with a very large number of graph components. For example, the ring $\mathcal{C}(\mathbb{R}; \mathbb{R})$ of continuous functions is indecomposable, but its spectrum has uncountably many graph components.

We shall study both the space of graph components and the structure of the individual graph components of prime spectra. From a purely topological point of view the study of the graph components of prime spectra coincides with the study of the graph components of spectral spaces. However, we are not only interested in topological questions. It is our goal to establish connections between properties of the graph components of the prime spectrum on the one hand and the arithmetic of a ring on the other hand. We start with a look at spectral spaces and then turn to prime spectra.

In Section 2 we introduce three algorithms that can be used to construct the graph component of a point in a spectral space. The termination of the algorithms plays a central role throughout. In Section 3 we present various examples of spectral spaces to illustrate theoretical results and to exhibit some complicated graph components. Section 4 contains basic information about the space of graph components. The space has the T_1-property if and only if each graph component is proconstructible, 4.2. Recall that every graph component carries a natural metric – the distance of two vertices is the minimal number of edges needed to connect the vertices. If there is a uniform bound for the diameter of the graph components then the space X/G is Hausdorff, 4.6. If a spectral space is normal (cf. [3], [22, Section 4]) then its space of maximal points is compact and is homeomorphic to the space of graph components, 4.7. The space of graph components is Boolean if and only if the topological components are graph connected, 4.9.

Section 5 starts the study of the connections between arithmetic properties of a ring and properties of the graph components of its prime spectrum. We consider classes of rings that are defined by properties of the graph components of their prime spectra. For example, one may consider the class of rings whose prime spectra have proconstructible graph components. Or, one may consider the class of rings whose prime spectra have graph components with diameter bounded by some fixed number

$K \in \mathbb{N}$. The main question that will be considered is whether such a class of rings is an elementary class in the sense of model theory. For model theoretic purposes we always use the language of ring theory, $\mathcal{L} = \{0, 1; +, -, \cdot\}$. A key step is to see how the termination of the algorithms of Section 2, which are concerned with $\mathrm{Spec}(A)$, corresponds to arithmetic conditions about A, 5.2. Ultraproducts of rings are used to show that various classes of rings that are defined by properties of the graph components of their prime spectrum are not elementary, e.g., the class of rings with graph connected prime spectrum, or the class of rings whose prime spectrum has only proconstructible graph components. However, the classes of rings that are defined by the property that the algorithms of Section 2 terminate after at most K steps, $K \in \mathbb{N}$, are elementary, 5.6.

Rings with graph connected prime spectrum are analyzed in Section 6. Two infinite families of elementary classes of rings are defined. The elements of each of these classes are characterized by the behavior of the algorithms of Section 2, cf. 6.6, 6.7, 6.8. Explicit axioms are exhibited. In Section 7 these results are extended to rings that are not necessarily indecomposable, but for which the topological components of the prime spectrum are graph connected. Several families of elementary classes of rings are defined. The main difference compared with the families of Section 6 is that now the axioms defining the families incorporate information about the existence of idempotents. Typically the axioms say that for certain pairs X, Y of subsets of $\mathrm{Spec}(A)$ there is an idempotent in A that separates the sets X and Y. Some elementary classes are characterized by the condition that the indecomposable factor rings of their elements belong to one of the classes of Section 6, cf. 7.1.

Not all classes of rings that are considered in the present paper are new. Some of them have already been studied in other contexts, such as von Neumann regular rings, or clean rings, or Gel'fand rings, or almost clean rings, 7.4, 5.7, 7.6.

1. Notation, terminology, and a review of spectral spaces

In this preliminary section we fix some general conventions, and then we give a short review of spectral spaces.

- In this paper 0 is a natural number.
- All rings are commutative, have a multiplicative unit and are reduced (or semiprime), i.e., there are no nontrivial nilpotents.
- Suppose that G is a simple graph, oriented or not. There is a natural metric on G: Given vertices $x, y \in G$, their distance is ∞ if there is no sequence of edges that connects the vertices with each other. If they can be connected then the distance is the length of the shortest connecting sequence. The ball of radius k about x is denoted by $B[x; k]$.
- Partially ordered sets will be called *posets*. Given a poset $(L, \leqslant)$, a subset $K \subseteq L$ is called an *upset* if $a \in K$, $a \leqslant b$ implies $b \in K$ and is called a *downset* if $b \in K$, $a \leqslant b$ implies $a \in K$. If $K \subseteq L$ is any subset then $\sigma(K) = \{b \in L \mid \exists a \in K\colon\ a \leqslant b\}$ is the smallest upset that contains K, and $\gamma(K) = \{a \in L \mid \exists b \in K\colon\ a \leqslant b\}$ is the smallest downset that contains K. Both maps preserve inclusion, are idempotent and send a set to a larger set, i.e., they are closure operators, [5, p. 42].
- Suppose that $L = (L, \leqslant)$ is a poset. Then one defines another partial order $\leqslant_{\mathrm{inv}}$ by setting $a \leqslant_{\mathrm{inv}} b$ if and only if $b \leqslant a$. The new partial order is the *inverse* or *dual* partial order of the original one. If L is a lattice with lattice operations $\vee$ and $\wedge$ then $(L, \leqslant_{\mathrm{inv}})$ is a lattice with lattice operations $\vee_{\mathrm{inv}}$ and $\wedge_{\mathrm{inv}}$ defined by $a \vee_{\mathrm{inv}} b = a \wedge b$, $a \wedge_{\mathrm{inv}} b = a \vee b$. The poset, or lattice, $(L, \leqslant_{\mathrm{inv}})$ is called the *inverse poset*, or *inverse lattice*, and is denoted by L_{inv}.
- Given a topological space X, the lattice of closed subsets is denoted by $\mathcal{A}(X)$.

Spectral spaces. Terminology and basic facts about prime spectra and spectral spaces will be used throughout. General references are [10] and [6]. The book [6] is in preparation and is not yet available. Therefore the material that is indispensable for reading the paper will be explained here.

If A is a ring then its set of prime ideals is denoted by $\mathrm{Spec}(A)$. If $a \in A$ then the sets $D(a) = \{p \in \mathrm{Spec}(A) \mid a \notin p\}$, $a \in A$, are a basis of the *Zariski topology* on $\mathrm{Spec}(A)$. One also defines $V(a) = \mathrm{Spec}(A) \setminus D(a)$. The *prime spectrum* or *Zariski spectrum* is the topological space $\mathrm{Spec}(A)$.

Given any subset $X \subseteq A$ one defines $V(X) = \bigcap_{a \in X} V(a)$, which is a closed subset of the spectrum. Every closed subset of $\mathrm{Spec}(A)$ can be written in this form. If $I = (X)$ is the ideal generated by X then $V(X) = V(I)$. The canonical homomorphism $\pi_I : A \to A/I$ is used to define the map $\mathrm{Spec}(\pi_I) : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$, $q \to \pi_I^{-1}(q)$, which is a homeomorphism onto the closed subset $V(I) \subseteq \mathrm{Spec}(A)$.

The space $\mathrm{Spec}(A)$ is a spectral space. By definition, a topological space X is *spectral* if the following conditions are satisfied, [10]:

- X is a quasi-compact T_0-space.
- The set of open and quasi-compact subsets is closed under finite intersections and is a basis of the topology.
- For very nonempty closed and irreducible set C there is a point $x \in X$ such that $C = \overline{\{x\}}$.

The set of open and quasi-compact subsets of X is denoted by $\mathring{\mathcal{K}}(X)$. This is a sublattice of the Boolean algebra $\mathfrak{P}(X)$, the power set of X. A subset $C \subseteq X$ is *constructible* if it belongs to the Boolean subalgebra of $\mathfrak{P}(X)$ generated by $\mathring{\mathcal{K}}(X)$; the Boolean algebra of constructible sets is denoted by $\mathcal{K}(X)$. The elements of $\mathring{\mathcal{K}}(X)$ are the constructible sets that are open. Therefore they are frequently referred to as the open and constructible sets. The set of complements of $\mathring{\mathcal{K}}(X)$ is the set $\overline{\mathcal{K}}(X)$ of closed and constructible sets.

The set $\overline{\mathcal{K}}(X)$ is the basis of another topology on X, which is called the *inverse topology*. If $\mathcal{T}$ is the topology of X then the inverse topology is denoted by $\mathcal{T}_{\mathrm{inv}}$, the set X with the inverse topology is denoted by X_{inv}. It is a remarkable fact that X_{inv} is a spectral space as well, [10, Proposition 8]. (The inverse topology has applications in real algebraic geometry, [19,20].) The constructible sets are the basis for another topology on X, which is called the *patch topology* [10] or *constructible topology*. The patch topology is denoted by $\mathcal{T}_{\mathrm{con}}$; the set X with the patch topology is denoted by X_{con}. This is a Boolean space, and its closed and open sets are exactly the constructible sets. Every Boolean space is a spectral space, as one sees easily from the definition. The space X itself is Boolean if and only if $X = X_{\mathrm{con}}$, if and only if $\mathring{\mathcal{K}}(X) = \mathcal{K}(X) = \overline{\mathcal{K}}(X)$. In general the patch topology is generated by the topology and the inverse topology, and $\mathring{\mathcal{K}}(X_{\mathrm{con}}) = \mathcal{K}(X)$. The elements of $\mathcal{A}(X_{\mathrm{con}})$ are called *proconstructible subsets*. We write $\widetilde{\mathcal{K}}(X) = \mathcal{A}(X_{\mathrm{con}})$.

If X is a spectral space and if $Y \subseteq X$ is a subset then Y carries the restriction of the topology of X. The subset Y is proconstructible if and only if the relative topology makes Y into a spectral space. The restriction map $\mathcal{K}(X) \to \mathcal{K}(Y)$, $C \to C \cap Y$ is a homomorphism of Boolean algebras. It restricts to surjective lattice homomorphisms $\mathring{\mathcal{K}}(X) \to \mathring{\mathcal{K}}(Y)$ and $\overline{\mathcal{K}}(X) \to \overline{\mathcal{K}}(Y)$.

Spectral spaces are T_0-spaces. Hence the specialization relation gives them the structure of posets. Thus, there are the closure operators $\sigma : \mathfrak{P}(X) \to \mathfrak{P}(X)$, $\sigma(C) = \{x \in X \mid \exists z \in C\colon\ z \rightsquigarrow x\}$ and $\gamma : \mathfrak{P}(X) \to \mathfrak{P}(X)$, $\gamma(C) = \{x \in X \mid \exists z \in C\colon\ x \rightsquigarrow z\}$. A subset $C \subseteq X$ is *closed under specialization* if $C = \sigma(C)$; it is *closed under generalization*, or *generically closed*, if $C = \gamma(C)$. The set C is closed for the topology if and only if it is proconstructible and closed under specialization. If the set C is proconstructible then $\overline{C} = \sigma(C)$, and it follows that $\sigma(C)$ is proconstructible. The same arguments, applied with the inverse topology, show that set $\gamma(C)$ is proconstructible as well. Thus σ and γ restrict to closure operators of $\widetilde{\mathcal{K}}(X)$. The restrictions are also denoted by σ and γ.

The set of maximal points of the poset $(X, \rightsquigarrow)$ is denoted by $\mathrm{Max}(X)$, the set of minimal points by $\mathrm{Min}(X)$. A point x is maximal if and only if the singleton set $\{x\}$ is closed. The minimal points are also called *generic points*. Every point specializes to some maximal point and has a generalization that is a generic point. The subspace $\mathrm{Max}(X) \subseteq X$ is always quasi-compact and T_1, but need not be Hausdorff. The subspace $\mathrm{Min}(X) \subseteq X$ is always Hausdorff, but need not be compact. A study of the spaces $\mathrm{Max}(\mathrm{Spec}(A))$ and $\mathrm{Min}(\mathrm{Spec}(A))$ (where A is a ring) is contained in [22].

One defines a *Priestley space* to be a Boolean space X together with a partial order $\leqslant$ such that the following condition is satisfied, [17, p. 218]:

- If $x \nleqslant y$ then there is a closed and open set U such that $x \notin U$ and $y \in U$ and $v \leqslant u$, $u \in U$ implies $v \in U$.

Given any spectral space X, the pair $(X_{\mathrm{con}}, \rightsquigarrow)$ is a Priestley space. Conversely, if $(X, \leqslant)$ is a Priestley space then the closed and open subsets of X that are downsets for the partial order are a basis for a topology of X. The space X is spectral and its specialization order is the original partial order $\leqslant$. Starting with a spectral space X, the spectral space derived from the Priestley space $(X_{\mathrm{con}}, \rightsquigarrow)$ coincides with X. Thus, spectral spaces and Priestley spaces are the same objects, only the presentation is different. Priestley spaces provide a convenient method for constructing examples of spectral spaces, which will be used in Section 3.

Every finite poset, equipped with the discrete topology, is a Priestley space, hence defines a spectral space. The open subsets of the spectral space are the downsets of the poset. Suppose that X is a spectral space and that $\leqslant$ is the inverse partial order of the specialization order. Then $(X_{\mathrm{con}}, \leqslant)$ is a Priestley space, and X_{inv} is the corresponding spectral space.

The spectral spaces form a category, which is denoted by $\mathtt{Spec}$. The morphisms have to be specified: Let X, Y be spectral spaces. A map $f : X \to Y$ of the underlying sets is a *spectral map* if it is continuous for two of the three topologies $\mathcal{T}$, $\mathcal{T}_{\mathrm{inv}}$, $\mathcal{T}_{\mathrm{con}}$, [10]. (In fact, if a map is continuous for two of these topologies, then it is continuous for all of them.) Equivalently, f is continuous for the patch topology and is monotonic for the specialization order.

Spectral spaces are intimately connected with *bounded distributive lattices* (distributive lattices that have a largest element, 'top', $\top$, and a smallest element, 'bottom', $\perp$). The bounded distributive lattices form a category, $\mathtt{BDLat}$, which is antiequivalent to the category $\mathtt{Spec}$. This is the content of the Stone duality for distributive lattices, [9, Chapter B], [17, Chapter 10]. We outline the constructions: Given a bounded distributive lattice L, one defines $\mathrm{Spec}(L)$ to be the set of prime filters of L with the topology that is generated by the closed sets $V(a) = \{\mathfrak{p} \in \mathrm{Spec}(L) \mid a \in \mathfrak{p}\}$, $a \in L$. Then $\mathrm{Spec}(L)$ is a spectral space. Starting with a spectral space X, the lattice $\overline{\mathcal{K}}(X)$ is bounded and distributive. There is a canonical lattice isomorphism $L \to \overline{\mathcal{K}}(\mathrm{Spec}(L))$, which is given by $a \to V(a)$. Similarly, there is a canonical homeomorphism $X \to \mathrm{Spec}(\overline{\mathcal{K}}(X))$, which is given by $x \to \mathfrak{p}(x) = \{C \in \overline{\mathcal{K}}(X) \mid x \in C\}$. A homomorphism $\varphi : L \to M$ of bounded distributive lattices defines a spectral map $\mathrm{Spec}(\varphi) : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$, which sends a prime filter $\mathfrak{q} \subset M$ to the prime filter $\varphi^{-1}(\mathfrak{q}) \subset L$. A spectral map $f : X \to Y$ defines a lattice homomorphism $\overline{\mathcal{K}}(f) : \overline{\mathcal{K}}(Y) \to \overline{\mathcal{K}}(X)$, which sends a closed and constructible set $C \subseteq Y$ to the closed and constructible set $f^{-1}(C) \subseteq X$.

The bounded distributive lattice L is a Boolean algebra if and only if its spectrum is a Boolean space.

The connected components of a spectral space. The subset $\mathring{\mathcal{K}}(X) \cap \overline{\mathcal{K}}(X) = \overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X) \subseteq \mathcal{K}(X)$ is the Boolean algebra of closed and open subsets of X. Every point $x \in X$ defines the ultra filter $u(x) = \{C \in \overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X) \mid x \in C\}$, and $\mathrm{T}[x] = \bigcap_{C \in u(x)} C$. (Recall that prime filters are ultra filters since we deal with a Boolean algebra.) Conversely, if $\mathfrak{u}$ is an ultrafilter, then $\bigcap_{C \in \mathfrak{u}} C$ is a (by compactness) nonempty connected set and is the connected component of each of its elements. Thus, we have defined a surjective map $u : X \to \mathrm{Spec}(\overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X))$, $x \to u(x)$, whose fibers are the connected components. Hence u factors through $p_{\mathrm{T}} : X \to X/\mathrm{T}$, $u = \bar{u} \circ p_{\mathrm{T}}$ (with $\bar{u} : X/\mathrm{T} \to \mathrm{Spec}(\overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X))$). The map u corresponds, under Stone duality, to the inclusion homomorphism $\overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X) \hookrightarrow \overline{\mathcal{K}}(X)$ of lattices. Thus, u is a spectral map, and $\bar{u}$ is continuous and bijective. The space X/T is Hausdorff, hence compact. One concludes that $\bar{u}$ is a homeomorphism.

Idempotents. The connected components of the prime spectrum of a ring are determined by the idempotents. The set of idempotents of the ring A is denoted by $E(A)$. This is a Boolean algebra – multiplication is the restriction of the multiplication of A, addition is given by $(e, f) \to e + f - 2 \cdot e \cdot f$. If $I \subseteq A$ is an ideal then $I \cap E(A)$ is an ideal of the Boolean algebra. If I is a prime ideal then $I \cap E(A)$ is a prime ($=$ maximal) ideal. This construction yields a spectral map $p_E : \mathrm{Spec}(A) \to \mathrm{Spec}(E(A))$. Here we consider $E(A)$ as a ring, not as a lattice. Accordingly, the points of the spectrum are prime ideals, not prime filters.

The ring A is indecomposable if there are no nontrivial idempotents, i.e., if $E(A)$ is the trivial Boolean algebra. Nontrivial idempotents correspond to partitions of $\mathrm{Spec}(A)$ into nonempty closed and open sets – if $e \in E(A)$ then $\mathrm{Spec}(A) = D(e) \cup D(1 - e)$. Thus, A is indecomposable if and only if $\mathrm{Spec}(A)$ is connected.

Suppose that $\mathfrak{i} \subseteq E(A)$ is an ideal of the Boolean algebra. Then $J_{\mathfrak{i}} = \mathfrak{i} \cdot A$ is an ideal of the ring, and $J_{\mathfrak{i}} \cap E(A) = \mathfrak{i}$. The factor ring $A/J_{\mathfrak{i}}$ is reduced. Moreover, $E(A/J_{\mathfrak{i}})$ is canonically isomorphic to $E(A)/\mathfrak{i}$. The ring $A/J_{\mathfrak{i}}$ is indecomposable if and only if $\mathfrak{i} \subset E(A)$ is maximal. In particular, the map p_E is surjective. The factor ring $A/J_{\mathfrak{i}}$ is indecomposable if and only if $V(\mathfrak{i}) = V(J_{\mathfrak{i}}) \subseteq \operatorname{Spec}(A)$ (which is homeomorphic to $\operatorname{Spec}(A/J_\iota)$) is a connected subset. Therefore the connected component of a prime ideal $p \in \operatorname{Spec}(A)$ can be determined as follows: The restriction $\mathfrak{p} = p \cap E(A)$ is a prime ideal. The closed subset $V(J_{\mathfrak{p}}) \subseteq \operatorname{Spec}(A)$, i.e., the fiber $p_E^{-1}(\{\mathfrak{p}\}) = V(J_{\mathfrak{p}})$ of the map p_E, is the connected component of p.

Suppose that $C \subseteq \operatorname{Spec}(A)$ is a constructible subset that contains a fiber $p_E^{-1}(\mathfrak{p})$. Then $p_E(\operatorname{Spec}(E) \setminus C)$ is a closed subset of $\operatorname{Spec}(E(A))$ that does not contain $\mathfrak{p}$. It follows that there is an idempotent $e \in E(A)$ such that $p_E^{-1}(\mathfrak{p}) \subseteq D(e) \subseteq C$.

The prime ideals of $E(A)$ yield an embedding $\Pi_A : A \to \prod_{\mathfrak{p} \in \operatorname{Spec}(E(A))} A/J_{\mathfrak{p}}$. We record the following simple facts:

Fact 1.1. *Let A be a ring, and pick an element $a \in A$.*

(a) *$a \in A^\times$ if and only if $\Pi_A(a) \in (\prod_{\mathfrak{p} \in \operatorname{Spec}(E(A))} A/J_{\mathfrak{p}})^\times$.*
(b) *a is regular if and only if $\Pi_A(a)$ is regular.*
(c) *a is an idempotent if and only if $\Pi_A(a)$ is an idempotent.*

Fact 1.2. *Let A be a ring, let $\mathfrak{i}, \mathfrak{j} \subseteq E(A)$ be ideals and set $\mathfrak{k} = \mathfrak{i} + \mathfrak{j}$. Then:*

(a) *$\mathfrak{i} \cap \mathfrak{j} = \{0\}$ if and only if $J_{\mathfrak{i}} \cap J_{\mathfrak{j}} = \{0\}$.*
(b) *$J_{\mathfrak{k}} = J_{\mathfrak{i}} + J_{\mathfrak{j}}$.*
(c) *The canonical map $\psi : A \to A/J_{\mathfrak{i}} \times_{A/J_{\mathfrak{k}}} A/J_{\mathfrak{j}}$ is an isomorphism if $\mathfrak{i} \cap \mathfrak{j} = \{0\}$.*

2. The graph components of a spectral space

Let X be a spectral space. We want to determine the graph component $\Gamma[x]$ of a point $x \in X$. Several simple algorithms are exhibited that produce the graph component. The algorithms use two actions on the subsets of X, namely the maps $\sigma : \mathfrak{P}(X) \to \mathfrak{P}(X)$ and $\gamma : \mathfrak{P}(X) \to \mathfrak{P}(X)$, cf. Section 1. The compositions of the maps γ and σ are denoted by $\rho = \gamma \circ \sigma$ and $\tau = \sigma \circ \gamma$.

The graph component of a point $x \in X$ is closed under specialization and generalization since $\Gamma[x]$ contains all points that can be reached through a combination of specializations and generalizations starting with x. It is clearly the smallest set that contains x and is closed under specialization and generalization. The topological component of x is also closed under specialization and generalization, hence $\Gamma[x] \subseteq \mathrm{T}[x]$. The topological component is the intersection of all closed and open subsets of X that contain x.

Given a point $x \in X$, its graph component is the union of the balls about x with finite radius, $\Gamma[x] = \bigcup_{k \in \mathbb{N}} B[x; k]$. The balls can be constructed recursively using the closure operators σ and γ. Consider the following algorithm that starts with any subset $X' \subseteq X$:

Basic Algorithm

- $B[X'; 0] = X'$.
- Suppose that the set $B[X'; k]$ has been constructed.
- $B[X'; k+1] = \sigma(B[X'; k]) \cup \gamma(B[X'; k])$.

The sets $B[X'; k]$ will be called *balls of radius k about X'*. The union of the balls is denoted by $\Gamma[X']$. If X' is proconstructible then each ball is proconstructible, and $\Gamma[X']$ is the union of a countable increasing sequence of proconstructible sets. The construction of the graph component $\Gamma[x]$ is the special case $X' = \{x\}$. We shall exhibit examples below to show that very little can be said about the graph components in complete generality.

We record the following simple fact:

Proposition 2.1. *Suppose that $X' \subseteq X$ is a subset in a spectral space and that $K \in \mathbb{N}$. Consider the conditions*:

(a) *Each graph component $\Gamma[x]$, $x \in X'$, has diameter at most K.*
(b) *The Basic Algorithm applied to X' terminates after at most K steps* (which means that $B[X'; K] = B[X'; K+1]$).
(c) *If X' is proconstructible then $\Gamma[X']$ is proconstructible.*

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold.*

In particular, a graph component is proconstructible if its diameter is finite. We shall show with examples that the converse of this statement is not true, i.e., there is a spectral space with a proconstructible graph component that does not have finite diameter, 3.4. Moreover, graph components are not proconstructible in general, 3.5, 3.6.

Proposition 2.2. *Let X be a spectral space, and suppose that $\Gamma \subseteq X$ is a constructible graph component. Then Γ is a topological component.*

Proof. The graph component is constructible and closed under specialization, hence it is closed, and it is closed under generalization, hence it is open. A closed and open set is a union of topological components. On the other hand, the graph component Γ is topologically connected. $\square$

The converse of 2.2 is false: If a graph component is a topological component then it does not have to be constructible. Every infinite Boolean space yields an example. The singleton subsets are the topological components and the graph components. If they are all constructible then they are all open, and the space is discrete. This is impossible since it is infinite and compact.

If the spectral space X is graph connected (i.e., there is only one graph component) then the graph component is constructible and, of course, proconstructible. On the other hand, suppose that X is any spectral space, and let $\Gamma \subseteq X$ be a graph component. If Γ is proconstructible then the subspace Γ of X is spectral, [6], and is graph connected. Thus, if one wants to analyze a single proconstructible graph component, without regard to the ambient spectral space, then it suffices to look at graph connected spectral spaces.

We shall be concerned mainly with the following two algorithms, which are variants of the Basic Algorithm. Their advantage compared with the Basic Algorithm is that, in the case of the prime spectrum of a ring, they are easier to analyze using ring elements. Again, let $X' \subseteq X$ be a subset.

1*st Algorithm*

- Define $C_0[X'] = X'$.
- Suppose that $C_0[X'], \dots, C_k[X']$ have been defined.
- Define $C_{k+1}[X'] = \sigma(C_k[X'])$ if k is even.
- Define $C_{k+1}[X'] = \gamma(C_k[X'])$ if k is odd.

2*nd Algorithm*

- Define $D_0[X'] = X'$.
- Suppose that $D_0[X'], \dots, D_k[X']$ have been defined.
- Define $D_{k+1}[X'] = \gamma(D_k[X'])$ if k is even.
- Define $D_{k+1}[X'] = \sigma(D_k[X'])$ if k is odd.

If the subset X' is closed then the Basic Algorithm coincides with the 2nd Algorithm. If the subset X' is generically closed then the Basic Algorithm coincides with the 1st Algorithm.

Lemma 2.3. *Let $X' \subseteq X$ be a subset of a spectral space. Then $C_k[X'], D_k[X'] \subseteq B[X';k] \subseteq C_{k+1}[X']$, $D_{k+1}[X']$.*

Proof. The proof is by induction on k. We do it for the sequence $(C_k[X'])_{k\in\mathbb{N}}$. To start with, note that $X' = C_0[X'] = B[X';0] \subseteq C_1[X']$. If $C_k[X'] \subseteq B[X';k] \subseteq C_{k+1}[X']$ then

$$\begin{aligned} C_{k+1}\big[X'\big] &= \sigma\big(C_k\big[X'\big]\big) \subseteq \sigma\big(B\big[X';k\big]\big) \cup \gamma\big(B\big[X';k\big]\big) = B\big[X',k+1\big] \\ &\subseteq \sigma\big(C_{k+1}\big[X'\big]\big) \cup \gamma\big(C_{k+1}\big[X'\big]\big) = \gamma\big(C_{k+1}\big[X'\big]\big) = C_{k+2}\big[X'\big] \end{aligned}$$

if k is even, and

$$\begin{aligned} C_{k+1}\big[X'\big] &= \gamma\big(C_k\big[X'\big]\big) \subseteq \sigma\big(B\big[X';k\big]\big) \cup \gamma\big(B\big[X';k\big]\big) = B\big[X',k+1\big] \\ &\subseteq \sigma\big(C_{k+1}\big[X'\big]\big) \cup \gamma\big(C_{k+1}\big[X'\big]\big) = \sigma\big(C_{k+1}\big[X'\big]\big) = C_{k+2}\big[X'\big] \end{aligned}$$

if k is odd. $\square$

By 2.3 one may consider the sequences $(C_k[X'])_{k\in\mathbb{N}}$ and $(D_k[X'])_{k\in\mathbb{N}}$ as approximations of the sequence of balls about X':

$$\begin{tikzcd}
C_0[X'] \arrow[rr] \arrow[dr, leftrightarrow, "="] \arrow[dd, leftrightarrow, "="'] & & C_1[X'] \arrow[rr] \arrow[dr] & & C_2[X'] \arrow[rr] \arrow[dr] & & \cdots \\
& B[X';0] \arrow[rr] \arrow[ur] \arrow[dr] & & B[X';1] \arrow[rr] \arrow[ur] \arrow[dr] & & B[X';2] \arrow[rr] \arrow[ur] \arrow[dr] & & \cdots \\
D_0[X'] \arrow[rr] \arrow[ur, leftrightarrow, "="'] & & D_1[X'] \arrow[rr] \arrow[ur] & & D_2[X'] \arrow[rr] \arrow[ur] & & \cdots
\end{tikzcd}$$

The arrows in the diagram indicate containment relations. If the set X' is generically closed then $C_k[X'] = B[X';k] = D_{k+1}[X']$. If the set X' is closed then $D_k[X'] = B[X';k] = C_{k+1}[X']$. Trivially, $\Gamma[X'] = \bigcup_{k\in\mathbb{N}} C_k[X'] = \bigcup_{k\in\mathbb{N}} D_k[X']$. If the Basic Algorithm terminates after at most K steps then the 1st Algorithm and the 2nd Algorithm terminate after at most $K+1$ steps. Conversely, if the 1st Algorithm or the 2nd Algorithm terminates after at most K steps then so does the Basic Algorithm.

Now consider a graph component Γ and suppose that its diameter is finite, say K. Given a point $x \in \Gamma$, the largest distance from x to any point of Γ is somewhere between $\lceil \frac{K}{2} \rceil$ and K. Thus, the algorithms terminate at $K+1$ at the latest, but do not terminate before $\lceil \frac{K}{2} \rceil$.

The different algorithms, and the sequences produced by them, are closely related to each other. But they differ with regard to the point of termination. The next result (without proof) shows this phenomenon for a few simple cases.

Proposition 2.4. *Let X be a graph connected spectral space.*

(a) *The following conditions are equivalent*:
  (i) *X has only one point.*
  (ii) *The* 1*st Algorithm, when applied to a generic point, terminates after* 0 *steps.*
  (iii) *The* 2*nd Algorithm, when applied to a maximal point, terminates after* 0 *steps.*

(b) *The following conditions are equivalent*:
  (i) *X has only one generic point.*
  (ii) *The* 1*st Algorithm, when applied to a generic point, terminates after at most* 1 *step.*

(c) *The following conditions are equivalent*:
  (i) *X has only one maximal point.*
  (ii) *The* 2*nd Algorithm, when applied to a maximal point, terminates after at most* 1 *step.*

(d) *The following conditions are equivalent*:
  (i) *Every generic point of X specializes to every maximal point.*
  (ii) *Every maximal point of X generalizes to every generic point.*
  (iii) *The* 1*st Algorithm, when applied to a generic point, terminates after at most* 2 *steps.*
  (iv) *The* 2*nd Algorithm, when applied to a maximal point, terminates after at most* 2 *steps.*

3. Examples of graph components in a spectral space

This section contains a collection of examples of spectral spaces. They illustrate various phenomena and will be used to produce counterexamples to several questions that arise in the study of graph components. We shall mostly use the method of Priestly spaces to verify that the constructions do, in fact, yield spectral spaces.

Before starting with the examples we give a brief explanation of how they are presented. Finite spectral spaces are usually described by drawing the specialization poset. Infinite spectral spaces are described by drawing the specialization poset and by an explanation of the patch topology. Specialization is indicated by arrows.

The first example shows how the different algorithms work in a finite spectral space:

Example 3.1. Consider the following poset:

$$\begin{tikzcd}
 & z_1 & & z_2 & & \\
y_1 \arrow[ur, squiggly] & & y_2 \arrow[ul, squiggly] \arrow[ur, squiggly] & & & \\
 & x_1 \arrow[ul, squiggly] \arrow[ur, squiggly] & & & & x_3 \arrow[uull, squiggly]
\end{tikzcd}$$

The diameter of the graph is 3. The following diagrams show the progress of the algorithms, starting with the points y_1, y_2 and x_3. Inclusions are proper if they are not said to be equal.

$$\begin{tikzcd}
 & C_1[y_1] \arrow[dr] & & C_2[y_1] \arrow[dr] & & C_3[y_1] \arrow[dr] & & C_4[y_1] \\
\{y_1\} \arrow[ur] \arrow[dr] & & B[y_1;1] \arrow[ur] \arrow[dr] & & B[y_1;2] \arrow[ur] \arrow[dr] & & B[y_1;3] \arrow[ur, "="] & \\
 & D_1[y_1] \arrow[ur] & & D_2[y_1] \arrow[ur] & & D_3[y_1] \arrow[ur, "="'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C_1[y_2] \arrow[dr] & & C_2[y_2] \arrow[dr, "="] & & \\
\{y_2\} \arrow[ur] \arrow[dr] & & B[y_2;1] \arrow[ur] \arrow[dr] & & B[y_2;2] \arrow[dr, "="] & \\
 & D_1[y_2] \arrow[ur] & & D_2[y_2] \arrow[ur] & & D_3[y_2]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & C_1[x_3] \arrow[dr, "="] & & C_2[x_3] \arrow[dr, "="] & & C_3[x_3] \arrow[dr, "="] & & \\
\{x_3\} \arrow[ur] \arrow[dr, "="'] & & B[x_3;1] \arrow[ur] \arrow[dr, "="'] & & B[x_3;2] \arrow[ur] \arrow[dr, "="'] & & B[x_3;3] \arrow[dr, "="] & \\
 & D_1[x_3] \arrow[ur] & & D_2[x_3] \arrow[ur] & & D_3[x_3] \arrow[ur] & & D_4[x_3]
\end{tikzcd}$$

The next example shows several spectral spaces for which the 1st Algorithm and the 2nd Algorithm terminate differently.

Example 3.2. Consider the spectral spaces

$X_{2\cdot n+2}$

$$\begin{tikzcd}[column sep=small]
 & x_1 & & x_3 & \cdots & x_{2\cdot n-1} & & x_{2\cdot n+1} & \\
x_0 \arrow[ur, squiggly] & & x_2 \arrow[ul, squiggly] \arrow[ur, squiggly] & & \cdots & & x_{2\cdot n} \arrow[ul, squiggly] \arrow[ur, squiggly] & & x_{2\cdot n+2} \arrow[ul, squiggly]
\end{tikzcd}$$

$Y_{2\cdot n+2}$

$$\begin{tikzcd}[column sep=small]
y_0 & & y_2 & & \cdots & & y_{2\cdot n} & & y_{2\cdot n+2} \\
 & y_1 \arrow[ul, squiggly] \arrow[ur, squiggly] & & y_3 \arrow[ul, squiggly] & \cdots & y_{2\cdot n-1} \arrow[ur, squiggly] & & y_{2\cdot n+1} \arrow[ul, squiggly] \arrow[ur, squiggly] &
\end{tikzcd}$$

$Z_{2\cdot n+3}$

$$\begin{tikzcd}[column sep=small]
z_0 & & z_2 & & \cdots & & z_{2\cdot n+2} & \\
 & z_1 \arrow[ul, squiggly] \arrow[ur, squiggly] & & z_3 \arrow[ul, squiggly] & \cdots & z_{2\cdot n+1} \arrow[ur, squiggly] & & z_{2\cdot n+3} \arrow[ul, squiggly]
\end{tikzcd}$$

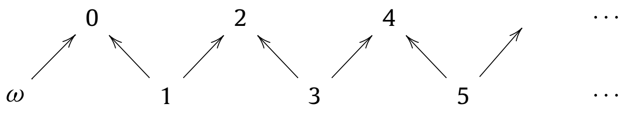
- If the 1st Algorithm is applied to x_0 or to $x_{2\cdot n+2}$ then it terminates after $2\cdot n+2$ steps. If it is applied to any other generic point of $X_{2\cdot n+2}$ then it terminates earlier. If the 2nd Algorithm is applied to x_1 or to $x_{2\cdot n+1}$ then it terminates after $2\cdot n+1$ steps. If it is applied to any other maximal point of $X_{2\cdot n+2}$ then it terminates earlier.
- If the 1st Algorithm is applied to y_1 or to $y_{2\cdot n+1}$ then it terminates after $2\cdot n+1$ steps. If it is applied to any other generic point of $Y_{2\cdot n+2}$ then it terminates earlier. If the 2nd Algorithm is applied to y_0 or to $y_{2\cdot n+2}$ then it terminates after $2\cdot n+2$ steps. If it is applied to any other maximal point of $Y_{2\cdot n+2}$ then it terminates earlier.
- If the 1st Algorithm is applied to $z_{2\cdot n+3}$ then it terminates after $2\cdot n+3$ steps. If it is applied to any other generic point of $Z_{2\cdot n+3}$ then it terminates earlier. If the 2nd Algorithm is applied to z_0 then it terminates after $2\cdot n+3$ steps. If it is applied to any other maximal point of $Z_{2\cdot n+3}$ then it terminates earlier.

Example 3.3. *Normal spectral spaces.* Let X be a normal spectral space, i.e., every point of X specializes to a unique maximal point, say $x \to \mu(x) \in \operatorname{Max}(X)$, [3], [22, Section 4]. Then $\Gamma[x] = B[x;2] = C_2[x] = D_2[x] = D_1[\mu(x)]$. The diameter of a graph component is at most 2. It has the value 1 if and only if each graph component is totally ordered under specialization.

Normal spectral spaces abound in real algebra: Real spectra of rings, [2, Chapter 7], are normal spectral spaces. They even have a much stronger property – the set of specializations of a point is a chain. Spectral spaces with this property are sometimes called *completely normal*. A poset is a *root system* if the principal upsets (those generated by elements of the set) are totally ordered. Thus, completely normal spectral spaces are spectral spaces that are root systems. The prime spectrum of a ring of continuous functions, more generally of a real closed ring, [18,21], is homeomorphic to its real spectrum, hence is a normal spectral space.

The next example shows a graph connected spectral space with infinite diameter. The example was developed in a discussion with Marcus Tressl.

Example 3.4. Let X be the one-point compactification of the discrete space $\mathbb{N}$. The additional point is denoted by ω. It is well known (and easy to prove) that X is a Boolean space. The following diagram exhibits a partial order on X:



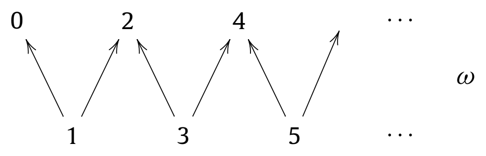
We wish to show that these data define a Priestley space. Pick two points $x, y \in X$, $x \nleqslant y$. We must find a downset that is open and closed in the Boolean topology and contains y, but not x. There are several cases to consider:

- Suppose that $y \in \mathbb{N}$ is odd: The set $\{y\}$ meets the requirements.
- Suppose that $2 \leqslant y \in \mathbb{N}$ is even: The set $\{y-1, y, y+1\}$ meets the requirements.
- Suppose that $y = 0$: In this case $2 \leqslant x \in \mathbb{N}$. The set $\{0, 1, 2 \cdot x + 1, 2 \cdot x + 2, \dots, \omega\}$ has the desired properties.
- Suppose that $y = \omega$: The set $\{2 \cdot x + 1, 2 \cdot x + 2, \dots, \omega\}$ has the desired properties.

We have established that X is a spectral space. It is clearly graph connected, and its diameter as a graph is infinite.

The next example is a modification of the previous one. It shows a graph component that is not proconstructible.

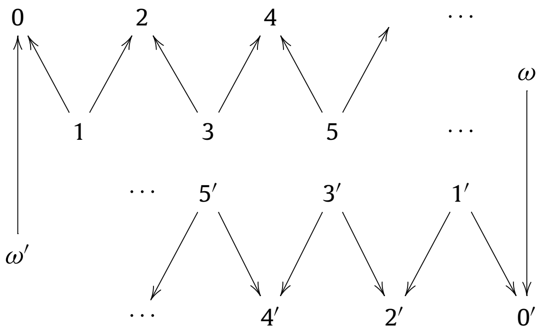
Example 3.5. Let X be the one-point compactification of the discrete space $\mathbb{N}$. The additional point is denoted by ω. This is a Boolean space. The following diagram presents a partial order that makes X into a spectral space:



This spectral space has two graph components, $\mathbb{N}$ and $\{\omega\}$. The graph component $\mathbb{N}$ has infinite diameter and is not proconstructible since the point ω is not isolated.

The next example is another modification of the previous ones. Its significance will be explained in Section 4.

Example 3.6. Let $\mathbb{N}$ and $\mathbb{N}'$ be two copies of the discrete space of natural numbers. The elements of $\mathbb{N}$ are denoted by $0, 1, \dots$, those of $\mathbb{N}'$ are denoted by $0', 1', \dots$. Let X be the one-point compactification of $\mathbb{N}$, X' the one-point compactification of $\mathbb{N}'$. The additional points are denoted by ω and ω'. Let Y be the topological sum of X and X', which is a Boolean space. Again, we exhibit a partial order on Y:

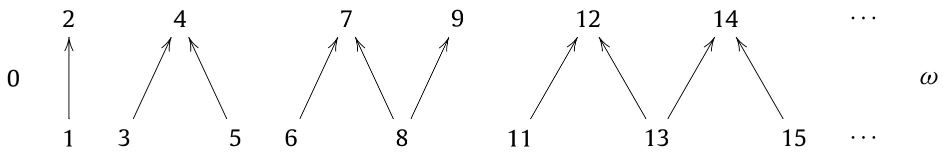


The same arguments as above show that these data determine a spectral space. There are two graph components, both with infinite diameter. They are not proconstructible.

Example 3.7. *A spectral space whose graph components have finite diameter, and the set of diameters is unbounded.* As above, let X be the one-point compactification of the discrete space $\mathbb{N}$. Again, X serves as the patch space of a spectral space. The specialization relation is defined as follows: For each $p \in \mathbb{N}$, the numbers $\binom{p+1}{2}, \binom{p+1}{2}+1, \ldots, \binom{p+2}{2}-1$ form a graph component, and the partial order in the component is given by the diagrams:

- $\binom{p+1}{2} \to \binom{p+1}{2}+1 \leftarrow \cdots \to \binom{p+2}{2}-2 \leftarrow \binom{p+2}{2}-1$ if p is even,
- $\binom{p+1}{2} \to \binom{p+1}{2}+1 \leftarrow \cdots \leftarrow \binom{p+2}{2}-2 \to \binom{p+2}{2}-1$ if p is odd.

The specialization relation is depicted in the following diagram:



The pth component, $p \in \mathbb{N}$, has diameter p. The additional point ω is a component of its own.

4. The space of graph components of a spectral space

This section contains basic information about the topology of the space X/G of graph components of a spectral space. Most of the results are concerned with separation properties.

First we record a trivial fact:

Proposition 4.1. *For every spectral space X the space X/G is quasi-compact.*

Proof. The space X/G is the image of the quasi-compact space X under the continuous map p_{G}. $\square$

By definition of the quotient topology, a subset $M \subseteq X/\mathrm{G}$ is closed (or open) if and only if $p_{\mathrm{G}}^{-1}(M) \subseteq X$ is closed (or open).

The space of graph components may be indiscrete, even if there are several components. For an example, consider 3.6: The space Y/G has two points, as there are two graph components. The graph components are $\Gamma[\omega]$ and $\Gamma[\omega']$. There are exactly two closed subsets in Y that are unions of graph components, namely $\emptyset$ and Y. Thus, Y/G carries the indiscrete topology.

The space X/G of 3.5 has two points as well, $\Gamma[0]$ and $\Gamma[\omega]$. This time $p_{\mathrm{G}}^{-1}(\Gamma[\omega]) = \{\omega\}$ is closed and $p_{\mathrm{G}}^{-1}(\Gamma[0])$ is not closed. It follows that X/G is a T_0-space with the specialization $\Gamma[0] \to \Gamma[\omega]$.

Proposition 4.2. *Let X be a spectral space. The space X/G is a T_1-space if and only if every graph component is proconstructible.*

Proof. A topological space has the T_1-property if and only if every singleton subset is closed. First suppose that X/G is a T_1-space, and let Γ be a graph component. The singleton subset $\{\Gamma\} \subseteq X/\mathrm{G}$ is closed, hence so is its inverse image $\Gamma \subseteq X$. Closed sets are proconstructible. Thus, the graph components are proconstructible.

Now suppose that the graph components are proconstructible. They are closed under specialization, hence they are closed. The definition of the quotient topology shows that each graph component, as a point of the quotient space, is closed. $\square$

Proposition 4.2 applies if each graph component can be constructed from one (or any) of its points with finitely steps using the algorithms of Section 2. This is the case if and only if each graph component has finite diameter, 2.1. Note that there exist spectral spaces with proconstructible graph components of infinite diameter, 3.4.

We give several different descriptions of the topology of X/G for the case that the graph components are proconstructible.

Proposition 4.3. *Suppose that X is a spectral space whose graph components are proconstructible. The topology of X is denoted by $\mathcal{T}$, the inverse topology by $\mathcal{T}_{\mathrm{inv}}$ and the patch topology by $\mathcal{T}_{\mathrm{con}}$. Then the quotient topology on X/G with respect to the map p_{G} is the same for each of the four topologies $\mathcal{T}_{\mathrm{con}}$, $\mathcal{T}$, $\mathcal{T}_{\mathrm{inv}}$ and $\mathcal{T} \cap \mathcal{T}_{\mathrm{inv}}$ on X.*

Proof. The four quotient spaces have the same underlying set, but the topologies may be different, to start with. We use the standard notation X_{con} and X_{inv} and write $X' = (X, \mathcal{T} \cap \mathcal{T}_{\mathrm{inv}})$. Then the quotient spaces are X/G, $X_{\mathrm{con}}/\mathrm{G}$, $X_{\mathrm{inv}}/\mathrm{G}$ and X'/G. There is a commutative diagram of continuous maps, all of which are the identity on the underlying sets:

$$\begin{tikzcd} & X/\mathrm{G} \arrow[dr, "h"] & \\ X_{\mathrm{con}}/\mathrm{G} \arrow[ur, "f"] \arrow[dr, "g"'] & & X'/\mathrm{G} \\ & X_{\mathrm{inv}}/\mathrm{G} \arrow[ur, "k"'] & \end{tikzcd}$$

If $h \circ f = k \circ g$ is a closed map then it is a homeomorphism (being bijective and continuous). The inverses of f, g, h, k exist since the maps are bijective. If $h \circ f$ is a homeomorphism then they are also continuous since $f^{-1} = (h \circ f)^{-1} \circ h$, $h^{-1} = f \circ (h \circ f)^{-1}$, etc.

To prove closedness of $h \circ f = k \circ g$, let $M \subseteq X_{\mathrm{con}}/\mathrm{G}$ be a closed set. Then $p_{\mathrm{G}}^{-1}(M) \subseteq (X, \mathcal{T}_{\mathrm{con}})$ is closed, hence proconstructible. It is a union of graph components. Therefore it is closed under specialization, which implies that it is closed for $\mathcal{T}$. It is also closed under generalization, which implies that it is closed for $\mathcal{T}_{\mathrm{inv}}$. But then it is closed for $\mathcal{T} \cap \mathcal{T}_{\mathrm{inv}}$, and one concludes that $h \circ f(M)$ is closed as a subset of X'/G. $\square$

Proposition 4.3 suggests a lattice theoretic interpretation of the space of graph components.

The set $\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$ is a sublattice of the power set $\mathfrak{P}(X)$. Its elements are the proconstructible sets that are closed under specialization and generalization. Note that usually this is not a Boolean algebra since the complement of a proconstructible set is proconstructible if and only if the set is constructible. This observation implies that $\overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X)$ is the largest Boolean algebra that is contained in $\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. Suppose now that the graph components of X are proconstructible. Then each $\Gamma[x]$ is an element, in fact an atom, of the lattice $\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. The lattice is atomic since every nonempty $C \in \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$ is a union of graph components; the graph components are exactly the atoms. One defines a map $v : X \to \mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X))$ by $x \to \{C \in \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X) \mid x \in C\}$. The main properties of the map v are summarized in

Proposition 4.4. *Let X be a spectral space with proconstructible graph components. The map v is defined as above. Then*:

(a) *The fibers of v are the graph components.*
(b) *The image of v is the set* $\mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)))$.
(c) *v is continuous.*
(d) *v induces a homeomorphism* $\bar{v} : X/\mathrm{G} \to \mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)))$.

Proof. (a) is trivial.

(b). The maximal points of $\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X))$ are the ultra filters of $\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. Given $x \in X$, $v(x)$ is the ultrafilter determined by the atom $\Gamma[x] \in \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. Thus, the image of v is contained in $\mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)))$. For the other inclusion, pick an ultrafilter $\mathfrak{u} \subset \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. Compactness implies that $D = \bigcap_{C \in \mathfrak{u}} C$ is nonempty. It is clear that $D \in \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. Thus, D contains a connected component $\Gamma[x]$. It follows that $\mathfrak{u} \subseteq v(x)$. One concludes that $\mathfrak{u} = v(x)$ since $\mathfrak{u}$ and $v(x)$ are both ultrafilters.

(c). Pick an element $C \in \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$ and consider the basic closed set $V(C) \subseteq \mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X))$. It is easy to check that $v^{-1}(V(C)) = C$, which is closed in X, and this proves that v is continuous.

(d). It follows from (a), (b) and (c) that there is a continuous and bijective map $\bar{v} : X/\mathrm{G} \to \mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)))$ such that $v = \bar{v} \circ p_{\mathrm{G}}$. It remains to show that $\bar{v}$ is closed: Pick a closed set $D \subseteq X/\mathrm{G}$ and set $C = p_{\mathrm{G}}^{-1}(D)$. This is a closed subset of X that is a union of graph components, hence belongs to $\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$. The definition of v implies that $\bar{v}(D) = v(C) = V(C) \cap \mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)))$, where $V(C)$ is the basic closed subset of $\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X))$ defined by C. $\square$

Corollary 4.5. *Let X be a spectral space with proconstructible graph components. The notation of the preceding paragraphs remains in force. For the definition of the map $\bar{u}$, see* Section 1, the part about connected components of a spectral space. *Let $\iota : \overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X) \to \mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)$ be the inclusion homomorphism. Then the following diagram is commutative*:

$$
\begin{tikzcd}
 & X \arrow[dl, "p_{\mathrm{G}}"'] \arrow[dr, "p_{\mathrm{T}}"] & \\
X/\mathrm{G} \arrow[rr, "p_{\mathrm{T,G}}"] \arrow[d, "\bar{v}"', "\simeq"] & & X/\mathrm{T} \arrow[d, "\simeq"', "\bar{u}"] \\
\mathrm{Max}(\mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X))) \arrow[r, "\subseteq"] & \mathrm{Spec}(\mathcal{A}(X_{\mathrm{inv}}) \cap \mathcal{A}(X)) \arrow[r, "\mathrm{Spec}(\iota)"] & \mathrm{Spec}(\overline{\mathcal{K}}(X_{\mathrm{inv}}) \cap \overline{\mathcal{K}}(X))
\end{tikzcd}
$$

As a follow-up to 4.2 we give a sufficient condition for X/G to be Hausdorff:

Proposition 4.6. *Let X be a spectral space whose graph components have bounded diameter. Then X/G is Hausdorff.*

Proof. Recall that the graph components are proconstructible if they have finite diameter, 2.1. Pick points $x, y \in X$ such that the graph components $\Gamma[x]$ and $\Gamma[y]$ are distinct. We must produce two closed sets $C, D \subseteq X$ such that

- $\Gamma[x] \subseteq C$, $\Gamma[y] \cap C = \emptyset$, and
- $\Gamma[y] \subseteq D$, $\Gamma[x] \cap D = \emptyset$, and
- C and D are both unions of graph components, and
- $C \cup D = X$.

The graph components $\Gamma[x]$ and $\Gamma[y]$ are proconstructible (since they have finite diameter, 2.1) and are closed under specialization and generalization. Let $\mathcal{U}$ be the set of open and constructible neighborhoods of $\Gamma[x]$, let $\mathcal{V}$ be the set of open and constructible neighborhoods of $\Gamma[y]$. Then $\Gamma[x] = \bigcap_{U \in \mathcal{U}} U$, $\Gamma[y] = \bigcap_{V \in \mathcal{V}} V$ and $\bigcap_{U \in \mathcal{U}} U \cap \bigcap_{V \in \mathcal{V}} V = \emptyset$. Quasi-compactness implies that there are disjoint sets $U \in \mathcal{U}$ and $V \in \mathcal{V}$. We define $C' = X \setminus V$ and $D' = X \setminus U$. Note that $\Gamma[x] \subseteq C'$, $\Gamma[y] \cap C' = \emptyset$, $\Gamma[y] \subseteq D'$, $\Gamma[x] \cap D' = \emptyset$ and $C' \cup D' = X$. Since C' and D' are constructible we know from 2.1 that $C = \Gamma[C']$ and $D = \Gamma[D']$ are proconstructible. They are also closed under specialization, hence they are closed. It is clear from the construction that C and D have the desired properties. $\square$

Next we consider normal spectral spaces. Various different characterizations are given in [3] and in [22, Sections 4, 5]. In the present context the most useful characterization is: A spectral space is normal if every point specializes to only one maximal point. It is immediately clear that every connected component has diameter at most 2. Recall that the space of maximal points of a normal spectral space is Hausdorff, [3], [22, Section 4].

Corollary 4.7. *Let X be a normal spectral space. Then* $\mathrm{Max}(X)$ *is homeomorphic to* X/G.

Proof. The canonical map $p_{\mathrm{G}}: X \to X/\mathrm{G}$ restricts to a continuous and bijective map $\mathrm{Max}(X) \to X/\mathrm{G}$. Both spaces are compact, hence the map is a homeomorphism. $\square$

Corollary 4.8. *Every compact space is the space of graph components of a suitable spectral space.*

Proof. Pick a compact space K and consider the ring $\mathcal{C}(K; \mathbb{R})$ of continuous functions. The Zariski spectrum is a normal spectral space (even a root system), which we denote by X. The space $\mathrm{Max}(X)$ is canonically homeomorphic to K, [7]. Thus, 4.7 implies that K is homeomorphic to X/G. $\square$

The next result gives a characterization of those spectral spaces whose space of graph components is Boolean:

Proposition 4.9. *Let X be a spectral space. Then X/G is Boolean if and only if the graph components coincide with the topological components.*

Proof. One direction of the equivalence is clear since the space of topological components of a spectral space is always Boolean. Conversely, suppose that the space of graph components is Boolean. Let $\Gamma[x]$ and $\Gamma[y]$ be two distinct graph components. Then there is a closed and open set $C \subseteq X/\mathrm{G}$ with $\Gamma[x] \in C$ and $\Gamma[y] \notin C$. The set $p_{\mathrm{G}}^{-1}(C) \subseteq X$ is closed and open, and $\Gamma[x] \subseteq p_{\mathrm{G}}^{-1}(C)$, $\Gamma[y] \cap p_{\mathrm{G}}^{-1}(C) = \emptyset$. It follows that $\Gamma[x]$ and $\Gamma[y]$ belong to different connected components of X. $\square$

The behavior of graph components under spectral maps is quite simple: Let $f : X \to Y$ be a spectral map between spectral spaces. If $C \subseteq X$ is topologically connected, or graph connected, then $f(C) \subseteq Y$ is topologically connected, or graph connected. Thus, f induces continuous maps $f_{\mathrm{T}} : X/\mathrm{T} \to Y/\mathrm{T}$ and $f_{\mathrm{G}} : X/\mathrm{G} \to Y/\mathrm{G}$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_{\mathrm{T}}"'] & Y \arrow[d, "p_{\mathrm{T}}"] \\ X/\mathrm{T} \arrow[r, "f_{\mathrm{T}}"] & Y/\mathrm{T} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_{\mathrm{G}}"'] & Y \arrow[d, "p_{\mathrm{G}}"] \\ X/\mathrm{G} \arrow[r, "f_{\mathrm{G}}"] & Y/\mathrm{G} \end{tikzcd}$$

5. The graph components of prime spectra

Let A be a ring. The graph components of its prime spectrum, $\mathrm{Spec}(A)$, are determined by the specialization relation between the prime ideals. Specialization of prime ideals is the same as containment. In the present section we develop methods to study the graph components and their properties with model theoretic methods. Difficulties arise since it is impossible to speak in 1st order model theory about individual prime ideals of a ring. We use the algorithms of Section 2 to overcome the problems. These methods are used to prove axiomatizability of several classes of rings that are defined by properties of the graph components of their prime spectrum, 5.5, 5.6.

There are several 1st order formulas that will be used over and over again. Therefore we introduce the following abbreviations:

- $\Theta_{2\cdot n} = \Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n}) \equiv \bigwedge_{i=1}^{n-1} a_{2\cdot i} \cdot a_{2\cdot i+1} = 0 \wedge \bigwedge_{i=1}^{n} a_{2\cdot i-1} + a_{2\cdot i} = 1$.
- $\Theta_{2\cdot n+1} = \Theta_{2\cdot n+1}(a_1, \ldots, a_{2\cdot n+1}) \equiv \Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n}) \wedge a_{2\cdot n} \cdot a_{2\cdot n+1} = 0$.
- $\Phi_{2\cdot n} = \Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \equiv a_0 \cdot a_1 = 0 \wedge \Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n})$.
- $\Phi_{2\cdot n+1} = \Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \equiv a_0 \cdot a_1 = 0 \wedge \Theta_{2\cdot n+1}(a_1, \ldots, a_{2\cdot n+1})$.
- $\Phi_\infty = \Phi_\infty(a_0, a_1, \ldots) \equiv \bigwedge_{i\in\mathbb{N}} a_{2\cdot i} \cdot a_{2\cdot i+1} = 0 \wedge \bigwedge_{i\in\mathbb{N}} a_{2\cdot i-1} + a_{2\cdot i} = 1$.
- $\Psi(x) \equiv \exists z\colon\ z \cdot x = 1$.

We point out how these and a few other formulas are connected to the prime spectrum of a ring:

- The formula $a = 0$ is equivalent to $V(a) = \mathrm{Spec}(A)$.
- The formula $\Psi(a)$ is equivalent to $D(a) = \mathrm{Spec}(A)$.
- The formula $a_0 \cdot a_1 = 0$ is equivalent to $D(a_0) \subseteq V(a_1)$.
- The formula $a_1 + a_2 = 1$ implies $V(a_1) \subseteq D(a_2)$.
- The formula $\Phi_\infty(a_0, a_1, \ldots)$ implies $D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq V(a_3) \subseteq \cdots$.

Every graph component $\Gamma \subseteq \mathrm{Spec}(A)$ is determined by each of its elements. Therefore Γ can be constructed using the 1st Algorithm or using he 2nd Algorithm of Section 2, starting with any element of Γ. As Γ contains both minimal prime ideals and maximal ideals one can start the construction either with a minimal prime ideal or with a maximal ideal. As a rule, we use the 1st Algorithm if we start with a minimal prime ideal, the 2nd Algorithm if we start with a maximal ideal.

Let $p \subseteq A$ be a minimal prime ideal. The first step of the 1st Algorithm constructs the set $\sigma(\{p\})$ of specializations of p, the second step constructs the set $\gamma(\sigma(\{p\}))$ of generalizations of the elements of $\sigma(\{p\})$, and so on. The exact set of specializations or generalizations can be described in terms of ring elements. However, the description is not finitary, hence it is not suitable for our model theoretic purposes. Thus, instead of considering the exact set of specializations or generalizations we use constructible subsets of $\mathrm{Spec}(A)$ that contain the specializations of p, the generalizations of the specializations of p, and so on. This leads us to consider sequences $a_0, a_1, \ldots \in A$ such that the following containment relations are satisfied:

$$D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq V(a_3) \subseteq \cdots.$$

We cannot speak about the initial prime ideal p in a finitary manner. Therefore we replace it with the open and constructible set $D(a_0)$, which has a finitary description by ring elements. In the next step the closed set $\sigma(\{p\})$ is replaced by a closed and constructible set that contains $D(a_0)$, and so on. If $p \in D(a_0)$ then $\sigma(\{p\}) \subseteq V(a_1)$, $\rho(\{p\}) \subseteq D(a_2)$, $\sigma \circ \rho(\{p\}) \subseteq V(a_3)$, and so on. It follows that $\Gamma[p] \subseteq \bigcup_{i\in\mathbb{N}} D(a_{2\cdot i}) = \bigcup_{i\in\mathbb{N}} V(a_{2\cdot i+1})$. Thus, the process does not determine the graph component $\Gamma[p]$, but a sequence of constructible sets, alternately open and closed, that covers the graph component. Such a sequence will be called a *covering sequence* for the graph component. Its union is an approximation of the graph component.

The construction of an entire covering sequence is not finitary either. But at least each step of the construction can be described with finitely many statements about the ring elements involved. This can be done as follows.

The containment $D(a_{2\cdot i}) \subseteq V(a_{2\cdot i+1})$ is equivalent to the condition $D(a_{2\cdot i}) \cap D(a_{2\cdot i+1}) = \emptyset$. Since the ring A is reduced this is equivalent to the 1st order statement $a_{2\cdot i} \cdot a_{2\cdot i+1} = 0$.

The containment $V(a_{2\cdot i+1}) \subseteq D(a_{2\cdot i+2})$ is equivalent to $D(a_{2\cdot i+1}) \cup D(a_{2\cdot i+2}) = \mathrm{Spec}(A)$. Equivalently, the ideal $(a_{2\cdot i+1}, a_{2\cdot i+2})$ is the entire ring, i.e., $\exists x, y$: $a_{2\cdot i+1} \cdot x + a_{2\cdot i+2} \cdot y = 1$.

Lemma 5.1. *Let A be a ring. Suppose that $a_0, a_1, \ldots \in A$ is a sequence of elements such that*

$$D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq V(a_3) \subseteq \cdots.$$

Then there is a sequence $b_0, b_1, \ldots \in A$ such that

$$D(b_0) \subseteq V(b_1) \subseteq D(b_2) \subseteq V(b_3) \subseteq \cdots$$

and $D(a_0) = D(b_0)$, $V(a_k) \subseteq V(b_k)$, holds for each $k \in \mathbb{N}$, and $b_{2\cdot i+2} = 1 - b_{2\cdot i+1}$ for each $i \in \mathbb{N}$.

Proof. One defines $b_0 = a_0$ and modifies every pair $a_{2\cdot i+1}$, $a_{2\cdot i+2}$ to meet the requirements. Suppose that $a_{2\cdot i+1} \cdot x_{2\cdot i+1} + a_{2\cdot i+2} \cdot y_{2\cdot i+2} = 1$. One defines $b_{2\cdot i+1} = a_{2\cdot i+1} \cdot x_{2\cdot i+1}$ and $b_{2\cdot i+2} = a_{2\cdot i+2} \cdot y_{2\cdot i+2}$. It remains to check that all the conditions are satisfied:

- $b_{2\cdot i+2} = 1 - b_{2\cdot i+1}$ holds trivially.
- $V(a_k) \subseteq V(b_k)$ holds for each $k \in \mathbb{N}$ since b_k is a multiple of a_k.
- $D(b_0) = D(a_0) \subseteq V(a_1) \subseteq V(b_1) \subseteq D(b_2) \subseteq D(a_2) \subseteq V(a_3) \subseteq V(b_3) \subseteq \cdots$. □

Because of 5.1 the study of covering sequences amounts to the study of sequences $a_0, a_1, \ldots \in A$ such that the infinite conjunction $\Phi_\infty(a_0, a_1, \ldots)$ is satisfied.

So far the entire discussion referred to the 1st Algorithm. Symmetrically, one can use the 2nd Algorithm in the same way, starting with a maximal ideal. This leads to the consideration of sequences $a_1, a_2, \ldots \in A$ such that

$$V(a_1) \subseteq D(a_2) \subseteq V(a_3) \subseteq D(a_4) \subseteq \cdots.$$

As in 5.1, such a sequence can be modified to obtain a sequence $b_1, b_2, \ldots \in A$ such that

$$V(b_1) \subseteq D(b_2) \subseteq V(b_3) \subseteq D(b_4) \subseteq \cdots$$

and $V(a_k) \subseteq V(b_k)$, holds for each $1 \leqslant k \in \mathbb{N}$, $b_{2\cdot i+2} = 1 - b_{2\cdot i+1}$ for each $i \in \mathbb{N}$. The sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ can both be extended by adding any term a_0 or b_0 such that $a_0 \cdot a_1 = 0$ and $b_0 \cdot b_1 = 0$. For example, one may choose $a_0 = 0 = b_0$. Thus the study of the second type of coverings of the graph components also amounts to the study of sequences $a_0, a_1, \ldots \in A$ that satisfy $\Phi_\infty(a_0, a_1, \ldots)$.

Theorem 5.2. *Let A be a ring, let $\Gamma \subseteq \operatorname{Spec}(A)$ be a graph component.*

(a) *If Γ is proconstructible then, for each sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ and $\Gamma \cap D(a_0) \neq \emptyset$, there is some $i \in \mathbb{N}$ such that $\Gamma \subseteq D(a_{2\cdot i})$.*

(b) *The following conditions are equivalent*:
   - (i) *The diameter of Γ is finite and the 1st Algorithm, when applied to a minimal prime ideal $p \in \Gamma$, terminates after at most K steps.*
   - (ii) *For each sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ and $\Gamma \cap D(a_0) \neq \emptyset$ either $\Gamma \subseteq D(a_K)$ if K is even, or $\Gamma \subseteq V(a_K)$ if K is odd.*

(c) *The following conditions are equivalent*:
   - (i) *The diameter of Γ is finite and the 2nd Algorithm, when applied to a maximal ideal $m \in \Gamma$, terminates after at most K steps.*
   - (ii) *For each sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ and $\Gamma \cap V(a_1) \neq \emptyset$ either $\Gamma \subseteq V(a_{K+1})$ if K is even, or $\Gamma \subseteq D(a_{K+1})$ if K is odd.*

Proof. (a). The graph component is contained in $\bigcup_{i\in\mathbb{N}} D(a_{2\cdot i})$ since $\Gamma \cap D(a_0) \neq \emptyset$. The claim follows since the proconstructible set Γ is quasi-compact and the covering sets $D(a_{2\cdot i})$ are open.

(b). (i) $\Rightarrow$ (ii). Pick a minimal prime ideal $p \in \Gamma \cap D(a_0)$. Then $\{p\} \subseteq D(a_0)$, $\sigma(\{p\}) \subseteq V(a_1)$, $\rho(\{p\}) \subseteq D(a_2)$, and so on. Now suppose that $K = 2 \cdot k$. Then $\Gamma = \rho^k(\{p\}) \subseteq D(a_{2\cdot k})$. If $K = 2 \cdot k + 1$ then $\Gamma = \sigma \circ \rho^k(\{p\}) \subseteq V(a_{2\cdot k+1})$.

(ii) $\Rightarrow$ (i). Pick a minimal prime ideal $p \in \Gamma$ and assume that the 1st Algorithm does not terminate after at most K steps. We suppose that $K = 2 \cdot k$. The case of odd K can be done similarly. The sequence

$$\{p\} \subset \sigma\big(\{p\}\big) \subset \rho\big(\{p\}\big) \subset \cdots \subset \rho^k\big(\{p\}\big) \subset \sigma \circ \rho^k\big(\{p\}\big)$$

ascends properly. There is a maximal ideal $m \in \sigma \circ \rho^k(\{p\}) \setminus \rho^k(\{p\})$. The 2nd Algorithm, starting with m, yields the properly increasing sequence

$$\{m\} \subset \gamma\big(\{m\}\big) \subset \tau\big(\{m\}\big) \subset \cdots \subset \tau^k\big(\{m\}\big) \subset \gamma \circ \tau^k\big(\{m\}\big).$$

Note that $p \in \gamma \circ \tau^k(\{m\}) \setminus \tau^k(\{m\})$.

Recursively one constructs a sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ such that $\Gamma \nsubseteq D(a_K)$:

- The set $\gamma \circ \tau^k(\{m\})$ is proconstructible and $\tau^k(\{m\})$ is a closed subset (closed even in $\operatorname{Spec}(A)$) that does not contain p. Therefore there is some $a_0 \in A$ such that $p \in D(a_0)$ and $D(a_0) \cap \tau^k(\{m\}) = \emptyset$.
- Suppose that $a_0, \ldots, a_{2\cdot j}$ have been constructed with $\bigwedge_{i=0}^{j-1} a_{2\cdot i} \cdot a_{2\cdot i+1} = 0 \wedge \bigwedge_{i=0}^{j-1} a_{2\cdot i+1} + a_{2\cdot i+2} = 1$ and $D(a_{2\cdot j}) \cap \tau^{k-j}(\{m\}) = \emptyset$.
- The set $D(a_{2\cdot j})$ is constructible and disjoint from the generically closed and proconstructible set $\gamma \circ \tau^{k-j-1}(\{m\})$. Moreover, the set $\tau^{k-j-1}(\{m\})$ is closed in $\operatorname{Spec}(A)$ and is contained in $\gamma \circ \tau^{k-j-1}(\{m\})$. For each maximal ideal $n \in \tau^{k-j-1}(\{m\})$ there is an element $c_n \in A \setminus n$ such that

$D(c_n) \cap D(a_{2\cdot j}) = \emptyset$. The sets $D(c_n)$ form an open cover of the quasi-compact set $\tau^{k-j-1}(\{m\})$. Thus, there is a finite subcover $\tau^{k-j-1}(\{m\}) \subseteq \bigcup_{l=1}^{L} D(c_{n_l})$. The ideal $I = \bigcap_{q\in\tau^{k-j-1}(\{m\})} q \subseteq A$ corresponds to the closed set $\tau^{k-j-1}(\{m\})$. In the factor ring A/I, the ideal generated by the elements $c_{n_1} + I, \ldots, c_{n_L} + I$ is the entire ring. Hence there are elements $x_1, \ldots, x_L \in A$ such that $c_{n_1} \cdot x_1 + \cdots + c_{n_L} \cdot x_L + I = 1 + I$. We define $a_{2\cdot j+1} = c_{n_1} \cdot x_1 + \cdots + c_{n_L} \cdot x_L$ and $a_{2\cdot j+2} = 1 - a_{2\cdot j+1}$. Note that $D(a_{2\cdot j+1}) \subseteq \bigcup_{l=1}^{L} D(c_{n_l})$. It follows that $D(a_{2\cdot j}) \cap D(a_{2\cdot j+1}) = \emptyset$, i.e., $a_{2\cdot j} \cdot a_{2\cdot j+1} = 0$. Thus the sequence $a_0, \ldots, a_{2\cdot j+2}$ satisfies the condition $\bigwedge_{i=0}^{j} a_{2\cdot i} \cdot a_{2\cdot i+1} = 0 \wedge \bigwedge_{i=0}^{j} a_{2\cdot i+1} + a_{2\cdot i+2} = 1$. Moreover, $D(a_{2\cdot j+2}) \cap \tau^{k-j-1}(\{m\}) = \emptyset$.
- The recursive construction yields the sequence $a_0, \ldots, a_{2\cdot k}$ with the property $\bigwedge_{i=0}^{k-1} a_{2\cdot i} \cdot a_{2\cdot i+1} = 0 \wedge \bigwedge_{i=0}^{k-1} a_{2\cdot i+1} + a_{2\cdot i+2} = 1$ and with $m \notin D(a_{2\cdot k})$. We extend the sequence by setting $a_{2\cdot j+1} = 0$ and $a_{2\cdot j+2} = 1$ for $j > k$.

The infinite sequence $a_0, a_1, \ldots \in A$ satisfies $\Phi_\infty(a_0, a_1, \ldots)$, and $p \in \Gamma \cap D(a_0)$, $m \in \Gamma \setminus D(a_K)$. This contradicts condition (ii), and the proof is finished.

(c) is proved in the same way as (b), *mutatis mutandis*. □

Corollary 5.3. *Let A be a ring whose prime spectrum is graph connected. Then the following conditions are equivalent*:

(a) *The* 1*st Algorithm, starting with a minimal prime ideal, terminates after at most K steps.*
(b) *Every sequence $\emptyset \neq D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq \cdots$ becomes stationary after at most K steps.*

The proof of 5.2 shows the advantage of the 1st Algorithm and the 2nd Algorithm compared with the Basic Algorithm for the computation of graph components. The termination of the 1st Algorithm and the 2nd Algorithm corresponds well to the termination of covering sequences of the graph components. There is no such simple correspondence for the Basic Algorithm.

Concerning 5.2 (a) we note that there exists a ring A with the following property: The prime spectrum has a graph component Γ that is not proconstructible. But if $a_0, a_1, \ldots \in A$ is any sequence with $\Phi_\infty(a_0, a_1, \ldots)$ and $\Gamma \cap D(a_0) \neq \emptyset$ then there is some $i \in \mathbb{N}$ such that $\Gamma \subseteq D(a_{2\cdot i})$. For example, if $\operatorname{Spec}(A)$ is homeomorphic to the spectral space Y of 3.6 then it has this property.

Covering sequences of graph components can be infinite: Let A be a ring whose prime spectrum is homeomorphic to the spectral space X of 3.5. Then there are infinite sequences $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ such that the infinite graph component is not contained in any of the sets $D(a_{2\cdot i})$.

More generally: Suppose that Γ is a graph component that is not proconstructible. Suppose that there is a subset $C \subseteq \operatorname{Spec}(A)$ that is closed and generically closed and contains Γ as an open subset. Then there is a sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ such that Γ is not contained in any set $D(a_{2\cdot i})$.

So far the graph components of prime spectra have been studied by building them up from the inside, using the algorithms of Section 2 and the covering sequences of the present section. For many rings one can also produce the graph components from the outside, enclosing them in an intersection of sets that are open and constructible and also in an intersection of sets that are closed and constructible. This presentation of the graph components is closely related to the considerations that led to 4.5.

Consider a subset $C \subseteq \operatorname{Spec}(A)$. We define $I_C = \{a \in A \mid C \subseteq V(a)\}$ and $M_C = \{a \in A \mid C \subseteq D(a)\}$. Then I_C is a radical ideal and M_C is a saturated multiplicative subset. (A multiplicative set M is *saturated* if $a \in M$ and $a = b \cdot c$ implies $b, c \in M$. Intersections of saturated multiplicative sets are saturated. Therefore every multiplicative set is contained in a smallest saturated multiplicative set, its *saturation*. The saturation of M is denoted by $S(M)$.)

The set C is closed if and only if $C = V(I_C)$. It is proconstructible and generically closed if and only if $C = D(M_C) = \bigcap_{a\in M_C} D(a)$. It is closed and generically closed if and only if $C = V(I_C)$ and $C = D(M_C)$, [22, Section 5]. The sets I_C and M_C are disjoint if and only if $C \neq \emptyset$.

Proposition 5.4. *Let A be a ring. The closed and generically closed subsets of* Spec(A) *correspond bijectively to the pairs (I, M), where $I \subseteq A$ is a radical ideal, $M \subseteq A$ is a saturated multiplicative set, and the following conditions are satisfied*:

(a) $\forall a \in I\ \exists b \in M$: $a \cdot b = 0$.
(b) $\forall a \in M\ \exists b \in I$: $(a, b) = A$.

Proof. We start with a closed and generically closed set $C \subseteq \mathrm{Spec}(A)$. Then the pair (I_C, M_C) satisfies both conditions: First pick $a \in I_C$. Then $D(a) \cap C = \emptyset$. Since C is generically closed and $D(a)$ is proconstructible there exists an element $b \in M_C$ with $D(a) \cap D(b) = \emptyset$, which is equivalent to $a \cdot b = 0$. Now pick $a \in M_C$. Then $V(a) \cap C = \emptyset$. Since $C = V(I_C)$ there is an element $b \in I_C$ such that $V(a) \cap V(b) = \emptyset$, which is equivalent to $(a, b) = A$.

Now suppose that (I, M) is a pair that satisfies the conditions. We claim that $V(I) = D(M)$. Let $p \in V(I)$. For each $a \in M$ there is an element $b \in I$ such that $V(b) \subseteq D(a)$. Since $p \in V(I)$ it follows that $p \in D(a)$. As a varies in M, we see that $p \in D(M)$. Conversely, suppose that $p \in D(M)$. If $a \in I$ then there is an element $b \in M$ such that $D(b) \subseteq V(a)$, which implies $p \in V(a)$. As a varies in I we see that $p \in V(I)$.

The set $V(I)$ is closed, and the set $D(M)$ is generically closed. Thus $V(I) = D(M)$ is closed and generically closed.

If we start with a closed and generically closed set C then I_C is a radical ideal, M_C is a saturated multiplicative set, the conditions (a) and (b) are satisfied, and $V(I_C) = C = D(M_C)$.

Now suppose that the pair (I, M) satisfies all the hypotheses. The set $C = V(I) = D(M)$ is closed and generically closed. It defines the pair (I_C, M_C). Obviously, $I \subseteq I_C$ and $M \subseteq M_C$. Radical ideals correspond bijectively to the closed subsets of Spec(A). Thus, $I = I_C$. Now pick an element $a \in M_C$. Since $V(a) \cap D(M) = \emptyset$ there is an element $b \in M$ such that $V(a) \cap D(b) = \emptyset$. It follows that $b \in \sqrt{(a)}$, i.e., there is some $n \in \mathbb{N}$ and some $c \in A$ such that $c \cdot a = b^n \in M$. Since M is saturated we conclude that $a \in M$. $\square$

The set of pairs (I, M) as in 5.4 is denoted by $IM(A)$. There is a canonical bijection with $\mathcal{A}(\mathrm{Spec}(A)_{\mathrm{inv}}) \cap \mathcal{A}(\mathrm{Spec}(A))$, the lattice of closed and generically closed subsets of Spec(A). Using the bijection with $IM(A)$ the lattice structure can be transferred from $\mathcal{A}(\mathrm{Spec}(A)_{\mathrm{inv}}) \cap \mathcal{A}(\mathrm{Spec}(A))$ to $IM(A)$. But the partial order and the lattice operations in $IM(A)$ can also be explained directly.

One defines a partial order by $(I, M) \leqslant (I', M')$ if $I \subseteq I'$ and $M \subseteq M'$. Two elements $(I, M), (I', M') \in IM(A)$ have a least upper bound, namely $(\sqrt{I + I'}, S(M \cdot M'))$, and a greatest lower bound, namely $(I \cap I', M \cap M')$. Thus, $IM(A)$ is a lattice. The bijection given by $C \to (I_C, M_C)$, is an isomorphism between the lattices $\mathcal{A}(\mathrm{Spec}(A)_{\mathrm{inv}}) \cap \mathcal{A}(\mathrm{Spec}(A))$ and $IM(A)_{\mathrm{inv}}$.

Suppose that the graph components of Spec(A) are proconstructible. Then it follows from 4.5 that the graph components correspond to the maximal elements of the lattice $IM(A)$.

Two elements $(I, M), (I', M') \in IM(A)$ are complements of each other if and only if $I + I' = A$ and $I \cap I' = \{0\}$. Thus, there is an idempotent $e \in E(A)$ with $I = e \cdot A$ and $I' = (1 - e) \cdot A$. The corresponding multiplicative sets are the saturations $M = S(1 - e)$ and $M' = S(e)$.

We have encountered several properties that graph components of spectral spaces may or may not have, e.g., graph components may be proconstructible, or they may have finite diameter. Such properties define classes of rings: There is the class of rings whose prime spectra have proconstructible graph components. Or, there is the class of rings whose prime spectra have only graph components with finite diameter. We ask whether these classes, and many other classes of rings that are defined similarly, are elementary (in the sense of model theory).

The following example settles several of these questions:

Example 5.5. *A sequence of rings with graph connected prime spectra, and an ultraproduct that has a graph component that is not proconstructible.* Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of rings whose prime spectra have the following form:

$$\begin{tikzcd}
A_0 & p_{00} \\
A_1 & & p_{11} \\
& p_{10} \arrow[ur, squiggly] \\
A_2 & & p_{21} \\
& p_{20} \arrow[ur, squiggly] & & p_{22} \arrow[ul, squiggly] \\
A_3 & & p_{31} & & p_{33} \\
& p_{30} \arrow[ur, squiggly] & & p_{32} \arrow[ul, squiggly] \arrow[ur, squiggly] \\
\vdots & \vdots
\end{tikzcd}$$

Let $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$, and define $A = (\prod_{n\in\mathbb{N}} A_n)/\mathfrak{U}$. We shall produce a graph component of $\operatorname{Spec}(A)$ that is not proconstructible.

Let $q_n \subseteq A_n$ be a prime ideal for each $n \in \mathbb{N}$. Then $(A_n/q_n)_{n\in\mathbb{N}}$ is a sequence of domains, hence $(\prod_{n\in\mathbb{N}} A_n/q_n)/\mathfrak{U}$ is a domain as well. The componentwise surjective homomorphism $\pi_{(q_n)_n} : \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} A_n/q_n$ yields a surjective homomorphism $\pi^{\mathfrak{U}}_{(q_n)_n} : A \to (\prod_{n\in\mathbb{N}} A_n/q_n)/\mathfrak{U}$. The kernel is a prime ideal, which is denoted by $(q_n|n)^{\mathfrak{U}}$. If $(r_n)_{n\in\mathbb{N}}$ is another sequence of prime ideals and if $q_n \rightsquigarrow r_n$ for each n then $(q_n|n)^{\mathfrak{U}} \rightsquigarrow (r_n|n)^{\mathfrak{U}}$.

We are now in a position to produce large numbers of prime ideals in A. For each n we extend the list of prime ideals of A_n to an infinite sequence by defining $p_{nn} = p_{n,n+1} = p_{n,n+2} = \cdots$. For each $i \in \mathbb{N}$ one defines $p_i = (p_{ni}|n)^{\mathfrak{U}}$. The following diagram shows how the prime ideals specialize to each other:

$$\begin{tikzcd}
& p_1 & & p_3 & & p_5 & & \cdots \\
p_0 \arrow[ur, squiggly] & & p_2 \arrow[ul, squiggly] \arrow[ur, squiggly] & & p_4 \arrow[ul, squiggly] \arrow[ur, squiggly] & & & \cdots
\end{tikzcd}$$

Thus, the prime ideals in the sequence $(p_i)_{i\in\mathbb{N}}$ belong to the same graph component. It is claimed that this graph component is not proconstructible.

Theorem 5.2 is the main tool for proving the claim. For each n one constructs a sequence $(a_{ni})_{i\in\mathbb{N}}$ in the ring A_n:

- $D(a_{n0}) = \{p_{n0}\}$.
- $V(a_{n1}) = \{p_{n0}, p_{n1}\}$.
- $D(a_{n2}) = \{p_{n0}, p_{n1}, p_{n2}\}$.
- $\vdots$
- $D(a_{n,n}) = \operatorname{Spec}(A)$ if n is even, $V(a_{n,n}) = \operatorname{Spec}(A)$ if n is odd.
- For $k \geqslant 1$, $a_{n,n+k} = 1$ if $n+k$ is even, $a_{n,n+k} = 0$ if $n+k$ is odd.

It is always true that $a_{n,2\cdot j} \cdot a_{n,2\cdot j+1} = 0$. The sequences may be chosen such that $a_{n,2\cdot j+1} + a_{n,2\cdot j+2} = 1$, 5.1. For each i the element a_i is defined to be the canonical image of the sequence $(a_{ni})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$

in A. The sequence $(a_i)_{i\in\mathbb{N}}$ satisfies $\Phi_\infty(a_0, a_1, \ldots)$, and $p_0 \in \Gamma[p_0] \cap D(a_0)$. Thus, the hypotheses of 5.2 (a) are satisfied. If $\Gamma[p_0]$ is proconstructible then there must exist an index $2 \cdot j$ with $\Gamma[p_0] \subset D(a_{2\cdot j})$. However, the construction shows that $a_{2\cdot j} \in p_{2\cdot j+1}$, and it follows that the graph component is not proconstructible. □

Example 5.5 shows that several classes of rings are not elementary. In each case the proof uses the following argument: If a class of rings is elementary then it is closed under the formation of ultraproducts, [4,11]. So, if one considers a class of rings that contains the sequence $(A_n)_{n\in\mathbb{N}}$ of 5.5, and if the ultraproduct A does not belong to the class, then the class cannot be elementary.

- *The class of rings with graph connected prime spectrum*. Note that each A_n has a graph connected prime spectrum, but the prime spectrum of A is not graph connected (since there is a graph component that is not proconstructible).
- *The class of rings for which the topological components of the prime spectrum coincide with the graph components*. We use the same argument as in the preceding item. In addition, it is only necessary to note that the ring A is indecomposable. For, indecomposability is an elementary property and is therefore preserved by the formation of ultraproducts.
- *The class of rings whose prime spectrum has only proconstructible graph components.*
- *The class of rings whose prime spectrum has only graph components of finite diameter.*

The negative results show that elementary classes of rings, defined by conditions about the graph components of the prime spectrum, require strong finiteness conditions. The next result gives a positive answer in this direction:

Theorem 5.6. *Let $\mathcal{R}_k$ be the class of rings A such that the* 1*st Algorithm, starting with any minimal prime ideal, terminates after at most k steps. Let $\mathcal{S}_k$ be the class of rings A such that the* 2*nd Algorithm, starting with any maximal ideal, terminates after at most k steps. The classes $\mathcal{R}_K$ and $\mathcal{S}_K$ are elementary.*

Proof. We claim that the following formula axiomatizes the class $\mathcal{R}_{2\cdot k+1}$ (together with the usual axioms for reduced rings, of course):

$$\begin{aligned} \forall a_0, \ldots, a_{2\cdot k+1}\colon\quad & a_0 \neq 0 \wedge \Phi_{2\cdot k+1}(a_0, \ldots, a_{2\cdot k+1}) \wedge a_{2\cdot k+1} \neq 0 \to \\ & \exists b_1, \ldots, b_{2\cdot k+1}\ \exists c_1, \ldots, c_{2\cdot k+1}\colon b_{2\cdot k+1} + c_{2\cdot k+1} = 1 \wedge \\ & \quad \Phi_{2\cdot k+1}(a_0, b_1, \ldots, b_{2\cdot k+1}) \wedge \Phi_{2\cdot k+1}(a_{2\cdot k+1}, c_1, \ldots, c_{2\cdot k+1}). \end{aligned}$$

First suppose that $A \in \mathcal{R}_{2\cdot k+1}$. Pick a sequence $a_0, \ldots, a_{2\cdot k+1} \in A$ such that the condition

$$a_0 \neq 0 \wedge \Phi_{2\cdot k+1}(a_0, \ldots, a_{2\cdot k+1}) \wedge a_{2\cdot k+1} \neq 0$$

is satisfied. The sequence of ring elements yields a sequence of constructible sets in $\mathrm{Spec}(A)$:

$$\emptyset \neq D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq \cdots \subseteq V(a_{2\cdot k+1}) \subset \mathrm{Spec}(A).$$

The set $C = \Gamma[D(a_0)]$ is closed and generically closed and is contained in $V(a_{2\cdot k+1})$. The set $C' = \Gamma[D(a_{2\cdot k+1})]$ is closed and generically closed as well (reading the sequence backwards). Moreover, C and C' are disjoint. Let $(I, M), (I', M') \in IM(A)$ be the pairs that correspond to C and C', 5.4. There are elements $b_{2\cdot k+1} \in I$ and $c_{2\cdot k+1} \in I'$ such that $b_{2\cdot k+1} + c_{2\cdot k+1} = 1$. Using 5.4 one constructs sequences $b_{2\cdot k+1}, b_{2\cdot k}, \ldots, b_1$ and $c_{2\cdot k+1}, c_{2\cdot k}, \ldots, c_1$ such that

- $b_i \in I$, $c_i \in I'$ if i is odd,
- $b_i \in M$, $c_i \in M'$ if i is even,

- $b_{2\cdot j+1} \cdot b_{2\cdot j} = 0$, $c_{2\cdot j+1} \cdot c_{2\cdot j} = 0$ for $j = 1, \ldots, k$,
- $(b_{2\cdot j}, b_{2\cdot j-1}) = A$, $(c_{2\cdot j}, c_{2\cdot j-1}) = A$ for $j = 1, \ldots, k$.

For each j one writes $1 = b_{2\cdot j} \cdot x + b_{2\cdot j-1} \cdot y$. Then $b_{2\cdot j-1} \cdot y \in I$ and $V(b_{2\cdot j-1} \cdot y) \subseteq D(b_{2\cdot j} \cdot x)$. Therefore $b_{2\cdot j} \cdot x \in M$. Replacing $b_{2\cdot j}$ by $b_{2\cdot j} \cdot x$ and $b_{2\cdot j-1}$ by $b_{2\cdot j-1} \cdot y$ one may assume that $b_{2\cdot j} + b_{2\cdot j-1} = 1$ for each j. In the same way the sequence $c_{2\cdot k+1}, \ldots, c_1$ can be modified so that $c_{2\cdot j} + c_{2\cdot j-1} = 1$. Almost all the asserted conditions about the sequences are satisfied now. It remains to check that $a_0 \cdot b_1 = 0$ and $a_{2\cdot k+1} \cdot c_1 = 0$: By construction we know that $D(a_0) \subseteq C \subseteq V(b_1)$ and $D(a_{2\cdot k+1}) \subseteq C' \subseteq V(c_1)$, and this proves the last two assertions.

Conversely, suppose that A is a ring that satisfies the formula and assume, by way of contradiction, that $A \notin \mathcal{R}_{2\cdot k+1}$. There is a minimal prime ideal $p \subseteq A$ such that the inclusions

$$\{p\} \subset \sigma\big(\{p\}\big) \subset \rho\big(\{p\}\big) \subset \sigma \circ \rho\big(\{p\}\big) \subset \cdots \subset \sigma \circ \rho^k\big(\{p\}\big) \subset \rho^{k+1}\big(\{p\}\big)$$

are proper. By 5.2 (b) there is a sequence $a_0, a_1, \ldots \in A$ with $\Phi_\infty(a_0, a_1, \ldots)$ such that $p \in D(a_0)$ and $\Gamma[p] \nsubseteq V(a_{2\cdot k+1})$. Pick an element $q \in D(a_{2\cdot k+1}) \cap \rho^{k+1}(\{p\})$. The initial segment $a_0, \ldots, a_{2\cdot k+1}$ of the sequence satisfies the hypothesis of the formula. Hence there exist sequences $b_1, \ldots, b_{2\cdot k+1}$ and $c_1, \ldots, c_{2\cdot k+1}$ such that the conclusion is satisfied. It follows that

$$D(a_0) \subseteq V(b_1) \subseteq D(b_2) \subseteq \cdots \subseteq V(b_{2\cdot k+1}),$$
$$D(a_{2\cdot k+1}) \subseteq V(c_1) \subseteq D(c_2) \subseteq \cdots \subseteq V(c_{2\cdot k+1}).$$

Now $p \in D(a_0)$ implies $\sigma \circ \rho^k(\{p\}) \subseteq V(b_{2\cdot k+1})$, and $q \in D(a_{2\cdot k+1}) \subseteq V(c_{2\cdot k+1})$ holds by construction. Since $V(b_{2\cdot k+1}) \cap V(c_{2\cdot k+1}) = \emptyset$ we know that there is no element of $V(c_{2\cdot k+1})$ that specializes to an element of $V(b_{2\cdot k+1})$. However, q was picked inside $\gamma(V(b_{2\cdot k+1}))$. This contradiction finishes the proof for the class $\mathcal{R}_{2\cdot k+1}$.

The following three claims are proved similarly:

- If $k > 0$ then the class $\mathcal{R}_{2\cdot k}$ is axiomatized by the following formula:

$$\begin{aligned} \forall a_0, \ldots, a_{2\cdot k}\colon\quad & a_0 \neq 0 \wedge \Phi_{2\cdot k}(a_0, \ldots, a_{2\cdot k}) \wedge \neg\Psi(a_{2\cdot k}) \to \\ & \exists b_1, \ldots, b_{2\cdot k}\ \exists c_1, \ldots, c_{2\cdot k}\colon\ b_{2\cdot k} \cdot c_{2\cdot k} = 0 \wedge \\ & \quad \Phi_{2\cdot k}(a_0, b_1, \ldots, b_{2\cdot k}) \wedge \Phi_{2\cdot k}(a_{2\cdot k}, c_1, \ldots, c_{2\cdot k}). \end{aligned}$$

- The class $\mathcal{S}_{2\cdot k+1}$ is axiomatized by the following formula:

$$\begin{aligned} \forall a_1, \ldots, a_{2\cdot k+2}\colon\quad & \neg\Psi(a_1) \wedge \Phi_{2\cdot k+2}(0, a_1, \ldots, a_{2\cdot k+2}) \wedge \neg\Psi(a_{2\cdot k+2}) \to \\ & \exists b_2, \ldots, b_{2\cdot k+2}\ \exists c_2, \ldots, c_{2\cdot k+2}\colon\ b_{2\cdot k+2} \cdot c_{2\cdot k+2} = 0 \wedge \\ & \quad \Phi_{2\cdot k+2}(0, a_1, b_2, \ldots, b_{2\cdot k+2}) \wedge \Phi_{2\cdot k+2}(0, a_{2\cdot k+2}, c_2, \ldots, c_{2\cdot k+2}). \end{aligned}$$

- If $k > 0$ then the class $\mathcal{S}_{2\cdot k}$ is axiomatized by the following formula:

$$\begin{aligned} \forall a_1, \ldots, a_{2\cdot k+1}\colon\quad & \neg\Psi(a_1) \wedge \Phi_{2\cdot k+1}(0, a_1, \ldots, a_{2\cdot k+1}) \wedge a_{2\cdot k+1} \neq 0 \to \\ & \exists b_2, \ldots, b_{2\cdot k+1}\ \exists c_2, \ldots, c_{2\cdot k+1}\colon\ b_{2\cdot k+1} + c_{2\cdot k+1} = 1 \wedge \\ & \quad \Phi_{2\cdot k+1}(0, a_1, b_2, \ldots, b_{2\cdot k+1}) \wedge \Phi_{2\cdot k+1}(0, a_{2\cdot k+1}, c_2, \ldots, c_{2\cdot k+1}). \end{aligned}$$

Concerning the classes $\mathcal{R}_0$ and $\mathcal{S}_0$ we refer to Remark 5.7 below. $\square$

Remark 5.7. For very small values of k the classes $\mathcal{R}_k$ and $\mathcal{S}_k$ are well-known classes of rings:

- If $A \in \mathcal{R}_0$ or if $A \in \mathcal{S}_0$ then there are no specializations in $\operatorname{Spec}(A)$. Thus, A is von Neumann regular. On the other hand, all von Neumann regular rings clearly belong to $\mathcal{R}_0$ and $\mathcal{S}_0$. Thus, the class of von Neumann regular rings is $\mathcal{R}_0 = \mathcal{S}_0$.
- If $A \in \mathcal{S}_1$ then every prime ideal is contained in a unique maximal ideal. Such rings are called *normal* or *Gel'fand rings*, [22,3,12].
- If $A \in \mathcal{R}_1$ then every prime ideal contains a unique minimal prime ideal. Such rings are called *inversely normal*, [22,8].

Trivially, the following inclusion relations hold between the classes of rings: $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots$, and $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots$. Moreover, some of the classes coincide:

Proposition 5.8. *For each $k \in \mathbb{N}$, $\mathcal{R}_{2\cdot k} = \mathcal{S}_{2\cdot k}$.*

Proof. Suppose that $A \notin \mathcal{R}_{2\cdot k}$. Then there is a shortest path that starts with a minimal prime ideal p_0, has length $2 \cdot k + 1$ and zigzags between minimal prime ideals and maximal ideals:

$$\begin{tikzcd}
 & p_1 & & p_3 & \cdots & p_{2\cdot k-1} & & p_{2\cdot k+1} \\
p_0 \arrow[ur, rightsquigarrow] & & p_2 \arrow[ul, rightsquigarrow] \arrow[ur, rightsquigarrow] & & \cdots & & p_{2\cdot k} \arrow[ul, rightsquigarrow] \arrow[ur, rightsquigarrow] &
\end{tikzcd}$$

Traversing the path in the opposite direction, one obtains a shortest path of length $2 \cdot k + 1$ that starts with a maximal ideal. Thus, $A \notin \mathcal{S}_{2\cdot k}$.

The other inclusion is proved similarly. $\square$

It is easy to show with examples that the there are no other inclusion relations and equalities between the classes $\mathcal{R}_k$ and $\mathcal{S}_k$. This will also be obvious from the results in the next section.

6. Rings with graph connected prime spectrum

This section is devoted to a model theoretic study of graph connected rings, i.e., rings with graph connected prime spectrum. It is clear from 5.5 that neither the class of graph connected rings is elementary, nor the class of graph connected rings with a prime spectrum of finite diameter. In view of 5.6 it is not much of a surprise that there are elementary classes of rings with graph connected prime spectrum if one assumes in addition that there is a bound for the termination of the 1st or 2nd Algorithm. We present explicit axiomatizations for several infinite families of classes of rings.

We consider the following classes of rings:

- $A \in \mathcal{C}_{2\cdot n}$ if $A \models \forall a_0, \ldots, a_{2\cdot n}\colon\ \Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to a_0 = 0 \vee \Psi(a_{2\cdot n})$.
- $A \in \mathcal{C}_{2\cdot n+1}$ if $A \models \forall a_0, \ldots, a_{2\cdot n+1}\colon\ \Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to a_0 = 0 \vee a_{2\cdot n+1} = 0$.
- $A \in \mathcal{D}_{2\cdot n}$ if $A \models \forall a_0, \ldots, a_{2\cdot n}\colon\ \Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to \Psi(1 - a_0) \vee \Psi(a_{2\cdot n})$.
- $A \in \mathcal{D}_{2\cdot n+1}$ if $A \models \forall a_0, \ldots, a_{2\cdot n+1}\colon\ \Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to \Psi(1 - a_0) \vee a_{2\cdot n+1} = 0$.

First one notes a simple important consequence of the conditions that define the classes:

Proposition 6.1. *If A belongs to any one of the classes $\mathcal{C}_k$ or $\mathcal{D}_k$ then A is indecomposable.*

Proof. Assume that there is a nontrivial idempotent e. Then the sequence $a_{2\cdot i} = e$, $a_{2\cdot i+1} = 1 - e$, $i \in \mathbb{N}$ satisfies the condition $\Phi_\infty(a_0, a_1, \ldots)$. If A belongs to any one of the classes then it follows that

$e = a_0 = 0$ or $1 - e = a_{2\cdot n+1} = 0$ or $1 - e = 1 - a_0 \in A^\times$ or $e = a_{2\cdot n} \in A^\times$ for some n. But all these statements are false, hence there are no nontrivial idempotents. $\square$

The ring theoretic content of the sentences defining the classes is not obvious in most cases. We give an explanation in a few simple special cases; another case will be considered later, 6.9.

Proposition 6.2. *Let A be a reduced ring.*

(a) *The following conditions are equivalent*:
 (i) $A \in \mathcal{C}_0$.
 (ii) *A is a field.*
(b) *The following conditions are equivalent*:
 (i) $A \in \mathcal{C}_1$.
 (ii) *A is a domain.*
(c) *The following conditions are equivalent*:
 (i) $A \in \mathcal{D}_0$.
 (ii) *A is local.*
(d) *The following conditions are equivalent*:
 (i) $A \in \mathcal{D}_1$.
 (ii) *Every zero divisor of A belongs to the Jacobson radical.*
 (iii) *If $a \in A$ then a is regular or $1 - a$ is a unit.*

Proof. (a). The defining condition of $\mathcal{C}_0$ says that every element is 0 or a unit. But this is the same as being a field.

(b). The condition for $\mathcal{C}_1$ is that $a_0 \cdot a_1 = 0$ implies $a_0 = 0$ or $a_1 = 0$, which means that the ring is a domain.

(c). The condition for $\mathcal{D}_0$ is that, given any element a, the element itself is a unit or $1 - a$ is a unit, which means that the ring is local.

(d). (i) $\Rightarrow$ (iii). Suppose that $A \in \mathcal{D}_1$ and pick a zero divisor a. If $a \cdot b = 0$ with $b \neq 0$ then $1 - a$ is a unit. (iii) $\Rightarrow$ (ii). Suppose that $a \in A$ is a zero divisor. Assume there is a maximal ideal m that does not contain a. Then there is an element $b \in A$ such that $a \cdot b + m = 1 + m$ in A/m. The element $a \cdot b$ is a zero divisor as well, but $1 - a \cdot b$ is not a unit, a contradiction. (ii) $\Rightarrow$ (i). Pick elements $a_0, a_1 \in A$ such that $a_0 \cdot a_1 = 0$ and $a_1 \neq 0$. Then $a_0 = 0$ or a_0 is a zero divisor, hence a_0 belongs to the Jacobson radical. In both cases we conclude that $1 - a_0$ is a unit. $\square$

The different classes of rings are related to each other. We exhibit those connections between them that can be derived readily from the definitions. Later we shall see that there are no other inclusion relations besides those shown here.

Proposition 6.3. *For every $n \in \mathbb{N}$, $\mathcal{C}_{2\cdot n+2} = \mathcal{D}_{2\cdot n+1}$.*

Proof. Suppose that $A \in \mathcal{C}_{2\cdot n+2}$. Pick a sequence $a_0, \ldots, a_{2\cdot n+1}$ such that $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds. The inverted sequence satisfies the same condition, $\Phi_{2\cdot n+1}(a_{2\cdot n+1}, \ldots, a_0)$. We extend this sequence by the term $1 - a_0$, and then $\Phi_{2\cdot n+2}(a_{2\cdot n+1}, \ldots, a_0, 1 - a_0)$ holds. Since $A \in \mathcal{C}_{2\cdot n+2}$ we conclude that $a_{2\cdot n+1} = 0$ or $1 - a_0 \in A^\times$, which proves the assertion.

For the other inclusion we proceed similarly: Suppose that $A \in \mathcal{D}_{2\cdot n+1}$. Pick a sequence $a_0, \ldots, a_{2\cdot n+2}$ such that $\Phi_{2\cdot n+2}(a_0, \ldots, a_{2\cdot n+2})$ holds. Note that $a_{2\cdot n+2} = 1 - a_{2\cdot n+1}$. We shorten the sequence by omitting the last term. Then $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds. Inverting the shortened sequence we obtain $\Phi_{2\cdot n+1}(a_{2\cdot n+1}, \ldots, a_0)$. Thus, $a_0 = 0$ or $a_{2\cdot n+2} = 1 - a_{2\cdot n+1} \in A^\times$, and the proof is finished. $\square$

Proposition 6.4. *The arrows in the following diagram indicate inclusion relations between the classes $\mathcal{C}_k$ and $\mathcal{D}_k$:*

$$
\begin{tikzcd}
 & \mathcal{C}_1 \arrow[dr] & & \mathcal{C}_3 \arrow[dr] & & \mathcal{C}_5 & \cdots & \\
\mathcal{C}_0 \arrow[ur] \arrow[dr] & & \mathcal{C}_2=\mathcal{D}_1 \arrow[ur] \arrow[dr] & & \mathcal{C}_4=\mathcal{D}_3 \arrow[ur] \arrow[dr] & & & \cdots \\
 & \mathcal{D}_0 \arrow[ur] & & \mathcal{D}_2 \arrow[ur] & & \mathcal{D}_4 & \cdots &
\end{tikzcd}
$$

Proof. Suppose that $A \in \mathcal{C}_{2\cdot n}$. We claim that $A \in \mathcal{C}_{2\cdot n+1}$. Let $a_0, \ldots, a_{2\cdot n+1}$ be a sequence in A such that $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds. Then $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$ holds, and the hypothesis implies that $a_0 = 0$ (and we are done) or $a_{2\cdot n} \in A^\times$, which implies $a_{2\cdot n+1} = 0$ since $a_{2\cdot n} \cdot a_{2\cdot n+1} = 0$.

Suppose that $A \in \mathcal{C}_{2\cdot n+1}$. We claim that $A \in \mathcal{C}_{2\cdot n+2}$. Let $a_0, \ldots, a_{2\cdot n+2}$ be a sequence in A such that $\Phi_{2\cdot n+2}(a_0, \ldots, a_{2\cdot n+2})$ holds. Then $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds, and the hypothesis implies that $a_0 = 0$ (and we are done) or $a_{2\cdot n+1} = 0$, which implies $a_{2\cdot n+2} \in A^\times$ since $a_{2\cdot n+1} + a_{2\cdot n+2} = 1$.

Suppose that $A \in \mathcal{D}_{2\cdot n}$. We claim that $A \in \mathcal{D}_{2\cdot n+1}$. Let $a_0, \ldots, a_{2\cdot n+1}$ be a sequence in A such that $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds. Then $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$ holds, and the hypothesis implies that $1 - a_0 \in A^\times$ (and we are done) or $a_{2\cdot n} \in A^\times$, which implies $a_{2\cdot n+1} = 0$ since $a_{2\cdot n} \cdot a_{2\cdot n+1} = 0$.

Suppose that $A \in \mathcal{D}_{2\cdot n+1}$. We claim that $A \in \mathcal{D}_{2\cdot n+2}$. Let $a_0, \ldots, a_{2\cdot n+2}$ be a sequence in A such that $\Phi_{2\cdot n+2}(a_0, \ldots, a_{2\cdot n+2})$ holds. Then $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ holds, and the hypothesis implies that $1 - a_0 \in A^\times$ (and we are done) or $a_{2\cdot n+1} = 0$, which implies $a_{2\cdot n+2} \in A^\times$ since $a_{2\cdot n+1} + a_{2\cdot n+2} = 1$. □

The next few results relate the classes $\mathcal{C}_k$ and $\mathcal{D}_k$ to the termination of the algorithms of Section 2. The following lemma will be extremely useful:

Lemma 6.5. *Let X be a spectral space and suppose that $Y, Z \subseteq X$ are proconstructible, $Y = \gamma(Y)$, $Z = \sigma(Z)$. Then the following conditions are equivalent*:

(a) $\sigma(Y) \cap Z = \emptyset$.
(b) $Y \cap \gamma(Z) = \emptyset$.

Proof. (a) $\Rightarrow$ (b). Assume that $y \in Y \cap \gamma(Z)$. Then there is an element $z \in Z$ such that $y \rightsquigarrow z$. But then $z \in \sigma(Y) \cap Z$, a contradiction. (b) $\Rightarrow$ (a) is proved similarly. □

Theorem 6.6. *Let A be a reduced ring. The following conditions are equivalent*:

(a) $A \in \mathcal{C}_{2\cdot n}$ $(= \mathcal{D}_{2\cdot n-1}$ *if* $n > 0)$.
(b) *The prime spectrum is graph connected, and the* 1*st Algorithm, starting from a minimal prime ideal, always terminates after at most* $2 \cdot n$ *steps.*

If $n > 0$ *then* (a) *and* (b) *are also equivalent to*:

(c) *The prime spectrum is graph connected, and the* 2*nd Algorithm, starting from a maximal prime ideal, always terminates after at most* $2 \cdot n$ *steps.*

Proof. First we settle the case $n = 0$. By 2.4 and 6.2 the conditions (a) and (b) both say that A is a field. We may now assume that $n > 0$.

(b) $\Rightarrow$ (c). If (c) is false then there is path in $\mathrm{Spec}(A)$

$$
\begin{tikzcd}
x_0 & & x_2 & & \cdots & & x_{2\cdot n} & \\
 & x_1 \arrow[ul, rightsquigarrow] \arrow[ur, rightsquigarrow] & & x_3 \arrow[ul, rightsquigarrow] & \cdots & x_{2\cdot n-1} \arrow[ur, rightsquigarrow] & & x_{2\cdot n+1} \arrow[ul, rightsquigarrow]
\end{tikzcd}
$$

that zigzags between maximal and minimal prime ideals and is a shortest connection from the maximal ideal x_0 to the minimal prime ideal $x_{2\cdot n+1}$. Traversing the path in the opposite direction one obtains a shortest path from the minimal prime ideal $x_{2\cdot n+1}$ to the maximal ideal x_0. The path has length $2\cdot n+1$, which contradicts (b).

(c) $\Rightarrow$ (b). The same argument as in the proof of (b) $\Rightarrow$ (c) is applied to the path

$$\begin{array}{ccccccccccccc}
 & & x_1 & & & & x_3 & \cdots & x_{2\cdot n-1} & & & & x_{2\cdot n+1}\\
 & \nearrow & & \nwarrow & & \nearrow & & & & \nwarrow & & \nearrow & \\
x_0 & & & & x_2 & & & \cdots & & & x_{2\cdot n} & &
\end{array}$$

(b) $\Rightarrow$ (a). Pick $a_0,\dots,a_{2\cdot n}\in A$, $a_0\neq 0$, and suppose that $\Phi_{2\cdot n}(a_0,\dots,a_{2\cdot n})$ holds. We must prove that $D(a_{2\cdot n})=\mathrm{Spec}(A)$.

As $a_0\neq 0$ there is a minimal prime ideal $p\in D(a_0)$. Starting with p, one applies the 1st Algorithm and obtains $\mathrm{Spec}(A)=\rho^n(\{p\})$. Consider the sequence of inclusions given by the condition $\Phi_{2\cdot n}(a_0,\dots,a_{2\cdot n})$:

$$D(a_0)\subseteq V(a_1)\subseteq D(a_2)\subseteq\cdots\subseteq V(a_{2\cdot n-1})\subseteq D(a_{2\cdot n}).$$

Then $\rho(D(a_{2\cdot i}))\subseteq D(a_{2\cdot i+2})$ for all $i=0,\dots,n-1$. It follows that

$$\mathrm{Spec}(A)=\rho^n\bigl(\{p\}\bigr)\subseteq\rho^n\bigl(D(a_0)\bigr)\subseteq D(a_{2\cdot n}),$$

which proves the claim.

(a) $\Rightarrow$ (b). By 6.1 there are no nontrivial idempotents. Now assume that the assertion is false. Then there is a minimal prime ideal p such that $\rho^n(\{p\})\neq\mathrm{Spec}(A)$. Since $\rho^n(\{p\})$ is generically closed there is a maximal ideal $m\notin\rho^n(\{p\})$. It follows that $m\notin\sigma(\rho^{n-1}(\{p\}))$, hence 6.5, with $Y=\rho^{n-1}(\{p\})$ and $Z=\{m\}$, yields $\rho^{n-1}(\{p\})\cap\gamma(\{m\})=\emptyset$. We apply 6.5 again, this time with $Y=\gamma(\{m\})$ and $Z=\sigma\circ\rho^{n-2}(\{p\})$, and obtain $\tau(\{m\})\cap\sigma(\rho^{n-2}(\{p\}))=\emptyset$. The argument can be repeated and yields eventually $\sigma(\{p\})\cap\tau^{n-1}(\{m\})=\emptyset$. The set $\tau^{n-1}(\{m\})\subseteq\mathrm{Spec}(A)$ is closed. Therefore there is an ideal $I_2\subseteq A$ with $V(I_2)=\tau^{n-1}(\{m\})$. Since $V(p)\cap V(I_2)=\emptyset$ it follows that $A=I_2+p$. Both ideals are proper. There are elements $a_1\in p$ and $a_2\in I_2$ with $1=a_1+a_2$. Since a_1 belongs to the minimal prime ideal there is an element $a_0\neq 0$ such that $a_0\cdot a_1=0$.

We extend the sequence a_0, a_1, a_2: Since $a_2\in I_2$ it follows that $V(a_2)\supseteq V(I_2)=\tau^{n-1}(\{m\})$, or, equivalently, $D(a_2)\cap\tau^{n-1}(\{m\})=\emptyset$, which yields $D(a_2)\cap\gamma(\tau^{n-2}(\{m\}))=\emptyset$. We use 6.5 again with $Y=D(a_2)$ and $Z=\tau^{n-2}(\{m\})$ and get $\sigma(D(a_2))\cap\tau^{n-2}(\{m\})=\emptyset$. Since $\sigma(D(a_2))$ and $\tau^{n-2}(\{m\})$ are closed in $\mathrm{Spec}(A)$ there are ideals $I_3,I_4\subseteq A$ such that $V(I_3)=\sigma(D(a_2))$ and $V(I_4)=\tau^{n-2}(\{m\})$. The closed sets $V(I_3)$ and $V(I_4)$ are disjoint, hence there are elements $a_3\in I_3$ and $a_4\in I_4$ with $1=a_3+a_4$. Note that $V(a_3)\supseteq V(I_3)\supseteq D(a_2)$, which implies $a_2\cdot a_3=0$.

We continue recursively in this way and find a sequence $a_0,\dots,a_{2\cdot n}$ such that the condition $\Phi_{2\cdot n}(a_0,\dots,a_{2\cdot n})$ is satisfied. Now condition (a) implies that $a_0=0$, which is false by construction, or $a_{2\cdot n}\in A^\times$. On the other hand, $a_{2\cdot n}\in I_{2\cdot n}$. By construction, $m\in V(I_{2\cdot n})$, which implies $a_{2\cdot n}\in m$, a contradiction. $\square$

Theorem 6.7. *Let A be a reduced ring. The following conditions are equivalent*:

(a) $A\in\mathcal{C}_{2\cdot n+1}$.
(b) *The prime spectrum is graph connected, and the* 1*st Algorithm, starting from a minimal prime ideal, always terminates after at most* $2\cdot n+1$ *steps.*

Proof. If $n=0$ then both conditions say that A is a domain, 2.4, 6.2.

(b) $\Rightarrow$ (a). Pick $a_0,\dots,a_{2\cdot n+1}\in A$, $a_0\neq 0$, and suppose that $\Phi_{2\cdot n+1}(a_0,\dots,a_{2\cdot n+1})$ holds. We must prove that $V(a_{2\cdot n+1})=\mathrm{Spec}(A)$.

As $a_0 \neq 0$ there is a minimal prime ideal $p \in D(a_0)$. Starting with p, one applies the algorithm and obtains $\operatorname{Spec}(A) = \gamma(\rho^n(\{p\}))$. Consider the sequence of inclusions given by the condition $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$:

$$D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq \cdots \subseteq V(a_{2\cdot n-1}) \subseteq D(a_{2\cdot n}) \subseteq V(a_{2\cdot n+1}).$$

Then $\sigma(D(a_0)) \subseteq V(a_1)$ and $\tau(V(a_{2\cdot i-1})) \subseteq V(a_{2\cdot i+1})$ for all $i = 1, \ldots, n$. It follows that

$$\operatorname{Spec}(A) = \tau^n\bigl(\sigma\bigl(\{p\}\bigr)\bigr) \subseteq \tau^n\bigl(\sigma\bigl(D(a_0)\bigr)\bigr) \subseteq \tau^n\bigl(V(a_1)\bigr) \subseteq V(a_{2\cdot n+1}),$$

which proves the claim.

(a) $\Rightarrow$ (b). By 6.1 there are no nontrivial idempotents. Assume that the assertion is false. Then there is a minimal prime ideal p such that $\tau^n \circ \sigma(\{p\}) \neq \operatorname{Spec}(A)$. Since $\tau^n \circ \sigma(\{p\})$ is closed under specialization there is a minimal prime ideal $q \notin \tau^n \circ \sigma(\{p\})$. It follows that $q \notin \rho^n(\{p\})$. We apply 6.5 with $Y = \{q\}$ and $Z = \tau^{n-1} \circ \sigma(\{p\})$ and obtain $\sigma(\{q\}) \cap \tau^{n-1} \circ \sigma(\{p\}) = \emptyset$. Two more applications of 6.5, first with $Y = \rho^{n-1}(\{p\})$ and $Z = \sigma(\{q\})$, then with $Y = \rho(\{q\})$ and $Z = \tau^{n-2} \circ \sigma(\{p\})$, yield first $\rho(\{q\}) \cap \rho^{n-1}(\{p\}) = \emptyset$, then $\tau \circ \sigma(\{q\}) \cap \tau^{n-2} \circ \sigma(\{p\}) = \emptyset$. Iteration of this process eventually leads to $\tau^{n-1} \circ \sigma(\{q\}) \cap \sigma(\{p\}) = \emptyset$. As $\tau^{n-1} \circ \sigma(\{q\}) \subseteq \operatorname{Spec}(A)$ is closed there is an ideal $I_2 \subseteq A$ such that $\tau^{n-1} \circ \sigma(\{q\}) = V(I_2)$. Now $V(I_2) \cap V(p) = \emptyset$, and $A = p + I_2$ follows. There are elements $a_1 \in p$ and $a_2 \in I_2$ with $1 = a_1 + a_2$. Because a_1 belongs to a minimal prime ideal there is an element $a_0 \neq 0$ such that $a_0 \cdot a_1 = 0$.

We extend the sequence a_0, a_1, a_2 as in the proof of 6.6: Since $a_2 \in I_2$ it follows that $V(a_2) \supseteq V(I_2) = \tau^{n-1} \circ \sigma(\{q\})$, equivalently $D(a_2) \cap \tau^{n-1} \circ \sigma(\{q\}) = \emptyset$. It follows that $D(a_2) \cap \rho^{n-1}(\{q\}) = \emptyset$. Again, 6.5 with $Y = D(a_2)$ and $Z = \sigma \circ \rho^{n-2}(\{q\})$ yields $\sigma(D(a_2)) \cap \sigma \circ \rho^{n-2}(\{q\}) = \emptyset$. There are ideals $I_3, I_4 \subseteq R$ with $V(I_3) = \sigma(D(a_2))$ and $V(I_4) = \sigma \circ \rho^{n-2}(\{q\})$. Then $I_3 + I_4 = A$, and there are elements $a_3 \in I_3$, $a_4 \in I_4$ with $1 = a_3 + a_4$. It follows from $V(a_3) \supseteq V(I_3) \supseteq D(a_2)$ that $a_2 \cdot a_3 = 0$.

Eventually we arrive at a sequence $a_0, \ldots, a_{2\cdot n}$ that satisfies $\Phi_{2\cdot n}$. The final term $a_{2\cdot n}$ is picked from an ideal $I_{2\cdot n}$ with $V(I_{2\cdot n}) = \sigma(\{q\})$. As $V(q) = \sigma(\{q\})$ we may choose $I_{2\cdot n} = q$. As $a_{2\cdot n}$ belongs to a minimal prime ideal there is an element $a_{2\cdot n+1} \neq 0$ such that $a_{2\cdot n} \cdot a_{2\cdot n+1} = 0$. Now the condition $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ is satisfied, but $a_0 \neq 0$ and $a_{2\cdot n+1} \neq 0$, which is a contradiction, and the proof is finished. $\square$

Theorem 6.8. *Let A be a reduced ring. The following conditions are equivalent*:

(a) $A \in \mathcal{D}_{2\cdot n}$.
(b) *The prime spectrum is graph connected, and the 2nd Algorithm, starting from a maximal ideal, always terminates after at most $2 \cdot n + 1$ steps.*

Proof. The case $n = 0$ is treated separately again: Both conditions say that A is a local ring, 2.4, 6.2. We assume now that $n > 0$.

(b) $\Rightarrow$ (a). Pick $a_0, \ldots, a_{2\cdot n} \in A$, $D(1 - a_0) \neq \operatorname{Spec}(A)$, and suppose that $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$ holds. We must prove that $a_{2\cdot n} \in A^\times$.

As $D(1 - a_0) \neq \operatorname{Spec}(A)$ there is a maximal ideal $m \in V(1 - a_0)$. Note that $m \in D(a_0)$. Starting with m, one applies the algorithm and obtains $\operatorname{Spec}(A) = \rho^n \circ \gamma(\{m\})$. Consider the sequence of inclusions given by the condition $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$:

$$D(a_0) \subseteq V(a_1) \subseteq D(a_2) \subseteq \cdots \subseteq V(a_{2\cdot n-1}) \subseteq D(a_{2\cdot n}).$$

It follows that

$$\operatorname{Spec}(A) = \rho^n \circ \gamma\bigl(\{m\}\bigr) \subseteq \rho^n\bigl(D(a_0)\bigr) \subseteq D(a_{2\cdot n}),$$

hence $a_{2\cdot n}$ is a unit, as claimed.

(a) $\Rightarrow$ (b). By 6.1 there are no idempotents. Assume by way of contradiction that the claim is false. There is a maximal ideal m such that $\rho^n \circ \gamma(\{m\}) \neq \mathrm{Spec}(A)$. There is a maximal ideal $l \notin \rho^n \circ \gamma(\{m\})$, hence $l \notin \sigma \circ \rho^{n-1} \circ \gamma(\{m\})$. With 6.5 it follows that $\rho^{n-1} \circ \gamma(\{m\}) \cap \gamma(\{l\}) = \emptyset$. Continuing recursively, as in the proofs of 6.6 and 6.7 one uses 6.5 to show that $\rho^{n-1} \circ \gamma(\{l\}) \cap \gamma(\{m\}) = \emptyset$. Recall that $\gamma(\{m\}) = \bigcap_{c \in A \setminus \{m\}} D(c)$. Using $D(c) \cap D(c') = D(c \cdot c')$, $\rho^{n-1} \circ \gamma(\{l\}) \cap \bigcap_{c \in A \setminus \{m\}} D(c) = \emptyset$ and quasi-compactness of $\rho^{n-1} \circ \gamma(\{l\})$ one concludes that there is an element $c \in A \setminus \{m\}$ with $\rho^{n-1} \circ \gamma(\{l\}) \cap D(c) = \emptyset$. Since m is maximal there is an element $d \in R$ with $d \cdot c + m = 1 + m$. We define $a_0 = c \cdot d$. Then $m \in \mathrm{Spec}(A) \setminus D(1 - a_0)$.

Starting with a_0 we build a sequence $a_0, \ldots, a_{2 \cdot n}$ such that $\Phi_{2 \cdot n}(a_0, \ldots, a_{2 \cdot n})$ holds: Since $D(a_0) \cap \rho^{n-1} \circ \gamma(\{l\}) = \emptyset$ one can apply 6.5 and gets $\sigma(D(a_0)) \cap \sigma \circ \rho^{n-2} \circ \gamma(\{l\}) = \emptyset$. Let $I_1, I_2 \subseteq A$ be ideals with $V(I_1) = \sigma(D(a_0))$, $V(I_2) = \sigma \circ \rho^{n-2} \circ \gamma(\{l\})$. Then there are elements $a_1 \in I_1$ and $a_2 \in I_2$ with $1 = a_1 + a_2$. Note that $a_0 \cdot a_1 = 0$ since $D(a_0) \subseteq V(I_1) \subseteq V(a_1)$. Iteration of this construction yields the desired sequence.

Since $A \in \mathcal{D}_{2 \cdot n}$ it follows that $1 - a_0 \in A^\times$ (which is impossible by construction) or $a_{2 \cdot n} \in A^\times$. But, by construction $a_{2 \cdot n} \in I_{2 \cdot n}$ and $V(I_{2 \cdot n}) = \{l\}$, hence $a_{2 \cdot n} \in l$, which is a contradiction. $\square$

We have now established the connections between the classes $\mathcal{C}_k$ and $\mathcal{D}_k$ and the algorithms of Section 2, hence also with the classes $\mathcal{R}_k$ and $\mathcal{S}_k$ of Section 5. For each k, $\mathcal{C}_k \subseteq \mathcal{R}_k$ and $\mathcal{D}_k \subseteq \mathcal{S}_{k+1}$ are the subclasses of rings with graph connected prime spectrum.

We use some of the examples in Section 3 to show that there are no inclusion relations between the classes $\mathcal{C}_k$ and $\mathcal{D}_k$ beyond those exhibited in 6.4.

- Let A be a ring whose prime spectrum is isomorphic to the spectral space $X_{2 \cdot n+2}$ of 3.2. Then A belongs $\mathcal{D}_{2 \cdot n} \setminus \mathcal{C}_{2 \cdot n+1}$, hence also to $\mathcal{C}_{2 \cdot n+2} \setminus \mathcal{C}_{2 \cdot n+1}$ and $\mathcal{D}_{2 \cdot n} \setminus \mathcal{D}_{2 \cdot n-1}$.
- Let A be a ring whose prime spectrum is isomorphic to the spectral space $Y_{2 \cdot n+2}$ of 3.2. Then A belongs to $\mathcal{C}_{2 \cdot n+1} \setminus \mathcal{D}_{2 \cdot n}$, hence also to $\mathcal{D}_{2 \cdot n+1} \setminus \mathcal{D}_{2 \cdot n}$ and $\mathcal{C}_{2 \cdot n+1} \setminus \mathcal{C}_{2 \cdot n}$.
- Let A be a ring whose prime spectrum is isomorphic to the spectral space $Z_{2 \cdot n+3}$ of 3.2. Then A belongs $\mathcal{C}_{2 \cdot n+3} \cap \mathcal{D}_{2 \cdot n+2}$, but not to $\mathcal{C}_{2 \cdot n+2} = \mathcal{D}_{2 \cdot n+1}$.

Finally in this section, we give a characterization of the rings in the class $\mathcal{C}_3$, cf. 6.2. A ring A is called *almost clean* if every element is a sum of a regular element and an idempotent, [13, Definition 11].

Proposition 6.9. *A ring A belongs to the class $\mathcal{C}_3$ if and only if it is indecomposable and almost clean.*

Proof. Suppose that $A \in \mathcal{C}_3$. Then A is indecomposable by 6.1. Now pick an element $a \in A$ that is not regular. We claim that $1 - a$ is regular. Assume that this is false. Then there are minimal prime ideals $p, q \subseteq A$ such that $1 - a \in p$ and $a \in q$. The 1st Algorithm provides a path from p to q that has length at most 3. Since both prime ideals are minimal and distinct, the length of the path is, in fact, 2. Thus, there is a maximal ideal m that contains both p and q. Note that $p \in V(1 - a)$ implies $m \in V(1 - a)$. As $1 \notin m$ it follows that $a \notin m$. On the other hand, $a \in q \subseteq m$, a contradiction. Now one writes $a = (a - 1) + 1$, which is the sum of a regular element and an idempotent.

Conversely, suppose that A is indecomposable and does not belong to $\mathcal{C}_3$. There are minimal prime ideals p and q such that $q \notin \rho(\{p\})$. For each maximal ideal $m \in \sigma(\{p\})$ there is an element $a_m \in A$ with $a_m \notin m$ and $a \in q$. The sets $D(a_m)$ form an open and constructible cover of $\rho(\{p\})$. By quasi-compactness of $\rho(\{p\})$ there is a finite subcover, $\rho(\{p\}) \subseteq D(a_{m_1}) \cup \cdots \cup D(a_{m_r})$. The ideal generated by $a_{m_1} + p, \ldots, a_{m_r} + p$ in A/p is the entire ring. Therefore there are elements $b_1, \ldots, b_r \in A$ such that $b_1 \cdot a_{m_1} + \cdots + b_r \cdot a_{m_r} + p = 1 + p$. We abbreviate $c = b_1 \cdot a_{m_1} + \cdots + b_r \cdot a_{m_r}$. Then $c \in q$ since each a_{m_i} belongs to q. Moreover, $1 - c \in p$, and therefore $1 - c$ is a zero divisor as well. We have shown that c cannot be written as a sum of a regular element and an idempotent (since 0 and 1 are the only idempotents). Thus R is not almost clean. $\square$

7. Prime spectra with graph connected topological components

Now the results of Section 6 are extended to rings that are not necessarily indecomposable. There are no restrictions on the number of topological components of the prime spectra anymore, but it will be assumed that the topological components coincide with the graph components and that the graph components can be determined with a bounded number of iterations of the 1st or 2nd Algorithms. Again we determine several families of elementary classes of rings.

The formulas that were used to axiomatize the classes $\mathcal{C}_k$ and $\mathcal{D}_k$ (in Section 6) are modified to take idempotents into account. The following notation will be used.

Suppose that Φ is any formula in the language $\mathcal{L}$ and that e is an idempotent. One defines Φ^e to be the formula that is obtained by multiplying every term in Φ with e. In particular, we shall apply this notation to the formulas $\Phi_k^e(a_0, \ldots, a_k)$ and $\Psi^e(y)$ of Section 5.

We define the following classes of rings:

- $A \in \overline{\mathcal{C}_{2\cdot n}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_0, \ldots, a_{2\cdot n}\colon\quad & \Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to \\ & \exists e\colon e^2 = e \wedge \bigl(e \cdot a_0 = 0 \vee \Psi^e(a_{2\cdot n})\bigr) \wedge \bigl((1-e) \cdot a_0 = 0 \vee \Psi^{1-e}(a_{2\cdot n})\bigr).\end{aligned}$$

- $A \in \overline{\mathcal{C}_{2\cdot n+1}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_0, \ldots, a_{2\cdot n+1}\colon\quad & \Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to \\ & \exists e\colon e^2 = e \wedge (e \cdot a_0 = 0 \vee e \cdot a_{2\cdot n+1} = 0) \\ & \quad \wedge \bigl((1-e) \cdot a_0 = 0 \vee (1-e) \cdot a_{2\cdot n+1} = 0\bigr).\end{aligned}$$

- $A \in \overline{\mathcal{D}_{2\cdot n}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_0, \ldots, a_{2\cdot n}\colon\quad & \Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to \\ & \exists e\colon e^2 = e \wedge \bigl(\Psi^e(1-a_0) \vee \Psi^e(a_{2\cdot n})\bigr) \wedge \bigl(\Psi^{1-e}(1-a_0) \vee \Psi^{1-e}(a_{2\cdot n})\bigr).\end{aligned}$$

- $A \in \overline{\mathcal{D}_{2\cdot n+1}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_0, \ldots, a_{2\cdot n+1}\colon\quad & \Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to \\ & \exists e\colon e^2 = e \wedge \bigl(\Psi^e(1-a_0) \vee e \cdot a_{2\cdot n+1} = 0\bigr) \\ & \quad \wedge \bigl(\Psi^{1-e}(1-a_0) \vee (1-e) \cdot a_{2\cdot n+1} = 0\bigr).\end{aligned}$$

Similar to 6.3 it is not difficult to see that the classes $\overline{\mathcal{D}_{2\cdot n+1}}$ and $\overline{\mathcal{C}_{2\cdot n+2}}$ coincide. (In view of 6.3 this also follows from 7.1 below.) The classes $\mathcal{C}_k$ and $\mathcal{D}_k$ are the subclasses of $\overline{\mathcal{C}_k}$ and $\overline{\mathcal{D}_k}$ that consist of the rings with graph connected prime spectrum.

The conditions that define the classes of rings say that certain subsets of the spectrum of a ring can be separated by closed and open sets. We have seen in Section 6 that the formulas $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$ and $\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1})$ yield the sequences

$$D(a_0) \subseteq V(a_1) \subseteq \cdots \subseteq D(a_{2\cdot n}),$$

$$D(a_0) \subseteq V(a_1) \subseteq \cdots \subseteq V(a_{2\cdot n+1}).$$

We extend the sequences to the left by adding the subset $V(1-a_0) \subseteq D(a_0)$. Any set in a sequence is disjoint from the complement of any set that comes later in the sequence. The formulas say that the first set in a sequence can be separated from the complement of the last set by an idempotent. For example, if $A \in \overline{\mathcal{D}_{2\cdot n}}$ and if $\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n})$ is satisfied then there is an idempotent e such that $V(1-a_0) \subseteq V(e)$ and $V(a_{2\cdot n}) \subseteq V(1-e)$.

Theorem 7.1. *Let $\mathcal{K}$ be one of the classes $\mathcal{C}_k$ and $\mathcal{D}_k$, let $\overline{\mathcal{K}}$ be the corresponding class $\overline{\mathcal{C}_k}$ or $\overline{\mathcal{D}_k}$. Let A be a ring. Then the following conditions are equivalent*:

(a) $A \in \overline{\mathcal{K}}$.
(b) *For each prime ideal $\mathfrak{p} \subseteq E(A)$ the factor ring $A/J_\mathfrak{p}$ belongs to $\mathcal{K}$.*

Proof. We prove the equivalence for the case $\mathcal{K} = \mathcal{C}_{2\cdot n}$. Exactly the same arguments can be used to prove the other cases.

(a) $\Rightarrow$ (b). Let $\mathfrak{p} \subseteq E(A)$ be a prime ideal. Note that the ring $A/J_\mathfrak{p}$ is indecomposable. Pick elements $a_0 + J_\mathfrak{p}, \ldots, a_{2\cdot n} + J_\mathfrak{p} \in A/J_\mathfrak{p}$ and suppose that $\Phi_{2\cdot n}(a_0 + J_\mathfrak{p}, \ldots, a_{2\cdot n} + J_\mathfrak{p})$ holds. The set

$$C = \big\{p \in \operatorname{Spec}(A) \bigm| A/p \models \Phi_{2\cdot n}(a_0 + p, \ldots, a_{2\cdot n} + p)\big\}$$

is closed and constructible in $\operatorname{Spec}(A)$ and contains the fiber $p_E^{-1}(\mathfrak{p})$. Therefore there exists an element $e \in E(A)$ such that $A/p \models \Phi_{2\cdot n}(a_0 + p, \ldots, a_{2\cdot n} + p)$ for all $p \in V(e)$, and $p_E^{-1}(\mathfrak{p}) \subseteq V(e)$. Let $e' = 1 - e$ be the orthogonal complement of e. The canonical homomorphism $e' \cdot R \to \prod_{p \in V(e)} A/p$ is injective. It follows that $e' \cdot R \models \Phi_{2\cdot n}^{e'}(a_0, \ldots, a_{2\cdot n})$. One defines a sequence $b_0, \ldots, b_{2\cdot n} \in R$ by setting $b_{2\cdot i} = e' \cdot a_{2\cdot i}$ for $i = 0, \ldots, n$ and $b_{2\cdot i+1} = e' \cdot a_{2\cdot i+1} + e$ for $i = 0, \ldots, n-1$. Then $\Phi_{2\cdot n}(b_0, \ldots, b_{2\cdot n})$ holds in R. The hypothesis yields an idempotent $f \in E(A)$ such that

$$\big(f \cdot b_0 = 0 \vee \Psi^f(b_{2\cdot n})\big) \wedge \big((1-f) \cdot b_0 = 0 \vee \Psi^{1-f}(b_{2\cdot n})\big)$$

holds. Exactly one of f and $1-f$ belongs to $\mathfrak{p}$, say $f \in \mathfrak{p}$. Then $(1-f) \cdot b_0 = 0 \vee \Psi^{1-f}(b_{2\cdot n})$ holds in $(1-f) \cdot A$. As $A/J_\mathfrak{p}$ is a factor ring of $(1-f) \cdot A$ we conclude that $b_0 + J_\mathfrak{p} = 0 \vee \Psi(b_{2\cdot n} + J_\mathfrak{p})$ holds in $R/J_\mathfrak{p}$. This proves the assertion since $a_0 + J_\mathfrak{p} = b_0 + J_\mathfrak{p}$ and $a_{2\cdot n} + J_\mathfrak{p} = b_{2\cdot n} + J_\mathfrak{p}$.

(b) $\Rightarrow$ (a). Suppose that $A/J_\mathfrak{p} \in \mathcal{C}_{2\cdot n}$ for each prime ideal $\mathfrak{p} \subseteq E(A)$. We want to show that $A \in \overline{\mathcal{C}_{2\cdot n}}$. Consider a sequence $a_0, \ldots, a_{2\cdot n} \in A$ such that $\Phi(a_0, \ldots, a_{2\cdot n})$ holds. Then, for each $\mathfrak{p} \in \operatorname{Spec}(E(A))$ the condition $\Phi(a_0 + J_\mathfrak{p}, \ldots, a_{2\cdot n} + J_\mathfrak{p})$ holds in $R/J_\mathfrak{p}$. By hypothesis, the condition $a_0 + J_\mathfrak{p} = 0 + J_\mathfrak{p} \vee \Psi(a_{2\cdot n} + J_\mathfrak{p})$ holds in $R/J_\mathfrak{p}$. The sets

$$M_1 = \big\{\mathfrak{p} \in \operatorname{Spec}\big(E(A)\big) \bigm| A/J_\mathfrak{p} \models a_0 + J_\mathfrak{p} = 0 + J_\mathfrak{p}\big\},$$
$$M_2 = \big\{\mathfrak{p} \in \operatorname{Spec}\big(E(A)\big) \bigm| A/J_\mathfrak{p} \models \Psi(a_{2\cdot n} + J_\mathfrak{p})\big\}$$

are constructible and cover $\operatorname{Spec}(E(A))$. But they are not necessarily disjoint. For each $\mathfrak{p} \in M_1$ there is a closed and constructible set $V_\mathfrak{p} \subseteq \operatorname{Spec}(A)$ such that $p_E^{-1}(\mathfrak{p}) \subseteq V_\mathfrak{p}$ and $E/p \models a_0 + p = 0 + p$ for all $p \in V_\mathfrak{p}$. Thus, there exists an idempotent $e_\mathfrak{p}$ such that $p_E^{-1}(\mathfrak{p}) \subseteq V(e_\mathfrak{p}) \subseteq V_\mathfrak{p}$. The subsets $V(e_\mathfrak{p}) \subseteq \operatorname{Spec}(A)$ are closed and open and are contained in $p_E^{-1}(M_1)$. Similarly, for each $\mathfrak{p} \in M_2$ there is an idempotent $e_\mathfrak{p}$ such that $p_E^{-1}(\mathfrak{p}) \subseteq V(e_\mathfrak{p})$ and $V(e_\mathfrak{p}) \subseteq p_E^{-1}(M_2)$.

The closed and open subsets $V(e_\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(E(A))$, cover $\operatorname{Spec}(A)$. Hence, by compactness, there is a finite subcover, $\operatorname{Spec}(A) = V(e_{\mathfrak{p}_1}) \cup \cdots \cup V(e_{\mathfrak{p}_k})$. There is a complete set $f_1, \ldots, f_r$ of mutually orthogonal idempotents such that each $e_{\mathfrak{p}_i}$ is a sum of some of the f_j. Thus, for each $j = 1, \ldots, r$, at least one of the conditions $V(f_j) \subseteq p_E^{-1}(M_1)$ and $V(f_j) \subseteq p_E^{-1}(M_2)$ holds. We arrange the enumeration such that $V(f_j) \subseteq p_E^{-1}(M_1)$ for $j = 1, \ldots, s$ and $V(f_j) \subseteq p_E^{-1}(M_2)$ for $j = s+1, \ldots, r$ and define $g = f_1 + \cdots + f_s \in E(A)$.

The canonical homomorphisms $(1-g)\cdot A \to \prod_{g\in\mathfrak{p}} A/J_{\mathfrak{p}}$ and $g\cdot A \to \prod_{1-g\in\mathfrak{p}} A/J_{\mathfrak{p}}$ are injective. It follows from

$$\prod_{g\in\mathfrak{p}} A/J_{\mathfrak{p}} \models (a_0 + J_{\mathfrak{p}})_{g\in\mathfrak{p}} = (0 + J_{\mathfrak{p}})_{g\in\mathfrak{p}}$$

and

$$\prod_{1-g\in\mathfrak{p}} A/J_{\mathfrak{p}} \models \Psi\big((a_{2\cdot n} + J_{\mathfrak{p}})_{1-g\in\mathfrak{p}}\big)$$

that $(1-g)\cdot A \models (1-g)\cdot a_0 = 0$ and $g\cdot A \models \Psi^g(a_{2\cdot n})$, and the proof is finished. □

Corollary 7.2. *The notation is as in* 7.1. *Suppose that $A \in \overline{\mathcal{K}}$. For each ideal $\mathfrak{i} \subseteq E(A)$ the factor ring $A/J_{\mathfrak{i}}$ belongs to $\overline{\mathcal{K}}$ as well.*

Proof. Suppose that $\mathfrak{p}/\mathfrak{i} \subseteq E(A)/\mathfrak{i} = E(A/J_{\mathfrak{i}})$ is a prime ideal. Then $(A/J_{\mathfrak{i}})/(J_{\mathfrak{p}}/J_{\mathfrak{i}}) \simeq E/J_{\mathfrak{p}}$ belongs to $\mathcal{K}$, and the claim follows from 7.1. □

Corollary 7.3. *For any $k \in \mathbb{N}$ the class $\overline{\mathcal{C}_k}$ is contained in the class $\mathcal{R}_k$, and the class $\overline{\mathcal{D}_k}$ is contained in the class $\mathcal{S}_{k+1}$.*

In 6.2 and 6.9 we gave explanations of the ring theoretic meaning of the classes $\mathcal{C}_k$ and $\mathcal{D}_k$ for small values of k. Now we do the same for the classes $\overline{\mathcal{C}_k}$ and $\overline{\mathcal{D}_k}$.

Theorem 7.4. *Let A be a reduced ring.*

(a) *$A \in \overline{\mathcal{C}_0}$ if and only if A is von Neumann regular.*
(b) *$A \in \overline{\mathcal{D}_0}$ if and only if A is clean, i.e., every element is the sum of a unit and an idempotent.*
(c) *$A \in \overline{\mathcal{C}_1}$ if and only if A is a weak Baer ring, i.e., the annihilator ideal of every element is generated by idempotents.*

Proof. (a). Suppose that A is von Neumann regular. For each element $a \in A$ there is an element $b \in A$ such that $a^2 \cdot b = a$. Then $e = a\cdot b \in E(A)$. One checks that $(e\cdot a)\cdot(e\cdot b) = e$, i.e., $\Psi^e(a)$ is satisfied, and $(1-e)\cdot a = a - a^2\cdot b = 0$ is satisfied. We have shown that the defining condition of $\overline{\mathcal{C}_0}$ holds in R.

Conversely, suppose that $A/J_{\mathfrak{p}}$ is a field for each $\mathfrak{p} \in \operatorname{Spec}(E(A))$. Then the ideals $J_{\mathfrak{p}}$ are maximal in A. Let $p \subseteq A$ be any prime ideal. Then $J_{p\cap E(A)} \subseteq p$, and it follows that p is a maximal ideal. If every prime ideal of A is maximal then the ring is von Neumann regular.

(b). Let A be clean and pick an element $a \in A$. By hypothesis, there are a unit u and an idempotent e with $a = u + e$. Then $e\cdot(1-a) = e\cdot(1-e-u) = -e\cdot u$, which is a unit in $e\cdot R$, i.e., $\Psi^e(1-a)$ is satisfied, and $(1-e)\cdot a = (1-e)\cdot(u+e) = (1-e)\cdot u$, which is a unit in $(1-e)\cdot R$, i.e., $\Psi^{1-e}(a)$ is satisfied. It follows that $A \in \overline{\mathcal{D}_0}$.

Now suppose that $A \in \overline{\mathcal{D}_0}$. Pick an element $a \in R$. The condition $\Phi_0(a)$ is satisfied, and it follows that there is an idempotent e such that one of $e - e\cdot a$ and $e\cdot a$ is a unit in $e\cdot A$ and one of $(1-e) - (1-e)\cdot a$ and $(1-e)\cdot a$ is a unit in $(1-e)\cdot A$. There are four cases to consider:

- If $e - e\cdot a$ and $(1-e) - (1-e)\cdot a$ are units then their sum, which is $1-a$, is a unit in A, and $a = 1 + (a-1)$ is a sum of an idempotent and a unit.
- If $e\cdot a$ and $(1-e)\cdot a$ are units, then their sum, which is a, is a unit in R, and $a = a + 0$ is the desired representation.

- If $e - e \cdot a$ and $(1-e) \cdot a$ are units then $a = ((e \cdot a - e) + (1-e) \cdot a) + e$ is a sum of a unit and an idempotent.
- If $e \cdot a$ and $(1-e) - (1-e) \cdot a$ are units then $a = (e \cdot a + ((1-e) \cdot a - (1-e))) + (1-e)$ is a sum of a unit and an idempotent.

(c). Suppose that A is a weak Baer ring and that $a_0 \cdot a_1 = 0$ in A. The hypothesis yields an idempotent e such that $a_0 \cdot e = 0$ and $a_1 = b \cdot e$ for some element $b \in A$. Clearly, one may choose $b = a_1$. Then $(1-e) \cdot a_1 = 0$ holds, and A satisfies the defining condition of the class $\overline{\mathcal{C}_1}$.

Conversely, suppose that $A \in \overline{\mathcal{C}_1}$. Pick an element $a \in A$. We must show that $\mathrm{Ann}(a)$ is generated by idempotents. So, suppose that $a \cdot b = 0$. The definition of the class $\overline{\mathcal{C}_1}$ yields an idempotent e with $e \cdot a = 0$ or $e \cdot b = 0$ and $(1-e) \cdot a = 0$ or $(1-e) \cdot b = 0$. If $e \cdot a = 0$ and $(1-e) \cdot a = 0$ then $a = 0$, and the annihilator ideal is generated by the idempotent 1. If $e \cdot b = 0$ and $(1-e) \cdot b = 0$ then $b = 0$, and b is a multiple of the idempotent 0, which belongs to $\mathrm{Ann}(a)$. Now suppose that $a \neq 0$, $b \neq 0$, $e \cdot a = 0$ and $(1-e) \cdot b = 0$. Then $b = b \cdot (e + (1-e)) = b \cdot e$, i.e., $b \in e \cdot A \subseteq \mathrm{Ann}(a)$. Finally, if $(1-e) \cdot a = 0$ and $e \cdot b = 0$ then $b = b \cdot (1-e) \in (1-e) \cdot A \subseteq \mathrm{Ann}(a)$. $\square$

The notion of clean rings was first introduced by Nicholson, [16, p. 271]. They have received a considerable amount of attention in the literature, in particular for rings of continuous functions, see e.g., [1,13,14], [15, Section 1]. There is a long list of equivalent conditions that characterize clean rings, cf. [15, Theorem 1.7]. One of the conditions is: The ring is Gel'fand, and the space of maximal ideals is Boolean. This characterization can be recovered easily from 7.4: If A is clean, i.e., belongs to $\overline{\mathcal{D}_0}$, then it belongs to $\mathcal{S}_1$, hence is normal, 5.7, and the map $\mathrm{Max}(\mathrm{Spec}(A)) \to \mathrm{Spec}(A) \xrightarrow{p_E} \mathrm{Spec}(E(A))$ is continuous and bijective, hence a homeomorphism (since the spaces are compact). Conversely, if the ring is Gel'fand and $\mathrm{Max}(A)$ is Boolean then the map $\mathrm{Max}(\mathrm{Spec}(A)) \to \mathrm{Spec}(A) \xrightarrow{p_E} \mathrm{Spec}(E(A))$ is a homeomorphism and every ring $A/J_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Spec}(E(A))$, is local.

If A is a ring of continuous functions, $A = \mathcal{C}(X;\mathbb{R})$, then $\mathrm{Spec}(A)$ is a root system and, hence, is normal. So the question of whether or not A is clean depends only on the existence of enough idempotents. Let $A^* = \mathcal{C}^*(X;\mathbb{R})$ be the ring of bounded continuous functions. Both rings have the same idempotents, and their spaces of maximal ideal are canonically homeomorphic to each other and are homeomorphic to the Stone–Čech compactification βX [7,21]. It follows that they are both clean or both unclean. One concludes that they are clean if and only if βX is Boolean. (This fact has already been proved in [13, Theorem 13].)

The formulas that define the classes $\overline{\mathcal{C}_{2\cdot n}}$, $\overline{\mathcal{C}_{2\cdot n+1}}$ and $\overline{\mathcal{D}_{2\cdot n}}$ assert the existence of idempotents. We present one of several other ways to introduce idempotents. The formulas $\Theta_{2\cdot n}$ and $\Theta_{2\cdot n+1}$ of Section 5 will be used. Consider the following classes of rings:

- $A \in \overline{\mathcal{E}_{2\cdot n}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_1, \ldots, a_{2\cdot n-1}\colon\quad & \Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1}) \to \\ & \exists e\colon e^2 = e \wedge \forall a_0, a_{2\cdot n}\colon \big(\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to \\ & \big(e \cdot a_0 = 0 \vee \Psi^e(a_{2\cdot n})\big) \wedge \big((1-e) \cdot a_0 = 0 \vee \Psi^{1-e}(a_{2\cdot n})\big)\big).\end{aligned}$$

- $A \in \overline{\mathcal{E}_{2\cdot n+1}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_1, \ldots, a_{2\cdot n}\colon\quad & \Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n}) \to \\ & \exists e\colon e^2 = e \wedge \forall a_0, a_{2\cdot n+1}\colon \big(\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to \\ & (e \cdot a_0 = 0 \vee e \cdot a_{2\cdot n+1} = 0) \wedge \big((1-e) \cdot a_0 = 0 \vee (1-e) \cdot a_{2\cdot n+1} = 0\big)\big).\end{aligned}$$

- $A \in \overline{\mathcal{F}_{2\cdot n}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_1, \ldots, a_{2\cdot n-1}\colon\quad &\Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1}) \to\\ &\exists e\colon\ e^2 = e \wedge \forall a_0, a_{2\cdot n}\colon\ \bigl(\Phi_{2\cdot n}(a_0, \ldots, a_{2\cdot n}) \to\\ &\bigl(\Psi^e(1-a_0) \vee \Psi^e(a_{2\cdot n})\bigr) \wedge \bigl(\Psi^{1-e}(1-a_0) \vee \Psi^{1-e}(a_{2\cdot n})\bigr)\bigr).\end{aligned}$$

- $A \in \overline{\mathcal{F}_{2\cdot n+1}}$ if it satisfies the following condition:

$$\begin{aligned}\forall a_1, \ldots, a_{2\cdot n}\colon\quad &\Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n}) \to\\ &\exists e\colon\ e^2 = e \wedge \forall a_0, a_{2\cdot n+1}\colon\ \bigl(\Phi_{2\cdot n+1}(a_0, \ldots, a_{2\cdot n+1}) \to\\ &\bigl(\Psi^e(1-a_0) \vee e \cdot a_{2\cdot n+1} = 0\bigr) \wedge \bigl(\Psi^{1-e}(1-a_0) \vee (1-e) \cdot a_{2\cdot n+1} = 0\bigr)\bigr).\end{aligned}$$

As before, it is not difficult to show that $\overline{\mathcal{F}_{2\cdot n+1}} = \overline{\mathcal{E}_{2\cdot n+2}}$; the details can be left to the reader. Moreover, the inclusions $\overline{\mathcal{E}_{2\cdot n}} \subseteq \overline{\mathcal{C}_{2\cdot n}}$, $\overline{\mathcal{E}_{2\cdot n+1}} \subseteq \overline{\mathcal{C}_{2\cdot n+1}}$ and $\overline{\mathcal{F}_{2\cdot n}} \subseteq \overline{\mathcal{D}_{2\cdot n}}$ are obvious. The intersection of the class of rings with graph connected prime spectrum with the class $\overline{\mathcal{E}_k}$, or with the class $\overline{\mathcal{F}_k}$, yields the class $\mathcal{C}_k$, or the class $\mathcal{D}_k$.

Similar to $\overline{\mathcal{C}_k}$ and $\overline{\mathcal{D}_k}$, the defining conditions of $\overline{\mathcal{E}_k}$ and $\overline{\mathcal{F}_k}$ can be interpreted as separation properties for subsets of spectra by closed and open sets. The conditions $\Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1})$ and $\Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n})$ yield sequences

$$V(a_1) \subseteq D(a_2) \subseteq \cdots \subseteq V(a_{2\cdot n-1}),$$
$$V(a_1) \subseteq D(a_2) \subseteq \cdots \subseteq D(a_{2\cdot n}).$$

The interior of a subset $M \subseteq \operatorname{Spec}(A)$ is denoted by $\operatorname{int}(M)$, the interior with respect to the inverse topology is denoted by $\operatorname{int}_{\mathrm{inv}}(M)$.

Lemma 7.5. *Suppose that A is a ring and $a \in A$. Then*:

(a) $\operatorname{int}(V(a)) = \bigcup_{c\in \operatorname{Ann}(a)} D(c)$.
(b) $\operatorname{int}_{\mathrm{inv}}(\operatorname{int}(V(a))) = \bigcup_{c\in \operatorname{Ann}(a)} V(1-c)$.

Proof. (a). If $c \in \operatorname{Ann}(a)$ then $D(c) \subseteq V(a)$, hence $D(c) \subseteq \operatorname{int}(V(a))$. Conversely, let $p \in \operatorname{int}(V(a))$. Then there is a basic open set $D(c)$ with $p \in D(c)$ and $D(c) \cap D(a) = \emptyset$. It follows that $c \in \operatorname{Ann}(a)$, which proves the other inclusion.

(b). First suppose that $c \in \operatorname{Ann}(a)$. Then $V(1-c) \subseteq D(c)$. The set $V(1-c)$ is open in the inverse topology, hence, using (a), $V(1-c) \subseteq \operatorname{int}_{\mathrm{inv}}(D(c)) \subseteq \operatorname{int}_{\mathrm{inv}}(\operatorname{int}(V(a)))$. Now suppose that $p \in \operatorname{int}_{\mathrm{inv}}(\operatorname{int}(V(a)))$. Then $\overline{\{p\}} \subseteq \operatorname{int}(V(a))$. By (a) and compactness there is a finite subset $F \subseteq \operatorname{Ann}(a)$ with $\overline{\{p\}} \subseteq \bigcup_{c\in F} D(c)$. The ideal generated in A/p by the elements $c + p$, $c \in F$, is the entire ring. Therefore there are elements $b_c \in A$ such that $\sum_{c\in F} b_c \cdot c + p = 1 + p$. Then $b = \sum_{c\in F} b_c \cdot c \in \operatorname{Ann}(a)$, and $p \in V(1-b)$. This proves the claim. □

If $A \in \overline{\mathcal{E}_{2\cdot n}}$ and if $\Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1})$ holds then there is an idempotent e such that $D(a_0) \subseteq V(e)$ for any $a_0 \in \operatorname{Ann}(a_1)$, i.e., $\operatorname{int}(V(a_1)) \subseteq V(e)$, 7.5 (a), and $V(1-a_{2\cdot n-1}) \subseteq V(1-e)$. Or, if $A \in \overline{\mathcal{F}_{2\cdot n}}$ and if $\Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1})$ then there is an idempotent e such that $V(1-a_0) \subseteq V(e)$ for each $a_0 \in \operatorname{Ann}(a_1)$, i.e., $\operatorname{int}_{\mathrm{inv}}(\operatorname{int}(V(a_1))) \subseteq V(e)$, 7.5 (b), and $V(1-a_{2\cdot n-1}) \subseteq V(1-e)$. Or, if $A \in \overline{\mathcal{E}_{2\cdot n+1}}$ and if $\Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n})$ then there is an idempotent e such that $D(a_0) \subseteq V(e)$ for any $a_0 \in \operatorname{Ann}(a_1)$, i.e., $\operatorname{int}(V(a_1)) \subseteq V(e)$, 7.5 (a), and $D(a_{2\cdot n+1}) \subseteq V(1-e)$ for any $a_{2\cdot n+1} \in \operatorname{Ann}(a_{2\cdot n})$, i.e., $\operatorname{int}(V(a_{2\cdot n})) \subseteq V(1-e)$, 7.5 (a).

For small values of k we describe the classes $\overline{\mathcal{E}_k}$ and $\overline{\mathcal{F}_k}$ in more conventional ring theoretic terms again:

Theorem 7.6. *Suppose that A is a ring.*

(a) *$A \in \overline{\mathcal{E}_0}$ if and only if A is a field or is a direct product of two fields.*
(b) *$A \in \overline{\mathcal{F}_0}$ if and only if A is a local ring or is a direct product of two local rings.*
(c) *$A \in \overline{\mathcal{E}_1}$ if and only if A is a domain or a direct product of two domains.*
(d) *$A \in \overline{\mathcal{E}_3}$ if and only if A is almost clean.*

Proof. First note that the conditions $\Theta_{2\cdot n-1}(a_1, \ldots, a_{2\cdot n-1})$, $n = 0, 1$, and $\Theta_{2\cdot n}(a_1, \ldots, a_{2\cdot n})$, $n = 0$, are always trivially satisfied. We claim that in rings that belong to $\overline{\mathcal{E}_0}$ or $\overline{\mathcal{E}_1}$ or $\overline{\mathcal{F}_0}$ there cannot be three nontrivial mutually orthogonal idempotents.

Assume that there are three nontrivial mutually orthogonal idempotents e_1, e_2, e_3 such that $1 = e_1 + e_2 + e_3$. The hypothesis of the conditions that define the classes $\overline{\mathcal{E}_0}$, $\overline{\mathcal{E}_1}$ and $\overline{\mathcal{F}_0}$ is satisfied, as we have noted. Therefore there is an idempotent e as stated.

First consider the class $\overline{\mathcal{E}_0}$. If we choose $a_0 = e_1$ then $e \cdot e_1 = 0 \vee \exists z\colon\ e = z \cdot e_1$ implies that $e \in (e_2 + e_3) \cdot A$ or $e \in e_1 \cdot A$, and $(1-e) \cdot e_1 = 0 \vee \exists z\colon\ 1 - e = z \cdot e_1$ implies that $1 - e \in (e_2 + e_3) \cdot A$ or $1 - e \in e_1 \cdot A$. Since e and $1-e$ cannot belong to the same proper ideal we see that $e \in (e_2 + e_3) \cdot A$ and $1 - e \in e_1 \cdot A$, or vice versa. It follows that $e = e_2 + e_3$, $1 - e = e_1$, or vice versa. The same argument can be repeated with $a_0 = e_2$ and yields $e = e_2$ or $e = e_1 + e_3$, which is a contradiction. The cases $\overline{\mathcal{F}_0}$ and $\overline{\mathcal{E}_1}$ are done in exactly the same way.

(a). If A is a field then the idempotent $e = 1$ has the desired properties. If $A = B \times C$ is a direct product of two fields, then the idempotent $e = (1, 0)$ satisfies the conditions. Thus, in both cases, $A \in \overline{\mathcal{E}_0}$.

Now suppose that $A \in \overline{\mathcal{E}_0}$. If A is indecomposable then it is a field. If A is not indecomposable then let e be one of the two nontrivial idempotents. Both e and $1 - e$ generate prime ideals in $E(A)$, and $A \simeq A/e \cdot A \times A/(1-e) \cdot A$. As $\overline{\mathcal{E}_0} \subseteq \overline{\mathcal{C}_0}$ we conclude from 7.1 and 6.2 that $A/e \cdot A$ and $A/(1-e) \cdot A$ are both fields.

(b) and (c) are done in the same way as (a).

(d). First suppose that A is almost clean. Pick an element $a_1 \in A$ and define $a_2 = 1 - a_1$. There is an idempotent e such that $c = e - a_1$ is a regular element. We prove that this idempotent has the desired properties: Pick elements a_0, a_3 such that $a_0 \cdot a_1 = 0$ and $a_2 \cdot a_3 = 0$. Let $p \subseteq A$ be a minimal prime ideal. If $a_0 \in p$ then $a_0 \cdot (1 - e) \in p$. If $a_0 \notin p$ then $a_1 \in p$. As $c = e - a_1 \notin p$ we see that $e \notin p$. But then $1 - e \in p$, and again we obtain $a_0 \cdot (1 - e) \in p$. Thus, $a_0 \cdot (1 - e)$ belongs to every minimal prime ideal, which implies that $a_0 \cdot (1 - e) = 0$. Exactly the same argument shows that $a_{2\cdot n+1} \cdot e = 0$. Thus, the defining condition of $\overline{\mathcal{E}_3}$ is satisfied.

Conversely, let $A \in \overline{\mathcal{E}_3}$. Pick an element $a \in A$, and define $a_1 = a$, $a_2 = 1 - a$. If one of a and $1 - a$ is a regular element then there is nothing to prove. So, suppose that this is not the case. Then there is an idempotent e as in the definition of the class. We may assume that $\operatorname{int}(V(a)) \subseteq V(e)$ and $\operatorname{int}(V(1-a)) \subseteq V(1-e)$. We claim that $(1 - e) - a$ is a regular element.

Assume that $(1-e) - a \in p$, p a minimal prime ideal. If $a \in p$ then $p \in \operatorname{int}(V(a))$, hence $p \in V(e) = D(1-e)$. It follows that $(1-e) - a \notin p$, a contradiction. If $a \notin p$ then $1 - e \notin p$, hence $e \in p$. But then $1 - a \in p$, which implies $p \in \operatorname{int}(V(1-a)) \subseteq V(1-e)$, i.e., $1 - e \in p$ – again a contradiction.

It has been shown that $(1-e) - a$ does not belong to any minimal prime ideal, hence is a regular element. We can now write $a = (a - (1-e)) + (1-e)$, a sum of a regular element and an idempotent, i.e., A is almost clean. □

We close with some observations concerning the separation of subsets of $\operatorname{Spec}(A)$ by idempotents. Recall: If $M \subseteq \operatorname{Spec}(A)$ is a proconstructible subset then $\gamma(M)$ is the closure of M for the inverse topology.

Given an element $a \in A$, the closed and constructible sets $V(a)$ and $V(1-a)$ are disjoint. Consider the following sequences of subsets:

$$\begin{tikzcd}
\gamma(V(a)) & \gamma(V(1-a)) \\
V(a) \arrow[u] & V(1-a) \arrow[u] \\
\mathrm{int}(V(a)) \arrow[u] & \mathrm{int}(V(1-a)) \arrow[u] \\
\mathrm{int}_{\mathrm{inv}}(\mathrm{int}(V(a))) \arrow[u] & \mathrm{int}_{\mathrm{inv}}(\mathrm{int}(V(1-a))) \arrow[u]
\end{tikzcd}$$

Some of the classes of rings that occurred here can be characterized via separation of the sets in the diagram. Note that, given an idempotent e and a proconstructible set M, the containment relations $M \subseteq V(e)$, $\gamma(M) \subseteq V(e)$ and $\overline{M} \subseteq V(e)$ are equivalent to each other.

- The ring A is clean, i.e., belongs to $\overline{\mathcal{D}_0}$, if and only if there is an idempotent that separates $\gamma(V(a))$ and $\gamma(V(1-a))$.
- The ring A belongs to $\overline{\mathcal{E}_2}$ if and only if the sets $\mathrm{int}(V(a))$ and $\gamma(V(1-a))$ can be separated by an idempotent.
- The ring A is almost clean, i.e., belongs to $\overline{\mathcal{E}_3}$, if and only if the sets $\mathrm{int}(V(a))$ and $\mathrm{int}(V(1-a))$ can be separated by an idempotent.
- The ring A belongs to $\overline{\mathcal{F}_2}$ if and only if $\mathrm{int}_{\mathrm{inv}}(\mathrm{int}(V(a)))$ and $\gamma(V(1-a))$ can be separated with an idempotent.

The separation properties yield the following result, which generalizes some of the equivalences of [13, Theorem 13].

Proposition 7.7. *A ring A is clean if and only if it is Gel'fand and almost clean.*

Proof. If A is clean then it is normal, cf. the observations following the proof of 7.4, and is clearly almost clean. Conversely, suppose that A is normal and almost clean. Pick an element $a \in A$. The sets $\gamma(V(a))$ and $\gamma(V(1-a))$ are closed and generically closed (by normality of $\mathrm{Spec}(A)$) and are disjoint. It follows that $\gamma(V(a)) = \bigcap_{c \in D(V(a))} D(c)$, $\gamma(V(1-a)) = \bigcap_{d \in D(V(1-a))} D(d)$, where $D(V(x)) = \{y \in A \mid V(x) \subseteq D(y)\}$. Compactness implies that there are $c \in D(V(a))$ and $d \in D(V(1-a))$ with $D(c) \cap D(d) = \emptyset$. One can modify c such that $c + I = 1 + I$, where $I \subseteq A$ is the radical ideal with $V(I) = \gamma(V(a))$. Now, for the element c, there is an idempotent e such that $\mathrm{int}(V(c)) \subseteq V(e)$ and $\mathrm{int}(V(1-c)) \subseteq V(1-e)$. By construction $\gamma(V(a)) \subseteq V(1-c)$, hence $\gamma(V(a)) \subseteq \mathrm{int}(V(1-c))$, and $\gamma(V(1-a)) \subseteq D(d) \subseteq V(c)$, hence $\gamma(V(1-a)) \subseteq \mathrm{int}(V(c))$. It follows that $\gamma(V(a)) \subseteq V(1-e)$ and $\gamma(V(1-a)) \subseteq V(e)$, and the proof is finished. $\square$

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