Journal of Algebra 344 (2011) 284–295

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The composition factors of the functor of permutation modules

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ARTICLE INFO

Article history:
Received 22 December 2010
Available online 18 July 2011
Communicated by Michel Broué

Keywords:
Biset functor
Permutation modules
Composition factors

ABSTRACT

Let k be a field, let $\Pi_k(G)$ be the Grothendieck group of permutation kG-modules, where G is a finite group, and let $\mathbb{C}\Pi_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \Pi_k(G)$. In this article, we find all the composition factors of the biset functor $\mathbb{C}\Pi_k$.

1. Introduction

Let k be a field of characteristic p, where p is a prime number or 0.

Let G be a finite group, let $B(G)$ be the Grothendieck group of G-sets (Definition 11) and let $\Pi_k(G)$ be the Grothendieck group of permutation kG-modules (Definition 13). We can define a map $\theta_G : B(G) \to \Pi_k(G)$ which is natural and surjective by definition. Now if we tensor everything with $\mathbb{C}$ and if G varies, $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ become $\mathbb{C}$-linear biset functors and $\theta : \mathbb{C}B \to \mathbb{C}\Pi_k$ is a natural transformation.

Recall that the simple biset functors $S_{H,V}$ are parametrized by pairs (H, V), where H is a finite group and V a simple $\mathbb{C}\,\mathrm{Out}(H)$-module. If $k = \mathbb{Q}$ then we have that $\mathbb{C}\Pi_{\mathbb{Q}} = \mathbb{C}R_{\mathbb{Q}}$, where $R_{\mathbb{Q}}(G)$ is the ordinary Grothendieck group of $\mathbb{Q}G$-modules, and Serge Bouc proves that $\mathbb{C}R_{\mathbb{Q}} = S_{\mathbf{1},\mathbb{C}}$ is a simple biset functor [1, Proposition 4.4.8].

We want to generalize this to an arbitrary field k. More precisely, we want to find the composition factors of $\mathbb{C}\Pi_k$. In order to do this, we need the composition factors of $\mathbb{C}B$. They were determined by Serge Bouc and they are the simple functors $S_{H,\mathbb{C}}$, where H is a B-group. A B-group is defined by a technical condition (see Definition 16).

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[1] Part of a PhD thesis, supervision of Prof. Jacques Thévenaz.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.06.032

Recall that a p-hypo-elementary group is the semi-direct product of a p-group with a cyclic p'-group. If $p = 0$, this means that the group is cyclic.

In this article, we will prove the following two theorems:

Theorem 1. *The composition factors of the functor $\mathbb{C}\Pi_k$ are the simple functors $S_{H,\mathbb{C}}$, where H is a p-hypo-elementary B-group and where $\mathbb{C}$ is the trivial $\mathbb{C}\operatorname{Out}(H)$-module.*

Theorem 2. *Let $G \cong P \rtimes C_n$ be a p-hypo-elementary group (P is a p-group and C_n a cyclic p'-group). Then G is a B-group if and only if:*

(i) *P is elementary abelian*;
(ii) *In a decomposition of P as a direct sum of simple $\mathbb{F}_p C_n$-modules, every simple $\mathbb{F}_p C_n$-module appears at most one time, except the trivial module, which appears* 0 *or* 2 *times*;
(iii) *The action of C_n on P is faithful.*

Serge Bouc proved that any finite group G has a largest quotient $\beta(G) = G/N$ which is a B-group, uniquely determined by G. In the course of the proof of Theorem 1, we also prove the following result:

Theorem 3. *Let G be a finite group. Then $\beta(G)$ is p-hypo-elementary if and only if G itself is p-hypo-elementary.*

This article begins with some background results. Then we define a natural transformation θ between the Burnside functor $\mathbb{C}B$ and the functor of permutation modules $\mathbb{C}\Pi_k$. Using this map and the classification of the composition factors of $\mathbb{C}B$ obtained by Serge Bouc [2] (see also Chapter 5 of [1]), we will find the composition factors of the functor of permutation modules $\mathbb{C}\Pi_k$. So we will have the proof of Theorem 1. Then we make a classification of the p-hypo-elementary groups which are B-groups (Theorem 2).

All groups are supposed finite, all vector spaces are finite dimensional and all modules are finitely generated left modules. Let G and H be finite groups. We write $G \gg H$ if H is isomorphic to a quotient of G. By $H \leqslant_G G$, we denote a subgroup H of G, up to conjugacy (in G). All G-sets and all (H, G)-bisets are finite. We denote by $[U]$ the isomorphism class of U (where U can be a group, a vector space, a module, a G-set, an (H, G)-biset, ...).

2. Background on biset functors

2.1. The category of biset functors

Definition 4. Let G and H be finite groups. Then $B(H, G)$ is the Grothendieck group of the isomorphism classes of finite (H, G)-bisets (for the disjoint union).

Notation 5. Let G be a group. We denote by Id_G the (G, G)-biset G where the two actions are defined by left and right multiplication in G. We also denote by Id_G the image of Id_G in $B(G, G)$.

Definition 6. (See [1], Definition 3.1.6.) We define the category $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are all finite groups;
- If G and H are finite groups, then

$$\mathrm{Hom}_{\mathcal{C}}(G, H) = \mathbb{C} \otimes_{\mathbb{Z}} B(H, G);$$

- The composition of morphisms in $\mathcal{C}$ is the $\mathbb{C}$-linear extension of the composition in $B(H,G)$, defined by $v \circ u = v \times_H u$ for all finite groups G, H and K, for all morphisms $u \in B(H,G)$ and for all morphisms $v \in B(K,H)$;
- For any finite group G, the identity morphism of G in $\mathcal{C}$ is equal to $\mathrm{Id}_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathrm{Id}_G$.

Definition 7. (See [1], Definition 3.2.2.) A *biset functor defined on* $\mathcal{C}$ with values in $\mathbb{C}-\mathbf{mod}$ is a $\mathbb{C}$-linear functor from $\mathcal{C}$ to the category $\mathbb{C}-\mathbf{mod}$ of all finite dimensional $\mathbb{C}$-vector spaces.

Biset functors over $\mathcal{C}$, with values in $\mathbb{C}-\mathbf{mod}$, are the objects of a category, denoted by $\mathcal{F}$, where morphisms are natural transformations of functors, and composition of morphisms is composition of natural transformations.

Remark 8. For convenience, we have tensored everything with $\mathbb{C}$. But actually we only need a field of characteristic 0 which contains all roots of unity.

Proposition 9. *(See [1], Proposition 3.2.8.) The category $\mathcal{F}$ is a $\mathbb{C}$-linear abelian category. In particular, if θ is a morphism of biset functors and G is a finite group, then*

$$(\mathrm{Ker}\,\theta)(G) = \mathrm{Ker}\,\theta_G, \qquad (\mathrm{Coker}\,\theta)(G) = \mathrm{Coker}\,\theta_G.$$

The simple objects of the category $\mathcal{F}$ are labeled by pairs (G,V), where G is a finite group and V a simple $\mathbb{C}\,\mathrm{Out}(G)$-module. We denote by $S_{G,V}$ the simple functor associated to (G,V). If $F \in \mathcal{F}$ is a simple functor, then $F \cong S_{G,V}$ where G is the smallest group (unique up to isomorphism) such that $F(G) \neq \{0\}$ and $V = F(G)$. We can define a notion of isomorphism on those pairs such that two simple functors are isomorphic if and only if the corresponding pairs are isomorphic [1, Theorem 4.3.10].

Proposition 10. *(See [1], Lemma 4.3.9.) Let G be a finite group and V a simple $\mathbb{C}\,\mathrm{Out}(G)$-module. If H is a finite group such that $S_{G,V}(H) \neq \{0\}$, then G is isomorphic to a subquotient of H.*

2.2. Three biset functors

In this section, we want to define three biset functors.

Definition 11. Let G be a finite group. Then $B(G)$ is the Grothendieck group of the set of isomorphism classes of finite G-sets (for disjoint union). Then $B(G)$ is a ring (called the Burnside ring of G), where the multiplication is defined by

$$[U] \cdot [V] = [U \times V]$$

for all G-sets U and V (extended to $B(G)$ by bilinearity).

Let G et H be two finite groups. For every (finite) (H,G)-biset U, we can define the following map:

$$\begin{aligned} B\big([U]\big):\quad & B(G) \to B(H), \\ & [V] \mapsto [U \times_G V] \end{aligned}$$

for every (finite) G-set V. This extends by $\mathbb{C}$-linearity to a map $\mathbb{C}B([U]) : \mathbb{C}B(G) \to \mathbb{C}B(H)$, where $\mathbb{C}B(G) = \mathbb{C} \otimes_{\mathbb{Z}} B(G)$.

Now we can define $\mathbb{C}B(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H,G)$. Let $u = \sum_{i=1}^{n} \lambda_i [U_i]$ where $\lambda_i \in \mathbb{C}$ and U_i is an (H,G)-biset, for every $i = 1,\dots,n$. Then $B(u) = \sum_{i=1}^{n} \lambda_i B([U_i])$. This defines a structure of biset functor $\mathbb{C}B$.

If G is a finite group, then $\{[G/K] \mid K \leqslant_G G\}$ is a $\mathbb{C}$-basis of $\mathbb{C}B(G)$. But in the rest of this article, we will use another $\mathbb{C}$-basis, which is the following:

Theorem 12. *(See Gluck [3], Yoshida [4].) Let G be a finite group. If H is a subgroup of G, denote by e_H^G the element of $\mathbb{C}B(G)$ defined by*

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K| \mu(K, H)[G/K],$$

where μ is the Möbius function of the poset of subgroups of G.

Then $e_H^G = e_K^G$ if the subgroups H and K are conjugate in G, and the elements e_H^G, for $H \leqslant_G G$, are the primitive idempotents of the $\mathbb{C}$-algebra $\mathbb{C}B(G)$.

In particular $\{e_H^G \mid H \leqslant_G G\}$ is a $\mathbb{C}$-basis of $\mathbb{C}B(G)$.

In the next two definitions, the Grothendieck groups are taken with respect to direct sums, that is, with respect to the relations $[M \oplus N] - [M] - [N]$.

Definition 13. We define $\Pi_k(G)$ as the Grothendieck group of the set of isomorphism classes of permutation kG-modules with respect to direct sums. For every (H, G)-biset U we define:

$$\begin{aligned} \Pi_k([U]): \quad & \Pi_k(G) \to \Pi_k(H), \\ & [kP] \mapsto [kU \otimes_{kG} kP] = \big[k(U \times_G P)\big] \end{aligned}$$

for every permutation kG-module kP. As for B, we can extend scalars to $\mathbb{C}$ and define $\mathbb{C}\Pi_k(u) : \mathbb{C}\Pi_k(G) \to \mathbb{C}\Pi_k(H)$ for $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$, where $\mathbb{C}\Pi_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \Pi_k(G)$. This defines a structure of biset functor $\mathbb{C}\Pi_k$.

Definition 14. We define $\mathrm{pp}_k(G)$ as the Grothendieck group of the set of isomorphism classes of p-permutation kG-modules (i.e. direct sums of indecomposable trivial source kG-modules) with respect to direct sums. For every (H, G)-biset U we define:

$$\begin{aligned} \mathrm{pp}_k([U]): \quad & \mathrm{pp}_k(G) \to \mathrm{pp}_k(H), \\ & [M] \mapsto [kU \otimes_{kG} M] \end{aligned}$$

for every trivial source kG-module M. As for B and Π_k, we can extend scalars to $\mathbb{C}$ and define $\mathbb{C}\,\mathrm{pp}_k(u) : \mathbb{C}\,\mathrm{pp}_k(G) \to \mathbb{C}\,\mathrm{pp}_k(H)$ for $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$, where $\mathbb{C}\,\mathrm{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{pp}_k(G)$. This defines a structure of biset functor $\mathbb{C}\,\mathrm{pp}_k$.

Moreover, the functor $\mathbb{C}\Pi_k$ is a subfunctor of $\mathbb{C}pp_k$.

Remark 15. In the case $p = 0$, every kG-module is a p-permutation kG-module and $\mathbb{C}\,\mathrm{pp}_k(G)$ is the ordinary Grothendieck group of kG-modules.

2.3. The Burnside biset functor

We describe in this section the composition factors of the functor $\mathbb{C}B$.

Definition 16. (See [1], Notation 5.2.2 and Definition 5.4.6.) If N is a normal subgroup of G, we define the number $m_{G,N}$ by

$$m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X| \mu(X,G) \in \mathbb{Q},$$

where μ is the Möbius function of the poset of subgroups of G.

A finite group G is a *B-group* (over $\mathbb{C}$) if for every non-trivial normal subgroup N of G, we have $m_{G,N} = 0$.

We denote by B-gr($\mathcal{C}$) the class of all finite B-groups and by [B-gr($\mathcal{C}$)] a set of representatives of isomorphism classes of finite B-groups.

A subset $\mathcal{A}$ of [B-gr($\mathcal{C}$)] is *closed* if for $G \in \mathcal{A}$ and $H \in$ [B-gr($\mathcal{C}$)] with $H \gg G$, we have $H \in \mathcal{A}$.

Remark 17. (See [1], Example 5.2.3.) We have that $m_{G,N} = m_{G,N\Phi(G)}$, for all normal subgroups N of G. In particular, $m_{G,\Phi(G)} = m_{G,\mathbf{1}} = 1$. This implies that if G is a B-group, then the Frattini subgroup $\Phi(G)$ of G is trivial.

Definition 18. (See [1], Theorem 5.4.11.) Let G be a finite group. Then $\beta(G)$ is defined to be the quotient G/N of G, where N is a normal subgroup of G such that $m_{G,N} \neq 0$ and G/N is a B-group.

Remark 19. In the above definition, $\beta(G)$ is well defined, up to group isomorphism. But the normal subgroup N is in general not unique.

Notation 20. (See [1], Notation 5.4.3.) Let G be a finite group. Then $\mathbf{e}_G$ denote the subfunctor of $\mathbb{C}B$ generated by $e_G^G \in \mathbb{C}B(G)$, where e_G^G is the idempotent defined in Theorem 12.

Theorem 21. *(See [1], Proposition 5.5.1.)*

1. *Let G be a B-group. Then the subfunctor $\mathbf{e}_G$ of $\mathbb{C}B$ has a unique maximal subfunctor, equal to*

$$\mathbf{j}_G = \sum_{\substack{H \in [\text{B-gr}(\mathcal{C})] \\ H \gg G,\, H \ncong G}} \mathbf{e}_H,$$

   and the quotient $\mathbf{e}_G/\mathbf{j}_G$ is isomorphic to the simple functor $S_{G,\mathbb{C}}$.
2. *If $F \subseteq F'$ are subfunctors of $\mathbb{C}B$ such that F'/F is simple, then there exists a unique $G \in$* [B-gr($\mathcal{C}$)] *such that $\mathbf{e}_G \subseteq F'$ and $\mathbf{e}_G \nsubseteq F$. In particular, $\mathbf{e}_G + F = F'$, $\mathbf{e}_G \cap F = \mathbf{j}_G$, and $F'/F \cong S_{G,\mathbb{C}}$.*

Remark 22. (See [1], Remark 5.5.2.) The "composition factors" (i.e. the simple subquotients) of the Burnside functor $\mathbb{C}B$ on $\mathcal{C}$ are exactly the functors $S_{G,\mathbb{C}}$, where G is an object of $\mathcal{C}$ (i.e. a finite group) which is a B-group.

Theorem 23. *There is an isomorphism of lattices between the poset of subfunctors of $\mathbb{C}B$ and the poset of closed subsets of* [B-gr($\mathcal{C}$)].

Proof. This bijection is a consequence of Theorem 5.4.14 and Proposition 5.5.3 of [1]. □

To be more precise, here is a description of this bijection: Let $\mathcal{A}$ be a closed subset of [B-gr($\mathcal{C}$)]. We want to define the subfunctor $F_{\mathcal{A}}$ of $\mathbb{C}B$ associated to the set $\mathcal{A}$. We set $\mathcal{B} = \mathcal{B}_{\mathcal{A}}$ by

$$\mathcal{B} = \{G \in \mathcal{C} \mid \beta(G) \in \mathcal{A}\} = \{G \in \mathcal{C} \mid \exists\, H \in \mathcal{A},\ G \gg H\}.$$

Then, for every group G, we have

$$F_{\mathcal{A}}(G) = \bigoplus_{\substack{H \leqslant_G G \\ H \in \mathcal{B}}} \mathbb{C} e_H^G = \bigoplus_{\substack{H \leqslant_G G \\ \beta(H) \in \mathcal{A}}} \mathbb{C} e_H^G.$$

More precisely, the set $\{e_H^G \mid H \leqslant_G G, H \in \mathcal{B}\} = \{e_H^G \mid H \leqslant_G G, \beta(H) \in \mathcal{A}\}$ is a $\mathbb{C}$-basis of $F_{\mathcal{A}}(G)$.

Conversely, if F is a subfunctor of $\mathbb{C}B$, we define the associated closed subset $\mathcal{A}$ of [B-gr($\mathcal{C}$)] by

$$\mathcal{A} = \left\{ H \in \left[\text{B-gr}(\mathcal{C})\right] \;\middle|\; e_H^H \in F(H) \right\}.$$

Remark that the functor $\mathbb{C}B$ corresponds to the set [B-gr($\mathcal{C}$)].

3. The biset functor of permutation modules

We want to construct a morphism between $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ and then use this morphism and the composition factors of $\mathbb{C}B$ to find those of $\mathbb{C}\Pi_k$.

Proposition 24. *There is a morphism of biset functor (i.e. a $\mathbb{C}$-linear natural transformation) θ between $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ such that*

$$\theta_G\big([G/L]\big) = \big[k(G/L)\big]$$

for all finite group G and every subgroup L of G, where we denote by θ_G the map $\theta(G) : \mathbb{C}B(G) \to \mathbb{C}\Pi_k(G)$ and $[G/L]$, $[k(G/L)]$ are the isomorphism classes of the G-set G/L and the kG-module $k(G/L)$, respectively.

Proof. We extend by $\mathbb{C}$-linearity the definition of θ_G to $\mathbb{C}B(G)$ and then it is easy to check that θ is well defined and a $\mathbb{C}$-linear natural transformation. $\square$

Remark 25. We know that $(\operatorname{Coker}\theta)(G) = \operatorname{Coker}\theta_G$ for all finite group G (Proposition 9), consequently the image of the natural transformation θ is $\mathbb{C}\Pi_k$.

Definition 26. A group H is said to be *p-hypo-elementary* (or *cyclic modulo p*) if the quotient $H/O_p(H)$ is cyclic ($O_p(H)$ is the largest normal p-subgroup of G); in other words, H has a normal p-subgroup for which the quotient is a cyclic p'-group.

If $p = 0$, a 0-*hypo-elementary group* is a cyclic group.

We denote by $\mathcal{H}$ the set of all finite p-hypo-elementary groups.

The aim now is to determine the kernel of θ. We will use the fact that $(\operatorname{Ker}\theta)(G) = \operatorname{Ker}\theta_G$ (Proposition 9) and that the set $\{e_H^G \mid H \leqslant_G G\}$ is a basis of $\mathbb{C}B(G)$ for all finite groups G. To do this, we need the following lemma, due to Conlon. We denote by $\bar{k}$ the algebraic closure of k.

Lemma 27. *Let G be a finite group and E be the set of conjugacy classes of pairs (H, g), where H is a p-hypo-elementary subgroup of G and g a generator of $H/O_p(H)$. Then we have an isomorphism*

$$\mathbb{C}\,\mathrm{pp}_{\bar{k}}(G) \cong \bigoplus_{(H,g)\in E} \mathbb{C}.$$

Moreover, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{C}B(G) \arrow[r, "\cong"] \arrow[d, "\theta_G"] & \bigoplus_{H\leqslant_G G}\mathbb{C} \arrow[d, "\lambda"] \\ \mathbb{C}\,\mathrm{pp}_{\bar{k}}(G) \arrow[r, "\cong"] & \bigoplus_{(H,g)\in E}\mathbb{C} \end{tikzcd}$$

We still write e_H^G for the primitive idempotent in $\bigoplus_{H\leqslant_G G}\mathbb{C}$ which is the image of the primitive idempotent $e_H^G \in \mathbb{C}B(G)$. We write $\varepsilon_{H,g}$ for the primitive idempotents in $\bigoplus_{(H,g)\in E}\mathbb{C}$. The map λ sends e_H^G to $\sum_{(H,g)\in E}\varepsilon_{H,g}$ if H is p-hypo-elementary, and zero otherwise.

Remark 28. The map θ_G is defined between $\mathbb{C}B(G)$ and $\mathbb{C}\Pi_{\bar{k}}(G)$, but as $\mathbb{C}\Pi_{\bar{k}}$ is a subfunctor of $\mathbb{C}\,\mathrm{pp}_{\bar{k}}$, we can extend it to $\mathbb{C}\,\mathrm{pp}_{\bar{k}}$ by composing with the inclusion.

Proof. A proof of this result in characteristic $p \neq 0$ can be found in [5], p. 188. The case $p = 0$ is completely straightforward. □

Proposition 29. *Let G be a finite group. The set*

$$B_{\mathrm{Ker}} = \{e_H^G \mid H \leqslant_G G,\ H \notin \mathcal{H}\}$$

is a basis of $\mathrm{Ker}\,\theta_G$*. Moreover, the set*

$$B_{\mathrm{Im}} = \{\theta(e_H^G) \mid H \leqslant_G G,\ H \in \mathcal{H}\}$$

is a basis of $\mathrm{Im}\,\theta_G$*.*

Proof. We considered the following composition of applications

$$\mathbb{C}B(G) \to \mathbb{C}\Pi_k(G) \xrightarrow{f} \mathbb{C}\Pi_{\bar{k}}(G) \xrightarrow{\iota} \mathbb{C}\,\mathrm{pp}_{\bar{k}}(G),$$

where f is the scalar extension from k to $\bar{k}$ and ι is the inclusion. Clearly, the map ι is injective. Let M and N be two permutation kG-modules such that $f([M]) = f([N])$, that is $\bar{k}\otimes_k M \cong \bar{k}\otimes_k N$. As M and N are finitely generated, this implies that $L\otimes_k M \cong L\otimes_k N$, for some finite dimensional field extension L of k. But, by Exercise 2, p. 138 of [6], this implies that $M \cong N$. So, we have that f is also injective.

Now the composition map $g = \iota\circ f\circ\theta_G$ is exactly the map θ_G of Lemma 27, and so B_{Ker} is a subset of $\mathrm{Ker}\,g$ and the set $\{g(e_H^G) \mid H \leqslant_G G, H \in \mathcal{H}\}$ is linearly independent. As $\iota\circ f$ is injective, this implies that B_{Ker} is a subset of $\mathrm{Ker}\,\theta_G$ and the set B_{Im} is linearly independent. We already know that B_{Ker} is a linearly independent set. If n is the number of conjugacy classes of subgroups of G, we have that:

$$n = \dim_{\mathbb{C}}\mathbb{C}B(G) = \dim_{\mathbb{C}}\mathrm{Ker}\,\theta_G + \dim_{\mathbb{C}}\mathrm{Im}\,\theta_G \geqslant |B_{\mathrm{Ker}}| + |B_{\mathrm{Im}}| = n$$

and so we must have equality, which proves the result. □

We now have a basis of $\mathrm{Ker}\,\theta_G$ and $\mathrm{Im}\,\theta_G$ for every finite group G and we will use this to study the image of the composition factors of $\mathbb{C}B$.

Let G be a finite B-group. If G is not a p-hypo-elementary group, then $e_G^G \in \mathrm{Ker}\,\theta_G$, hence $\mathbf{e}_G \subseteq \mathrm{Ker}\,\theta$. So we can assume that G is a p-hypo-elementary group. Now the functor $\theta(\mathbf{e}_G)/\theta(\mathbf{j}_G)$ is a quotient of the simple functor $\mathbf{e}_G/\mathbf{j}_G \cong S_{G,\mathbb{C}}$. Hence it is either 0 or isomorphic to $S_{G,\mathbb{C}}$. It is zero if and only if $\theta(\mathbf{e}_G) = \theta(\mathbf{j}_G)$, i.e. if $\mathbf{e}_G \subseteq \mathbf{j}_G + \operatorname{Ker}\theta$. But $e_G^G \in \mathbf{e}_G(G)$, and $e_G^G \notin \mathbf{j}_G(G) + \operatorname{Ker}\theta_G$, as $G \in \mathcal{H}$. Hence $\theta(\mathbf{e}_G) \neq \theta(\mathbf{j}_G)$ and $\theta(\mathbf{e}_G)/\theta(\mathbf{j}_G) \cong S_{G,\mathbb{C}}$.

Now we found the image of every composition functor of $\mathbb{C}B$ in $\mathbb{C}\Pi_k$ and as every composition factor has a preimage in $\mathbb{C}B$, this gives us the complete list of composition factors of $\mathbb{C}\Pi_k$. So we have proved the following theorem.

Theorem 30. *The composition factors of the functor $\mathbb{C}\Pi_k$ are the simple functors $S_{H,\mathbb{C}}$, where H is a p-hypo-elementary B-group and where $\mathbb{C}$ is the trivial $\mathbb{C}\operatorname{Out}(H)$-module.*

Remarks 31.

1. With the same method, we can find an infinite sequence of subfunctors of $\mathbb{C}\Pi_k$ such that every successive quotient is simple. But to do this we need to make a choice. This sequence is finite if we evaluate it in a finite group.
2. With the same method, we can find the description of all the subfunctors of $\mathbb{C}\Pi_k$. We use the description of the subfunctors of $\mathbb{C}B$ and we obtain a bijection between the subfunctor of $\mathbb{C}\Pi_k$ and the closed subset of $[\text{B-gr}(\mathcal{C})] \cap \mathcal{H}$.
3. If $p = 0$, the unique B-group which is cyclic is $\mathbf{1}$ so we obtain that $\mathbb{C}\Pi_k \cong S_{\mathbf{1},\mathbb{C}}$, which is Proposition 4.4.8 of [1].

Moreover, the above proof implies the next theorem:

Theorem 32. *Let G be a finite group. Then $\beta(G)$ is p-hypo-elementary if and only if G itself is p-hypo-elementary.*

Proof. Let G be a finite group and H be a subgroup of G. By Corollary 29, we know that e_H^G is an element of $\ker\theta_G$ if and only if $H \notin \mathcal{H}$.

On the other hand, using the bijection between the subfunctors of $\mathbb{C}B(G)$ and the closed subsets of $[\text{B-gr}(\mathcal{C})]$, we find that the kernel $\operatorname{Ker}\theta$ corresponds to the set $\mathcal{N} = [\text{B-gr}(\mathcal{C})] \cap \{G \mid G \notin \mathcal{H}\}$. But this implies that e_H^G is in $\ker\theta_G$ if and only if there exists $L \in \mathcal{N}$ such that $H \gg L$, which is equivalent to $\beta(H) \gg L$. But this implies that $e_H^G \in \ker\theta_G$ if and only if $\beta(H) \in \mathcal{N}$ (because $\mathcal{N}$ is closed), that is, if and only if $\beta(H) \notin \mathcal{H}$.

If we put together those two results, we obtain that $H \notin \mathcal{H}$ if and only if $\beta(H) \notin \mathcal{H}$, which proves that $H \in \mathcal{H}$ if and only if $\beta(H) \in \mathcal{H}$. □

4. B-groups and p-hypo-elementary groups

Now, to make precise Theorem 30, we want to find which p-hypo-elementary groups are also B-groups. Recall that the rational number $m_{G,N}$ is defined in Definition 16.

Proposition 33. *(See [1], Proposition 5.6.4.) If N is a minimal normal abelian subgroup of G, then*

$$m_{G,N} = 1 - \frac{|K_G(N)|}{|N|}$$

where $K_G(N)$ is the set of complements of N in G.

In particular, if the group G is solvable, then G is a B-group if and only if $|K_G(N)| = |N|$ for all minimal normal subgroups N of G.

Remark 34. As all p-hypo-elementary groups are solvable, we will use the second part of this proposition to determine which p-hypo-elementary groups are B-groups.

The aim is to find necessary and sufficient conditions for a p-hypo-elementary group to be a B-group. In order to do this, we need some results that can be stated in a more general case. We suppose that $G = P \rtimes H$ where P is a p-group of order p^m and H a p'-group of order n. This notation is kept throughout this section.

Proposition 35. *The Frattini subgroup $\Phi(G)$ contains $\Phi(P)$.*

Proof. Let M be a maximal subgroup of G. We have to show that $\Phi(P)$ is contained in M. Clearly, P is the unique p-Sylow subgroup of G, consequently P is the set of p-elements of G. So if we set $S = M \cap P$, then S is a normal p-Sylow subgroup of M. If $S = P$, then it is clear that $\Phi(P) \subseteq P \subseteq M$, so we can suppose that $S \neq P$. As $\Phi(P)$ is a characteristic subgroup of P and P is a normal subgroup of G, G normalizes $\Phi(P)$. As a consequence, $\Phi(P) \cdot S$ and $\Phi(P) \cdot M$ are subgroups. We now have the following diagram:

$$\begin{tikzcd}
 & G \arrow[dl, no head] \arrow[dr, no head] & & \\
P \arrow[dr, no head] & & \Phi(P) \cdot M \arrow[dl, no head] \arrow[dr, no head] & \\
 & \Phi(P) \cdot S \arrow[dl, no head] \arrow[dr, no head] & & M \arrow[dl, no head] \\
\Phi(P) & & S &
\end{tikzcd}$$

As M is maximal we have that $\Phi(P) \cdot M$ is either M or G. But

$$\big(\Phi(P) \cdot M\big) \cap P = \Phi(P) \cdot S \neq P$$

(because $S \neq P$) so $\Phi(P) \cdot M$ must be M, which implies that $\Phi(P) \subseteq M$. $\square$

Corollary 36. *If G is a B-group, then P is an elementary abelian group.*

Proof. This is a consequence of Proposition 35 and Remark 17. $\square$

So we suppose now that P is an elementary abelian group. So P is an $\mathbb{F}_p$-vector space on which H acts, namely an $\mathbb{F}_p H$-module. As H is a p'-group, P is a semi-simple $\mathbb{F}_p H$-module.

Proposition 37. *Suppose that H is a cyclic group of order n. If G is a B-group, then the group H acts faithfully on P.*

Proof. The case $n = 1$ is clear, so we can suppose that $n \geqslant 2$. Let d be the divisor of n such that $\operatorname{Ker}\varphi = C_d$, where $\varphi : H \to \operatorname{Aut}(P)$ is the action of H on P. To show that the action is faithful, we have to show that $d = 1$.

So we suppose that $d > 1$. As C_d acts trivially on P and H is abelian, C_d is a central subgroup of G, and so in particular a normal subgroup. As $d > 1$, there exists a minimal normal subgroup N of C_d (it is even central). But then N has at most one complement in G: If C is a complement of N in G, then C contains the unique Sylow p-subgroup P of G. Consequently, we have that $C = P \rtimes L$, where L is a subgroup of H. But then L is a complement of N in H, which is cyclic, so there is at most one possibility for L.

But N should have $|N| > 1$ complements in G because G is a B-group (Proposition 33). So we must have that $d = 1$, that is, the action is faithful. □

Lemma 38. *Let G be a finite group and let H, K and L be subgroups of G such that $H \leqslant K \leqslant H \cdot L$. Then $K = H(K \cap L)$.*

Proof. Clear. □

Proposition 39. *If H acts faithfully on P, then a minimal normal subgroup N of G is always contained in P.*

Proof. Let N be a minimal normal (non-trivial) subgroup of G. If $N \cap P \neq 1$ then by minimality of N, we have that $N \subseteq P$. We can now suppose that $N \cap P = 1$. Then, as N and P are normal in G, $[N, P] = 1$, which implies that $N \subseteq C_G(P)$.

We have that $P \leqslant C_G(P) \leqslant P \cdot H$ so by Lemma 38 we have that $C_G(P) = P \cdot (C_G(P) \cap H) = P \cdot C_H(P)$. But as H acts faithfully on P, $C_H(P) = 1$ so that $N \subseteq C_G(P) = P$. This is impossible because this implies that $N = 1$, which contradicts the assumption on N. □

Proposition 40. *Let N be a normal subgroup of G contained in P. Then every complement of N is of the form $S \rtimes Q$, where S is a normal subgroup of G which is a complement of N in P and Q is a subgroup of G conjugate to H.*

Proof. Let C be a complement of N in G (i.e. $N \cap C = 1$ and $N \cdot C = G$). We define $S = C \cap P$, which is a normal p-Sylow subgroup of C. By the Schur–Zassenhaus theorem [7, Theorem 7.41], there exists a subgroup Q of C such that $C = S \rtimes Q$. Notice that the order of Q is n so that Q is conjugate to H (by the second part of the Schur–Zassenhaus theorem [7], Theorem 7.42). Now S is normal in C and in P (which is abelian) so also in $G = P \rtimes Q$.

Conversely, let $C = S \rtimes Q$ such that S is a normal subgroup of G which is a complement of N in P and Q is a subgroup of G of order n (hence conjugate to H). Clearly, we have that $N \cap C = N \cap S = 1$ and $N \cdot C = N \cdot (S \rtimes Q) = (N \cdot S) \rtimes Q = P \rtimes Q = G$, which proves that C is a complement of N in G. □

Remark 41. We will need those results in the following case: If G is a p-hypo-elementary group (i.e. H is cyclic) and a B-group, then, by Proposition 37, the action of H on P is faithful. Let N be a minimal normal subgroup of G. Then, by Proposition 39, N is contained in P and Proposition 40 applies.

For the rest of this part, $G = P \rtimes H$ is a p-hypo-elementary group. This means that H is a cyclic group. We suppose that it is a B-group, so we know that P is elementary abelian, P is a semi-simple $\mathbb{F}_p H$-module and H acts faithfully on P. We now decompose P into his isotypic components (see [6], pp. 46–47 for a definition):

$$P \cong \bigoplus_{i=1}^{t} P_i$$

and for each isotypic component, there exists a simple $\mathbb{F}_p H$-module S_i and an integer m_i such that

$$P_i \cong \bigoplus_{j=1}^{m_i} S_i.$$

We can suppose that S_1 is the trivial $\mathbb{F}_p H$-module (if necessary, we add $P_1 = \{0\}$). For each $1 \leqslant i \leqslant t$, there exists an integer s_i such that $|S_i| = p^{s_i}$.

We are now able to find the number of complements of a minimal normal subgroup N of G and find the necessary and sufficient condition such that this number is $|N|$.

Let N be a minimal normal subgroup of G. By Proposition 39, N is a subgroup of P on which H acts, i.e. an $\mathbb{F}_pH$-submodule of P. Furthermore, the minimality of N implies that N is simple. So there exists an integer $1 \leqslant l \leqslant t$ such that $N \cong S_l$. Then N is a submodule of P_l. We know by Proposition 40 that a complement of N in G is of the form $C \rtimes Q$ where C is a normal subgroup of G which is a complement of N in P and Q is a cyclic subgroup of G of order n. So C is a complement of N in P not only as a group but also as a module. Our first step will be to determine the number of possibilities for C. In order to do this, we use the fact that the isotypic components are unique up to isomorphism [6, pp. 46–47] which implies that a complement of N in P (as a module) is of the form

$$H_l \oplus \bigoplus_{\substack{i=1\\ i\neq l}}^{t} P_i$$

where H_l is a complement (as a module) of N in P_l.

By using the isomorphism $\mathbb{F}_pH \cong \mathbb{F}_p[t]/\langle t^n\rangle$ and the Chinese remainder theorem, we can see that the simple $\mathbb{F}_pH$-modules can be seen as fields and this field structure contains the structure of $\mathbb{F}_pH$-module. So we can see $N \cong S_l$ as the field $\mathbb{F}_{p^{s_l}}$ and the module P_l as a vector space over this field. Moreover, in order to find the number of complements of N in P_l, it is sufficient to count the number of complements as $\mathbb{F}_{p^{s_l}}$-vector space. This is an easy exercise: N has

$$p^{s_l(m_l-1)}$$

complements in P_l and so also in P.

As every complement of N is of the form $C \rtimes Q$ and we know the number of possibilities for C, we only need to find the number of possibilities for Q (when C is fixed). In fact, this is equal to the number of complements of P in G divided by the number of complements of C in $C \rtimes Q$, where Q is an arbitrary subgroup of G conjugate to H. In order to make this calculation, we need the next lemma:

Lemma 42. *The number of complements of P in G is equal to p^{m-m_1}.*

Proof. We have to find the number of conjugates of H. Let E be the set of conjugates of H. By the Schur–Zassenhaus theorem [7, Theorem 7.41], we know that P acts transitively on E. Let S be the stabilizer of H in P. Then the number of conjugates of H is equal to $|P|/|S|$. So, in order to conclude, we have to determine the order of S. After some easy calculation, we can find that $S = P_1$ and consequently the number of conjugates of H is p^{m-m_1}. □

This gives us the two numbers we have to divide: The number of complements of P in G is p^{m-m_1} and the number of complements of C in $C \rtimes Q$ for a fixed Q is:

$$\begin{cases} p^{m-s_1-(m_1-1)} = p^{m-m_1} & \text{if } l = 1, \\ p^{m-s_l-m_1} & \text{if } l \neq 1. \end{cases}$$

So the number of possibilities for Q is

$$\begin{cases} 1 & \text{if } l = 1, \\ p^{s_l} & \text{if } l \neq 1. \end{cases}$$

To conclude, the number of complements of N in G is:

$$\begin{cases} p^{m_1-1} & \text{if } l=1, \\ p^{s_l m_l} & \text{if } l \neq 1. \end{cases}$$

But for G to be a B-group, we must have that the number of complements of N is $|N| = p^{s_l}$. So if $l=1$, $m_1 = 2$ (or $m_1 = 0$ if $P_1 = 0$) and if $l \neq 1$ then $m_l = 1$, which concludes the proof of the following theorem:

Theorem 43. *Let $G \cong P \rtimes C_n$ be a p-hypo-elementary group (P is a p-group and C_n a cyclic p'-group). Then G is a B-group if and only if:*

(i) *P is elementary abelian*;
(ii) *In a decomposition of P as a direct sum of simple $\mathbb{F}_p C_n$-modules, every simple $\mathbb{F}_p C_n$-module appears at most one time, except the trivial module, which appears* 0 *or* 2 *times*;
(iii) *The action of C_n on P is faithful.*

Remarks 44.

1. The above theorem can be generalized to other groups ($P \rtimes H$, where P is a p-group and H a solvable p'-group). For more details, see [8].
2. If $p = 0$, we have proved that the only cyclic B-group is **1**. So we recover a known result given in [2], Example 7.2.5.

Acknowledgment

We thank the referee for useful suggestions.

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