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Quasi-abelian crossed modules and nonabelian cohomology

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ABSTRACT

We extend the work of M. Borovoi on the nonabelian Galois cohomology of linear reductive algebraic groups over number fields to a general base scheme. As an application, we obtain new results on the arithmetic of such groups over global function fields.

1. Introduction

Let G be a linear reductive algebraic group over a number field K. In [1,2], M. Borovoi studied the Galois cohomology sets $H^i(K, G)$ $(i = 1, 2)$ by relating them to certain "abelian cohomology groups" $H^i_{\mathrm{ab}}(K, G)$. In particular, it was shown in [1] that there exists a *surjective* abelianization map $\mathrm{ab}^1 : H^1(K, G) \to H^1_{\mathrm{ab}}(K, G)$. Later, in [3, Proposition 6.6], a new proof of this fact was given which suggested to the author the existence of a prolongation, at least for certain types of crossed modules, of L. Breen's fundamental exact sequence [4, (4.2.2)] (reproduced here as Proposition 2.4). It was expected that such an extension of Breen's sequence would provide, via work of J.-C. Douai, a common explanation for the validity of both [3, Proposition 6.5], and [2, Theorem 5.5]. The purpose of this paper is to establish such a prolongation of Breen's sequence over any base scheme S, thus

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confirming the above expectation (see Theorem 1.1 below). In particular, we extend Borovoi's abelian cohomology theory to this general setting. To be precise, let S be any scheme, let S_{fl} be the small fppf site over S and let G be a reductive group scheme over S. Write G^{der} for the derived group of G, $\widetilde{G}$ for the simply-connected central cover of G^{der} and μ for the fundamental group of G^{der}. Then G defines a crossed module $(\widetilde{G} \to G)$ on S_{fl} which is quasi-isomorphic to the abelian crossed module $(Z(\widetilde{G}) \to Z(G))$ (thus it is *quasi-abelian*, in the sense of Definition 3.2). The abelian (flat) cohomology groups $H^i_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$ of G are by definition the fppf hypercohomology groups $\mathbb{H}^i(S_{\mathrm{fl}}, Z(\widetilde{G}) \to Z(G))$. Now let $H^2(S_{\mathrm{fl}}, \widetilde{G})$ be the second cohomology set introduced by J. Giraud in [21, Chapter IV]. It contains a distinguished element $\varepsilon_{\widetilde{G}}$, the so-called *unit class*, and a subset $H^2(S_{\mathrm{fl}}, \widetilde{G})'$ of *neutral classes* containing $\varepsilon_{\widetilde{G}}$. We regard $H^2(S_{\mathrm{fl}}, \widetilde{G})$ as a pointed set with basepoint $\varepsilon_{\widetilde{G}}$. Analogous definitions apply to $H^2(S_{\mathrm{fl}}, G)$. Then, by using ideas of L. Breen [4], J. Giraud [21] and M. Borovoi [2], we are able to define abelianization maps $\mathrm{ab}^i : H^i(S_{\mathrm{fl}}, G) \to H^i_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$ for $i = 0, 1, 2$ (which generalize those defined in [1,2]) so that the following theorem holds. To simplify the notation, let $H^i(G)$ and $H^i_{\mathrm{ab}}(G)$ denote $H^i(S_{\mathrm{fl}}, G)$ and $H^i_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$, respectively.

Theorem 1.1. *Let G be a reductive group scheme over a scheme S. Then there exists a sequence of flat (fppf) cohomology sets*

$$\begin{aligned} 1 \to \mu(S) \to \widetilde{G}(S) \to G(S) \xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(G) \to H^1(\widetilde{G}) \to H^1(G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(G) \xrightarrow{\delta_1} H^2(\widetilde{G}) \\ \to H^2(G) \xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(G) \to H^3\big(Z(\widetilde{G})\big) \to H^3\big(Z(G)\big) \to \cdots, \end{aligned}$$

which is an exact sequence of pointed sets at every term except $H^1_{\mathrm{ab}}(G)$, where a class $y \in H^1_{\mathrm{ab}}(G)$ is in the image of ab^1 if, and only if, $\delta_1(y) \in H^2(\widetilde{G})'$.

The above theorem is, in fact, a corollary of Theorem 4.2, which is the main theorem of the paper. When S is the spectrum of a number field K, the theorem shows, as expected, that one and the same fact underlie the validity of both [3, Proposition 6.6], and [2, Theorem 5.5]. Namely, that all classes in $H^2(K_{\mathrm{fl}}, \widetilde{G})$ are neutral. See Section 5.

The theorem yields the following *integral* version of [1, Theorem 5.7].

Corollary 1.2. *Let G be a reductive group scheme over the spectrum S of the ring of integers of a number field. Then the first abelianization map $\mathrm{ab}^1 : H^1(S_{\mathrm{fl}}, G) \to H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$ is surjective.*

The following is a brief summary of the paper. In Section 2 we review basic facts from Breen's nonabelian cohomology theory of crossed modules. In Section 3 we introduce quasi-abelian crossed modules and establish a part of the sequence appearing in Theorem 4.2. The basic reference for this section is the Giraud–Grothendieck nonabelian cohomology theory [21]. In Section 4, following M. Borovoi, we define the map ab^2 and obtain the latter part of the sequence appearing in Theorem 4.2. In Section 5, which concludes the paper, we discuss applications of Theorem 1.1 to linear reductive algebraic groups over certain types of fields, especially of positive characteristic. Additional applications are discussed in [22].

2. Preliminaries

Let E be a site and let $\widetilde{E}$ be the topos defined by E. We begin by recalling the basic properties of the nonabelian cohomology theory of crossed modules developed in [4].[2]

[2] In contrast to [4], we work with left crossed modules and right torsors throughout. See [4, p. 416], for the equivalence of both approaches. Thus references to results from [4] below are, in fact, to their opposite-hand versions.

Definition 2.1. A (left) crossed module on E consists of a homomorphism $\partial: F \to G$ of groups of $\widetilde{E}$ together with a left action of G on F, denoted by $(g, f) \mapsto {}^g f$, such that

$$\partial\big({}^g f\big) = g\partial(f)g^{-1}$$

and

$${}^{\partial(f)} f' = ff'f^{-1},$$

for every $g \in G$ and $f, f' \in F$.

The definition immediately implies that $\operatorname{Ker}\partial$ is central in F and $\operatorname{Im}\partial$ is normal in G. Further, ∂ is a G-homomorphism for the given action of G on F and the left action of G on itself via inner automorphisms.

Crossed modules will be regarded as complexes of length two $(F \xrightarrow{\partial} G)$, with F and G placed in degrees -1 and 0, respectively.

Examples 2.2.

(i) If $\partial: F \to G$ is a homomorphism of abelian groups of $\widetilde{E}$ and G acts trivially on F, then $(F \xrightarrow{\partial} G)$ is a crossed module. Such crossed modules are called *abelian*.

(ii) If G is a group of $\widetilde{E}$, F is a normal subgroup of G and G acts on F by conjugation, then $(F \hookrightarrow G)$ is a crossed module.

(iii) Let S be a scheme and let E be a standard site on S, i.e., every representable presheaf on E is a sheaf and finite fibered products exist in E. Examples include the small fppf and étale sites over S [10, Proposition IV.6.3.1(iii)]. Let G be a reductive S-group scheme with derived group G^{der}, let $\widetilde{G}$ be the simply-connected central cover of G^{der} and let $\partial: \widetilde{G} \to G$ be the composition $\widetilde{G} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G$ (see [23, §4], for the existence and basic properties of $\widetilde{G}$). Then there exists a canonical "conjugation" action of G on $\widetilde{G}$ so that the complex $(\widetilde{G} \xrightarrow{\partial} G)$ of representable sheaves on E is a crossed module on E. See [5, Example 1.9, p. 28]. Further, if $Z(G)$ denotes the center of G, then the induced action of $Z(G)$ on $\widetilde{G}$ is trivial.

Let $(F \xrightarrow{\partial} G)$ be a crossed module. For $i = -1, 0, 1$, let $H^i(E, F \to G)$ be the sets $H^i(\widetilde{E}, F \to G^{\mathrm{o}})$ defined in [4, p. 426], where G^{o} is the opposite group of G [21, Definition III.1.1.3, p. 106]. If $(F \to G)$ is an *abelian* crossed module (see Example 2.2(i)), the pointed sets $H^i(E, F \to G)$ coincide with the usual flat hypercohomology groups of the complex of abelian groups $(F \to G)$. Further, if G is a group of $\widetilde{E}$, then the pointed sets $H^i(E, 1 \to G)$, where $i = 0, 1$, agree with the usual cohomology sets $H^i(E, G)$. See [4, p. 427, line 3]. In particular, $H^1(E, 1 \to G) = H^1(E, G)$ is the set of isomorphism classes of right G-torsors on E. We have [4, (4.2.1)],

$$H^{-1}(E, F \xrightarrow{\partial} G) = H^0(E, \operatorname{Ker}\partial), \tag{2.1}$$

which is an abelian group. Further, $H^0(E, F \to G)$ is the set of isomorphism classes of "(F, G)-torsors", i.e., pairs (Q, t) where Q is an F-bitorsor and $t: Q \to G$ is an F-equivariant map for the right action of F on G via ∂. This set is naturally equipped with a group structure given by the contracted product of F-bitorsors. See [4, p. 432]. In order to describe $H^1(E, F \to G)$, we first note that the crossed module $(F \to G)$ is functorially associated to the opposite $\mathcal{C}$ of the gr-stack of (F, G)-torsors on E. See [4, Theorem 4.6, p. 433]. Then, by [4, Theorem 6.2, p. 440], $H^1(E, F \to G)$ is in natural bijection with the set $H^1(\mathcal{C})$ of equivalence classes of (right) $\mathcal{C}$-torsors on E. The sets thus defined behave functorially with respect to inverse images, i.e., if $u: E' \to E$ is a morphism of sites and $i = 0, 1$, then there exist maps

$$h^i(u, F \to G): H^i(E, F \to G) \to H^i\big(E', u^*F \to u^*G\big) \tag{2.2}$$

which coincide with those defined in [21, Proposition V.1.5.1, p. 316], when $(F \to G) = (1 \to G)$. See [4, §6]. For ease of notation, we will sometimes write $H^i(G)$ and $H^i(F \to G)$ for $H^i(E, G)$ and $H^i(E, F \to G)$, respectively.

A morphism of crossed modules $(F_1 \xrightarrow{\partial_1} G_1) \to (F_2 \xrightarrow{\partial_2} G_2)$ (see [4, p. 416], for the definition) is called a *quasi-isomorphism* if the induced group homomorphisms $\operatorname{Ker}\partial_1 \to \operatorname{Ker}\partial_2$ and $\operatorname{Coker}\partial_1 \to \operatorname{Coker}\partial_2$ are isomorphisms. Since quasi-isomorphic crossed modules define equivalent gr-stacks (cf. [25, XVIII, 1.4.12]), a quasi-isomorphism of crossed modules $(F_1 \to G_1) \to (F_2 \to G_2)$ induces bijections

$$H^i(E, F_1 \to G_1) \xrightarrow{\sim} H^i(E, F_2 \to G_2),$$

for $i = -1, 0, 1$.

Examples 2.3.

(i) Let $(F \xrightarrow{\partial} G)$ be a crossed module on E such that ∂ is *injective*. Then $(F \xrightarrow{\partial} G)$ is quasi-isomorphic to $(1 \to \operatorname{Coker}\partial)$ and there exist canonical bijections

$$H^i(E, F \xrightarrow{\partial} G) \simeq H^i(E, \operatorname{Coker}\partial)$$

for $i = 0, 1$.

(ii) Let $(F \xrightarrow{\partial} G)$ be a crossed module on E such that ∂ is *surjective*. Then there exists a quasi-isomorphism $(\operatorname{Ker}\partial \to 1) \to (F \xrightarrow{\partial} G)$ which induces bijections

$$H^i(E, F \xrightarrow{\partial} G) \xrightarrow{\sim} H^{i+1}(E, \operatorname{Ker}\partial),$$

where $i = -1, 0, 1$. Note that $H^2(E, \operatorname{Ker}\partial)$ is the usual second cohomology group of the abelian group $\operatorname{Ker}\partial$ of $\widetilde{E}$.

Let $(F \xrightarrow{\partial} G)$ be a crossed module as above. Then ∂ induces a map $\partial^{(1)} : H^1(F) \to H^1(G)$ which maps the class of an F-torsor Q to the class of the G-torsor $Q \wedge^F G$ [21, Proposition III.1.3.6, p. 116].

Proposition 2.4. *Let $(F \xrightarrow{\partial} G)$ be a crossed module of E. Then there exists an exact sequence of pointed sets*

$$1 \to H^{-1}(F \to G) \to H^0(F) \to H^0(G) \xrightarrow{\psi_0} H^0(F \to G) \xrightarrow{\delta'_0} H^1(F) \xrightarrow{\partial^{(1)}} H^1(G) \xrightarrow{\psi_1} H^1(F \to G),$$

where the maps ψ_i, $i = 0, 1$, are induced by the embedding of crossed modules $(1 \to G) \hookrightarrow (F \to G)$ and the map δ'_0 is defined below.

Proof. See [4, (4.2.2)]. $\square$

Remarks 2.5.

(a) The map δ'_0 is defined as follows. If a class $c \in H^0(F \to G)$ is represented by an (F, G)-torsor (Q, t), where Q is an F-bitorsor, then $\delta'_0(c) \in H^1(F)$ is represented by Q regarded only as a right F-torsor. See [4, p. 414, line −10].

(b) The map ψ_0 (which is denoted α in [4, (2.16.1), p. 414]) is a homomorphism of groups. See [4, p. 432]. Thus it defines a right action of $H^0(G)$ on $H^0(F \to G)$ and it follows without difficulty from (a) and Proposition 2.4 that δ'_0 induces an injection

$$H^0(F \to G)/H^0(G) \to H^1(F).$$

(c) The group $H^0(F \to G)$ acts on the right on the set $H^1(F)$ as follows. If $p \in H^1(F)$ is represented by an F-torsor P and $c \in H^0(F \to G)$ is represented by an (F, G)-torsor (Q, t), then $p \cdot c \in H^1(F)$ is represented by $P \wedge^F Q$. This action is compatible with the map δ_0', i.e., $\delta_0'(c_1 c_2) = \delta_0'(c_1) \cdot c_2$ for all $c_1, c_2 \in H^0(F \to G)$. In particular, the action is transitive if, and only if, δ_0' is surjective. In addition, the given action is compatible with inverse images, i.e., if $u : E' \to E$ is a morphism of sites and $h^i(u, F \to G)$ are the maps (2.2), then

$$h^1(u, F)(p \cdot c) = h^1(u, F)(p) \cdot h^0(u, F \to G)(c)$$

in $H^1(E', u^*F)$. This follows from the fact that $u^*(P \wedge^F Q) \simeq u^*P \wedge^{u^*F} u^*Q$ [21, p. 316, line −4].

3. Quasi-abelian crossed modules

If A is a group of $\widetilde{E}$, $H^2(E, A)$ will denote the second cohomology set of A defined in [21, Definition IV.3.1.3, p. 247].[3] It contains a distinguished element ε_A, namely the class of the gerbe TORS(A) of A-torsors on E, which is called the *unit class*. This class is contained in a subset $H^2(E, A)' \subset H^2(E, A)$ of *neutral classes* [21, Definition IV.3.1.1, p. 247]. For convenience, we will sometimes write $H^2(A)$ and $H^2(A)'$ for $H^2(E, A)$ and $H^2(E, A)'$, respectively. Both $H^2(A)$ and $H^2(A)'$ will be regarded as pointed sets with basepoint ε_A.

Let $(F \xrightarrow{\partial} G)$ be a crossed module on E such that $G = \operatorname{Im} \partial \cdot \operatorname{Cent}_G(\operatorname{Im} \partial)$, where $\operatorname{Cent}_G(\operatorname{Im} \partial)$ is the centralizer in G of $\operatorname{Im} \partial$, and the induced action of $Z(G)$ on $Z(F)$ is trivial. By restricting ∂ to $Z(F)$, we obtain a map $\partial' : Z(F) \to Z(\operatorname{Im} \partial)$. On the other hand, the equality $G = \operatorname{Im} \partial \operatorname{Cent}_G(\operatorname{Im} \partial)$ implies that $Z(\operatorname{Im} \partial) = \operatorname{Im} \partial \cap Z(G)$. Thus ∂ induces a map $\partial_Z : Z(F) \to Z(G)$, namely the composition

$$Z(F) \xrightarrow{\partial'} Z(\operatorname{Im} \partial) \hookrightarrow Z(G),$$

and $(Z(F) \xrightarrow{\partial_Z} Z(G))$ is an abelian crossed module (see Example 2.2(i)). Further, there exists an embedding of crossed modules

$$\big(Z(F) \xrightarrow{\partial_Z} Z(G)\big) \hookrightarrow (F \xrightarrow{\partial} G). \tag{3.1}$$

Note that, since $\operatorname{Ker} \partial \subset Z(F)$, we have $\operatorname{Ker} \partial_Z = \operatorname{Ker} \partial' = \operatorname{Ker} \partial$.

Definition 3.1. Let $(F \xrightarrow{\partial} G)$ be a crossed module on E such that $G = \operatorname{Im} \partial \operatorname{Cent}_G(\operatorname{Im} \partial)$ and $Z(G)$ acts trivially on $Z(F)$. Let $i \geqslant -1$ be an integer. The i-th *abelian cohomology group of* $(F \to G)$ is by definition the hypercohomology group

$$H^i_{\mathrm{ab}}(E, F \to G) = \mathbb{H}^i\big(E, Z(F) \xrightarrow{\partial_Z} Z(G)\big),$$

where ∂_Z is as defined above.

For $i = 0, 1$, the embedding of crossed modules (3.1) defines maps

$$\varphi_i : H^i_{\mathrm{ab}}(E, F \to G) \to H^i(E, F \to G). \tag{3.2}$$

Further, the short exact sequence of complexes

$$0 \to \big(0 \to Z(G)\big) \xrightarrow{j} \big(Z(F) \xrightarrow{\partial_Z} Z(G)\big) \xrightarrow{\pi} \big(Z(F) \to 0\big) \to 0 \tag{3.3}$$

[3] For an excellent summary of Giraud's theory, see [8, §1.2], and [9, §1].

induces an exact sequence of abelian groups

$$\cdots \to H^i\big(Z(G)\big) \xrightarrow{j^{(i)}} H^i_{\mathrm{ab}}(F \to G) \xrightarrow{\pi^{(i)}} H^{i+1}\big(Z(F)\big) \xrightarrow{\partial_Z^{(i+1)}} H^{i+1}\big(Z(G)\big) \to \cdots. \tag{3.4}$$

Definition 3.2. A crossed module $(F \xrightarrow{\partial} G)$ on E is called *quasi-abelian* if the following conditions hold:

(i) the induced action of $Z(G)$ on F is trivial,[4]
(ii) $G = (\mathrm{Im}\,\partial) \cdot Z(G)$, and
(iii) the map $\partial' : Z(F) \to Z(\mathrm{Im}\,\partial)$ induced by ∂ is surjective.

Clearly, an abelian crossed module is quasi-abelian.

Example 3.3. The crossed module $(\widetilde{G} \to G)$ considered in Example 2.2(iii) is a quasi-abelian crossed module on S_{fl} (but perhaps not on $S_{\text{ét}}$ if μ is non-smooth). Indeed, the induced action of $Z(G)$ on $\widetilde{G}$ is trivial, $G = G^{\mathrm{der}} Z(G)$ by [10, XXII, 6.2.3], and $\partial' : Z(\widetilde{G}) = \widetilde{G} \times_{G^{\mathrm{der}}} Z(G^{\mathrm{der}}) \to Z(G^{\mathrm{der}})$ is a surjective map of fppf sheaves.

Definition 3.2 is motivated by

Proposition 3.4. *Let $(F \to G)$ be a quasi-abelian crossed module on E. Then the embedding of crossed modules (3.1) is a quasi-isomorphism.*

Proof. Clearly, conditions (i) and (ii) of Definition 3.2 imply that (3.1) is defined. Further, since $Z(\mathrm{Im}\,\partial) = \mathrm{Im}\,\partial \cap Z(G)$, there exists a canonical exact sequence

$$0 \to \mathrm{Coker}\,\partial' \to \mathrm{Coker}\,\partial_Z \xrightarrow{f} \mathrm{Coker}\,\partial \to 0,$$

where the map f is induced by (3.1). Now, since $\mathrm{Coker}\,\partial' = 0$ by condition (iii) of Definition 3.2, f is an isomorphism. Since $\mathrm{Ker}\,\partial_Z = \mathrm{Ker}\,\partial$ as noted above, the proof is complete. □

Remark 3.5. Conditions (i) and (ii) of Definition 3.2 imply that $(Z(F) \to Z(G))$ coincides with the *center* of $(F \to G)$, as defined in [24, p. 171]. Thus, the following is a seemingly more general version of Definition 3.2: a crossed module is quasi-abelian if it is quasi-isomorphic to its center.

Corollary 3.6. *Let $(F \to G)$ be a quasi-abelian crossed module on E. Then the maps (3.2) are bijections.*

When $(F \to G)$ is quasi-abelian, there exists a useful variant of (3.4). Namely, the short exact sequence of complexes

$$0 \to \big(Z(F) \overset{\partial'}{\twoheadrightarrow} Z(\mathrm{Im}\,\partial)\big) \to \big(Z(F) \xrightarrow{\partial_Z} Z(G)\big) \to (0 \to \mathrm{Coker}\,\partial) \to 0$$

induces an exact sequence of abelian groups

$$\cdots \to H^{i-1}(\mathrm{Coker}\,\partial) \to H^{i+1}(\mathrm{Ker}\,\partial) \to H^i_{\mathrm{ab}}(F \to G) \xrightarrow{t^{(i)}_{\mathrm{ab}}} H^i(\mathrm{Coker}\,\partial) \to \cdots. \tag{3.5}$$

[4] I thank M. Borovoi for pointing out the need to assume that $Z(G)$ acts trivially on all of F. If this is not the case, then certain desirable properties need not hold.

The map $t_{\mathrm{ab}}^{(i)}$ was first considered in [2, §6.1], when $i=2$. We will write $c^{(i)}: H^i(Z(G)) \to H^i(\operatorname{Coker}\partial)$ for the canonical map induced by the projection $Z(G) \to \operatorname{Coker}\partial_Z = \operatorname{Coker}\partial$. Then $c^{(i)}$ factors as

$$H^i\big(Z(G)\big) \xrightarrow{j^{(i)}} H^i_{\mathrm{ab}}(F \to G) \xrightarrow{t_{\mathrm{ab}}^{(i)}} H^i(\operatorname{Coker}\partial), \tag{3.6}$$

where $j^{(i)}$ is the map appearing in (3.4).

Next, if G is a group of $\widetilde{E}$, we *define* $\operatorname{Inn}(G)$ to be the quotient $G/Z(G)$. It is canonically isomorphic to the group of inner automorphisms of G. Now let $(F \xrightarrow{\partial} G)$ be a quasi-abelian crossed module on E. Then there exists an exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Ker}\partial' \arrow[r] \arrow[d, equal] & Z(F) \arrow[r, "\partial'"] \arrow[d, hook] & Z(\operatorname{Im}\partial) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \operatorname{Ker}\partial \arrow[r] & F \arrow[r, "\partial"] & \operatorname{Im}\partial \arrow[r] & 1.
\end{tikzcd}$$

Consequently, ∂ induces an isomorphism

$$\operatorname{Inn}(F) \stackrel{\mathrm{def}}{=} F/Z(F) \simeq \operatorname{Im}\partial/Z(\operatorname{Im}\partial) = \operatorname{Im}\partial/\operatorname{Im}\partial \cap Z(G).$$

On the other hand, by Definition 3.2(ii),

$$\operatorname{Im}\partial/\operatorname{Im}\partial \cap Z(G) \simeq \operatorname{Im}\partial Z(G)/Z(G) = G/Z(G) = \operatorname{Inn}(G).$$

We conclude that ∂ induces an isomorphism $\bar{\partial}: \operatorname{Inn}(F) \xrightarrow{\sim} \operatorname{Inn}(G)$. Now, by [21, Proposition IV.4.2.12(iii), p. 285], the exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Z(F) \arrow[r] \arrow[d, "\partial_Z"] & F \arrow[r] \arrow[d, "\partial"] & \operatorname{Inn}(F) \arrow[r] \arrow[d, "\bar{\partial}", "\simeq"'] & 1 \\
1 \arrow[r] & Z(G) \arrow[r] & G \arrow[r] & \operatorname{Inn}(G) \arrow[r] & 1
\end{tikzcd}$$

induces an exact commutative diagram

$$\begin{tikzcd}
H^1(\operatorname{Inn}(F)) \arrow[r, "d_F"] \arrow[d, "\bar{\partial}^{(1)}", "\simeq"'] & H^2(Z(F)) \arrow[d, "\partial_Z^{(2)}"] \\
H^1(\operatorname{Inn}(G)) \arrow[r, "d_G"] & H^2(Z(G)),
\end{tikzcd} \tag{3.7}$$

where d_F and d_G are the second coboundary maps of [21, IV.4.2.2, p. 280], and the left-hand vertical map is a bijection by [21, IV.3.1.6.2, p. 250]. By [21, Proposition IV.5.2.8, p. 300], the map d_G (respectively, d_F) may be described as follows. Let $p \in H^1(\operatorname{Inn}(G))$, choose an $\operatorname{Inn}(G)$-torsor P representing p and let $(H, u: \operatorname{lien}(H) \xrightarrow{\sim} \operatorname{lien}(G))$ be the representative of $\operatorname{lien}(G)$ defined by P. Thus H is the twist of G by P, where $\operatorname{Inn}(G)$ acts on G in the natural way, and u is the isomorphism defined in [21, proof of Corollary IV.1.1.7.3, p. 188, lines 6–8]. Then $d_G(p)$ is the class of the $Z(G)$-gerbe $\mathrm{BITORS}(H, G)(u)$ of H–G-bitorsors Q on E such that the isomorphism $\pi(Q)$ of [21, Corollary IV.5.2.6, p. 298], equals u^{-1}.

Now, by [21, Proposition IV.4.2.8(i), p. 283][5] (see also [21, Remark IV.4.2.10, p. 284][6]), and [21, Proposition IV.3.2.6, p. 255], there exist exact sequences of pointed sets

$$H^1(G) \xrightarrow{b_G^{(1)}} H^1\bigl(\mathrm{Inn}(G)\bigr) \xrightarrow{d_G} H^2\bigl(Z(G)\bigr) \tag{3.8}$$

and

$$H^1(G) \xrightarrow{b_G^{(1)}} H^1\bigl(\mathrm{Inn}(G)\bigr) \xrightarrow{n_G} H^2(G)' \to 1, \tag{3.9}$$

where $b_G : G \to \mathrm{Inn}(G)$ is the canonical map and the map n_G is defined as follows: if p and (H, u) are as above, then $n_G(p) \in H^2(G)'$ is the class of the gerbe $\mathrm{TORS}(H)$ of H-torsors on E, which is a G-gerbe via u.

Next, condition (ii) of Definition 3.2 implies, in fact, that $\mathrm{Cent}_G(\mathrm{Im}\,\partial) = Z(G)$. We conclude that the morphism of liens $\mathrm{lien}(\partial) : \mathrm{lien}(F) \to \mathrm{lien}(G)$ satisfies the hypotheses of [21, Proposition IV.3.1.5, p. 249]. Consequently, ∂ induces a map $\partial^{(2)} : H^2(F) \to H^2(G)$ which maps $H^2(F)'$ into $H^2(G)'$. On the other hand, by [21, Theorem IV.3.3.3(i), p. 257], there exists a simply-transitive action of $H^2(Z(G))$ on $H^2(G)$ given by the map

$$H^2\bigl(Z(G)\bigr) \times H^2(G) \to H^2(G), \quad (x, r) \mapsto x \cdot r, \tag{3.10}$$

where $x \cdot r$ is the class of the contracted product $X \wedge^Z R$, where $Z = Z(G)$ and X and R are representatives of x and r, respectively. By [21, Corollary IV.3.3.4(ii), p. 258], the map $\partial^{(2)}$ is compatible with $\partial_Z^{(2)}$ and the actions (3.10), i.e., the following diagram commutes

$$\begin{array}{ccc} H^2(Z(F)) \times H^2(F) & \longrightarrow & H^2(F) \\ \Big\downarrow {\scriptstyle (\partial_Z^{(2)}, \partial^{(2)})} & & \Big\downarrow {\scriptstyle \partial^{(2)}} \\ H^2(Z(G)) \times H^2(G) & \longrightarrow & H^2(G). \end{array} \tag{3.11}$$

Proposition 3.7. *Let d_G and n_G be the maps appearing in (3.7) and (3.9), respectively. Then, for every $p \in H^1(\mathrm{Inn}(G))$,*

$$n_G(p) = d_G(p) \cdot \varepsilon_G.$$

Proof. As noted above, $n_G(p)$ and ε_G are represented by $\mathrm{T}H := \mathrm{TORS}(H)$ and $\mathrm{T}G := \mathrm{TORS}(G)$, respectively. On the other hand, $d_G(p)$ is represented by the $Z(G)$-gerbe $\mathrm{BITORS}(H, G)(u)$. Now, by [21, Theorem IV.3.3.3(ii), p. 257], the unique element $x \in H^2(Z(G))$ such that $n_G(p) = x \cdot \varepsilon_G$ is represented by the $Z(G)$-gerbe $\mathrm{HOM}_G(\mathrm{T}G, \mathrm{T}H)$. Thus, it suffices to check that there exists a $Z(G)$-equivalence of $Z(G)$-gerbes $\mathrm{HOM}_G(\mathrm{T}G, \mathrm{T}H) \simeq \mathrm{BITORS}(H, G)(u)$. This follows from [21, Proposition IV.5.2.5(iii), p. 297]. $\square$

The proposition has the following corollary, previously noted in [8, p. 584].

Corollary 3.8. *The image of $d_G : H^1(\mathrm{Inn}(G)) \to H^2(Z(G))$ is the set of all elements $x \in H^2(Z(G))$ such that $x \cdot \varepsilon_G \in H^2(G)$ is neutral. In particular, d_G is surjective if, and only if, every class of $H^2(G)$ is neutral.*

[5] When applying this proposition recall that $H^2(Z(G))' = \{0\}$ by [21, IV.3.3.2.2, p. 257].

[6] Note that the exact sequence appearing in [21] contains an unfortunate misprint: the "map" $a^{(2)}$ appearing there is only a relation (as in [21, Definition IV.3.1.4, p. 248]), even when A is central in B.

Proof. This follows from the proposition, the existence of (3.10) and the exactness of (3.9). □

Now, for $i = 0, 1$, we define the i-th *abelianization map*

$$\mathrm{ab}^i : H^i(G) \to H^i_{\mathrm{ab}}(F \to G) \tag{3.12}$$

as the composite

$$H^i(G) \xrightarrow{\psi_i} H^i(F \to G) \xrightarrow{\varphi_i^{-1}} H^i_{\mathrm{ab}}(F \to G),$$

where ψ_i is the map of Proposition 2.4 and φ_i is the bijection (3.2). Further, let $\delta_0 : H^0_{\mathrm{ab}}(F \to G) \to H^1(F)$ be the composite

$$H^0_{\mathrm{ab}}(F \to G) \xrightarrow{\varphi_0} H^0(F \to G) \xrightarrow{\delta_0'} H^1(F),$$

where δ_0' is the map described in Remark 2.5(a). Thus δ_0 maps the class of a $(Z(F), Z(G))$-torsor (Q, t) to the class of the F-torsor $Q \wedge^Z F$, where $Z = Z(F)$.

Remarks 3.9.

(a) If $(F \xrightarrow{\partial} G)$ is a quasi-abelian crossed module on E such that ∂ is *surjective*, then $H^1_{\mathrm{ab}}(F \to G)$ can be identified with $H^2(\operatorname{Ker} \partial)$ (see Example 2.3(ii)). Under this identification, ab^1 corresponds to the coboundary map $H^1(G) \to H^2(\operatorname{Ker} \partial)$ induced by the central extension $1 \to \operatorname{Ker} \partial \to F \to G \to 1$.

(b) The right action of $H^0(F \to G)$ on $H^1(F)$ described in Remark 2.5(c) induces, via φ_0, a right action of $H^0_{\mathrm{ab}}(F \to G)$ on $H^1(F)$ which can be described as follows. If $p \in H^1(F)$ is represented by an F-torsor P and $c \in H^0_{\mathrm{ab}}(F \to G)$ is represented by a $(Z(F), Z(G))$-torsor (Q, t), then $p \cdot c \in H^1(F)$ is represented by the F-torsor $P \wedge^F (Q \wedge^Z F) \simeq P \wedge^Z Q$, where $Z = Z(F)$ (for the isomorphism, see [21, III, 1.3.1.3, p. 115, 1.3.5, p. 116 and 2.4.5, p. 149]). As in Remark 2.5(c), the above action is compatible with inverse images and with the map δ_0. In particular, the given action is transitive if, and only if, the map δ_0 is surjective.

The following statement is immediate from Proposition 2.4 and the definitions of ab^i and δ_0.

Proposition 3.10. *Let $(F \to G)$ be a quasi-abelian crossed module on E. Then there exists an exact sequence of pointed sets*

$$\begin{aligned} 1 \to H^{-1}(F \to G) \to H^0(F) \to H^0(G) &\xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(F \to G) \\ &\xrightarrow{\delta_0} H^1(F) \xrightarrow{\partial^{(1)}} H^1(G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(F \to G). \end{aligned}$$

We now discuss twisting. Let P be a (right) G-torsor. For any G-object X of E, let ${}^P X$ be the twist of X by P [21, Proposition III.2.3.7, p. 146]. Now let ${}^P\partial : {}^P F \to {}^P G$ be the twist of ∂ by P [21, III.2.3.3.1, p. 142]. Further, let $\theta_P : H^1(G) \xrightarrow{\sim} H^1({}^P G)$ be the bijection defined in [21, Remark III.2.6.3, p. 154]. If P^{o} denotes the G-torsor opposite to P [21, III.1.5.5.2, p. 122], and Q represents the class $q \in H^1(G)$, then $\theta_P(q)$ is represented by the ${}^P G$-torsor $Q \wedge^G P^{\mathrm{o}}$ [21, Proposition III.2.6.1(i), p. 153]. Let ${}^P\mathrm{ab}^1$ denote the composite

$$H^1({}^P G) \xrightarrow{{}^P\psi_1} H^1({}^P F \to {}^P G) \xrightarrow{{}^P\varphi_1^{-1}} H^1_{\mathrm{ab}}({}^P F \to {}^P G) = H^1_{\mathrm{ab}}(F \to G).$$

Proposition 3.11. *Let P be a G-torsor and let $p \in H^1(G)$ be its class. Then there exists a commutative diagram*

$$\begin{tikzcd} H^1(G) \arrow[r, "\mathrm{ab}^1"] \arrow[d, "\theta_P"] & H^1_{\mathrm{ab}}(F \to G) \arrow[d, "r_P"] \\ H^1({}^P G) \arrow[r, "{}^P\mathrm{ab}^1"] & H^1_{\mathrm{ab}}(F \to G), \end{tikzcd}$$

where r_P is given by $r_P(x) = x - \mathrm{ab}^1(p)$ for $x \in H^1_{\mathrm{ab}}(F \to G)$.

Proof. (C. Demarche) Let $\mathcal{C}$ be the gr-stack associated to $(F \to G)$, so that $H^1(F \to G) \simeq H^1(\mathcal{C})$, and let $\mathcal{P}$ be a $\mathcal{C}$-torsor representing the class in $H^1(\mathcal{C})$ which corresponds to $\psi_1(p) \in H^1(F \to G)$. Similarly, let $\mathcal{C}^{\mathrm{ab}}$ be the gr-stack associated to $(Z(F) \to Z(G))$, so that $H^1(Z(F) \to Z(G)) \simeq H^1(\mathcal{C}^{\mathrm{ab}})$, and let $\mathcal{P}^{\mathrm{ab}}$ be a $\mathcal{C}^{\mathrm{ab}}$-torsor representing the class in $H^1(\mathcal{C}^{\mathrm{ab}})$ corresponding to $\mathrm{ab}^1(p) = \varphi_1^{-1}(\psi_1(p)) \in H^1(Z(F) \to Z(G))$. Consider the diagram

$$\begin{tikzcd} H^1(G) \arrow[r, "\theta_P"] \arrow[d, "\psi_1"] & H^1({}^P G) \arrow[d, "{}^P\psi_1"] \\ H^1(F \to G) = H^1(\mathcal{C}) \arrow[r, "\theta_{\mathcal{P}}"] \arrow[d, "\varphi_1^{-1}"] & H^1({}^P F \to {}^P G) = H^1({}^{\mathcal{P}}\mathcal{C}) \arrow[d, "{}^P\varphi_1^{-1}"] \\ H^1(Z(F) \to Z(G)) = H^1(\mathcal{C}^{\mathrm{ab}}) \arrow[r, "\theta_{\mathcal{P}^{\mathrm{ab}}}"] \arrow[d, "="] & H^1(Z({}^P F) \to Z({}^P G)) = H^1({}^{\mathcal{P}^{\mathrm{ab}}}\mathcal{C}^{\mathrm{ab}}) \arrow[d, "="] \\ H^1(Z(F) \to Z(G)) \arrow[r, "r_P"] & H^1(Z(F) \to Z(G)). \end{tikzcd}$$

The left- and right-hand vertical compositions equal ab^1 and ${}^P\mathrm{ab}^1$, respectively. Now, by functoriality of twisting [21, III.2.6.3.2, p. 155], the above diagram commutes except perhaps for the bottom square. But by [21, Remark III.2.6.3, p. 154] (which extends easily to commutative gr-stacks), the bottom square is commutative as well. This completes the proof. □

By [21, Remark III.3.4.4(2), p. 166], there exists an action of $H^1(Z(G))$ on $H^1(G)$ given by the map

$$H^1\big(Z(G)\big) \times H^1(G) \to H^1(G), \quad (p, q) \mapsto p \cdot q, \tag{3.13}$$

where $p \cdot q$ is the class of $P \wedge^Z Q$, where $Z = Z(G)$ and P and Q are representatives of p and q, respectively. Now recall from (3.4) the map $j^{(1)} : H^1(Z(G)) \to H^1_{\mathrm{ab}}(F \to G)$.

Corollary 3.12. *For any $p \in H^1(Z(G))$ and $q \in H^1(G)$,*

$$\mathrm{ab}^1(p \cdot q) = j^{(1)}(p) + \mathrm{ab}^1(q).$$

Proof. (After C. Demarche) Let Q be a representative of q. By the proposition

$${}^Q\mathrm{ab}^1\big(\theta_Q(p \cdot q)\big) = r_Q\big(\mathrm{ab}^1(p \cdot q)\big) = \mathrm{ab}^1(p \cdot q) - \mathrm{ab}^1(q).$$

On the other hand, by [21, Proposition III.3.4.5(ii), p. 167], $\theta_Q(p\cdot q)=({}^{Q}i)^{(1)}(p)$, where ${}^{Q}i: Z(G)\to {}^{Q}G$ is the Q-twist of the canonical embedding $i: Z(G)\to G$. The corollary now follows from the commutativity of the diagram

$$\begin{tikzcd} H^1(Z(G)) \arrow[r, "j^{(1)}"] \arrow[d, "({}^{Q}i)^{(1)}"] & H^1(Z(F)\to Z(G)) \arrow[d, "{}^{Q}\varphi_1"] \\ H^1({}^{Q}G) \arrow[r, "{}^{Q}\psi_1"] & H^1({}^{Q}F\to {}^{Q}G). \end{tikzcd} \qquad \square$$

The following result is similar to [21, Lemma III.3.3.4, p. 163].

Lemma 3.13. *Let P be an F-torsor and let $Q=P\wedge^F G$. Then there exists an exact commutative diagram*

$$\begin{tikzcd} H^0(G) \arrow[r, "\mathrm{ab}^0"] & H^0_{\mathrm{ab}}(F\to G) \arrow[r, "\delta_0"] & H^1(F) \arrow[r, "\partial^{(1)}"] \arrow[d, "\theta_P"] & H^1(G) \arrow[r, "\mathrm{ab}^1"] \arrow[d, "\theta_Q"] & H^1_{\mathrm{ab}}(F\to G) \arrow[d, equal] \\ H^0({}^{Q}G) \arrow[r, "{}^{Q}\mathrm{ab}^0"] & H^0_{\mathrm{ab}}(F\to G) \arrow[r, "{}^{P}\delta_0"] & H^1({}^{P}F) \arrow[r, "{}^{P}\partial^{(1)}"] & H^1({}^{Q}G) \arrow[r, "{}^{Q}\mathrm{ab}^1"] & H^1_{\mathrm{ab}}(F\to G). \end{tikzcd} \tag{3.14}$$

Furthermore, if $p\in H^1(F)$ is the class of P and c is any class in $H^0_{\mathrm{ab}}(F\to G)$, then

$$\theta_P(p\cdot c)={}^{P}\delta_0(c).$$

Proof. The commutativity of the left-hand square is [21, III.2.6.3.2, p. 155], and that of right-hand square is a particular case of Proposition 3.11. Now, if $c\in H^0_{\mathrm{ab}}(F\to G)$ is represented by a $(Z(F),Z(G))$-torsor (Q,t) then, by Remark 3.9(b), $\theta_P(p\cdot c)$ is represented by the ${}^{P}F$-torsor $(Q\wedge^Z P)\wedge^F P^0\simeq Q\wedge^{Z}{}^{P}F$, where $Z=Z(F)$. See [21, Corollary III.1.6.5(1), p. 125]. This completes the proof. $\square$

Proposition 3.14. *Let $(F\xrightarrow{\partial} G)$ be a quasi-abelian crossed module on E.*

(a) *Let $p\in H^1(F)$, let P be an F-torsor representing p and let $Q=P\wedge^F G$. Then the stabilizer of p in $H^0_{\mathrm{ab}}(F\to G)$ is the image of ${}^{Q}\mathrm{ab}^0: H^0({}^{Q}G)\to H^0_{\mathrm{ab}}(F\to G)$.*
(b) *The map $\partial^{(1)}$ induces an injection*

$$H^1(F)/H^0_{\mathrm{ab}}(F\to G)\to H^1(G)$$

whose image is the kernel of ab^1.

Proof. If $c\in H^0_{\mathrm{ab}}(F\to G)$ satisfies $p\cdot c=p$ then, by the lemma, ${}^{P}\delta_0(c)=\theta_P(p\cdot c)=\theta_P(p)$ is the unit class of $H^1({}^{P}F)$ [21, Remark III.2.6.3, p. 154], whence c is in the image of ${}^{Q}\mathrm{ab}^0$ by the exactness of the bottom row of diagram (3.14). Assertion (a) follows. That the image of the map in (b) is the kernel of ab^1 is immediate from the exactness of the top row of (3.14). To prove its injectivity, let $p,p'\in H^1(F)$ be such that $\partial^{(1)}(p)=\partial^{(1)}(p')$. Then the commutativity of (3.14) and the exactness of its bottom row show that $\theta_P(p')={}^{P}\delta_0(c)=\theta_P(p\cdot c)$ for some $c\in H^0_{\mathrm{ab}}(F\to G)$. Thus $p'=p\cdot c$, which completes the proof. $\square$

Corollary 3.15. *Let P be a G-torsor and let $p \in H^1(G)$ be its class. Then the map $\theta_P^{-1} \circ {}^P\partial^{(1)} : H^1({}^PF) \to H^1(G)$ induces a bijection*

$$H^1\big({}^PF\big)/H^0_{\mathrm{ab}}(F \to G) \xrightarrow{\sim} \big\{p' \in H^1(G)\colon \mathrm{ab}^1\big(p'\big) = \mathrm{ab}^1(p)\big\}.$$

Proof. Injectivity follows from the injectivity of θ_P^{-1} and part (b) of the proposition applied to the crossed module $({}^PF \xrightarrow{{}^P\partial} {}^PG)$. To prove surjectivity, let $p' \in H^1(G)$ be such that $\mathrm{ab}^1(p') = \mathrm{ab}^1(p)$ and let P' and P be representatives of p' and p, respectively. Then, by Proposition 3.11,

$${}^P\mathrm{ab}^1\big(\theta_P\big(P'\big)\big) = r_{{}_P}\big(\mathrm{ab}^1\big(p'\big)\big) = \mathrm{ab}^1\big(p'\big) - \mathrm{ab}^1(p) = 0.$$

Thus, by Proposition 3.10 applied to $({}^PF \xrightarrow{{}^P\partial} {}^PG)$, $\theta_P(P')$ is in the image of ${}^P\partial^{(1)}$. This completes the proof. $\square$

Recall from (3.7) and (3.8) the maps $\bar{\partial}^{(1)}$ and $b_G^{(1)}$ and let $\tilde{b}_G^{(1)} : H^1(G) \to H^1(\mathrm{Inn}(F))$ be the composition

$$H^1(G) \xrightarrow{b_G^{(1)}} H^1\big(\mathrm{Inn}(G)\big) \xrightarrow{\bar{\partial}^{(1)-1}} H^1\big(\mathrm{Inn}(F)\big). \tag{3.15}$$

Recall also from (3.4) the map $\pi^{(1)} : H^1_{\mathrm{ab}}(F \to G) \to H^2(Z(F))$.

Lemma 3.16. *The diagram*

$$\begin{array}{ccc} H^1(G) & \xrightarrow{\mathrm{ab}^1} & H^1_{\mathrm{ab}}(F \to G) \\ \big\downarrow{\scriptstyle \tilde{b}_G^{(1)}} & & \big\downarrow{\scriptstyle \pi^{(1)}} \\ H^1(\mathrm{Inn}(F)) & \xrightarrow{d_F} & H^2(Z(F)) \end{array}$$

commutes.

Proof. (After C. Demarche) Let p be a class in $H^1(G)$, let P be a G-torsor representing p and let Q be an $\mathrm{Inn}(F)$-torsor representing $\tilde{b}_G^{(1)}(p)$. Then $(d_F \circ \tilde{b}_G^{(1)})(p)$ is represented by the $Z(F)$-gerbe $K(Q)$ of liftings of Q to F [21, IV.4.2.2, p. 280]. Now recall the gr-stack $\mathcal{C}^{\mathrm{ab}}$ associated to $(Z(F) \to Z(G))$, so that $H^1_{\mathrm{ab}}(F \to G) \simeq H^1(\mathcal{C}^{\mathrm{ab}})$, and let $\mathcal{P}^{\mathrm{ab}}$ be a $\mathcal{C}^{\mathrm{ab}}$-torsor representing the class in $H^1(\mathcal{C}^{\mathrm{ab}})$ corresponding to $\mathrm{ab}^1(p)$. Then $\pi^{(1)}(\mathrm{ab}^1(p))$ is represented by the $Z(F)$-gerbe $K(\mathcal{P}^{\mathrm{ab}})$ of liftings of $\mathcal{P}^{\mathrm{ab}}$ to $Z(G)$ (see (3.3)). Thus, by [21, Corollary IV.2.2.7, p. 216], it suffices to define a $Z(F)$-morphism of $Z(F)$-gerbes $m : K(Q) \to K(\mathcal{P}^{\mathrm{ab}})$. Let R be a lift of Q to F, let $r \in H^1(F)$ be its class and set $t = \partial^{(1)}(r) \in H^1(G)$. The commutativity of the diagram

$$\begin{array}{ccc} H^1(F) & \xrightarrow{\partial^{(1)}} & H^1(G) \\ \big\downarrow{\scriptstyle b_F^{(1)}} & & \big\downarrow{\scriptstyle b_G^{(1)}} \\ H^1(\mathrm{Inn}(F)) & \xrightarrow{\bar{\partial}^{(1)}} & H^1(\mathrm{Inn}(G)) \end{array}$$

shows that $b_G^{(1)}(t) = \bar{\partial}^{(1)}(b_F^{(1)}(r)) = \bar{\partial}^{(1)}(\tilde{b}^{(1)}(p)) = b_G^{(1)}(p)$. Thus, by [21, Proposition III.3.4.5, (iii) and (iv), p. 167], there exists a class $z \in H^1(Z(G))$, which is uniquely determined modulo the image of the coboundary map ${}^{P}d_G : H^0({}^{P}\mathrm{Inn}(G)) \to H^1(Z(G))$, such that $p = z \cdot t$. Note that, since ${}^{P}d_G$ factors as

$$H^0\big({}^{P}\mathrm{Inn}(G)\big) \xrightarrow{({}^{P}\bar{\partial}^{(0)})^{-1}} H^0\big({}^{P}\mathrm{Inn}(F)\big) \xrightarrow{{}^{P}d_F} H^1\big(Z(F)\big) \xrightarrow{\partial_Z^{(1)}} H^1\big(Z(G)\big),$$

z is uniquely determined modulo the image of $\partial_Z^{(1)} \circ {}^{P}d_F$. Let X be a $Z(G)$-torsor representing z. Since $\mathrm{ab}^1(t) = \mathrm{ab}^1(\partial^{(1)}(r)) = 0$, Corollary 3.12 yields $j^{(1)}(z) = \mathrm{ab}^1(z \cdot t) = \mathrm{ab}^1(p)$. Thus X is a lift of $\mathcal{P}^{\mathrm{ab}}$ to $Z(G)$ and we set $m(R) = X$. It is not difficult to check that $\mathrm{lien}(m)$ is the identity of $Z(F)$, which completes the proof. $\square$

We now recall from (3.4) the map $\pi^{(1)} : H^1_{\mathrm{ab}}(F \to G) \to H^2(Z(F))$ and define

$$\delta_1 : H^1_{\mathrm{ab}}(F \to G) \to H^2(F) \tag{3.16}$$

by the formula $\delta_1(y) = \pi^{(1)}(y) \cdot \varepsilon_F$ for every $y \in H^1_{\mathrm{ab}}(F \to G)$.

Proposition 3.17. *A class $y \in H^1_{\mathrm{ab}}(F \to G)$ is in the image of* ab^1 *if, and only if, $\delta_1(y) \in H^2(F)'$.*

Proof. Let $q \in H^1(G)$. By Lemma 3.16 and Proposition 3.7 (applied to F),

$$\delta_1\big(\mathrm{ab}^1(q)\big) = \pi^{(1)}\big(\mathrm{ab}^1(q)\big) \cdot \varepsilon_F = d_F\big(\tilde{b}^{(1)}(q)\big) \cdot \varepsilon_F = n_F\big(\tilde{b}^{(1)}(q)\big) \in H^2(F)'.$$

Conversely, let $y \in H^1_{\mathrm{ab}}(F \to G)$ be such that $\delta_1(y) = \pi^{(1)}(y) \cdot \varepsilon_F \in H^2(F)'$. Then $\pi^{(1)}(y) = d_F(x)$ for some $x \in H^1(\mathrm{Inn}(F))$ by the exactness of (3.9) and Proposition 3.7. Now, by the exactness of (3.4) and the commutativity of (3.7),

$$0 = \partial_Z^{(2)}\big(\pi^{(1)}(y)\big) = \partial_Z^{(2)}\big(d_F(x)\big) = d_G\big(\bar{\partial}^{(1)}(x)\big).$$

We conclude that $\bar{\partial}^{(1)}(x) = b^{(1)}(z)$ for some $z \in H^1(G)$ by the exactness of (3.8). Thus $x = \tilde{b}^{(1)}(z)$ by the definition of $\tilde{b}^{(1)}$ (3.15), and Lemma 3.16 yields

$$\pi^{(1)}(y) = d_F(x) = d_F\big(\tilde{b}^{(1)}(z)\big) = \pi^{(1)}\big(\mathrm{ab}^1(z)\big).$$

Now the exactness of (3.4) shows that $y - \mathrm{ab}^1(z) = j^{(1)}(w)$ for some $w \in H^1(Z(G))$, whence

$$y = j^{(1)}(w) + \mathrm{ab}^1(z) = \mathrm{ab}^1(w \cdot z),$$

by Corollary 3.12. The proof is now complete. $\square$

Proposition 3.18. *The sequence*

$$H^1_{\mathrm{ab}}(F \to G) \xrightarrow{\delta_1} H^2(F) \xrightarrow{\partial^{(2)}} H^2(G)$$

is exact.

Proof. Let $y \in H^1_{\mathrm{ab}}(F \to G)$. By the definition of δ_1, the commutativity of (3.11) and the exactness of (3.4), we have

$$\partial^{(2)}\big(\delta_1(y)\big) = \partial_Z^{(2)}\big(\pi^{(1)}(y)\big) \cdot \varepsilon_G = 0 \cdot \varepsilon_G = \varepsilon_G.$$

Conversely, assume that $s=z\cdot\varepsilon_F\in H^2(F)$ (where $z\in H^2(Z(F))$) is such that $\partial^{(2)}(s)=\partial_Z^{(2)}(z)\cdot\varepsilon_G=\varepsilon_G$. Then $\partial_Z^{(2)}(z)=0$, whence $z=\pi^{(1)}(y)$ for some $y\in H^1_{\rm ab}(F\to G)$ by the exactness of (3.4). Thus $s=\pi^{(1)}(y)\cdot\varepsilon_F=\delta_1(y)\in\operatorname{Im}\delta_1$. $\square$

4. The main theorem

Let $(F\xrightarrow{\partial}G)$ be a quasi-abelian crossed module on E. The *second abelianization map* of $(F\to G)$ is the map

$$\mathrm{ab}^2: H^2(G)\to H^2_{\rm ab}(F\to G) \tag{4.1}$$

defined as follows: if $s\in H^2(G)$ and x is the unique element of $H^2(Z(G))$ such that $s=x\cdot\varepsilon_G$ (3.10), then $\mathrm{ab}^2(s)=j^{(2)}(x)$, where $j^{(2)}: H^2(Z(G))\to H^2_{\rm ab}(F\to G)$ is the map appearing in (3.4).

We will also need the map

$$t:=t^{(2)}_{\rm ab}\circ\mathrm{ab}^2: H^2(G)\to H^2(\operatorname{Coker}\partial), \tag{4.2}$$

where $t^{(2)}_{\rm ab}$ is the map appearing in (3.5). If s and x are as above, then $t(s)=t^{(2)}_{\rm ab}(j^{(2)}(x))=c^{(2)}(x)$, where $c^{(2)}: H^2(Z(G))\to H^2(\operatorname{Coker}\partial)$ is the canonical map induced by the projection $Z(G)\to\operatorname{Coker}\partial_Z=\operatorname{Coker}\partial$ (see (3.6)).

We now extract from (3.4) the subsequence

$$H^2_{\rm ab}(F\to G)\xrightarrow{\pi^{(2)}}H^3\big(Z(F)\big)\xrightarrow{\partial_Z^{(3)}}H^3\big(Z(G)\big)\to\cdots. \tag{4.3}$$

Then the following holds.

Proposition 4.1. *The sequence*

$$H^2(F)\xrightarrow{\partial^{(2)}}H^2(G)\xrightarrow{\mathrm{ab}^2}H^2_{\rm ab}(F\to G)\xrightarrow{\pi^{(2)}}H^3\big(Z(F)\big)\xrightarrow{\partial_Z^{(3)}}H^3\big(Z(G)\big)\to\cdots$$

is exact.

Proof. Since (4.3) is exact, we need only check the exactness of the given sequence at $H^2(G)$ and $H^2_{\rm ab}(F\to G)$. Exactness at $H^2_{\rm ab}(F\to G)$ is not difficult: we have $\pi^{(2)}\circ\mathrm{ab}^2=\pi^{(2)}\circ j^{(2)}=0$, and if $y\in H^2_{\rm ab}(F\to G)$ is such that $\pi^{(2)}(y)=0$, then (by the exactness of (3.4)) there exists an element $x\in H^2(Z(G))$ such that $y=j^{(2)}(x)=\mathrm{ab}^2(x\cdot\varepsilon_G)\in\operatorname{Im}\mathrm{ab}^2$. To check exactness at $H^2(G)$, let $t=y\cdot\varepsilon_F\in H^2(F)$, where $y\in H^2(Z(F))$. Then, by the commutativity of (3.11), $\partial_Z^{(2)}(y)$ is the unique element x of $H^2(Z(G))$ such that $\partial^{(2)}(t)=x\cdot\varepsilon_G$. Consequently $\mathrm{ab}^2(\partial^{(2)}(t))=j^{(2)}(x)=j^{(2)}(\partial_Z^{(2)}(y))=0$, since (3.4) is exact. On the other hand, if $s\in H^2(G)$ is such that $\mathrm{ab}^2(s)=j^{(2)}(x)=0$, where x is the unique element of $H^2(Z(G))$ such that $s=x\cdot\varepsilon_G$, then $x=\partial_Z^{(2)}(y)$ for some $y\in H^2(Z(F))$ by the exactness of (3.4). Thus, again by the commutativity of (3.11),

$$s=\partial_Z^{(2)}(y)\cdot\varepsilon_G=\partial^{(2)}(y\cdot\varepsilon_F)\in\operatorname{Im}\partial^{(2)}.$$

This completes the proof. $\square$

Combining the above proposition with Propositions 3.10, 3.17 and 3.18, we obtain the main result of this paper:

Theorem 4.2. *Let $(F \xrightarrow{\partial} G)$ be a quasi-abelian crossed module on E. Then*

$$\begin{aligned} 1 &\to H^{-1}(F \to G) \to H^0(F) \to H^0(G) \xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(F \to G) \\ &\xrightarrow{\delta_0} H^1(F) \xrightarrow{\partial^{(1)}} H^1(G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(F \to G) \xrightarrow{\delta_1} H^2(F) \xrightarrow{\partial^{(2)}} H^2(G) \\ &\xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(F \to G) \xrightarrow{\pi^{(2)}} H^3\big(Z(F)\big) \xrightarrow{\partial_Z^{(3)}} H^3\big(Z(G)\big) \to \cdots \end{aligned}$$

is an exact sequence of pointed sets at every term except $H^1_{\mathrm{ab}}(F \to G)$, where a class $y \in H^1_{\mathrm{ab}}(F \to G)$ is in the image of ab^1 if, and only if, $\delta_1(y) \in H^2(F)'$.

Remark 4.3. The exact sequence of the theorem is compatible with inverse images, i.e., if $u : E' \to E$ is a morphism of sites, then there exists an exact commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, "\partial^{(1)}"] & H^1(E, G) \arrow[r, "\mathrm{ab}^1"] \arrow[d] & H^1_{\mathrm{ab}}(E, F \to G) \arrow[r, "\delta_1"] \arrow[d] & H^2(E, F) \arrow[r, "\partial^{(2)}"] \arrow[d] & \cdots \\ \cdots \arrow[r, "(\partial^{(1)})'"] & H^1(E', u^*G) \arrow[r, "(\mathrm{ab}^1)'"] & H^1_{\mathrm{ab}}(E', u^*F \to u^*G) \arrow[r, "\delta_1'"] & H^2(E', u^*F) \arrow[r, "(\partial^{(2)})'"] & \cdots \end{tikzcd}$$

whose rows are the exact sequences of the theorem over E and over E' and whose vertical maps are given by (2.2) and [21, Definition V.1.5.1, p. 316]. This follows from the definitions of the various horizontal maps involved, Remark 3.9(b) and [21, Proposition V.1.5.2(ii), p. 317].

Regarding the map ab^2, the following holds.

Lemma 4.4. $H^2(G)' \subset \operatorname{Ker} \mathrm{ab}^2$.

Proof. By the theorem, we need to check that $H^2(G)'$ is contained in the image of $\partial^{(2)}$. Let $s \in H^2(G)'$. By the exactness of (3.9) and Proposition 3.7, there exists an element $p \in H^1(\mathrm{Inn}(G))$ such that $s = d_G(p) \cdot \varepsilon_G$. On the other hand, by the surjectivity of $\bar{\partial}^{(1)}$ and the commutativity of diagram (3.7), there exists an element $q \in H^1(\mathrm{Inn}(F))$ such that

$$d_G(p) = d_G\big(\bar{\partial}^{(1)}(q)\big) = \partial_Z^{(2)}\big(d_F(q)\big).$$

Thus, by the commutativity of (3.11),

$$s = d_G(p) \cdot \varepsilon_G = \partial^{(2)}\big(d_F(q) \cdot \varepsilon_F\big) \in \operatorname{Im} \partial^{(2)}. \qquad \square$$

Since $\partial^{(2)} : H^2(F) \to H^2(G)$ maps $H^2(F)'$ into $H^2(G)'$, Theorem 4.2 and the above lemma yield inclusions

$$\partial^{(2)}\big(H^2(F)'\big) \subset H^2(G)' \subset \operatorname{Ker} \mathrm{ab}^2 = \partial^{(2)}\big(H^2(F)\big).$$

Thus, the following is an immediate consequence of Theorem 4.2.

Corollary 4.5. *Let $(F \to G)$ be a quasi-abelian crossed module on E such that every class of $H^2(F)$ is neutral. Then*

(i) *the abelianization map $\mathrm{ab}^1 : H^1(G) \to H^1_{\mathrm{ab}}(F \to G)$ is surjective, and*
(ii) *a class $s \in H^2(G)$ is neutral if, and only if, $\mathrm{ab}^2_G(s) = 0$.*

5. Applications

Let S be a scheme and let S_{fl} be the small fppf site over S. An S-group scheme G is called *reductive* (respectively, *semisimple*) if it is affine and smooth over S and its geometric fibers are *connected* reductive (respectively, semisimple) algebraic groups [10, XIX, Definition 2.7]. As noted in Example 3.3, the composition $\partial : \widetilde{G} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G$ defines a quasi-abelian crossed module $(\widetilde{G} \xrightarrow{\partial} G)$ on S_{fl}. We have $\operatorname{Ker}\partial = \mu$ and $\operatorname{Coker}\partial = G^{\mathrm{tor}}$ as sheaves on S_{fl}, where μ is the fundamental group of G and $G^{\mathrm{tor}} = G/G^{\mathrm{der}}$ is the coradical of G [10, XXII, 6.2]. Set $H^i_{\mathrm{ab}}(S_{\mathrm{fl}}, G) = H^i_{\mathrm{ab}}(S_{\mathrm{fl}}, \widetilde{G} \to G)$, which will also be denoted by $H^i_{\mathrm{ab}}(G)$ to simplify some statements. If G has trivial fundamental group (respectively, if G is semisimple), then $H^i_{\mathrm{ab}}(G) = H^i(G^{\mathrm{tor}})$ (respectively, $H^i_{\mathrm{ab}}(G) = H^{i+1}(\mu)$). See Examples 2.3. If K is a field and G is a (connected) reductive algebraic group over K, $H^1(K, G)$ will denote the first Galois cohomology set of G. Note that there exists a canonical bijection $H^1(K, G) \simeq H^1(K_{\mathrm{fl}}, G)$ [28, Remark III.4.8(a), p. 123].[7] If G is, in addition, commutative and $i \geqslant 1$, then $H^i(K, G)$ will denote the i-th Galois cohomology group of G.

By (2.1) and Theorem 4.2, the following holds.

Theorem 5.1. *Let G be a reductive group scheme over a scheme S. Then there exists a sequence of flat (fppf) cohomology sets*

$$\begin{aligned} 1 &\to \mu(S) \to \widetilde{G}(S) \to G(S) \xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(G) \xrightarrow{\delta_0} H^1(\widetilde{G}) \xrightarrow{\partial^{(1)}} H^1(G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(G) \\ &\xrightarrow{\delta_1} H^2(\widetilde{G}) \xrightarrow{\partial^{(2)}} H^2(G) \xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(G) \to H^3\big(Z(\widetilde{G})\big) \to H^3\big(Z(G)\big) \to \cdots, \end{aligned}$$

which is an exact sequence of pointed sets at every term except $H^1_{\mathrm{ab}}(G)$, where a class $y \in H^1_{\mathrm{ab}}(G)$ is in the image of ab^1 *if, and only if, $\delta_1(y) \in H^2(\widetilde{G})'$.*

Recall that, if G is a reductive group scheme, G^{ad} is the standard notation for $\operatorname{Inn}(G)$.

Definition 5.2. A scheme S is called *of Douai type* if, for every semisimple and simply-connected S-group scheme G, the coboundary map

$$d_G : H^1\big(S_{\mathrm{fl}}, G^{\mathrm{ad}}\big) \to H^2\big(S_{\mathrm{fl}}, Z(G)\big)$$

(induced by the central extension $1 \to Z(G) \to G \to G^{\mathrm{ad}} \to 1$) is surjective. Equivalently (see Corollary 3.8), every class of $H^2(S_{\mathrm{fl}}, G)$ is neutral. If $S = \operatorname{Spec} A$ is affine, then we will also say that A is of Douai type.

Remark 5.3. By Remark 3.9(a), the map d_G appearing in the above definition can be identified with the first abelianization map ab^1 for the adjoint group G^{ad}.

Examples 5.4. The following are examples of schemes of Douai type.

(i) $S = \operatorname{Spec} K$, where K is a complete discretely-valued field with finite residue field. See [11, Chapter VI, Theorems 1.4, p. 80, 1.6, p. 81, and 2.1, p. 87]. See also [12, Theorem 1.1 and Remark on p. 322].

(ii) $S = \operatorname{Spec} K$, where K is a global field, i.e., K is either a number field or a function field in one variable over a finite field. See [11, Theorem VIII.1.2, p. 108]. See also [13].

[7] In fact, the natural map $H^1(S_{\mathrm{\acute{e}t}}, G) \to H^1(S_{\mathrm{fl}}, G)$ is bijective for any S by the smoothness of G. See [28].

(iii) S is a nonempty open subscheme of either the spectrum of the ring of integers of a number field or a smooth, complete and irreducible curve over a finite (respectively, algebraically closed) field. See [14, Theorem 1.1 and Remark (b), p. 326], and note that Lemma 1.1 from [14] holds over any S as above by, e.g., [29, proof of Proposition II.2.1, p. 202].

(iv) $S = \operatorname{Spec} K$, where K is a function field in one variable over a quasi-finite field k of positive characteristic which is algebraic over its prime subfield k_0 and satisfies $[k : k_0] = \prod p^{n_p}$, where $n_p < \infty$ for every prime p. See [16, proof of Proposition 1.1 and Remark 1.2].

(v) S is a smooth, complete and irreducible curve over a quasi-finite field k with function field K, where k and K are as in (iv). See [16, proof of Lemma 3.1 and Remark 3.2(1) and (3)].

(vi) $S = \operatorname{Spec} K$, where K is a field of characteristic zero and of (Galois) cohomological dimension $\leqslant 2$ such that, for central simple algebras over finite extensions of K, exponent and index coincide. Indeed, it is shown in [7, proof of Theorem 2.1(a)], that $d_G : H^1(K_{\mathrm{fl}}, G^{\mathrm{ad}}) \to H^2(K_{\mathrm{fl}}, Z(G))$ is surjective for every semisimple and simply-connected K-group G. Note that, by [27], examples of such fields include the “fields of types (gl), (ll) and (sl)” considered in [7] (see below for the definitions of types (gl) and (ll)). See [7, Theorems 1.3–1.5].

(vii) S is a regular and integral two-dimensional scheme equipped with a proper birational morphism $S \to \operatorname{Spec} A$, where A is an excellent, Henselian, two-dimensional local domain with algebraically closed residue field of characteristic 0. See [18].

(viii) S is a projective, smooth and geometrically irreducible curve over a p-adic field. See the forthcoming paper [19].

Theorem 5.5. *Let S be a scheme of Douai type and let G be a reductive group scheme over S.*

(i) *The group $H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$ acts on the right on the set $H^1(S_{\mathrm{fl}}, \widetilde{G})$ compatibly with the map $\delta_0 : H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \to H^1(S_{\mathrm{fl}}, \widetilde{G})$ and there exists an exact sequence of pointed sets*

$$1 \to H^1(S_{\mathrm{fl}}, \widetilde{G})/H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \xrightarrow{\bar{\partial}^{(1)}} H^1(S_{\mathrm{fl}}, G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \to 1,$$

where the map $\bar{\partial}^{(1)}$ (which is induced by $\partial^{(1)}$) is injective.

(ii) *A class $\xi \in H^2(S_{\mathrm{fl}}, G)$ is neutral if, and only if, $\mathrm{ab}^2_G(\xi) = 0$.*

Proof. The surjectivity of ab^1 and assertion (ii) are immediate from Corollary 4.5 and Definition 5.2. The action mentioned in (i) is defined in Remark 3.9(b), and the exactness of the sequence in (i) follows from Proposition 3.14(b) and the surjectivity of ab^1. $\square$

Remarks 5.6.

(a) The exact sequence in part (i) of the theorem is compatible with inverse images, i.e., if $S' \to S$ is a morphism of schemes of Douai type, then the diagram

$$\begin{tikzcd}
1 \to H^1(S_{\mathrm{fl}}, \widetilde{G})/H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \arrow[r] \arrow[d] & H^1(S_{\mathrm{fl}}, G) \arrow[r] \arrow[d] & H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \arrow[r] \arrow[d] & 1 \\
1 \to H^1(S'_{\mathrm{fl}}, \widetilde{G})/H^0_{\mathrm{ab}}(S'_{\mathrm{fl}}, G) \arrow[r] & H^1(S'_{\mathrm{fl}}, G) \arrow[r] & H^1_{\mathrm{ab}}(S'_{\mathrm{fl}}, G) \arrow[r] & 1
\end{tikzcd}$$

commutes. This follows from Remark 4.3, and the fact that the action of $H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$ on $H^1(S_{\mathrm{\acute{e}t}}, \widetilde{G})$ is compatible with inverse images. See Remark 3.9(b).

(b) Let S and G be as in the theorem and let $\mathcal{C}^{\mathrm{ab}}$ be the gr-stack associated to $(Z(\widetilde{G}) \to Z(G))$, so that there exists a bijection $H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G) \simeq H^1(\mathcal{C}^{\mathrm{ab}})$. For each class $\xi \in H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G)$, let $\mathcal{P}^{\mathrm{ab}}_{\xi}$ be a

$\mathcal{C}^{\mathrm{ab}}$-torsor representing its image in $H^1(\mathcal{C}^{\mathrm{ab}})$ and let P_ξ be a lift of $\mathcal{P}_\xi^{\mathrm{ab}}$ to G. Then part (i) of the theorem and Corollary 3.15 yield a (non-canonical) bijection

$$H^1(S_{\mathrm{fl}}, G) \simeq \coprod_{\xi \in H^1_{\mathrm{ab}}(S_{\mathrm{fl}}, G)} H^1\big(S_{\mathrm{fl}}, {}^{P_\xi}\widetilde{G}\big)/H^0_{\mathrm{ab}}(S_{\mathrm{fl}}, G).$$

(c) The theorem holds if S is any scheme and G is a reductive group scheme over S such that every class of $H^2(S_{\mathrm{fl}}, \widetilde{G})$ is neutral. For example, if S is a ruled surface of the type considered in [15, Corollary 3.14, p. 76],[8] and G is *split*, then the theorem holds for G (and part (i) generalizes [15, Corollary 3.15]). However, we have chosen to work with schemes S of Douai type because we want our statements to apply uniformly to all reductive group schemes over S.
(d) By Example 5.4(iii), the theorem holds if S is the spectrum of the ring of integers of a number field. Thus Corollary 1.2 of the Introduction is contained in part (i) of the theorem.

Let S be a scheme and let L be an S_{fl}-lien which is locally represented by a reductive group scheme over S. Let $\mathrm{ab}^2_L : H^2(S_{\mathrm{fl}}, L) \to H^2_{\mathrm{ab}}(S_{\mathrm{fl}}, L)$ be the map defined in [17, p. 23]. It is not difficult to check that, if $L = \mathrm{lien}(G)$ is represented by a group G of the topos $\widetilde{S}_{\mathrm{fl}}$, then ab^2_L coincides with the map ab^2_G considered above. The following result generalizes [2, Theorem 5.5].

Corollary 5.7. *Let S be a scheme of Douai type and let L be an S_{fl}-lien which is locally represented by a reductive S-group scheme. Then a class $\xi \in H^2(S_{\mathrm{fl}}, L)$ is neutral if, and only if, $\mathrm{ab}^2_L(\xi) = 0$.*

Proof. Assume that L is locally represented by a reductive S-group scheme G. By [17, Proposition 1.2, p. 22] (see also [12, Lemma 1.1], and [11, V.3.1, p. 74]), L is, in fact, globally represented by a quasi-split form G_L of G, i.e., $L \simeq \mathrm{lien}(G_L)$. The result now follows by applying part (ii) of the proposition to G_L. □

Theorem 5.8. *Let K be a field of Douai type and of Galois cohomological dimension $\leqslant 2$. Let G be a (connected) reductive algebraic group over K.*

(i) *If $H^1(K, H)$ is trivial for every semisimple and simply-connected K-group H,*[9] *then the first abelianization map $\mathrm{ab}^1 : H^1(K, G) \to H^1_{\mathrm{ab}}(K_{\mathrm{fl}}, G)$ is bijective.*
(ii) *There exists an exact sequence of pointed sets*

$$1 \to H^2(K_{\mathrm{fl}}, G)' \to H^2(K_{\mathrm{fl}}, G) \xrightarrow{t} H^2\big(K, G^{\mathrm{tor}}\big) \to 1,$$

where t is the map (4.2).

Proof. (i) The hypothesis and Corollary 3.15 show that ab^1 is injective. Since it is surjective by Theorem 5.5(i), it is in fact bijective.

(ii) Since K has cohomological dimension $\leqslant 2$ and both μ and $Z(\widetilde{G})$ are commutative and finite K-group schemes, $H^i(K_{\mathrm{fl}}, \mu) = H^i(K_{\mathrm{fl}}, Z(\widetilde{G})) = 0$ for every $i \geqslant 3$ by [31, Theorem 4, p. 593]. Thus Theorem 5.1 and Theorem 5.5(ii) yield an exact sequence

$$1 \to H^2(K_{\mathrm{fl}}, G)' \to H^2(K_{\mathrm{fl}}, G) \xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \to 1.$$

On the other hand, (3.5) shows that $t^{(2)}_{\mathrm{ab}} : H^2_{\mathrm{ab}}(K_{\mathrm{fl}}, G) \to H^2(K, G^{\mathrm{tor}})$ is an isomorphism. Thus, since $t = t^{(2)}_{\mathrm{ab}} \circ \mathrm{ab}^2$, the sequence of the statement is indeed exact. □

[8] We do not know if these schemes are of Douai type.
[9] By Serre's conjecture II (see [20]), this is expected to follow from the hypothesis $\operatorname{cd} K \leqslant 2$.

Remarks 5.9.

(a) A field which is either a complete and discretely-valued field with finite residue field or a global field without real primes satisfies the hypotheses, and therefore the conclusions, of the theorem. See Examples 5.4(i) and (ii), [30, Theorems 6.4 and 6.6, pp. 284 and 286], [6, Theorem 4.7(ii)], and [26, Theorem A, p. 125].
(b) The conclusion in part (i) of the theorem for the fields of types (gl), (ll) and (sl) mentioned in Example 5.4(vi) was previously established in [3, Theorem 6.7] (provided that G contains no factors of type E_8 in the (gl) case).
(c) Let K be either a global function field or the completion of such a field at one of its primes. Let G be a *semisimple* algebraic group over K. Then ab^1 can be identified with the coboundary map $H^1(K,G) \to H^2(K_{\mathrm{fl}},\mu)$ induced by the central extension $1 \to \mu \to \widetilde{G} \to G \to 1$ (see Remark 3.9(a)). Thus part (i) of the theorem generalizes [32, Theorem A, p. 458] (from semisimple to arbitrary connected reductive groups over K).

Now let K be a global field and let G be a (connected) reductive algebraic group over K. Set

$$Ш^1(K,G) = \mathrm{Ker}\left[H^1(K,G) \to \prod_{\text{all } v} H^1(K_v,G)\right]$$

and

$$Ш^1_{\mathrm{ab}}(K,G) = \mathrm{Ker}\left[H^1_{\mathrm{ab}}(K_{\mathrm{fl}},G) \to \prod_{\text{all } v} H^1_{\mathrm{ab}}(K_{v,\mathrm{fl}},G)\right].$$

Corollary 5.10. *Let G be a (connected) reductive algebraic group over a global field K. Then the abelianization map* $\mathrm{ab}^1 : H^1(K,G) \to H^1_{\mathrm{ab}}(K_{\mathrm{fl}},G)$ *induces a bijection*

$$Ш^1(K,G) \simeq Ш^1_{\mathrm{ab}}(K,G).$$

Proof. The number field case is due to M. Borovoi [1, Theorem 5.13]. The function field case is obtained by applying Theorem 5.8(i) over K and over the various completions of K. See Remark 5.9(a). □

Remarks 5.11.

(a) A result similar to the above is known to hold over the fields of type (ll) mentioned in Example 5.4(vi). See [3, Theorem 7.1].
(b) The corollary generalizes the function field case of [16, Corollary 2.2], from semisimple to arbitrary (connected) reductive groups.

We now recall from [7] that a field K is called *of type (gl)* if it is the function field of a smooth, projective and connected surface over an algebraically closed field k of characteristic zero. It is called *of type (ll)* if it is the field of fractions of an excellent, Henselian, two-dimensional local domain A with residue field k as above. If X is a smooth projective model of K over k (respectively, if X is a regular and integral two-dimensional scheme equipped with a proper birational morphism $X \to \operatorname{Spec} A$) and v is a discrete valuation associated to a point of codimension 1 on X, K_v will denote the completion of K at v. See [7, §1], for a description of these fields. Let G be a (connected) reductive algebraic group over a field of type (gl) or (ll). For any prime v of K, let $\mathrm{res}_v : H^2(K_{\mathrm{fl}},G) \to H^2(K_{v,\mathrm{fl}},G)$ and $\mathrm{res}_{\mathrm{ab},v} : H^2_{\mathrm{ab}}(K_{\mathrm{fl}},G) \to H^2_{\mathrm{ab}}(K_{v,\mathrm{fl}},G)$ be the maps of pointed sets induced by $\operatorname{Spec} K_v \to \operatorname{Spec} K$. Further, we will write $\mathrm{ab}^2_v : H^2(K_{v,\mathrm{fl}},G) \to H^2_{\mathrm{ab}}(K_{v,\mathrm{fl}},G)$ for the second abelianization map associated to $G \times_K K_v$.

The next corollary is analogous to [2, Theorem 6.8].[10]

Corollary 5.12. *Let K be either a field of type (gl), (ll) or a global function field. Let G be a (connected) reductive algebraic group over K such that $\text{Ш}^2(K, G^{\text{tor}}) = 0$.*[11] *Then a class $\xi \in H^2(K_{\text{fl}}, G)$ is neutral if, and only if, $\text{res}_v(\xi) \in H^2(K_{v,\text{fl}}, G)$ is neutral for every prime v of K.*

Proof. This is immediate from Theorem 5.8(ii) (applied over K and over K_v for every prime v of K). □

The following proposition complements the results of [2] and concludes this paper.

Proposition 5.13. *Let G be a (connected) reductive algebraic group over a number field K. Then a class $\xi \in H^2_{\text{ab}}(K_{\text{fl}}, G)$ is in the image of ab^2 if, and only if, $\text{res}_{\text{ab},v}(\xi) \in H^2_{\text{ab}}(K_{v,\text{fl}}, G)$ is in the image of ab^2_v for every real prime v of K.*

Proof. If ξ is in the image of ab^2, then $\text{res}_{\text{ab},v}(\xi)$ is in the image of ab^2_v for every prime v of K by the commutativity of the diagram

$$\begin{array}{ccc} H^2(K_{\text{fl}}, G) & \xrightarrow{\text{ab}^2} & H^2_{\text{ab}}(K_{\text{fl}}, G) \\ \downarrow{\scriptstyle \text{res}_v} & & \downarrow{\scriptstyle \text{res}_{\text{ab},v}} \\ H^2(K_{v,\text{fl}}, G) & \xrightarrow{\text{ab}^2_v} & H^2_{\text{ab}}(K_{v,\text{fl}}, G) \end{array}$$

(see Remark 4.3). On the other hand, by Proposition 4.1, $\text{Im}\,\text{ab}^2$ is a subgroup of $H^2_{\text{ab}}(K_{\text{fl}}, G)$ and the corresponding quotient group $\text{Coker}\,\text{ab}^2$ injects as a subgroup of $H^3(K, Z(\widetilde{G}))$. The proposition now follows from the fact that the canonical map

$$H^3\big(K, Z(\widetilde{G})\big) \to \prod_{v \text{ real}} H^3\big(K_v, Z(\widetilde{G})\big)$$

is an isomorphism [29, Theorem I.4.10(c), p. 70]. □

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[10] Case (ll) of Corollary 5.12 is implicit in [7, proof of Theorem 5.5].
[11] Sufficient conditions for the vanishing of $\text{Ш}^2(K, G^{\text{tor}})$ can be found in [7, Theorem 5.5(ii)–(iv)], and [2, Theorem 7.3(ii)–(vi)].

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