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Journal of Algebra

www.elsevier.com/locate/jalgebra



Bar operators for quasiparabolic conjugacy classes in a Coxeter group

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ARTICLE INFO

Article history:

Received 19 March 2015

Available online 27 January 2016

Communicated by Gunter Malle

Keywords:

Quasiparabolic sets

Coxeter groups

Iwahori–Hecke algebras

Twisted involutions

Bar operators

Canonical bases

Kazhdan–Lusztig bases

 W -graphs

ABSTRACT

The action of a Coxeter group W on the set of left cosets of a standard parabolic subgroup deforms to define a module \mathcal{M}^J of the group's Iwahori–Hecke algebra \mathcal{H} with a particularly simple form. Rains and Vazirani have introduced the notion of a *quasiparabolic set* to characterize W -sets for which analogous deformations exist; a motivating example is the conjugacy class of fixed-point-free involutions in the symmetric group. Deodhar has shown that the module \mathcal{M}^J possesses a certain antilinear involution, called the bar operator, and a certain basis invariant under this involution, which generalizes the Kazhdan–Lusztig basis of \mathcal{H} . The well-known significance of this basis in representation theory makes it natural to seek to extend Deodhar's results to the quasiparabolic setting. In general, the obstruction to finding such an extension is the existence of an appropriate quasiparabolic analogue of the “bar operator.” In this paper, we consider the most natural definition of a quasiparabolic bar operator, and develop a theory of “quasiparabolic Kazhdan–Lusztig bases” under the hypothesis that such a bar operator exists. Giving content to this theory, we prove that a bar operator in the desired sense does exist for quasiparabolic W -sets given by twisted conjugacy classes of twisted involutions. Finally, we prove

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¹ This research was conducted with support from the National Science Foundation.

several results classifying the quasiparabolic conjugacy classes in a Coxeter group.

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1. Introduction

Let (W, S) be a Coxeter system with length function $\ell : W \rightarrow \mathbb{N}$, and let $\mathcal{H} = \mathcal{H}(W, S)$ be its Iwahori–Hecke algebra: this is the $\mathbb{Z}[v, v^{-1}]$ -algebra \mathcal{H} , with a basis given by the symbols H_w for $w \in W$, whose multiplication is uniquely determined by the condition that

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w) \\ H_{sw} + (v - v^{-1}) \cdot H_w & \text{if } \ell(sw) < \ell(w) \end{cases} \quad \text{for } s \in S \text{ and } w \in W.$$

Observe that H_1 (which we typically write as 1 or omit) is the multiplicative unit of \mathcal{H} and that H_s is invertible for each $s \in S$. There exists a unique ring homomorphism $\mathcal{H} \rightarrow \mathcal{H}$ with $v \mapsto v^{-1}$ and $H_s \mapsto H_s^{-1}$; we denote this map by $H \mapsto \bar{H}$, and refer to it as the *bar operator* of \mathcal{H} .

Certain representations of W admit natural and interesting deformations to modules of the algebra \mathcal{H} . For example, \mathcal{H} viewed as a left module over itself clearly deforms the regular representation of W . For another example, suppose $J \subset S$ is a subset of simple generators and let $X = W/W_J$ be the set of left cosets of the standard parabolic subgroup $W_J = \langle J \rangle$ in W . Define the height of a coset to be the minimal length of any of its elements, i.e., set

$$\text{ht}(\mathcal{C}) = \min_{w \in \mathcal{C}} \ell(w) \quad \text{for a left coset } \mathcal{C} \in W/W_J.$$

Fix $u \in \{-v^{-1}, v\}$. For each choice of u , there is a unique \mathcal{H} -module structure on the free $\mathbb{Z}[v, v^{-1}]$ -module generated by W/W_J in which $H_s \in \mathcal{H}$ for $s \in S$ acts on cosets $\mathcal{C} \in W/W_J$ by the formula

$$H_s : \mathcal{C} \mapsto \begin{cases} s\mathcal{C} & \text{if } \text{ht}(s\mathcal{C}) > \text{ht}(\mathcal{C}) \\ s\mathcal{C} + (v - v^{-1}) \cdot \mathcal{C} & \text{if } \text{ht}(s\mathcal{C}) < \text{ht}(\mathcal{C}) \\ u \cdot \mathcal{C} & \text{if } \text{ht}(s\mathcal{C}) = \text{ht}(\mathcal{C}). \end{cases} \tag{1.1}$$

Denote these \mathcal{H} -modules by \mathcal{M}^J (when $u = v$) and \mathcal{N}^J (when $u = -v^{-1}$), respectively. Note that if we specialize the parameter v to 1, then \mathcal{M}^J and \mathcal{N}^J become the modules of the group ring $\mathbb{Z}W$ given by respectively inducing the trivial and sign representations of W_J to W .

The formulas above are well-defined if we replace $X = W/W_J$ by the set of cosets of any subgroup $H \subset W$. However, the assertion that (1.1) defines an \mathcal{H} -module structure only holds for some choices of H and not for others, in a fashion which is not yet very well understood. The following is therefore a natural question: given a W -set X with a height function $\text{ht} : X \rightarrow \mathbb{N}$, when does the free $\mathbb{Z}[v, v^{-1}]$ -module generated by X have an \mathcal{H} -module structure described by the obvious analogue of (1.1)? Rains and Vazirani [27] identify a simple set of conditions which are sufficient for this phenomenon to occur, and call W -sets satisfying these conditions *quasiparabolic*. We review the precise definition in Section 2.1; informally, a W -set is quasiparabolic if it has a “Bruhat order” which is compatible with its height function and which satisfies a few technical properties exactly analogous to the Bruhat order on W . The W -set of left cosets of any standard parabolic subgroup is quasiparabolic. More exotically, some but not all conjugacy classes of involutions in a Coxeter group are quasiparabolic, relative to the height function $\frac{1}{2}\ell$.

Let $M \mapsto \overline{M}$ denote a \mathbb{Z} -linear map $\mathcal{M}^J \rightarrow \mathcal{M}^J$. We call such a map a *bar operator* of \mathcal{M}^J if it fixes the unique coset in W/W_J of height zero and satisfies

$$\overline{HM} = \overline{H} \cdot \overline{M} \quad \text{for all } H \in \mathcal{H} \text{ and } M \in \mathcal{M}^J. \tag{1.2}$$

Define a *bar operator* of \mathcal{N}^J analogously. In [5,6], Deodhar shows that \mathcal{M}^J and \mathcal{N}^J both admit unique bar operators, and proves that each module has a unique basis of elements invariant under the bar operator which is congruent to the “standard basis” of cosets W/W_J modulo $v^{-1}\mathbb{Z}[v^{-1}]$ -linear combinations of standard basis elements. These new bases are the *parabolic Kazhdan–Lusztig bases* of \mathcal{M}^J and \mathcal{N}^J ; when $J = \emptyset$, they both may be identified with the well-known *Kazhdan–Lusztig basis* of \mathcal{H} introduced in [18].

Rains and Vazirani show that the free $\mathbb{Z}[v, v^{-1}]$ -module generated by a quasiparabolic set X may be given two \mathcal{H} -module structures, which we denote \mathcal{M} and \mathcal{N} , by a formula exactly analogous to (1.1). (We review the precise definitions in Section 2.2.) One naturally asks whether there exists a notion of a “quasiparabolic Kazhdan–Lusztig basis” for these modules, which specializes to Deodhar’s parabolic Kazhdan–Lusztig bases when $X = W/W_J$. The exploration of this question is the main topic of the present work. As motivation, we recall that the (parabolic) Kazhdan–Lusztig bases attached to

a Coxeter system display a number of remarkable properties not at all evident from their elementary definition, and have connections to a surprising variety of topics in representation theory. It seems reasonable to expect that some interesting properties and connections will likewise hold in the quasiparabolic setting; [27, §9] presents several phenomena along these lines.

The main obstruction to formulating a definition of a “quasiparabolic Kazhdan–Lusztig basis” is proving the existence a bar operator for the \mathcal{H} -modules \mathcal{M} and \mathcal{N} . For us, a *bar operator* is a \mathbb{Z} -linear map $\mathcal{M} \rightarrow \mathcal{M}$ (respectively, $\mathcal{N} \rightarrow \mathcal{N}$) which fixes elements of minimal height in each W -orbit in X and is compatible with the bar operator of \mathcal{H} in the sense of (1.2); see Definition 3.1. The following conjecture is equivalent to [27, Conjecture 8.4] by [27, Proposition 2.15]:

Conjecture. (See Rains and Vazirani [27].) *If X is a quasiparabolic set which is bounded below (in the sense that the heights of the elements in any given W -orbit are bounded below), then the corresponding modules \mathcal{M} and \mathcal{N} each have bar operators.*

In this paper, we develop a number of general consequences of this conjecture, and also prove that the conjecture holds in some motivating cases of interest. A more detailed outline of our results goes as follows. After stating some preliminaries in Section 2, we devote Section 3 to developing the general properties of bar operators, where we prove the following:

Theorem. (See Section 3.) *Suppose X is a quasiparabolic set which is bounded below. If either of the corresponding modules \mathcal{M} or \mathcal{N} has a bar operator, then both modules have unique bar operators which determine each other and are involutions.*

We write that X *admits a bar operator* if both of the corresponding modules \mathcal{M} and \mathcal{N} do; in this case, we prove that \mathcal{M} and \mathcal{N} each have a certain distinguished basis in the following sense:

Theorem. (See Theorem 3.14.) *Assume X is a quasiparabolic set which is bounded below and admits a bar operator. Then \mathcal{M} and \mathcal{N} each have a unique “canonical basis,” by which we mean a basis of elements invariant under the corresponding bar operator which is congruent to the “standard basis” X modulo $v^{-1}\mathbb{Z}[v^{-1}]$ -linear combinations of standard basis elements.*

These bases generalize Deodhar’s parabolic Kazhdan–Lusztig bases, and in Section 3.2 we show that they retain many of the same properties. In Section 3.3 we prove that the canonical bases of \mathcal{M} and \mathcal{N} define two ways of viewing the quasiparabolic set X as a W -graph.

For the preceding theorems to be of interest we must have other examples of quasiparabolic sets with bar operators, besides the motivating case $X = W/W_J$. In Section 4 we describe a source of such quasiparabolic sets. Let $\theta : W \rightarrow W$ be a group

automorphism with $\theta(S) = S$. Then W acts on itself by the twisted conjugation $w : x \mapsto w \cdot x \cdot \theta(w)^{-1}$; an orbit under this action is a *twisted conjugacy class*; and an element $x \in W$ is a *twisted involution* (relative to θ) if $x^{-1} = \theta(x)$.

Theorem. (See [Theorem 4.19](#).) *Any twisted conjugacy class of twisted involutions (relative to θ) which is quasiparabolic (relative to the height function $\frac{1}{2}\ell$) admits a bar operator.*

This result applies, in particular, to Rains and Vazirani’s motivating example of the conjugacy class of fixed-point-free involutions in the symmetric group, which thus index two “quasiparabolic Kazhdan–Lusztig bases.” In Sections 4.1 and 4.2 we prove several results which control which twisted conjugacy classes are quasiparabolic. Among these are the following statements, which show that the previous theorem’s restriction to the case of twisted involutions is not so limiting:

Theorem. (See [Corollary 4.7](#).) *In an arbitrary Coxeter group, all (ordinary) conjugacy classes which are quasiparabolic (relative to the height function $\frac{1}{2}\ell$) consist of involutions.*

Theorem. (See [Theorem 4.9](#).) *In a finite Coxeter group, all twisted conjugacy classes which are quasiparabolic (relative to the height function $\frac{1}{2}\ell$) consist of twisted involutions.*

There can exist quasiparabolic twisted conjugacy classes which do not consist of twisted involutions; we construct examples in a necessarily infinite Coxeter group in Section 4.2. In the last section of the paper, we list a number of open questions and problems.

2. Preliminaries

In this section (W, S) denotes an arbitrary Coxeter system with length function ℓ . We write \leq for the Bruhat order on W . Recall that if $s \in S$ and $w \in W$ then $sw < w$ if and only if $\ell(sw) = \ell(w) - 1$.

2.1. Quasiparabolic sets

Rains and Vazirani introduce the following definitions in [\[27, §2\]](#).

Definition 2.1. A *scaled W -set* is a W -set X with a height function $\text{ht} : X \rightarrow \mathbb{Q}$ satisfying

$$|\text{ht}(x) - \text{ht}(sx)| \in \{0, 1\} \quad \text{for all } s \in S \text{ and } x \in X.$$

Denote the set of reflections in W by $R = \{wsw^{-1} : w \in W \text{ and } s \in S\}$.

Definition 2.2. A scaled W -set (X, ht) is *quasiparabolic* if both of the following properties hold:

(QP1) If $\text{ht}(rx) = \text{ht}(x)$ for some $(r, x) \in R \times X$ then $rx = x$.

(QP2) If $\text{ht}(rx) > \text{ht}(x)$ and $\text{ht}(srx) < \text{ht}(sx)$ for some $(r, x, s) \in R \times X \times S$ then $rx = sx$.

Example 2.3. The set W with height function $\text{ht} = \ell$ is quasiparabolic relative to its action on itself by left (also, by right) multiplication and also when viewed as a scaled $W \times W$ -set relative to the action $(x, y) : w \mapsto xwy^{-1}$; see [27, Theorem 3.1].

Example 2.4. Let $J \subset S$ and define $W^J = \{w \in W : ws > w \text{ for all } s \in J\}$. It is well-known that any element $w \in W$ has a unique factorization $w = uv$ with $u \in W^J$ and $v \in W_J = \langle J \rangle$. Define

$$s \bullet w = \begin{cases} sw & \text{if } sw \in W^J \\ w & \text{otherwise} \end{cases} \quad \text{for } s \in S \text{ and } w \in W^J.$$

Then $\bullet : S \times W^J \rightarrow W^J$ extends to an action of W on W^J , which is isomorphic to the natural action of W on W/W_J . The W -set W^J is quasiparabolic relative to the height function $\text{ht} = \ell$. This example is fundamental, and motivates the name “quasiparabolic.”

Example 2.5. A conjugacy class in W is a scaled W -set relative to conjugation and the height function $\text{ht} = \ell/2$. This scaled W -set is sometimes but not always quasiparabolic; see Section 4.1.

We restate [27, Corollary 2.13] as the lemma which follows this definition:

Definition 2.6. An element x in a scaled W -set X is W -minimal (respectively, W -maximal) if $\text{ht}(sx) \geq \text{ht}(x)$ (respectively, $\text{ht}(sx) \leq \text{ht}(x)$) for all $s \in S$.

Lemma 2.7. (See Rains and Vazirani [27].) *If a scaled W -set is quasiparabolic, then each of its orbits contains at most one W -minimal element and at most one W -maximal element. These elements, if they exist, have minimal (respectively, maximal) height in their W -orbits.*

Remark 2.8. This property is enough to nearly classify the quasiparabolic conjugacy classes in the symmetric group. Assume that $W = S_n$ and $S = \{s_i = (i, i + 1) : i = 1, \dots, n - 1\}$. Suppose $\mathcal{K} \subset S_n$ is a quasiparabolic conjugacy class (relative to the height function $\text{ht} = \ell/2$). Since \mathcal{K} is finite, it contains a unique W -minimal element by Lemma 2.7. As every permutation is conjugate in S_n to its inverse (which has the same length), \mathcal{K} must consist of involutions. There are $1 + \lfloor n/2 \rfloor$ such conjugacy classes: $\{1\}$ and the conjugacy classes of $s_1 s_3 s_5 \cdots s_{2k-1}$ for positive integers k with $2k \leq n$. While $\{1\}$ is trivially quasiparabolic, the conjugacy class of $s_1 s_3 s_5 \cdots s_{2k-1}$ is quasiparabolic only if $2k = n$, since otherwise $s_2 s_4 s_6 \cdots s_{2k}$ belongs to the same conjugacy class but

has the same (minimal) length. The only remaining conjugacy class, consisting of the fixed-point-free involutions in S_n for n even, is quasiparabolic by [27, Theorem 4.6].

For the rest of this section, (X, ht) denotes a fixed quasiparabolic W -set. The following lemma is a consequence of [27, Theorem 2.8].

Lemma 2.9. (See Rains and Vazirani [27].) *Suppose $x_0 \in X$ is a W -minimal element. The set*

$$\mathcal{R}_{\text{ht}}(x) \stackrel{\text{def}}{=} \{w \in W : x = wx_0 \text{ such that } \text{ht}(x) = \ell(w) + \text{ht}(x_0)\} \tag{2.1}$$

is then nonempty for any element x in the W -orbit of x_0 .

Additionally, we have this definition from [27, §5], which attaches to X a certain partial order:

Definition 2.10. The *Bruhat order* on a quasiparabolic W -set X is the weakest partial order \leq with $x \leq rx$ for all $x \in X$ and $r \in R$ with $\text{ht}(x) \leq \text{ht}(rx)$.

Remark 2.11. If (X, ht) is one of the quasiparabolic W -sets in Examples 2.3 or 2.4, then the Bruhat order coincides with the usual Bruhat order on W restricted to X . If X is a quasiparabolic conjugacy class in W as in Example 2.5, then the Bruhat order on W restricts to an order which is equal to or stronger than the Bruhat order on X (viewed as a quasiparabolic set). In all known examples these two orders actually coincide, but showing whether this holds in general is an open problem; see the remarks following [27, Proposition 5.17] and also Conjecture 5.4. If these two orders were always equal, it would follow from [27, Proposition 5.16] that any quasiparabolic conjugacy class is a graded poset with respect to the order induced by the usual Bruhat order, a property which does not hold for arbitrary conjugacy classes in Coxeter groups.

It follows immediately from the definition that if $x, y \in X$ then $x < y$ implies $\text{ht}(x) < \text{ht}(y)$. Rains and Vazirani develop in [27, Section 5] several other general properties of the Bruhat order. Among other facts, they show that the set X is a graded poset relative to \leq , and that the length of every maximal chain in the Bruhat order between $x \leq y$ is $\text{ht}(y) - \text{ht}(x)$ [27, Proposition 5.16]. We note explicitly the following lemma (which appears as [27, Lemma 5.7]) for use later:

Lemma 2.12. (See Rains and Vazirani [27].) *Let $x, y \in X$ such that $x \leq y$ and $s \in S$. Then*

$$sy \leq y \Rightarrow sx \leq y \quad \text{and} \quad x \leq sx \Rightarrow x \leq sy.$$

2.2. Hecke algebra modules

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and recall that the *Iwahori–Hecke algebra* of (W, S) is the \mathcal{A} -algebra

$$\mathcal{H} = \mathcal{H}(W, S) = \mathcal{A}\text{-span}\{H_w : w \in W\}$$

defined in the introduction. For background on this algebra, see, for example, [2,17–19]. Observe that $H_s^{-1} = H_s + (v^{-1} - v)$ and that $H_w = H_{s_1} \cdots H_{s_k}$ whenever $w = s_1 \cdots s_k$ is a reduced expression. Hence every basis element H_w for $w \in W$ is invertible.

Rains and Vazirani show that the permutation representation of W on a quasiparabolic set deforms to a well-behaved representation of \mathcal{H} . In detail, for any scaled W -set (X, ht) let

$$\mathcal{M} = \mathcal{M}(X, \text{ht}) = \mathcal{A}\text{-span}\{M_x : x \in X\} \quad \text{and} \quad \mathcal{N} = \mathcal{N}(X, \text{ht}) = \mathcal{A}\text{-span}\{N_x : x \in X\}$$

denote the free \mathcal{A} -modules with bases given by the symbols M_x and N_x for $x \in X$. We call $\{M_x\}_{x \in X}$ and $\{N_x\}_{x \in X}$ the *standard bases* of \mathcal{M} and \mathcal{N} , respectively. We view the \mathcal{A} -modules \mathcal{M} and \mathcal{N} as distinct \mathcal{H} -modules according to the following result, which appears as [27, Theorem 7.1].

Theorem 2.13. (See Rains and Vazirani [27].) *Assume (X, ht) is a quasiparabolic W -set.*

(a) *There is a unique \mathcal{H} -module structure on \mathcal{M} such that for all $s \in S$ and $x \in X$*

$$H_s M_x = \begin{cases} M_{sx} & \text{if } \text{ht}(sx) > \text{ht}(x) \\ M_{sx} + (v - v^{-1})M_x & \text{if } \text{ht}(sx) < \text{ht}(x) \\ vM_x & \text{if } \text{ht}(sx) = \text{ht}(x). \end{cases}$$

(b) *There is a unique \mathcal{H} -module structure on \mathcal{N} such that for all $s \in S$ and $x \in X$*

$$H_s N_x = \begin{cases} N_{sx} & \text{if } \text{ht}(sx) > \text{ht}(x) \\ N_{sx} + (v - v^{-1})N_x & \text{if } \text{ht}(sx) < \text{ht}(x) \\ -v^{-1}N_x & \text{if } \text{ht}(sx) = \text{ht}(x). \end{cases}$$

Remark 2.14. Our notation, which is patterned on Soergel’s conventions in [31, §3], translates to that of [27] on setting $H_s = v^{-1}T_{\pm}(s)$ and M_x (respectively, N_x) = $v^{-\text{ht}(x)}T(x)$.

Note that the \mathcal{H} -modules \mathcal{M} and \mathcal{N} are identical if $sx \neq x$ for all $s \in S$ and $x \in X$. Note that this occurs if and only if $w \in W$ has even length whenever $wx = x$ for some $x \in X$. Following Rains and Vazirani [27, Definition 3.4], we say that a W -set X is *even* if it has these equivalent properties.

Denote by $A_1 = \langle s_0 \rangle$ the unique Coxeter group with a single simple generator s_0 . Identifying s_0 with the nontrivial permutation of $\{1, 2\}$ gives an isomorphism $A_1 \cong S_2$.

We view the product group $W \times A_1$ as a Coxeter group relative to the generating set $S \cup \{s_0\}$. In [27, §3], Rains and Vazirani describe a construction which attaches to any scaled W -set (X, ht) an even scaled $W \times A_1$ -set $(\tilde{X}, \tilde{\text{ht}})$, with the property that (X, ht) is quasiparabolic if and only if $(\tilde{X}, \tilde{\text{ht}})$ is quasiparabolic. Following [27], we refer to $(\tilde{X}, \tilde{\text{ht}})$ as the *even double cover* of (X, ht) . This construction is useful for reducing certain arguments to the even case, and so we review it here.

Fix a scaled W -set (X, ht) and define $\tilde{X} = X \times \mathbb{F}_2$, where \mathbb{F}_2 is the field of two elements viewed as the set $\{0, 1\}$ with addition computed modulo 2. The groups W and A_1 each act on \tilde{X} by

$$w : (x, k) \mapsto (wx, k + \ell(w)) \quad \text{and} \quad s_0 : (x, k) \mapsto (x, k + 1)$$

for $w \in W$ and $x \in X$ and $k \in \mathbb{F}_2$. These actions commute with each other and so define an action of $W \times A_1$ on \tilde{X} . Define a height function $\tilde{\text{ht}}$ on \tilde{X} by the formula

$$\tilde{\text{ht}}(x, k) = \text{ht}(x) + \begin{cases} 0 & \text{if } \text{ht}(x) \equiv k \pmod{2} \\ 1 & \text{if } \text{ht}(x) \not\equiv k \pmod{2}. \end{cases}$$

Observe that if $x_0 \in X$ is W -minimal if and only if $(x_0, \text{ht}(x_0)) \in \tilde{X}$ is $W \times A_1$ -minimal. The following result appears as [27, Theorem 3.5].

Theorem 2.15. (See Rains and Vazirani [27].) *If (X, ht) is a scaled W -set then $(\tilde{X}, \tilde{\text{ht}})$ is an even scaled $W \times A_1$ -set, which is quasiparabolic if and only if (X, ht) is quasiparabolic.*

Remember that if (X, ht) is quasiparabolic then the $\mathcal{H}(W \times A_1, S \cup \{s_0\})$ -modules $\mathcal{M}(\tilde{X}, \tilde{\text{ht}})$ and $\mathcal{N}(\tilde{X}, \tilde{\text{ht}})$ are isomorphic by construction. In the following lemma, which appears as [27, Proposition 7.7], note that H_{s_0} is the generator of $\mathcal{H}(A_1, \{s_0\}) \subset \mathcal{H}(W \times A_1, S \cup \{s_0\})$.

Lemma 2.16. (See Rains and Vazirani [27].) *Suppose every orbit in X contains a W -minimal element. The \mathcal{A} -linear maps $\mathcal{M}(X, \text{ht}) \rightarrow \mathcal{M}(\tilde{X}, \tilde{\text{ht}})$ and $\mathcal{N}(X, \text{ht}) \rightarrow \mathcal{M}(\tilde{X}, \tilde{\text{ht}})$ with*

$$M_x \mapsto (H_{s_0} + v^{-1})M_{(x, \text{ht}(x))} \quad \text{and} \quad N_x \mapsto (H_{s_0} - v)M_{(x, \text{ht}(x))} \quad \text{for } x \in X$$

are then injective homomorphisms of $\mathcal{H}(W, S)$ -modules.

3. Bar operators, canonical bases, and W -graphs

Everywhere in this section (W, S) is an arbitrary Coxeter system; $\mathcal{H} = \mathcal{H}(W, S)$ is its Iwahori–Hecke algebra; (X, ht) is a fixed quasiparabolic W -set; and $\mathcal{M} = \mathcal{M}(X, \text{ht})$ and $\mathcal{N} = \mathcal{N}(X, \text{ht})$ are the corresponding \mathcal{H} -modules defined by Theorem 2.13.

3.1. Bar operators

We write $f \mapsto \bar{f}$ for the ring involution of $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with $v \mapsto v^{-1}$. A map $U \rightarrow V$ of \mathcal{A} -modules is \mathcal{A} -antilinear if $x \mapsto y$ implies $ax \mapsto \bar{a}y$ for all $a \in \mathcal{A}$. Recall that we also use the notation $f \mapsto \bar{f}$ to denote the bar operator of \mathcal{H} defined the beginning of the introduction.

Definition 3.1. A \mathbb{Z} -linear map $\mathcal{M} \rightarrow \mathcal{M}$, denoted $M \mapsto \overline{M}$, is a bar operator if

$$\overline{HM} = \overline{H} \cdot \overline{M} \quad \text{and} \quad \overline{M_{x_0}} = M_{x_0}$$

for all $(H, M) \in \mathcal{H} \times \mathcal{M}$ and all W -minimal $x_0 \in X$. An \mathcal{A} -antilinear map $\mathcal{N} \rightarrow \mathcal{N}$ is a bar operator if the same conditions hold, *mutatis mutandis*.

Although at this point there is no obvious obstruction to the modules \mathcal{M} and \mathcal{N} each having multiple bar operators, we will nevertheless always denote such maps by the notation $X \mapsto \overline{X}$. We will soon see that in the case which interest us, if a bar operator exists then it is unique, which justifies this convention.

All of our results concern quasiparabolic W -sets whose orbits each contain a (unique) W -minimal element. Without loss of generality, we can always assume that the height function on such a W -set has values all greater than some fixed number, since it makes no difference to translate the height function by a constant on any given orbit. We therefore refer to quasiparabolic W -sets whose orbits all have W -minimal elements as those which are *bounded below*.

Assume (X, ht) is bounded below. The set $\mathcal{R}_{\text{ht}}(x) \subset W$ given by (2.1) is then well-defined for all $x \in X$, and if $x_0 \in X$ is the W -minimal element in the orbit of x , then $H_w M_{x_0} = M_x$ and $H_w N_{x_0} = N_x$ for all $w \in \mathcal{R}_{\text{ht}}(x)$. Therefore, if modules \mathcal{M} and \mathcal{N} have bar operators, then

$$\overline{M_x} = \overline{H_w} M_{x_0} \quad \text{and} \quad \overline{N_x} = \overline{H_w} N_{x_0} \quad \text{for any } w \in \mathcal{R}_{\text{ht}}(x). \tag{3.1}$$

The right sides of these formulas are defined unambiguously once we fix a choice of $w \in \mathcal{R}_{\text{ht}}(x)$. Since the bar operator of \mathcal{H} is an involution, this implies the following:

Proposition 3.2. *Assume (X, ht) is bounded below.*

- (a) *If \mathcal{M} (respectively, \mathcal{N}) has a bar operator, then it is unique.*
- (b) *If \mathcal{M} (respectively, \mathcal{N}) has a (unique) bar operator, then it is an involution.*

While (3.1) explicitly describes what the bar operators on \mathcal{M} and \mathcal{N} must be if they exist, it is difficult to show that these formulas are well-defined. We can show the following, however.

Theorem 3.3. *Assume (X, ht) is bounded below. Then \mathcal{M} and \mathcal{N} both have bar operators if the \mathcal{A} -antilinear maps defined by (3.1) are well-defined, in the sense that*

$$\overline{H_a}M_{x_0} = \overline{H_b}M_{x_0} \quad \text{and} \quad \overline{H_a}N_{x_0} = \overline{H_b}N_{x_0} \tag{3.2}$$

whenever $x_0 \in X$ is W -minimal and $a, b \in \mathcal{R}_{\text{ht}}(x)$ for some $x \in Wx_0$.

Proof. First, we will show that the result holds in the case when (X, ht) is an even quasiparabolic set. We will then prove that if the module attached to the even double cover $(\tilde{X}, \tilde{\text{ht}})$ of (X, ht) admits a bar operator, then the modules \mathcal{M} and \mathcal{N} each admit bar operators as well. Finally, we will check that (3.2) holds for (X, ht) only if the analogous condition holds for $(\tilde{X}, \tilde{\text{ht}})$.

For the first step, assume that (X, ht) is even (so that $\mathcal{M} = \mathcal{N}$) and that there exists a well-defined \mathcal{A} -antilinear map $\mathcal{M} \rightarrow \mathcal{M}$, to be denoted $M \mapsto \overline{M}$, such that if x belongs to the orbit of the W -minimal element $x_0 \in X$, then $\overline{M_x} = \overline{H_w}M_{x_0}$ for any $w \in \mathcal{R}_{\text{ht}}(x)$. Clearly $\overline{M_{x_0}} = M_{x_0}$ if $x_0 \in X$ is W -minimal since then $\mathcal{R}_{\text{ht}}(x_0) = \{1\}$, so to show that this map is a bar operator, it remains just to check that $\overline{H_s \cdot M_x} = \overline{H_s} \overline{M_x}$ for $s \in S$ and $x \in X$. Let $x_0 \in X$ be the W -minimal element in the orbit of x and choose $w \in \mathcal{R}_{\text{ht}}(x)$. If $\text{ht}(sx) > \text{ht}(x)$ so that $H_s M_x = M_{sx}$, then clearly $\ell(sw) > \ell(w)$ and $sw \in \mathcal{R}_{\text{ht}}(sx)$, so we have $H_s H_w = H_{sw}$ and

$$\overline{H_s \cdot M_x} = \overline{H_s \cdot H_w \cdot M_{x_0}} = \overline{H_s H_w} M_{x_0} = \overline{H_{sw}} M_{x_0} = \overline{M_{sx}} = \overline{H_s} \overline{M_x}.$$

If $\text{ht}(sx) < \text{ht}(x)$, then $M_x = H_s M_{sx}$ so $\overline{M_x} = \overline{H_s \cdot M_{sx}}$ by what we have just shown, and hence

$$\overline{H_s \cdot M_x} = \overline{H_s \cdot H_s \cdot M_{sx}} = \overline{H_1 + (v - v^{-1})H_s} \cdot \overline{M_{sx}} = \overline{M_{sx} + (v - v^{-1})M_x} = \overline{H_s} \overline{M_x}.$$

Since (X, ht) is even, this suffices to show that $\mathcal{M} = \mathcal{N}$ has a bar operator.

For the second part of the proof, suppose the $\mathcal{H}(W \times A_1, S \cup \{s_0\})$ -module $\mathcal{M}(\tilde{X}, \tilde{\text{ht}})$ admits a bar operator, defined with respect to the quasiparabolic set $(\tilde{X}, \tilde{\text{ht}})$. Lemma 2.16 shows that \mathcal{M} and \mathcal{N} may be identified with \mathcal{H} -submodules of $\mathcal{M}(\tilde{X}, \tilde{\text{ht}})$, and we claim that the bar operator on the latter module restricts (via these identifications) to bar operators on \mathcal{M} and \mathcal{N} . This is straightforward to prove after noting that $H_{s_0} + v^{-1}$ and $H_{s_0} - v$ are bar invariant elements of $\mathcal{H}(W \times A_1, S \cup \{s_0\})$ which commute with all elements of the subalgebra $\mathcal{H}(W, S)$, and also that $x \in X$ is W -minimal if and only if $(x, \text{ht}(x)) \in \tilde{X}$ is $W \times A_1$ -minimal. We omit the details.

Finally suppose (3.2) holds. Fix a W -minimal element $x_0 \in X$ and let x belong to its orbit, and write \tilde{M}_{x_0} and \tilde{N}_{x_0} for the images of $M_{x_0} \in \mathcal{M}$ and $N_{x_0} \in \mathcal{N}$ in $\mathcal{M}(\tilde{X}, \tilde{\text{ht}})$ under the homomorphisms in Lemma 2.16. Observe that $\tilde{M}_{x_0} - \tilde{N}_{x_0} = (v + v^{-1})M_{(x_0, \text{ht}(x_0))}$ and note that if $k \in \mathbb{F}_2$ then $\mathcal{R}_{\tilde{\text{ht}}}^e(x, k) = \{s_0^e \cdot w : w \in \mathcal{R}_{\text{ht}}(x)\}$ where $e \in \{0, 1\}$ is such that $e \equiv k - \text{ht}(x) \pmod{2}$. Hence, if $a, b \in \mathcal{R}_{\tilde{\text{ht}}}^e(x, k)$, then $a = s_0^e \cdot a'$ and $b = s_0^e \cdot b'$ for some $a', b' \in \mathcal{R}_{\text{ht}}(x)$, so by Lemma 2.16

$$\begin{aligned} (v + v^{-1})\overline{H}_a M_{(x_0, \text{ht}(x_0))} &= \overline{H}_{s_0}^e \cdot \overline{H}_{a'} \tilde{M}_{x_0} - \overline{H}_{s_0}^e \cdot \overline{H}_{a'} \tilde{N}_{x_0} \\ &= \overline{H}_{s_0}^e \cdot \overline{H}_{b'} \tilde{M}_{x_0} - \overline{H}_{s_0}^e \cdot \overline{H}_{b'} \tilde{N}_{x_0} = (v + v^{-1})\overline{H}_b M_{(x_0, \text{ht}(x_0))}. \end{aligned}$$

From this, we conclude that $\overline{H}_a M_{(x_0, \text{ht}(x_0))} = \overline{H}_b M_{(x_0, \text{ht}(x_0))}$, which is what we wanted to show. \square

As an application, we recover the following result of Deodhar from [5, §2].

Corollary 3.4. (See Deodhar [5].) *If $(X, \text{ht}) = (W^J, \ell)$ for some $J \subset S$, then the corresponding \mathcal{H} -modules $\mathcal{M} = \mathcal{M}(X, \text{ht})$ and $\mathcal{N} = \mathcal{N}(X, \text{ht})$ both admit unique bar operators.*

Proof. The condition in Theorem 3.3 holds trivially since $\mathcal{R}_{\text{ht}}(x) = \{x\}$ for all $x \in W^J$. \square

Remark 3.5. We do not know of any examples of transitive, bounded quasiparabolic sets (X, ht) for which $\mathcal{R}_{\text{ht}}(x)$ is always a singleton set as in the preceding proof except those isomorphic to (W^J, ℓ) for some $J \subset S$; see Conjecture 5.1.

Recall that \leq denotes the Bruhat order on (X, ht) , as given in Definition 2.10.

Lemma 3.6. *Assume (X, ht) is bounded below and let $x \in X$.*

- (a) *If \mathcal{M} has a bar operator $M \mapsto \overline{M}$ then $\overline{M}_x \in M_x + \mathcal{A}\text{-span}\{M_w : w < x\}$.*
- (b) *If \mathcal{N} has a bar operator $N \mapsto \overline{N}$ then $\overline{N}_x \in N_x + \mathcal{A}\text{-span}\{N_w : w < x\}$.*

In particular, when defined, $\{\overline{M}_x\}_{x \in X}$ and $\{\overline{N}_x\}_{x \in X}$ are \mathcal{A} -bases of \mathcal{M} and \mathcal{N} , respectively.

Proof. We only prove part (a), as the proof of (b) is identical. If x is W -minimal then the desired containment is automatic, so assume x is not W -minimal and that $\overline{M}_{x'} \in M_{x'} + \mathcal{A}\text{-span}\{M_w : w < x'\}$ for all $x' < x$ in X . There is then $s \in S$ such that $\text{ht}(sx) < \text{ht}(x)$ (by the definition of W -minimal), so, using that $M_x = H_s M_{sx}$ and the inductive hypothesis, we have

$$\begin{aligned} \overline{M}_x &= \overline{H}_s \cdot \overline{M}_{sx} \in \overline{H}_s \left(M_{sx} + \sum_{w < sx} \mathcal{A} \cdot M_w \right) \\ &\subset M_x + \mathcal{A}\text{-span}\{M_{sx}\} + \mathcal{A}\text{-span}\{\overline{H}_s M_w : w < sx\}. \end{aligned}$$

Since $\overline{H}_s = H_s + v^{-1} - v$ we have $\overline{H}_s M_w \in \mathcal{A} \cdot M_w + \mathcal{A} \cdot M_{sw}$ for all $w \in X$. Thus \overline{M}_x has the desired unitriangular form provided that whenever $w \in X$ such that $w < sx < x$ we have $sw < x$; this property holds by Lemma 2.12. Finally, since all lower intervals in the poset (X, \leq) are finite, it follows from (a) and (b) that $\{\overline{M}_x\}_{x \in X}$ and $\{\overline{N}_x\}_{x \in X}$ are \mathcal{A} -bases of \mathcal{M} and \mathcal{N} . \square

Write $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ for the \mathcal{A} -linear with $\Theta(H_w) = (-1)^{\ell(w)} \overline{H_w}$ for $w \in W$; one checks that Θ is an \mathcal{A} -algebra automorphism. Next, define

$$\text{ht}_{\min}(x) = \min_{w \in W} \text{ht}(wx) \quad \text{for } x \in X.$$

Note that $\text{ht}_{\min}(x) = \text{ht}(x_0)$ if there exists a W -minimal element x_0 in the orbit of x , and that otherwise $\text{ht}_{\min}(x)$ is undefined. Finally, when (X, ht) is bounded below and \mathcal{N} (respectively, \mathcal{M}) has a bar operator, we define $\Phi_{\mathcal{M}\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ and $\Phi_{\mathcal{N}\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{N}$ as the \mathcal{A} -linear maps with

$$\Phi_{\mathcal{M}\mathcal{N}}(M_x) = (-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \overline{N_x} \quad \text{and} \quad \Phi_{\mathcal{N}\mathcal{M}}(N_x) = (-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \overline{M_x} \tag{3.3}$$

for $x \in X$. These maps are “ Θ -twisted homomorphisms” of \mathcal{H} -modules in the following sense.

Lemma 3.7. *Assume (X, ht) is bounded below. When respectively defined, the maps $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$ are bijections such that for all $(H, M, N) \in \mathcal{H} \times \mathcal{M} \times \mathcal{N}$ it holds that*

$$\Phi_{\mathcal{M}\mathcal{N}}(HM) = \Theta(H)\Phi_{\mathcal{M}\mathcal{N}}(M) \quad \text{and} \quad \Phi_{\mathcal{N}\mathcal{M}}(HN) = \Theta(H)\Phi_{\mathcal{N}\mathcal{M}}(N).$$

Proof. Both $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$ are bijections since, by the previous lemma, they each map a basis to a basis. Since Θ is an algebra automorphism, to show that $\Phi_{\mathcal{M}\mathcal{N}}$ has the desired property it is enough to check that $\Phi_{\mathcal{M}\mathcal{N}}(H_s M_x) = -(-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \overline{H_s N_x} = \Theta(H_s)\Phi_{\mathcal{M}\mathcal{N}}(M_x)$ for $s \in S$ and $x \in X$. This is straightforward; for example, if $\text{ht}(sx) = \text{ht}(x)$ then

$$\Phi_{\mathcal{M}\mathcal{N}}(H_s M_x) = \Phi_{\mathcal{M}\mathcal{N}}(v M_x) = (-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \overline{v^{-1} N_x} = -(-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \overline{H_s N_x}.$$

The calculations in the case when $\text{ht}(sx) > \text{ht}(x)$ and $\text{ht}(sx) < \text{ht}(x)$ are similar. An identical argument shows that the same property holds for $\Phi_{\mathcal{N}\mathcal{M}}$. \square

We now prove the following conceptually plausible, but technically nontrivial result.

Proposition 3.8. *Assume (X, ht) is bounded below. If either of the corresponding \mathcal{H} -modules \mathcal{M} or \mathcal{N} has a bar operator, then both modules have unique bar operators.*

Proof. Assume that \mathcal{M} has a bar operator; we will show that this implies that \mathcal{N} does as well. The converse implication holds by a symmetric argument. Let $\Phi_{\mathcal{M}\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be the map given before Lemma 3.7, and define $N \mapsto \overline{N}$ as the \mathcal{A} -antilinear map $\mathcal{N} \rightarrow \mathcal{N}$ with

$$\overline{N_x} = (-1)^{\text{ht}(x) - \text{ht}_{\min}(x)} \Phi_{\mathcal{N}\mathcal{M}}^{-1}(M_x) \quad \text{for } x \in X.$$

We check that this map has the defining properties of a bar operator. If $x \in X$ is W -minimal then $M_x = \overline{M_x}$ so by definition $\overline{N_x} = N_x$. In turn, if $s \in S$ and $x \in X$ then we claim that

$$\begin{aligned} \Phi_{\mathcal{N}\mathcal{M}}(\overline{H_s \cdot N_x}) &= -H_s \cdot \Phi_{\mathcal{N}\mathcal{M}}(\overline{N_x}) = \overline{\Theta(H_s)\Phi_{\mathcal{N}\mathcal{M}}(N_x)} = \overline{\Phi_{\mathcal{N}\mathcal{M}}(H_s N_x)} \\ &= \Phi_{\mathcal{N}\mathcal{M}}(\overline{H_s N_x}). \end{aligned}$$

To check this, observe that the first and third equalities hold by [Lemma 3.7](#); the second holds by definition since the bar operator on \mathcal{M} is an involution; and the last equality holds since by construction $\Phi_{\mathcal{N}\mathcal{M}}(\overline{N}) = \overline{\Phi_{\mathcal{N}\mathcal{M}}(N)}$ for all $N \in \mathcal{N}$. As $\Phi_{\mathcal{N}\mathcal{M}}$ is a bijection, we conclude that $\overline{H_s N_x} = \overline{H_s} \cdot \overline{N_x}$. Since the bar operator on \mathcal{H} is a ring involution, this suffices to show that $\overline{H_w} \cdot \overline{N_x} = \overline{H_w N_x}$ for all $w \in W$ and $x \in X$. We deduce by antilinearity that $\overline{HN} = \overline{H} \cdot \overline{N}$ for all $H \in \mathcal{H}$ and $N \in \mathcal{N}$. Hence the map $N \mapsto \overline{N}$ is a bar operator on \mathcal{N} , as desired. \square

Given a quasiparabolic W -set (X, ht) which is bounded below, we say that (X, ht) admits a bar operator if both (equivalently, either) of the modules \mathcal{M} and \mathcal{N} have a (unique) bar operator.

Remark 3.9. Assume (X, ht) is bounded below and admits a bar operator. Let

$$V = \mathcal{M} \text{ (respectively, } \mathcal{N}) \quad \text{and} \quad a_c = M_c \text{ (respectively, } N_c) \text{ for } c \in X.$$

Also define $\langle -, - \rangle : V \times V \rightarrow \mathcal{A}$ as the \mathcal{A} -sesquilinear inner product with

$$\langle a_c, \overline{a_{c'}} \rangle = \delta_{c,c'} \quad \text{for } c, c' \in X.$$

Combining [Definition 3.1](#), [Proposition 3.2](#), and [Lemma 3.6](#) shows that the bar operator on V together with the inner product $\langle -, - \rangle$ and the “standard basis” $\{a_c\}_{c \in X}$ are what Webster [\[37\]](#) calls a *pre-canonical structure*. When $(X, \text{ht}) = (W, \ell)$, this pre-canonical structure arises from a “categorification” of the Iwahori–Hecke algebra, via the theory of either intersection cohomology or Soergel bimodules; see [\[9,10,30,32\]](#). It would be interesting to have an interpretation along these lines for the pre-canonical structure attached to a general quasiparabolic W -set.

The following statement is clear from the proof of [Theorem 3.3](#).

Corollary 3.10. *Assume (X, ht) is bounded below. The quasiparabolic set (X, ht) then admits a bar operator if and only if its even double cover $(\tilde{X}, \tilde{\text{ht}})$ also admits a bar operator.*

When (X, ht) is the W -set of left cosets of a standard parabolic subgroup (see [Example 2.4](#)), the following proposition reduces to the main result of Deodhar’s paper [\[6, Theorem 2.1\]](#).

Proposition 3.11. *Assume the quasiparabolic W -set (X, ht) is bounded below and admits a bar operator, so that the maps $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$ are both defined.*

(a) *The following diagrams commute:*

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{M \mapsto \overline{M}} & \mathcal{M} \\
 \Phi_{\mathcal{M}\mathcal{N}} \downarrow & & \downarrow \Phi_{\mathcal{M}\mathcal{N}} \\
 \mathcal{N} & \xrightarrow{N \mapsto \overline{N}} & \mathcal{N}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{M \mapsto \overline{M}} & \mathcal{M} \\
 \Phi_{\mathcal{N}\mathcal{M}} \uparrow & & \uparrow \Phi_{\mathcal{N}\mathcal{M}} \\
 \mathcal{N} & \xrightarrow{N \mapsto \overline{N}} & \mathcal{N}
 \end{array}$$

(b) *The following diagrams commute:*

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{M \mapsto \overline{M}} & \mathcal{M} \\
 \Phi_{\mathcal{M}\mathcal{N}} \downarrow & & \uparrow \Phi_{\mathcal{M}\mathcal{N}}^{-1} \\
 \mathcal{N} & \xrightarrow{N \mapsto \overline{N}} & \mathcal{N}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{M \mapsto \overline{M}} & \mathcal{M} \\
 \Phi_{\mathcal{N}\mathcal{M}} \uparrow & & \downarrow \Phi_{\mathcal{N}\mathcal{M}}^{-1} \\
 \mathcal{N} & \xrightarrow{N \mapsto \overline{N}} & \mathcal{N}
 \end{array}$$

(c) *The maps $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$ are inverses of each other.*

Proof. Parts (a) and (b) follow from Proposition 3.2 and the definitions of $\Phi_{\mathcal{N}\mathcal{M}}$ and $\Phi_{\mathcal{M}\mathcal{N}}$, while part (c) follows from the definitions and part (a). \square

3.2. Canonical bases

Everywhere in this section we assume that (X, ht) is a quasiparabolic W -set which is bounded below and admits a bar operator; $\mathcal{M} = \mathcal{M}(X, \text{ht})$ and $\mathcal{N} = \mathcal{N}(X, \text{ht})$ are as in Theorem 2.13. We begin by recalling the following well-known theorem of Kazhdan and Lusztig [18]:

Theorem 3.12. (See Kazhdan and Lusztig [18].) *For each $w \in W$ there is a unique $\underline{H}_w \in \mathcal{H}$ with*

$$\underline{H}_w = \overline{H}_w \in H_w + \sum_{y < w} v^{-1} \mathbb{Z}[v^{-1}] \cdot H_y.$$

The elements $\{\underline{H}_w\}_{w \in W}$ form an \mathcal{A} -basis for \mathcal{H} , called the Kazhdan–Lusztig basis.

One checks that $\underline{H}_1 = H_1 = 1$ and $\underline{H}_s = H_s + v^{-1}$ for $s \in S$. Define $h_{x,y} \in \mathbb{Z}[v^{-1}]$ for $x, y \in W$ such that $\underline{H}_y = \sum_{x \in W} h_{x,y} H_x$. It follows by recent work of Elias and Williamson [9] that the polynomials $h_{x,y}$ actually always belong to $\mathbb{N}[v^{-1}]$. Moreover, when W is the Weyl group of a complex semisimple Lie algebra, these polynomials encode in a certain precise sense the multiplicities of simple modules in Verma modules

in the principal block of category \mathcal{O} ; this is the original *Kazhdan–Lusztig conjecture* [18, Conjecture 1.5].

Such phenomena suggest that it would be interesting to formulate an analogue of the Kazhdan–Lusztig basis for the modules \mathcal{M} and \mathcal{N} . For this purpose, we will need the following lemma:

Lemma 3.13. *Let $\mathcal{C} \subset \mathcal{A}$ be a subset such that $\{f \in \mathcal{C} : \bar{f} = f\} = \{0\}$; for example, $v^{-1}\mathbb{Z}[v^{-1}]$. Then 0 is the only element of \mathcal{M} (respectively, \mathcal{N}) which is both (i) invariant under the bar operator and (ii) a linear combination of standard basis elements with coefficients all in \mathcal{C} .*

Proof. Let $\varepsilon_x \in \mathcal{C}$ for $x \in X$ be such that the element $\varepsilon = \sum_{x \in X} \varepsilon_x M_x$ (respectively, $\sum_{x \in X} \varepsilon_x N_x$) has properties (i) and (ii). Suppose $\varepsilon \neq 0$; we argue by contradiction. Let x be maximal in (X, \leq) such that $\varepsilon_x \neq 0$. By Lemma 3.6, the coefficient of M_x (respectively, N_x) in $\bar{\varepsilon}$ is then $\bar{\varepsilon}_x$, so since $\bar{\varepsilon} = \varepsilon$ we must have $\bar{\varepsilon}_x = \varepsilon_x$; our hypothesis on \mathcal{C} now leads to the contradiction $\varepsilon_x = 0$. \square

The following generalizes Theorem 3.12 and also results of Deodhar from [5, §3].

Theorem 3.14. *Assume the quasiparabolic W -set (X, ht) is bounded below and admits a bar operator. For each $x \in X$ there are unique elements $\underline{M}_x \in \mathcal{M}(X, \text{ht})$ and $\underline{N}_x \in \mathcal{N}(X, \text{ht})$ with*

$$\underline{M}_x = \overline{\underline{M}_x} \in M_x + \sum_{w < x} v^{-1}\mathbb{Z}[v^{-1}] \cdot M_w \quad \text{and} \quad \underline{N}_x = \overline{\underline{N}_x} \in N_x + \sum_{w < x} v^{-1}\mathbb{Z}[v^{-1}] \cdot N_w$$

where both sums are over $w \in X$. The elements $\{\underline{M}_x\}_{x \in X}$ and $\{\underline{N}_x\}_{x \in X}$ form \mathcal{A} -bases for $\mathcal{M}(X, \text{ht})$, $\mathcal{N}(X, \text{ht})$, which we refer to as the canonical bases of these modules.

Proof. The theorem follows from the general fact (first proved using different terminology by Du [8]) that any pre-canonical structure whose index set (X, \leq) has finite lower intervals admits a unique canonical basis; compare Remark 3.9 with [25, Theorem 2.5]. For a self-contained proof, one can adapt, almost verbatim, the argument which Soergel gives to prove [31, Theorem 3.1]. \square

Define $m_{x,y}$ and $n_{x,y}$ for $x, y \in X$ as the polynomials in $\mathbb{Z}[v^{-1}]$ such that

$$\underline{M}_y = \sum_{x \in X} m_{x,y} M_x \quad \text{and} \quad \underline{N}_y = \sum_{x \in X} n_{x,y} N_x. \tag{3.4}$$

Let $\mu_m(x, y)$ and $\mu_n(x, y)$ denote the coefficients of v^{-1} in $m_{x,y}$ and $n_{x,y}$ respectively. Observe that if $x < y$ then $m_{x,y}$ and $n_{x,y}$ are both polynomials in v^{-1} without constant term, while if $x \not< y$ then $m_{x,y} = n_{x,y} = \delta_{x,y}$. When $(X, \text{ht}) = (W, \ell)$ as in Example 2.3, we have $m_{x,y} = n_{x,y} = h_{x,y}$.

Remark 3.15. A surprising property of the polynomials $h_{x,y}$ is that their coefficients are always nonnegative [9]. By contrast, $m_{x,y}$ and $n_{x,y}$ can each have both positive and negative coefficients. If $(X, \text{ht}) = (W^J, \ell)$ for some $J \subset S$ as in Example 2.4, then $\{m_{x,y}\} \subset \{h_{x,y}\} \subset \mathbb{N}[v^{-1}]$ (see [5, Proposition 3.4]), but even in this case the polynomials $n_{x,y}$ may still have negative coefficients.

The following theorem describes the action of \mathcal{H} on the basis elements \underline{M}_x and \underline{N}_x .

Theorem 3.16. *Let $s \in S$ and $x \in X$. Recall that $\underline{H}_s = H_s + v^{-1}$.*

(a) *In \mathcal{M} , the following multiplication formula holds:*

$$\underline{H}_s \underline{M}_x = \begin{cases} (v + v^{-1})\underline{M}_x & \text{if } \text{ht}(sx) \leq \text{ht}(x) \\ \underline{M}_{sx} + \sum_{sw \leq w < x} \mu_m(w, x)\underline{M}_w & \text{if } \text{ht}(sx) > \text{ht}(x). \end{cases}$$

(b) *In \mathcal{N} , the following multiplication formula holds:*

$$\underline{H}_s \underline{N}_x = \begin{cases} (v + v^{-1})\underline{N}_x & \text{if } \text{ht}(sx) < \text{ht}(x) \\ \underline{N}_{sx} + \sum_{sw < w < x} \mu_n(w, x)\underline{N}_w & \text{if } \text{ht}(sx) > \text{ht}(x) \\ \sum_{sw < w < x} \mu_n(w, x)\underline{N}_w & \text{if } \text{ht}(sx) = \text{ht}(x). \end{cases}$$

Proof. We first prove part (a); there are three cases to consider. First suppose $\text{ht}(sx) > \text{ht}(x)$. Using the definition of the module \mathcal{M} in Theorem 2.13, one checks that the linear combination $\underline{H}_s \underline{M}_x - \underline{M}_{sx} - \sum_{sw \leq w < x} \mu_m(w, x)\underline{M}_w$ is bar invariant and belongs to $v^{-1}\mathbb{Z}[v^{-1}]\text{-span}\{M_w : w \in X\}$, so it must be zero by Lemma 3.13.

Next suppose $sx = x$. The following identity then holds, since one can check that the difference between the two sides is a bar invariant linear combination of standard basis elements M_w with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$, and the only such element is zero by Lemma 3.13:

$$\underline{H}_s \underline{M}_x = (v + v^{-1})\underline{M}_x - \sum_{sw > w < x} \mu_m(w, x)\underline{M}_w. \tag{3.5}$$

Since $\underline{H}_s \underline{H}_s = (v + v^{-1})\underline{H}_s$, multiplying both sides of this equation by \underline{H}_s implies that

$$\sum_{sw > w < x} \mu_m(w, x)\underline{H}_s \underline{M}_w = 0.$$

By considering those $w \in X$ which are maximal in the Bruhat order such $sw > w < x$ and $\mu_m(w, x) \neq 0$, and then expanding the products $\underline{H}_s \underline{M}_w$, it becomes clear that the preceding equation can only hold if $\mu_m(w, x) = 0$ for all $w \in X$ with $sw > w < x$. We conclude from (3.5) that $\underline{H}_s \underline{M}_x = (v + v^{-1})\underline{M}_x$ when $sx = x$.

Finally suppose $\text{ht}(sx) < \text{ht}(x)$. What we have already shown implies $\underline{M}_x = \underline{H}_s \underline{M}_{sx} - \sum_{sw \leq w < sx} \mu_m(w, sx) \underline{M}_w$. Since $\underline{H}_s \underline{H}_s = (v + v^{-1}) \underline{H}_s$, we obtain by induction

$$\underline{H}_s \underline{M}_x = (v + v^{-1}) \underline{H}_s \underline{M}_{sx} - \sum_{sw \leq w < sx} \mu_m(w, sx) \underline{H}_s \underline{M}_w = (v + v^{-1}) \underline{M}_x.$$

This completes the proof of part (a).

The proof of part (b) is similar. The formula for $\underline{H}_s \underline{N}_x$ when $\text{ht}(sx) \neq \text{ht}(x)$ follows by arguments similar to the ones already given. When $sx = x$, one checks that $\underline{H}_s \underline{N}_x - \sum_{sw < w < x} \mu_n(w, x) \underline{N}_w$ is a bar invariant element of $v^{-1} \mathbb{Z}[v^{-1}]$ -span $\{N_w : w \in X\}$, and hence zero by Lemma 3.13. \square

Define $\tilde{m}_{x,y} = v^{\text{ht}(y) - \text{ht}(x)}$ and $\tilde{n}_{x,y} = v^{\text{ht}(y) - \text{ht}(x)} n_{x,y}$ for $x, y \in X$. The preceding theorem translates to the following recurrences, which one can use to compute these polynomials.

Corollary 3.17. *Let $x, y \in X$ and $s \in S$.*

(a) *If $sy = y$ then $\tilde{m}_{x,y} = \tilde{m}_{sx,y}$ and if $sy < y$ then*

$$\begin{aligned} \tilde{m}_{x,y} = \tilde{m}_{sx,y} &= \begin{cases} \tilde{m}_{x,sy} + v^2 \cdot \tilde{m}_{sx,sy} & \text{if } sx > x \\ v^2 \cdot \tilde{m}_{x,sy} + \tilde{m}_{sx,sy} & \text{if } sx \leq x \end{cases} \\ &- \sum_{\substack{x < t < sy \\ st \leq t}} \mu_m(t, sy) \cdot v^{\text{ht}(y) - \text{ht}(t)} \cdot \tilde{m}_{x,t}. \end{aligned}$$

(b) *If $sy < y$ then*

$$\begin{aligned} \tilde{n}_{x,y} = \tilde{n}_{sx,y} &= \begin{cases} \tilde{n}_{x,sy} + v^2 \cdot \tilde{n}_{sx,sy} & \text{if } sx > x \\ v^2 \cdot \tilde{n}_{x,sy} + \tilde{n}_{sx,sy} & \text{if } sx < x \\ 0 & \text{if } sx = x \end{cases} \\ &- \sum_{\substack{x < t < sy \\ st < t}} \mu_n(t, sy) \cdot v^{\text{ht}(y) - \text{ht}(t)} \cdot \tilde{n}_{x,t}. \end{aligned}$$

Proof. The assertion that $\tilde{m}_{x,y} = \tilde{m}_{sx,y}$ if $sy \leq y$ follows by comparing the coefficients of M_x in the identity $\underline{H}_s \underline{M}_y = (v + v^{-1}) \underline{M}_y$. The second equality in part (b) follows by comparing coefficients in the identity $\underline{H}_s \underline{M}_{sy} = \underline{M}_y + \sum_{st \leq t < y} \mu_m(t, y) \underline{M}_t$. The proof of part (c) is similar. \square

By definition $m_{x,y} = n_{x,y} = 0$ when $x \not\leq y$. When $x \leq y$, the following parity property holds:

Proposition 3.18. *If $x, y \in X$ with $x \leq y$ then*

$$v^{\text{ht}(y)-\text{ht}(x)}m_{x,y} = \tilde{m}_{x,y} \in 1 + v^2\mathbb{Z}[v^2] \quad \text{and} \quad v^{\text{ht}(y)-\text{ht}(x)}n_{x,y} = \tilde{n}_{x,y} \in \mathbb{Z}[v^2].$$

Consequently, $\mu_m(x, y) = \mu_n(x, y) = 0$ whenever $\text{ht}(y) - \text{ht}(x)$ is even.

Proof. If y is W -minimal then $x \leq y$ implies $x = y$ in which case $\tilde{m}_{x,y} = \tilde{n}_{x,y} = 1 \in 1 + v^2\mathbb{Z}[v^2]$. Alternatively, suppose y is not W -minimal, so that there exists some $s \in S$ such that $sy < y$. We may assume by induction that $\tilde{m}_{x',y'}$ and $\tilde{n}_{x',y'}$ respectively belong to $1 + v^2\mathbb{Z}[v^2]$ and $v^2\mathbb{Z}[v^2]$ for all $x', y' \in X$ with $x' \leq y' < y$. The coefficients $\mu_m(t, sy)$ and $\mu_n(t, sy)$ are then nonzero only for those $t \in X$ with $\text{ht}(y) - \text{ht}(t)$ even, so the recurrences in Corollary 3.17 imply via Lemma 2.12 that $\tilde{m}_{x,y} \in 1 + v^2\mathbb{Z}[v^2]$ and $\tilde{n}_{x,y} \in v^2\mathbb{Z}[v^2]$. \square

Finally, we clarify that nothing is gained or lost by preferring the indeterminate v^{-1} over v in Theorem 3.14. Define for $y \in X$ the elements

$$\underline{M}'_y = \sum_{x \in X} (-1)^{\text{ht}(y)-\text{ht}(x)} \cdot \overline{n_{x,y}} \cdot M_x \quad \text{and} \quad \underline{N}'_y = \sum_{x \in X} (-1)^{\text{ht}(y)-\text{ht}(x)} \cdot \overline{m_{x,y}} \cdot N_x. \tag{3.6}$$

Write $\varepsilon(x) = (-1)^{\text{ht}(x)-\text{ht}_{\min}(x)}$ for $x \in X$ and recall the definition of the maps $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$ from (3.3). We note the following lemma.

Lemma 3.19. *For each $x \in X$ it holds that*

$$\underline{M}'_x = \varepsilon(x) \cdot \Phi_{\mathcal{N}\mathcal{M}}(\underline{N}_x) \quad \text{and} \quad \underline{N}'_x = \varepsilon(x) \cdot \Phi_{\mathcal{M}\mathcal{N}}(\underline{M}_x).$$

Proof. We have $\underline{M}'_x = \varepsilon(x) \cdot \overline{\Phi_{\mathcal{N}\mathcal{M}}(\underline{N}_x)}$ and $\underline{N}'_x = \varepsilon(x) \cdot \overline{\Phi_{\mathcal{M}\mathcal{N}}(\underline{M}_x)}$ by the definition of the maps $\Phi_{\mathcal{M}\mathcal{N}}$ and $\Phi_{\mathcal{N}\mathcal{M}}$. Proposition 3.11(a) shows that these equations remain valid after erasing the bar operators on the right, since the canonical bases of \mathcal{M} and \mathcal{N} are bar invariant. \square

Since the maps $\Phi_{\mathcal{N}\mathcal{M}}$ and $\Phi_{\mathcal{M}\mathcal{N}}$ are bijections, $\{\underline{M}'_x\}_{x \in X}$ and $\{\underline{N}'_x\}_{x \in X}$ are \mathcal{A} -bases of \mathcal{M} and \mathcal{N} , respectively. These bases are uniquely characterized analogously to Theorem 3.14, as follows.

Corollary 3.20. *For each $x \in X$, the elements \underline{M}'_x and \underline{N}'_x are the unique ones in \mathcal{M} and \mathcal{N} with*

$$\underline{M}'_x = \overline{\underline{M}'_x} \in M_x + \sum_{w < x} v\mathbb{Z}[v] \cdot M_w \quad \text{and} \quad \underline{N}'_x = \overline{\underline{N}'_x} \in N_x + \sum_{w < x} v\mathbb{Z}[v] \cdot N_w.$$

Proof. In view of [Lemma 3.19](#), both \underline{M}'_x and \underline{N}'_x are bar invariant by [Proposition 3.11\(a\)](#), and they are given by unitriangular linear combinations of standard basis elements of the desired form by definition. The uniqueness of the elements with these properties follows from [Lemma 3.13](#). \square

Remark 3.21. To conclude this section, we explain more precisely how our results and notation connect to earlier work. Define $T_w = v^{\ell(w)}H_w \in \mathcal{H}$ for $w \in W$. Often, for example in [\[5,18,27\]](#), formulas involving \mathcal{H} are written in the terms of the basis $\{T_w\}$ rather than $\{H_w\}$.

- If $(X, \text{ht}) = (W, \ell)$ as in [Example 2.3](#), then $\mathcal{M} \cong \mathcal{N} \cong \mathcal{H}$ as left \mathcal{H} -modules and $m_{x,y} = n_{x,y} = h_{x,y}$ for all $x, y \in W$. In this case the bases $\{\underline{M}_w\}$ and $\{\underline{M}'_w\}$ of \mathcal{M} may be respectively identified with the bases of \mathcal{H} which are denoted $\{C'_w\}$ and $\{C_w\}$ in [\[18\]](#).
- If $(X, \text{ht}) = (W^J, \ell)$ for some $J \subset S$ as in [Example 2.4](#), then \mathcal{M} (respectively, \mathcal{N}) is isomorphic to the \mathcal{H} -module \mathcal{M}^J defined in [\[5\]](#) with $u = q$ (respectively $u = -1$). In this case the basis which Deodhar denotes $\{C_w^J\}$ corresponds to the basis $\{\underline{M}'_w\}$ (respectively, $\{\underline{N}'_w\}$).

3.3. *W*-graphs

Recall that $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let \mathcal{X} be an \mathcal{H} -module which is free as an \mathcal{A} -module. Given an \mathcal{A} -basis $V \subset \mathcal{X}$, consider the directed graph with vertex set V and with an edge from $x \in V$ to $y \in V$ whenever there exists $H \in \mathcal{H}$ such that the coefficient of y in Hx is nonzero. Each strongly connected component in this graph spans a quotient \mathcal{H} -module since its complement spans a submodule of \mathcal{X} . There is a natural partial order on the set of strongly connected components in any directed graph, and this order in our present context gives rise to a filtration of \mathcal{X} . For some choices of bases of V , this filtration can be interesting and nontrivial.

When this procedure is applied to the Kazhdan–Lusztig basis of \mathcal{H} (viewed as a left module over itself), the graph one obtains has a particular form, which serves as the prototypical example of a *W*-graph. The notion of a *W*-graph dates to Kazhdan and Lusztig’s paper [\[18\]](#), but our conventions in the following definitions have been adopted from Stembridge’s more recent work [\[34,35\]](#).

Definition 3.22. Let I be a finite set. An *I*-labeled graph is a triple $\Gamma = (V, \omega, \tau)$ where

- (i) V is a finite vertex set;
- (ii) $\omega : V \times V \rightarrow \mathcal{A}$ is a map;
- (iii) $\tau : V \rightarrow \mathcal{P}(I)$ is a map assigning a subset of I to each vertex.

We write $\omega(x \rightarrow y)$ for $\omega(x, y)$ when $x, y \in V$. One views Γ as a weighted directed graph on the vertex set V with an edge from x to y when the weight $\omega(x \rightarrow y)$ is nonzero.

Definition 3.23. Fix a Coxeter system (W, S) . An S -labeled graph $\Gamma = (V, \omega, \tau)$ is a W -graph if the free \mathcal{A} -module generated by V may be given an \mathcal{H} -module structure with

$$H_s x = \begin{cases} vx & \text{if } s \notin \tau(x) \\ -v^{-1}x + \sum_{y \in V; s \notin \tau(y)} \omega(x \rightarrow y)y & \text{if } s \in \tau(x) \end{cases} \quad \text{for } s \in S \text{ and } x \in V.$$

The prototypical W -graph defined by the Kazhdan–Lusztig basis of \mathcal{H} has several notable features; Stembridge [34,35] calls W -graphs with these features *admissible*. We introduce the following slight variant of Stembridge’s definition.

Definition 3.24. An I -labeled graph $\Gamma = (V, \omega, \tau)$ is *quasi-admissible* if

- (a) it is *reduced* in the sense that $\omega(x \rightarrow y) = 0$ whenever $\tau(x) \subset \tau(y)$;
- (b) its edge weights $\omega(x \rightarrow y)$ are all integers;
- (c) it is bipartite;
- (d) the edge weights satisfy $\omega(x \rightarrow y) = \omega(y \rightarrow x)$ whenever $\tau(x) \not\subset \tau(y)$ and $\tau(y) \not\subset \tau(x)$.

The I -labeled graph Γ is *admissible* if its integer edge weights are all nonnegative.

Let (X, ht) denote a fixed quasiparabolic W -set which is bounded below and admits a bar operator, so that canonical bases $\{\underline{M}_x\} \subset \mathcal{M} = \mathcal{M}(X, \text{ht})$ and $\{\underline{N}_x\} \subset \mathcal{N} = \mathcal{N}(X, \text{ht})$ given in Theorem 3.14 are well-defined. We show below that these bases induce two quasi-admissible W -graph structures on the set X . Define the maps $\mu_m, \mu_n : X \times X \rightarrow \mathbb{Z}$ as just before (3.4).

Lemma 3.25. Let $x, y \in X$ with $x < y$.

- (a) If there exists $s \in S$ with $sy \leq y$ and $sx > x$, then $\mu_m(x, y) = \delta_{sx, y}$.
- (b) If there exists $s \in S$ with $sy < y$ and $sx \geq x$, then $\mu_n(x, y) = \delta_{sx, y}$.

Proof. Suppose $s \in S$ is such that $sy \leq y$ (respectively, $sy < y$), so that Corollary 3.17 implies

$$m_{x, y} = v^{\text{ht}(x) - \text{ht}(sx)} m_{sx, y} \quad (\text{respectively, } n_{x, y} = v^{\text{ht}(x) - \text{ht}(sx)} n_{sx, y}). \quad (3.7)$$

If $sx = y > x$ then $m_{x, y} = n_{x, y} = v^{-1}$ so $\mu_m(x, y) = \mu_n(x, y) = 1$. Suppose alternatively that $sx \neq y$. Lemma 2.12 then implies that $sx < y$, and so $m_{sx, y}$ and $n_{sx, y}$ both belong to $v^{-1}\mathbb{Z}[v^{-1}]$. If $sx > x$ then it follows by (3.7) that $m_{x, y}$ and $n_{x, y}$ are contained in $v^{-2}\mathbb{Z}[v^{-1}]$ so necessarily $\mu_m(x, y) = \mu_n(x, y) = 0$. It remains only to show that if $sx = x$

then $\mu_n(x, y) = 0$; for this, we note that if $sy < y$ and $sx = x$ then [Corollary 3.17\(a\)](#) reduces to the formula

$$\mu_n(x, y) = - \sum_{\substack{x < t < sy \\ st < t}} \mu_n(t, sy)\mu_n(x, t).$$

We may assume by induction that $\mu_n(x, t) = 0$ for all $t \in X$ with $x < t < y$ and $st < t$, and so we conclude from this formula that $\mu_n(x, y) = 0$ as desired. \square

Define $\tau_m, \tau_n : X \rightarrow \mathcal{P}(S)$ as the maps with

$$\tau_m(x) = \{s \in S : sx \leq x\} \quad \text{and} \quad \tau_n(x) = \{s \in S : sx \geq x\}$$

and let $\omega_m : X \times X \rightarrow \mathbb{Z}$ be the map with

$$\omega_m(x \rightarrow y) = \begin{cases} \mu_m(x, y) + \mu_m(y, x) & \text{if } \tau_m(x) \not\subset \tau_m(y) \\ 0 & \text{if } \tau_m(x) \subset \tau_m(y). \end{cases}$$

Define $\omega_n : X \times X \rightarrow \mathbb{Z}$ by the same formula, but with μ_m and τ_m replaced by μ_n and τ_n .

Theorem 3.26. *Both $\Gamma_m = (X, \omega_m, \tau_m)$ and $\Gamma_n = (X, \omega_n, \tau_n)$ are quasi-admissible W -graphs.*

Proof. To see that Γ_n is a W -graph, observe that [Lemma 3.25\(b\)](#) implies that the formula in [Theorem 3.16\(b\)](#) for the action of $H_s \in \mathcal{H}$ on $\underline{N}_x \in \mathcal{N}$ for $s \in S$ and $x \in X$ can be written as

$$H_s \underline{N}_x = \begin{cases} v \underline{N}_x & \text{if } sx < x \\ -v^{-1} \underline{N}_x + \sum_{y \in X; sy < y} (\mu_n(x, y) + \mu_n(y, x)) \underline{N}_y & \text{if } sx \geq x. \end{cases}$$

One checks that this coincides with the \mathcal{H} -module structure described in [Definition 3.23](#) for the maps $\tau = \tau_n$ and $\omega = \omega_n$.

To prove that Γ_n is a W -graph, recall the definition of the elements $\underline{N}'_x \in \mathcal{N}$ for $x \in X$ from [\(3.6\)](#). By [Theorem 3.16](#) and [Lemmas 3.7, 3.19, and 3.25](#), it holds that if $s \in S$ and $x \in X$ then

$$H_s \underline{N}'_x = \begin{cases} v \underline{N}'_x + \sum_{y \in X; sy \leq y} (\mu_m(x, y) + \mu_m(y, x)) \underline{N}'_y & \text{if } sx > x \\ -v^{-1} \underline{N}'_x & \text{if } sx \leq x. \end{cases} \tag{3.8}$$

The matrix of the action of H_s on the basis $\{\underline{N}'_x\}_{x \in X} \subset \mathcal{N}$ is evidently the transpose of the action proscribed by [Definition 3.23](#) with $\tau = \tau_m$ and $\omega = \omega_m$. Since \mathcal{H} is the quotient of the free \mathcal{A} -algebra generated by $\{H_s : s \in S\}$ by relations which are invariant under taking transposes, it follows that Γ_m is a W -graph.

By Proposition 3.18, the division of X into elements of even and odd height affords a bipartition of Γ_m and Γ_n . Properties (a) and (c) in Definition 3.24 hold by construction, so we conclude that Γ_m and Γ_n are both quasi-admissible. \square

Remark 3.27. If $(X, \text{ht}) = (W, \ell)$ then $\Gamma_m = \Gamma_n$ and both of these graphs coincide with the original admissible W -graph structure on W described in [18]. If W is finite and $(X, \text{ht}) = (W^J, \ell)$ for some $J \subset S$ as in Example 2.4, then Γ_m and Γ_n are distinct but still admissible, and are isomorphic to the subgraphs of the W -graph on W induced on the respective vertex sets

$$W^{J, \max} = \{w \in W : ws < w \text{ for all } s \in J\} \quad \text{and} \\ W^J = \{w \in W : ws > w \text{ for all } s \in J\}.$$

This result does not seem to be well-known, and originates in work of Couillens [4]; see Chmutov’s thesis [3, §1.2.4] for an exposition, as well as the related papers of Howlett and Yin [12,13].

In the literature on W -graphs, strongly connected components (in a W -graph Γ) are referred to as *cells* as explained at the beginning of this section, the cells of Γ define a filtration of its corresponding \mathcal{H} -module, and so classifying the cells is a natural problem of interest. When $(X, \text{ht}) = (W, \ell)$ the cells of $\Gamma_m = \Gamma_n$ are the *left cells* of (W, S) , about which there exists a substantial literature; see [2, Chapter 6] for an overview. It is a natural open problem to study to cells of the W -graphs Γ_m and Γ_n defined in this section for more general quasiparabolic sets.

4. Quasiparabolic conjugacy classes

Rains and Vazirani mention two W -actions motivating their study of quasiparabolic sets in [27]: the action of W on cosets of standard parabolic subgroups, and the action of W on itself by (twisted) conjugation. The quasiparabolic W -set coming from the former example is relatively well understood, having been studied, for example, in [4–6,31]. This section is devoted to quasiparabolic twisted conjugacy classes, about which less is known.

4.1. Necessary properties

Let $\text{Aut}(W, S)$ denote the group of automorphisms of W which preserve the set of simple generators S , and for $\theta \in \text{Aut}(W, S)$ define sets $W^{\theta,+}$ and W^+ by

$$W^{\theta,+} = \{(x, \theta) : x \in W\} = W \times \{\theta\} \quad \text{and} \quad W^+ = W \times \text{Aut}(W, S).$$

One gives a group structure to the set W^+ via the multiplication formula

$$(x, \alpha)(y, \beta) = (x \cdot \alpha(y), \alpha\beta).$$

The group W^+ is a semidirect product $W \rtimes \text{Aut}(W, S)$, which we sometimes refer to as the *extended (Coxeter) group* of W . We view W and $\text{Aut}(W, S)$ as subgroups of W^+ by identifying $x \in W$ and $\theta \in \text{Aut}(W, S)$ with $(x, 1)$ and $(1, \theta)$, respectively. The group W acts by conjugation on W^+ , and for each $\theta \in \text{Aut}(W, S)$ the subset $W^{\theta,+} \subset W^+$ is a union of W -conjugacy classes. The conjugation action of W on $W^{\theta,+}$ coincides with the θ -twisted conjugation action of W on itself. We identify each ordinary conjugacy class in W with a W -conjugacy class in the set $W^{\text{id},+} \subset W^+$.

Extend the length function on W to W^+ by setting $\ell(x, \theta) = \ell(x)$. Any W -conjugacy class \mathcal{K} in W^+ is then a scaled W -set with respect to the height function $\text{ht}(w) = \frac{1}{2}\ell(w)$ for $w \in \mathcal{K}$. If this scaled W -set is quasiparabolic, then we say that \mathcal{K} is *quasiparabolic*.

Example 4.1. Consider the $W \times W$ -conjugacy class of $(1, \theta) \in (W \times W)^+$, where $\theta \in \text{Aut}(W \times W)$ is the automorphism $\theta : (x, y) \mapsto (y, x)$. This conjugacy class (with the height function $\frac{1}{2}\ell$) is isomorphic as a scaled W -set to the quasiparabolic set (W, ℓ) from [Example 2.3](#). Via this example, one can view our results concerning quasiparabolic conjugacy classes as generalizing constructions (e.g., the Kazhdan–Lusztig basis) attached to W itself.

The main object of this section is to say something about when a W -conjugacy class in W^+ is quasiparabolic. We will need the following lemma, which is similar to a property Rains and Vazirani check in the course of their proof of [\[27, Theorem 3.1\]](#).

Lemma 4.2. *Let $w \in W^+$ and $r \in R$ and $s \in S$.*

- (a) *If $\ell(wr) > \ell(w)$ and $\ell(sw) < \ell(sw)$ then $swr = w$.*
- (b) *If $\ell(rw) > \ell(w)$ and $\ell(rws) < \ell(ws)$ then $rws = w$.*

Proof. We only prove part (a) since the other part is equivalent via the identity $\ell(x) = \ell(x^{-1})$. Since R is preserved by every $\theta \in \text{Aut}(W, S)$, to prove part (a) for all $w \in W^+$ it suffices to check the given statement for $w \in W$. Proceeding, suppose $w \in W$ is such that $\ell(wr) > \ell(w)$ and $\ell(sw) < \ell(sw)$. Let $w = s_1s_2 \cdots s_k$ be a reduced expression; then $sw = ss_1s_2 \cdots s_k$ is also a reduced expression since $\ell(sw) = \ell(w) + 1$ as $\ell(sw) > \ell(sw) \geq \ell(wr) - 1 \geq \ell(w)$. Given that $\ell(sw) < \ell(sw)$, the Strong Exchange Condition [\[17, Theorem 5.8\]](#) implies that either $swr = w$ or $swr = ss_1 \cdots s_{i-1}s_{i+1} \cdots s_k$ for some $1 \leq i \leq k$. The latter case cannot occur, since it implies that $wr = s_1 \cdots s_{i-1}s_{i+1} \cdots s_k$ which in turn implies the contradiction $\ell(wr) \leq k - 1 < \ell(w)$. \square

Define $\text{Des}_L(w) = \{s \in S : \ell(sw) < \ell(w)\}$ for $w \in W^+$.

Theorem 4.3. *Fix $\theta \in \text{Aut}(W, S)$ and let $\mathcal{K} \subset W^{\theta,+}$ be a quasiparabolic W -conjugacy class. Suppose $w = (x, \theta) \in W^+$ is the unique W -minimal element of \mathcal{K} and define $J = \text{Des}_L(w)$.*

- (a) For all $s \in J$ it holds that $sws = w$.
- (b) The standard parabolic subgroup $W_J \subset W$ is finite and preserved by θ .
- (c) It holds that $x = w_J$ where w_J denotes the longest element in W_J .

Proof. If $x = 1$ then $J = \emptyset$ and parts (a)–(c) hold vacuously. Therefore assume $\ell(x) = \ell(w) > 0$. To prove part (a), note that if $s \in J$ then we have $\ell(w) \leq \ell(sws) \leq \ell(sw) + 1 = \ell(w)$ since w is minimal in its conjugacy class, so $\ell(sws) = \ell(w)$ which implies that $sws = w$ since the conjugacy class of w is quasiparabolic.

For the first assertion in part (b), observe that $x^{-1}wx = (\theta(x), \theta)$ is W -conjugate to w and has the same length, so since w is the unique minimal element in its conjugacy class we must have $x = \theta(x)$, which implies that $J = \theta(J)$.

Fix $k \geq 1$ and let $s_i \in J$ be such that $s_1s_2 \cdots s_k$ is a reduced expression. Define $w_0 = w$ and $w_i = w_{i-1}s_i = ws_1s_2 \cdots s_i$ for $i \geq 1$. We claim that $\ell(w_i) = \ell(w) - i$ for all $0 \leq i \leq k$. We prove this by induction on i ; the claim is true if $i \in \{0, 1\}$ by part (a), so assume $i \geq 2$ and that $\ell(w_j) = \ell(w) - j$ when $j < i$. By part (a), $s_1w_{i-1} = ws_2 \cdots s_{i-1}$ and $s_1w_{i-1}s_i = ws_2 \cdots s_{i-1}s_i$. Since $s_2 \cdots s_{i-1}$ and $s_2 \cdots s_{i-1}s_i$ are reduced expressions of length less than i , it follows that

$$\ell(s_1w_{i-1}) = \ell(w_{i-2}) > \ell(w_{i-1}) \quad \text{and} \quad \ell(s_1w_{i-1}s_i) = \ell(w_{i-1})$$

by our inductive hypothesis. Now observe that if $\ell(w_i) = \ell(w_{i-1}s_i) \neq \ell(w_{i-1}) - 1$ then $\ell(w_{i-1}s_i) = \ell(w_{i-1}) + 1 > \ell(s_1w_{i-1}s_i)$, in which case the preceding lemma (with $r = s_1$ and $s = s_i$) gives $s_1w_{i-1} = w_{i-1}s_i$ which implies that $s_2 \cdots s_{i-1} = s_1s_2 \cdots s_{i-1}s_i$. The last identity contradicts our assumption that $s_1 \cdots s_k$ is a reduced expression, so we must have $\ell(w_i) = \ell(w_{i-1}) - 1 = \ell(w) - i$ as desired. Our claim thus holds for all i by induction.

It follows from the claim just proved that if $z \in W_J$ then $\ell(wz) = \ell(w) - \ell(z)$. Since the length of wz is necessarily nonnegative, we deduce that W_J must be finite, which completes the proof of part (b). To prove part (c), let $s_i \in S$ be such that $x = s_1 \cdots s_k$ is a reduced expression. Since $w_J = \theta(w_J)$ by part (b), our claim implies that $\ell(xw_J) = \ell(x\theta(w_J)) = \ell(ww_J) = \ell(x) - \ell(w_J)$. We may therefore assume that for some $j \geq 1$ it holds that $s_j s_{j+1} \cdots s_k$ is a reduced expression for $w_J^{-1} = w_J$. We now argue that $j = 1$. To show this, observe that $s_1 \in J = \text{Des}_L(w_J)$, so by our claim and part (a) it follows that

$$\ell(s_1ww_J) = \ell(ws_1w_J) = \ell(w) - \ell(s_1w_J) > \ell(w) - \ell(w_J) = \ell(ww_J).$$

Thus $s_1 \notin \text{Des}_L(ww_J)$, which clearly only holds if $j = 1$, since ww_J has length $j - 1$ and if $j > 1$ then $ww_J = (s_1 \cdots s_{j-1}, \theta)$. We conclude that $x = w_J$ which proves part (c). \square

Given $w \in W^+$ and $H \subset W$ and $\theta \in \text{Aut}(W)$, define the following subgroups:

$$C_W(w) = \{x \in W : xw = wx\} \quad \text{and} \quad N_{W,\theta}(H) = \{x \in W : xH = H\theta(x)\}.$$

The first subgroup is the usual centralizer while the second is a twisted normalizer.

Corollary 4.4. *If $w = (x, \theta) \in W^+$ is the unique W -minimal element in quasiparabolic W -conjugacy class then $C_W(w) = N_{W,\theta}(W_J)$ where $J = \text{Des}_L(w)$.*

Proof. [Theorem 4.3](#) shows that x is both central and equal to the longest element w_J in W_J . Pfeiffer and Röhrle have shown that usual centralizer $C_W(w_J)$ is equal to the usual normalizer of W_J if w_J is central in W_J [[26](#), [Proposition 2.2](#)]; their proof of this fact carries over to our slightly more general twisted situation with almost no modification. \square

We state below three more corollaries, after introducing some more notation. First define

$$I^+ = I^+(W, S) \stackrel{\text{def}}{=} \{w \in W^+ : w^2 = 1\}.$$

We refer to elements of I^+ as *twisted involutions*. Observe that a pair $(x, \theta) \in W^+$ is a twisted involution if and only if $\theta^2 = 1$ and $\theta(x) = x^{-1}$; in this situation; the element $x \in W$ is sometimes referred to as a twisted involution relative to θ . Additionally, for $\theta \in \text{Aut}(W, S)$ define

$$\iota(\theta) \stackrel{\text{def}}{=} \{(x^{-1}\theta(x), \theta) \in W^+ : x \in W\}$$

Observe that $\iota(\theta)$ is the W -conjugacy class of $(1, \theta) \in W^+$, so $\iota(\text{id}) = \{1\} \subset W$ and if $\theta^2 = 1$ then $\iota(\theta) \subset I^+$. When $\theta^2 = 1$, Hultman [[16](#)] refers to the elements of $\iota(\theta)$ as *twisted identities*. Both I^+ and $\iota(\theta)$ have a number of interesting properties; see, for example, [[14–16,28,29,33](#)].

Corollary 4.5. *Let $\theta \in \text{Aut}(W, S)$ and let $\mathcal{K} \subset W^{\theta,+}$ be a quasiparabolic W -conjugacy class. The operation $w \mapsto w^2$ then defines a surjective map $\mathcal{K} \rightarrow \iota(\theta^2)$.*

Proof. Let $w = (x, \theta)$ be the unique minimal element in \mathcal{K} . By [Theorem 4.3](#), $x = w_J$ for a θ -invariant subset $J \subset S$, so $x = x^{-1} = \theta(x)$ and $w^2 = (1, \theta^2) \in \iota(\theta^2)$, so the corollary follows. \square

Corollary 4.6. *If $\theta^2 = 1$ then all quasiparabolic W -conjugacy classes in $W^{\theta,+}$ are subsets of I^+ .*

Proof. This follows from the preceding corollary since if $\theta^2 = 1$ then the preimage of $\iota(\theta^2) = \{1\}$ under $w \mapsto w^2$ is precisely I^+ . \square

Corollary 4.7. *All quasiparabolic conjugacy classes in W are subsets of $\{w \in W : w^2 = 1\}$.*

Proof. This follows from the previous corollary since $\{w \in W : w^2 = 1\} = W^{\text{id},+} \cap I^+$. \square

Remark 4.8. When W is finite, this last corollary follows more directly from the well-known fact (discussed, for example, in the introduction of [24]) that every element of W is conjugate to its inverse, so only conjugacy classes of involutions have unique minimal elements. In an infinite Coxeter group an element can fail to be conjugate to its inverse.

Our last result in this section is the following theorem promised in the introduction.

Theorem 4.9. *If W is finite, then all quasiparabolic conjugacy classes in W^+ are subsets of I^+ .*

We prove the theorem after stating two preliminary lemmas. Recall that a Coxeter system (W, S) is *irreducible* if no proper nonempty subset $J \subset S$ is such that $st = ts$ for all $s \in J$ and $t \in S \setminus J$. If $J \subset S$ then we write W_J for the subgroup which J generates; then (W_J, J) is itself a Coxeter system, whose length function coincides with the restriction of $\ell : W \rightarrow \mathbb{N}$. Define

$$\mathcal{J} = \mathcal{J}(W, S) = \{J : \emptyset \subsetneq J \subset S \text{ such that } (W_J, J) \text{ is irreducible}\}.$$

For each $J \in \mathcal{J}$ we denote by $\pi_J : W \rightarrow W_J$ the unique surjective homomorphism with $\pi_J(s) = s$ for $s \in J$ and $\pi_J(s) = 1$ for $s \in S \setminus J$. The map

$$w \mapsto (\pi_J(w))_{J \in \mathcal{J}}$$

is then an isomorphism of Coxeter systems $W \xrightarrow{\sim} \prod_{J \in \mathcal{J}} W_J$, where the product group is interpreted as a Coxeter system relative to the generating set given by the image of S .

Fix $\theta \in \text{Aut}(W, S)$ and note that θ permutes the set \mathcal{J} , in the sense that $\theta(J) \in \mathcal{J}$ for all $J \in \mathcal{J}$. Given $J \in \mathcal{J}$, let J_1, J_2, \dots, J_k be the distinct elements of the $\langle \theta \rangle$ -orbit of J , ordered such that $J = J_1$ and $\theta(J_i) = J_{i+1}$ (indices interpreted modulo k). Define $W_{J,\theta} = W_{J_1} \times W_{J_2} \times \dots \times W_{J_k}$ and let τ_θ be the automorphism of $W_{J,\theta}$ with

$$\tau_\theta(x_1, \dots, x_{k-1}, x_k) = (\theta(x_k), \theta(x_1), \dots, \theta(x_{k-1})) \quad \text{for } x_i \in W_{J_i}.$$

Note that $(W_{J,\theta}, K)$ is a Coxeter system when K is the smallest set preserved by τ_θ which contains $(s, 1, \dots, 1) \in W_{J,\theta}$ for all $s \in J$, and $\tau_\theta \in \text{Aut}(W_{J,\theta}, K)$. Define $\pi_{J,\theta} : W^{\theta,+} \rightarrow (W_{J,\theta})^{\tau_\theta,+}$ by

$$\pi_{J,\theta}(x, \theta) = ((\pi_{J_1}(x), \pi_{J_2}(x), \dots, \pi_{J_k}(x)), \tau_\theta) \quad \text{for } x \in W.$$

We now state two lemmas using this formalism.

Lemma 4.10. *Fix $\theta \in \text{Aut}(W, S)$ and let $\mathcal{K} \subset W^{\theta,+}$ be a W -conjugacy class.*

- (a) For each $J \in \mathcal{J}(W, S)$, the image $\pi_{J,\theta}(\mathcal{K})$ is a $W_{J,\theta}$ -conjugacy class.
- (b) \mathcal{K} is quasiparabolic if and only if $\pi_{J,\theta}(\mathcal{K})$ is quasiparabolic for every $J \in \mathcal{J}(W, S)$.

Proof. We just sketch the idea of a proof of this result, which is intuitively clear. Part (a) follows by elementary considerations. The “only if” direction of part (b) follows from [27, Proposition 2.6] (which states that a set which is quasiparabolic relative to the action of a Coxeter group is also quasiparabolic relative to any of the group’s standard parabolic subgroups) while the “if” direction follows from [27, Proposition 3.3] (which states that the Cartesian product of several quasiparabolic sets is a quasiparabolic set relative to the Cartesian product of the acting Coxeter groups). \square

Lemma 4.11. *Suppose $\theta \in \text{Aut}(W, S)$ transitively permutes $\mathcal{J} = \mathcal{J}(W, S)$. Assume $|\mathcal{J}| \geq 2$ and let $\mathcal{K} \subset W^{\theta,+}$ be a W -conjugacy class.*

- (a) If $|\mathcal{J}| > 2$ then \mathcal{K} is not quasiparabolic.
- (b) If $|\mathcal{J}| = 2$ then \mathcal{K} is quasiparabolic if and only if its minimal element is $(1, \theta) \in W^+$.

Hence, if \mathcal{K} is quasiparabolic then $\mathcal{K} \subset I^+$.

Proof. Let $k = |\mathcal{J}(W, S)|$. Since θ transitively permutes the elements of $\mathcal{J}(W, S)$, we can assume without loss of generality that $W = W' \times W' \times \dots \times W'$ (k factors) for some Coxeter system (W', S') and that θ acts on W by the formula $(w_1, \dots, w_{k-1}, w_k) \mapsto (w_k, w_1, \dots, w_{k-1})$.

Suppose \mathcal{K} is quasiparabolic, and let $w_i \in W'$ be the elements such that $w = ((w_1, \dots, w_k), \theta) \in \mathcal{K}$ is the unique element of minimal length. We then must have $w_1 = \dots = w_k = 1$, since if some $w_i \neq 1$ then there would exist $s \in S'$ with $sw_i < w_i$, and in this case one can check that if $t = (1, \dots, s, \dots, 1) \in S$ is the simple generator with 1 in all but the i th coordinate, then $twt \neq w$ has $\ell(twt) \leq \ell(w)$, contradicting Lemma 2.7. Hence \mathcal{K} must contain the element $(1, \theta)$, which is automatically minimal since it has length 0.

We now argue that the case $k \geq 3$ leads to contradiction. For this, choose any $r \in S'$, and define $s, t \in S$ by $s = (r, 1, 1, \dots)$ and $t = (1, r, 1, \dots)$. If $k \geq 3$ then the element $x = s(1, \theta)s = ((s, s, 1, \dots), \theta) \in \mathcal{K}$ has $txt = ((s, 1, s, \dots), \theta) \neq x$ but $\text{ht}(txt) = \text{ht}(x) = 1$. This contradicts (QP1) in the definition of a quasiparabolic set, so we conclude that $k = 2$, which proves part (a) and one direction of part (b). For the rest of part (b), it remains to check that the W -conjugacy class of $(1, \theta)$ is in fact quasiparabolic when $k = 2$. This follows as a standard exercise from properties of the Bruhat order of W ; alternatively, the desired claim is a consequence of a general criterion of Rains and Vazirani which we will restate below as Theorem 4.13. \square

Finally, we prove Theorem 4.9.

Proof of Theorem 4.9. Let $\theta \in \text{Aut}(W, S)$ and suppose $\mathcal{K} \subset W^{\theta,+}$ is a quasiparabolic W -conjugacy class. To show that $\mathcal{K} \subset I^+$, we reduce via Lemma 4.10 to the case when θ acts transitively on $\mathcal{J}(W, S)$. In this situation, Lemma 4.11 implies that if $\mathcal{J}(W, S)$ has $k \geq 2$ elements then $\mathcal{K} \subset I^+$ (and in fact $k = 2$). On the other hand, if $\theta^2 = 1$ then by Corollary 4.6 we likewise have $\mathcal{K} \subset I^+$. It thus only remains to show that $\mathcal{K} \subset I^+$ if (W, S) is irreducible and $\theta^2 \neq 1$. This actually leaves very little left to check: for if (W, S) is finite and irreducible and $\theta \in \text{Aut}(W, S)$ is not an involution, then by the classification results in [17, Chapter 2], (W, S) is necessarily of type D_4 and θ can be identified with the automorphism of order three described in [11, §6.2].

Explicitly, let W be the Coxeter group of D_4 , i.e. the group generated by the set of involutions $S = \{s_1, s_2, s_3, s_4\}$, where s_1, s_2, s_4 pairwise commute and $s_i s_3$ has order 3 for $i \in \{1, 2, 4\}$. Assume $\theta \in \text{Aut}(W, S)$ is not an involution. Then, after possibly relabeling the simple generators, we may assume that θ acts on S by mapping $s_1 \mapsto s_3$ and $s_3 \mapsto s_4$ and $s_4 \mapsto s_1$ and $s_2 \mapsto s_2$. Calculations of Geck, Kim, and Pfeiffer (see [11, Table I]) show that only two W -conjugacy classes in $W^{\theta,+}$ have unique elements of minimal length, namely, the conjugacy classes of $(1, \theta)$ and (s_2, θ) . One checks that both classes violate (QP1) in Definition 2.2: the first class contains $x = s_1(1, \theta)s_1 = (s_1 s_3, \theta)$ which has the same length as $s_3 x s_3 \neq x$, while the second class contains $y = s_1 s_2 s_3 (s_2, \theta) s_3 s_2 s_1$ which has the same length as $s_2 y s_2 \neq y$. We conclude that $\mathcal{K} \subset I^+$, as desired. \square

4.2. Sufficient conditions

Rains and Vazirani prove a useful sufficient condition for a W -conjugacy class of twisted involutions to be quasiparabolic. Recall that $R = \{wsw^{-1} : (w, s) \in W \times S\}$ is the set of reflections in W .

Definition 4.12. A twisted involution $w \in I^+$ is *perfect* if $(rw)^4 = 1$ for all $r \in R$.

Observe that if $w \in I^+$ is perfect then all elements in the W -conjugacy class of w are also perfect, so it makes sense to say that a W -conjugacy class of twisted involutions is *perfect* if any of its elements are. The following appears as [27, Theorem 4.6].

Theorem 4.13. (See Rains and Vazirani [27].) *All perfect conjugacy classes in I^+ are quasiparabolic.*

As Rains and Vazirani note in [27], it is straightforward to check that all fixed-point-free involutions in S_{2n} are perfect. Therefore:

Corollary 4.14. (See Rains and Vazirani [27].) *The conjugacy class of fixed-point-free involutions in the symmetric group S_{2n} is quasiparabolic for all n .*

Rains and Vazirani describe explicitly the perfect W -conjugacy classes in W^+ when W is finite in [27, Example 9.2]. There can exist quasiparabolic conjugacy classes in I^+ which are not perfect, however, even when W is finite. For example:

- If (W, S) has type F_4 , then we have checked with a computer that the conjugacy class of the nontrivial diagram automorphism in $\text{Aut}(W, S) \subset I^+$ has 72 elements and is quasiparabolic but not perfect.
- If (W, S) has type $I_2(2m)$, then the conjugacy classes of each simple generator are disjoint of size m , while the conjugacy class of the nontrivial diagram automorphism in $\text{Aut}(W, S) \subset I^+$ has size $2m$. All three conjugacy classes are quasiparabolic, but the first two are perfect only when $m \in \{1, 2\}$ while the third is perfect only when $m = 1$.

By appealing to [Theorem 4.9](#) and using a computer for the exceptional types, one can show that when (W, S) is an irreducible finite Coxeter system these are the only examples of quasiparabolic W -conjugacy classes in W^+ which are not perfect. Combining this with [Lemmas 4.10 and 4.11](#) and the discussion in [\[27, Example 9.2\]](#) would afford a classification of all quasiparabolic conjugacy classes in a finite (extended) Coxeter group. We do not pursue this topic here, however.

We can describe examples of quasiparabolic conjugacy classes which are not comprised of twisted involutions. A Coxeter system (W, S) is *universal* if st has infinite order for all distinct generators $s, t \in S$. Each element of a universal Coxeter group has a unique reduced expression.

Proposition 4.15. *Suppose (W, S) is a universal Coxeter system. Let $\mathcal{K} \subset W^+$ be a W -conjugacy class. The following are then equivalent:*

- (a) \mathcal{K} is quasiparabolic.
- (b) \mathcal{K} contains a unique minimal element.
- (c) \mathcal{K} contains an element $(x, \theta) \in W^+$ with $x = \theta(x)$ and $x \in \{1\} \cup S$.

Remark 4.16. Note in the situation of (c) that (x, θ) has length 0 or 1 and so is necessarily an element of minimal length in \mathcal{K} , as conjugation preserves length parity.

Proof. By [Lemma 2.7](#), (a) \Rightarrow (b) so we only need to show that (c) \Rightarrow (b) and (c) \Rightarrow (a). For the first implication, suppose $w = (x, \theta) \in W^+$ is the unique minimal element in its W -conjugacy class. Since the conjugate element $x^{-1}wx = (\theta(x), \theta)$ has the same length as w , we must have $x = \theta(x)$. We wish to show that $x \in \{1\} \cup S$. If $x \neq 1$ then there is a unique reduced expression $x = s_1s_2 \cdots s_k$ where $k \geq 1$. The conjugate element $s_1ws_1 = (s_2 \cdots s_k\theta(s_1), \theta)$ then has length $\ell(w)$ or $\ell(w) - 2$; since w is the unique minimal element in its conjugacy class, the latter case cannot occur and we must have $s_1s_2 \cdots s_k = s_2 \cdots s_k\theta(s_1)$. Both of these expressions are reduced, so they can be equal only if $k = 1$, in which case $x \in S$.

This shows that (b) \Rightarrow (c) and it remains only to show that (c) \Rightarrow (a). For this, suppose $w = (x, \theta) \in W^+$ such that $x = \theta(x) \in \{1\} \cup S$, so that $\ell(w) \in \{0, 1\}$. Since W is universal, the centralizer $C_W(w) = \{z \in W : zw = wz\}$ is given by

$$C_W(w) = W_J \quad \text{where} \quad J = \begin{cases} \{x\} & \text{if } x \in S \\ \{s \in S : \theta(s) = s\} & \text{if } x = 1. \end{cases}$$

It follows by [27, Proposition 2.15] that the W -conjugacy class of w is isomorphic as a scaled W -set (after translating the height function by $\frac{1}{2}\ell(w)$) to (W^J, ℓ) . Since the latter set is quasiparabolic, so is the former, and thus (c) \Rightarrow (a) as required. \square

Corollary 4.17. *Suppose (W, S) is a universal Coxeter system. Then all W -conjugacy classes in I^+ are quasiparabolic, but there exist quasiparabolic W -conjugacy classes in W^+ which are not contained in I^+ whenever $|S| \geq 3$.*

Proof. Let $\mathcal{K} \subset I^+$ be a W -conjugacy class and let $w = (x, \theta) \in \mathcal{K}$ be some minimal element. To show that \mathcal{K} is quasiparabolic it suffices by the proposition to show that w is the unique minimal element in \mathcal{K} . If $x = 1$ then this is clear, so suppose $x \neq 1$ and choose $s \in S$ such that $\ell(sw) < \ell(w)$ for some $s \in S$. The minimality of w implies $\ell(sws) = \ell(w)$, which implies $sw = ws$ by a straightforward argument using the (weak) Exchange Condition; see [15, Lemma 3.4]. The identity $sw = ws$ is equivalent to $sx = x\theta(s)$, which can hold only if $x = s = \theta(s)$ since W is universal. By the proposition we therefore conclude that w is the unique minimal element in \mathcal{K} as desired.

Proposition 4.15 shows that when W is universal the W -conjugacy class of $(1, \theta)$ is quasiparabolic for any $\theta \in \text{Aut}(W, S)$. This conjugacy class is not a subset of I^+ whenever $\theta^2 \neq 1$, which can occur if $|S| \geq 3$ since $\text{Aut}(W, S)$ is isomorphic to the group of permutations of S . \square

As noted in the proof of Proposition 4.15, if (W, S) is universal and $\mathcal{K} \subset W^+$ is a quasiparabolic W -conjugacy class, then $(\mathcal{K}, \frac{1}{2}\ell)$ may be identified (after possibly translating the height function) with the quasiparabolic set (W^J, ℓ) for some $J \subset S$. Thus, Corollary 3.4 implies the following:

Corollary 4.18. *If (W, S) is a universal Coxeter system, then all quasiparabolic W -conjugacy classes in W^+ admit bar operators.*

4.3. Bar operators for twisted involutions

Let (W, S) be any Coxeter system and write $I_{\text{QP}}^+ = I_{\text{QP}}^+(W, S)$ for the union of all quasiparabolic W -conjugacy classes in $I^+ \subset W^+$. This union is, by construction, a quasiparabolic set relative to the height function $\frac{1}{2}\ell$, and it is bounded below. Given $w \in W^+$, define

$$|w|_m = v^{\ell_{\min}(w)} \quad \text{and} \quad |w|_n = (-v)^{-\ell_{\min}(w)} = (-1)^{\ell(w)} / |w|_m$$

where $\ell_{\min}(w) = \min_{x \in W} \ell(xwx^{-1})$. Note that these quantities depend only on the W -conjugacy class of w . Our main result in this section is the following theorem from the introduction.

Theorem 4.19. *The quasiparabolic set $(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ admits a bar operator. The (unique) bar operators on the corresponding \mathcal{H} -modules $\mathcal{M} = \mathcal{M}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ and $\mathcal{N} = \mathcal{N}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ act by the formulas*

$$\overline{M_{(x,\theta)}} = |(x, \theta)|_m \cdot \overline{H_x} \cdot M_{(x^{-1}, \theta)} \quad \text{and} \quad \overline{N_{(x,\theta)}} = |(x, \theta)|_n \cdot \overline{H_x} \cdot N_{(x^{-1}, \theta)}$$

for $(x, \theta) \in I_{\mathbb{Q}\mathbb{P}}^+$.

Remark 4.20. In [20,22,23], Lusztig and Vogan study a module of the Iwahori–Hecke algebra of an arbitrary Coxeter system on the free \mathcal{A} -module generated by all of I^+ . They prove (see [20, Theorems 0.2 and 0.4]) that this module possesses a “bar operator” admitting a unique invariant “canonical basis.” The formula for their bar operator is the same (up to scaling factors) as the ones given in the preceding theorem, but there does not appear to be any simple relationship between Lusztig and Vogan’s I^+ -indexed canonical basis and the two canonical bases we obtain for $\mathcal{M}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ and $\mathcal{N}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$; see Problem 5.9. Indeed, the polynomial coefficients of these bases satisfy different degree bounds, and the two W -graph structures on $I_{\mathbb{Q}\mathbb{P}}^+$ afforded by Theorems 3.26 and 4.19 can each have cells which are not subsets of the two-sided cells in W (as defined in [18]). By contrast, the results of [22, Section 5] show that the cells of the natural “ W -graph” structure on I^+ induced by Lusztig and Vogan’s canonical basis are always contained in a two-sided cell.

Proof. We prove the statement for the module \mathcal{M} ; the proof for \mathcal{N} is very similar. First, we check that $\overline{M_w} = M_w$ if $w = (x, \theta) \in I_{\mathbb{Q}\mathbb{P}}^+$ is the unique element of minimal length in its W -conjugacy class. By Theorem 4.3 there exists a θ -invariant subset $J \subset S$ such that $x = x^{-1} = w_J$ and $s w s = w$ for all $s \in J$. In any reduced expression $x = s_1 s_2 \cdots s_k$ every factor satisfies $s_i \in J$ (see [17, Theorem 5.5]), and so

$$\overline{H_x} M_w = H_{s_1}^{-1} H_{s_2}^{-1} \cdots H_{s_k}^{-1} M_w = v^{-\ell(x)} M_w.$$

As $\ell(x) = \ell_{\min}(w)$ since $w = (x, \theta)$ is W -minimal, it follows that $\overline{M_w} = M_w$ as desired.

According to Definition 3.1, it now remains only to show that $\overline{HM} = \overline{H} \cdot \overline{M}$ for all $H \in \mathcal{H}$ and $M \in \mathcal{M}$. For this it suffices to check that

$$\overline{H_s M_{(x,\theta)}} = \overline{H_s} \cdot \overline{M_{(x,\theta)}} \quad \text{for } s \in S \text{ and } (x, \theta) \in I_{\mathbb{Q}\mathbb{P}}^+.$$

Set $x' = s x \theta(s)$ so that $s(x, \theta)s = (x', \theta)$, and observe that $s x < x$ if and only if $\theta(s)x^{-1} < x^{-1}$ since $(x, \theta) \in I^+$ implies $x^{-1} = \theta(x)$. As an abbreviation we define $\kappa = |(x, \theta)|_m = |(x', \theta)|_m$. There are now three cases to consider, according to the difference in length between x and x' :

(1) If $\ell(x') > \ell(x)$ then $H_s H_x = H_{x'} \overline{H_{\theta(s)}}$ and $H_{\theta(s)} M_{(x^{-1}, \theta)} = M_{(x'^{-1}, \theta)}$ so

$$\overline{H_s} \cdot \overline{M_{(x,\theta)}} = \kappa \cdot \overline{H_{x'}} \cdot H_{\theta(s)} \cdot M_{(x^{-1}, \theta)} = \kappa \cdot \overline{H_{x'}} \cdot M_{(x'^{-1}, \theta)} = \overline{M_{(x', \theta)}} = \overline{H_s M_{(x,\theta)}}.$$

(2) If $\ell(x') < \ell(x)$ then $H_s H_x = H_{sx} + (v - v^{-1})H_x = H_{x'} H_{\theta(s)} + (v - v^{-1})H_x$ so

$$\overline{H_s} \cdot \overline{M_{(x,\theta)}} = \kappa \cdot \overline{H_{x'}} \cdot \overline{H_{\theta(s)}} \cdot M_{(x^{-1},\theta)} + (v^{-1} - v) \cdot \overline{M_{(x,\theta)}}.$$

Since $\overline{H_{\theta(s)}} M_{(x^{-1},\theta)} = M_{(x',\theta)}$, the right side of the preceding identity is equal to

$$\overline{M_{(x',\theta)}} + (v^{-1} - v) \cdot \overline{M_{(x,\theta)}} = \overline{H_s M_{(x,\theta)}}.$$

(3) If $\ell(x') = \ell(x)$ then $x' = x$ by condition (QP1) in Definition 2.2, so we have $H_s H_x = H_x H_{\theta(s)}$ and $H_s M_{(x,\theta)} = v M_{(x,\theta)}$, and therefore

$$\overline{H_s} \cdot \overline{M_{(x,\theta)}} = \kappa \cdot \overline{H_s} \cdot \overline{H_x} \cdot M_{(x^{-1},\theta)} = \kappa \cdot \overline{H_x} \cdot \overline{H_s} \cdot M_{(x^{-1},\theta)} = v^{-1} \overline{M_{(x,\theta)}} = \overline{H_s M_{(x,\theta)}}.$$

Hence the given \mathcal{A} -antilinear map $\mathcal{M} \rightarrow \mathcal{M}$ is a bar operator, which is what we set out to prove. \square

Assume (W, S) is a finite Coxeter system, so that W has a longest element w_0 . Recall since the longest element is unique, we have $w_0 = w_0^{-1} = \theta(w_0)$ for all $\theta \in \text{Aut}(W, S)$. Write θ_0 for the inner automorphism of W given by

$$\theta_0 : w \mapsto w_0 w w_0.$$

This map is an automorphism of the poset (W, \leq) and in particular is length-preserving [2, Proposition 2.3.4(ii)]; thus it belongs to $\text{Aut}(W, S)$. In fact, θ_0 lies in the center of $\text{Aut}(W, S)$. Let

$$w_0^+ = (w_0, \theta_0) \in W^+.$$

Observe that this element is a central involution in W^+ , and so if $w = (x, \theta) \in I^+$ then $w w_0^+ = w_0^+ w = (x w_0, \theta \theta_0) \in I^+$. Relative to this notation, we have the following lemma.

Lemma 4.21. *The map $w \mapsto w w_0^+$ defines a Bruhat order-reversing involution of W^+ (also, of I^+) which induces an involution of the set of quasiparabolic conjugacy classes in W^+ .*

Proof. The map $w \mapsto w w_0^+$ is an involution of W^+ which preserves I^+ since w_0^+ is a central involution; the map reverses the Bruhat order on W^+ by [2, Proposition 2.3.4(i)]. Let $\mathcal{K} \subset W^+$ be a quasiparabolic W -conjugacy class. Since w_0^+ is central the set $\mathcal{K} w_0^+ = \{w w_0^+ : w \in \mathcal{K}\}$ is then also a W -conjugacy class and it remains only to show that it is quasiparabolic. This is straightforward from Definition 2.2 since \mathcal{K} is quasiparabolic and since for any $x \in W$ we have $\ell(x w w_0^+ x^{-1}) = \ell(x w x^{-1} w_0^+) = \ell(w_0) - \ell(x w x^{-1})$ by [2, Proposition 2.3.2(ii)]. \square

Let $\mathcal{M} = \mathcal{M}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ and $\mathcal{N} = \mathcal{N}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ as in [Theorem 4.19](#). For the rest of this section $m_{x,y}$ and $n_{x,y}$ for $x, y \in I_{\mathbb{Q}\mathbb{P}}^+$ denote the polynomials defined from the canonical bases of these particular modules as in [\(3.4\)](#). When W is finite, one can prove an inversion formula for these polynomials, analogous to [\[18, Theorem 3.1\]](#) concerning the original Kazhdan–Lusztig polynomials.

We introduce some notation which will be helpful in proving this result. Let \mathcal{M}^* be the \mathcal{A} -module of \mathcal{A} -linear maps $\mathcal{M} \rightarrow \mathcal{A}$. For $w \in I_{\mathbb{Q}\mathbb{P}}^+$ define $M_w^* \in \mathcal{M}^*$ as the \mathcal{A} -linear map with

$$M_w^*(M_{w'}) = \delta_{w,w'} \quad \text{for } w' \in I_{\mathbb{Q}\mathbb{P}}^+.$$

When W is finite, the set of elements $\{M_w^* : w \in I_{\mathbb{Q}\mathbb{P}}^+\}$ forms an \mathcal{A} -basis for \mathcal{M}^* . We view \mathcal{M}^* as an \mathcal{H} -module with respect to the action defined by

$$(HL)(m) = L(H^\dagger m) \quad \text{for } H \in \mathcal{H} \text{ and } L \in \mathcal{M}^* \text{ and } m \in \mathcal{M},$$

where $H \mapsto H^\dagger$ denotes the \mathcal{A} -algebra anti-automorphism of \mathcal{H} with $(H_w)^\dagger = H_{w^{-1}}$ for $w \in W$.

Theorem 4.22. *If W is finite, then for all $x, y \in I_{\mathbb{Q}\mathbb{P}}^+$ it holds that*

$$\sum_{w \in I_{\mathbb{Q}\mathbb{P}}^+} (-1)^{\frac{\ell(x) + \ell(w)}{2}} \cdot m_{x,w} \cdot n_{yw_0^+, ww_0^+} = \delta_{x,y}.$$

Remark 4.23. Recall that $m_{x,y} = n_{x,y} = 0$ unless x and y belong to the same W -conjugacy class, in which case $\ell(y) - \ell(x)$ is even, so the exponentiation of -1 in this formula is well-defined.

Remark 4.24. An analogous inversion formula, due to Douglass [\[7\]](#), exists for the polynomials $m_{x,y}$ and $n_{x,y}$ defined relative to the quasiparabolic set (W^J, ℓ) when W is finite (see [Example 2.4](#)); see [\[31, Proposition 3.9\]](#) for a restatement of this formula in notation closer to ours.

Proof. Let $\Upsilon : \mathcal{M} \rightarrow \mathcal{M}^*$ be the \mathcal{A} -linear map with $\Upsilon\left(\overline{M_{ww_0^+}}\right) = M_w^*$ for $w \in I_{\mathbb{Q}\mathbb{P}}^+$. [Lemmas 3.6 and 4.21](#) ensure that this map is a well-defined \mathcal{A} -linear bijection. Using the fact that $w \mapsto ww_0^+$ is an involution of $I_{\mathbb{Q}\mathbb{P}}^+$ which commutes with W -conjugation and which reverses the Bruhat order, it is straightforward to check that Υ is moreover an isomorphism of \mathcal{H} -modules. Next, denote by $L \mapsto \bar{L}$ the \mathcal{A} -antilinear map $\mathcal{M}^* \rightarrow \mathcal{M}^*$ with

$$\bar{L}(m) = \overline{L(\bar{m})} \quad \text{for } L \in \mathcal{M}^* \text{ and } m \in \mathcal{M}.$$

It follows by [Lemma 3.6](#) that $\overline{M_w^*} = M_w^*$ if $w \in I_{\mathbb{Q}\mathbb{P}}^+$ is W -maximal, and since $\overline{H^\dagger} = \overline{H}^\dagger$ for all $H \in \mathcal{H}$, one easily checks that $\overline{HL}(m) = (\overline{H} \cdot \bar{L})(m)$ for all $H \in \mathcal{H}$ and $L \in \mathcal{M}^*$

and $m \in \mathcal{M}$. From these properties and the fact that $w \mapsto ww_0^+$ is Bruhat order-reversing on $I_{\mathbb{Q}\mathbb{P}}^+$, it follows that map $M \mapsto \Upsilon^{-1}(\overline{\Upsilon(M)})$ is a bar operator on \mathcal{M} . Since the bar operator on \mathcal{M} is unique by Proposition 3.2, it must hold that $\overline{M} = \Upsilon^{-1}(\overline{\Upsilon(M)})$ or, equivalently, that

$$\Upsilon(\overline{M}) = \overline{\Upsilon(M)} \quad \text{for all } M \in \mathcal{M}. \tag{4.1}$$

Now recall the definition of the element $\underline{M}'_x \in \mathcal{M}$ for $x \in I_{\mathbb{Q}\mathbb{P}}^+$ from (3.6). Since $\ell(xw_0^+) - \ell(w_0^+) = \ell(w) - \ell(x)$ and since $\underline{M}'_{xw_0^+} = \overline{\underline{M}'_{xw_0^+}}$, it holds that $\Upsilon(\underline{M}'_{xw_0^+}) = \Upsilon(\overline{\underline{M}'_{xw_0^+}})$ which means that

$$\Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y) = \sum_{w \in I_{\mathbb{Q}\mathbb{P}}^+} (-1)^{\frac{\ell(x)+\ell(w)}{2}} \cdot n_{ww_0^+,xw_0^+} \cdot m_{w,y}.$$

Since $n_{x,y}$ and $m_{x,y}$ each belong to the set $\delta_{x,y} + v^{-1}\mathbb{Z}[v^{-1}]$, it follows that $\Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y) \in \delta_{x,y} + v^{-1}\mathbb{Z}[v^{-1}]$. On the other hand, $\Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y)$ must be invariant under the bar operator on \mathcal{A} since (4.1) combined with the bar invariance of the elements \underline{M}_x and \underline{M}'_x implies that

$$\overline{\Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y)} = \overline{\Upsilon(\underline{M}'_{xw_0^+})(\overline{\underline{M}_y})} = \Upsilon(\overline{\underline{M}'_{xw_0^+}})(\underline{M}_y) = \Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y).$$

The only way to reconcile these observations is to conclude that

$$\sum_{w \in I_{\mathbb{Q}\mathbb{P}}^+} (-1)^{\frac{\ell(x)+\ell(w)}{2}} \cdot n_{ww_0^+,xw_0^+} \cdot m_{w,y} = \Upsilon(\underline{M}'_{xw_0^+})(\underline{M}_y) = \delta_{x,y}.$$

This identity is equivalent to the statement of the theorem: the theorem asserts that a matrix identity of the form $AB = 1$ holds for two certain square matrices A and B whose rows and columns are indexed by $I_{\mathbb{Q}\mathbb{P}}^+$, and the preceding identity is the transpose of that equation. \square

Corollary 4.25. *If W is finite, then*

$$M_x = \sum_{w \in I_{\mathbb{Q}\mathbb{P}}^+} (-1)^{\frac{\ell(x)+\ell(w)}{2}} \cdot n_{xw_0^+,ww_0^+} \cdot \underline{M}_w \quad \text{and}$$

$$N_x = \sum_{w \in I_{\mathbb{Q}\mathbb{P}}^+} (-1)^{\frac{\ell(x)+\ell(w)}{2}} \cdot m_{xw_0^+,ww_0^+} \cdot \underline{N}_w$$

for all $x \in I_{\mathbb{Q}\mathbb{P}}^+$, where $\{M_x\}$ and $\{\underline{M}_x\}$ (respectively, $\{N_x\}$ and $\{\underline{N}_x\}$) denote the standard and canonical bases of the \mathcal{H} -module $\mathcal{M}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$ (respectively, $\mathcal{N}(I_{\mathbb{Q}\mathbb{P}}^+, \frac{1}{2}\ell)$).

Proof. Expand the canonical basis elements on the right as $\underline{M}_w = \sum_{y \in I_{\mathbb{Q}^p}^+} m_{y,w} M_y$ and $\underline{N}_w = \sum_{y \in I_{\mathbb{Q}^p}^+} n_{y,w} N_y$, interchange the order of summation, and then apply [Theorem 4.22](#). \square

5. Problems and conjectures

We mention some conjectures and problems related to our results. Recall the definition of the notation $\mathcal{R}_{\text{ht}}(x)$ from [\(2.1\)](#). As we noted in [Remark 3.5](#), it appears that the only bounded quasiparabolic sets which automatically admit bar operators are those arising from the parabolic case, in the sense of the following conjecture:

Conjecture 5.1. *If (X, ht) is a quasiparabolic W -set which is transitive and bounded below, and if $|\mathcal{R}_{\text{ht}}(x)| = 1$ for all $x \in X$, then $(X, \text{ht}) \cong (W^J, \ell)$ for some $J \subset S$.*

[\[1, Theorem 3.11.4\]](#) summarizes a number of interpretations of the “parabolic Kazhdan–Lusztig bases” of $\mathcal{M}(W^J, \ell)$ and $\mathcal{N}(W^J, \ell)$ in a representation theoretic context. Such interpretations lead to the following problem, which is related to the discussion in [\[27, §9\]](#) and [\[29, §10\]](#).

Problem 5.2. Find a geometric or representation-theoretic interpretation of the quasiparabolic conjugacy classes in W^+ , of the corresponding modules \mathcal{M} and \mathcal{N} , and their canonical bases.

Remark 5.3. As mentioned by an anonymous referee, there are a few case of quasiparabolic conjugacy classes for which the corresponding \mathcal{H} -modules and their bases do admit natural representation-theoretic interpretations. These are the conjugacy classes of the fixed-point-free involutions in S_{2n} , of the longest element of type D_4 in the Weyl group of type E_6 , and of a perfect involution with maximal proper centralizer in type D_n . These classes correspond to the real semisimple Lie groups $\text{SU}^*(2n)$, E_6^{-26} , and $\text{SO}(1, 2n - 1)$, and the associated \mathcal{H} -module basis elements correspond to standard representations of these Lie groups whose central characters are the same as that of the trivial representation; see [\[21,36\]](#).

The following conjecture is stated implicitly in [\[27, §5\]](#), and proved in the special case of W -conjugacy classes of automorphisms $\theta \in \text{Aut}(W, S) \subset W^+$ which are perfect involutions [\[27, Proposition 5.17\]](#). This conjecture seems to closely parallel the main result of [\[29\]](#).

Conjecture 5.4. *The “Bruhat order” on a quasiparabolic W -conjugacy class in W^+ as given by [Definition 2.10](#) coincides with the restriction of the usual Bruhat order on W^+ .*

As Rains and Vazirani note in [\[27\]](#), the criterion that any perfect conjugacy class of twisted involutions is quasiparabolic is often inadequate in applications involving infinite Coxeter groups.

Problem 5.5. Formulate a version of [Theorem 4.13](#) which can be used to prove that (interesting) conjugacy classes in W^+ are quasiparabolic when W is infinite. Classify the quasiparabolic conjugacy classes in W^+ when (W, S) is an affine Weyl group.

It appears that quasiparabolic W -conjugacy classes in I^+ may be characterized by a simpler set of conditions than the ones in [Definition 2.2](#). Specifically, we conjecture the following:

Conjecture 5.6. Any conjugacy class in I^+ which satisfies property (QP1) in [Definition 2.2](#) (relative to the height function $\text{ht} = \frac{1}{2}\ell$) also satisfies (QP2), and hence is quasiparabolic.

A lot of useful technical machinery has been developed for twisted involutions in a Coxeter group; see, for example, [\[14–16,28,29,33\]](#). One reason to expect the preceding conjecture to be true is that it reduces via this machinery to the following second conjecture, which can be viewed as a plausible “strong exchange condition” for twisted involutions, analogous to Hultman’s “(weak) exchange condition” [\[15, Proposition 3.10\]](#). Recall here that $R = \{wsw^{-1} : (w, s) \in W \times S\}$.

Conjecture 5.7. Let $\mathcal{K} \subset I^+$ be a W -conjugacy class such that $\ell(rwr) = \ell(w)$ implies $rwr = w$ for all $(r, w) \in R \times \mathcal{K}$. Then $\ell(rwr) < \ell(w)$ implies $rwr < w$ for all $(r, w) \in R \times \mathcal{K}$.

Our results in [Section 3.3](#) lead to the following problem.

Problem 5.8. Describe the cells of the W -graphs Γ_m and Γ_n attached via [Theorem 3.26](#) to a quasiparabolic conjugacy class in a finite or affine Weyl group.

In the classical cases, this problem is of interest just in view of the elegant combinatorial description of the left cells in the symmetric group (see [\[2, Chapter 6\]](#)). More generally, it would be especially interesting to connect information about the cells in Γ_m and Γ_n to [Problem 5.2](#).

As discussed in [Remark 4.20](#), Lusztig and Vogan [\[20,22,23\]](#) have recently studied an Iwahori–Hecke algebra module spanned by the entire set of twisted involutions I^+ in a Coxeter group, which admits a “bar operator” given formally by nearly the same definition as for the bar operators in [Theorem 4.19](#). Despite this, it remains unclear whether the canonical bases corresponding to these bar operators have any simple relationship.

Problem 5.9. How are the bases $\{\underline{M}_x\}_{x \in I^+_{\mathbb{Q}P}}$ and $\{\underline{N}_x\}_{x \in I^+_{\mathbb{Q}P}}$ defined by [Theorems 3.14 and 4.19](#) related to the canonical basis indexed by I^+ studied in [\[20,22,23\]](#)?

Acknowledgments

I thank Daniel Bump, Michael Chmutov, John Stembridge, and Zhiwei Yun for helpful discussions.

References

- [1] A.A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* 9 (2) (1996) 473–527.
- [2] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Grad. Texts in Math., vol. 231, Springer, New York, 2005.
- [3] M.S. Chmutov, The structure of W -graphs arising in Kazhdan–Lusztig theory, Ph.D. thesis, Department of Mathematics, University of Michigan, 2014.
- [4] M. Coullens, Généralisation parabolique des polynômes et des bases de Kazhdan–Lusztig, *J. Algebra* 213 (2) (1999) 687–720.
- [5] V.V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan–Lusztig polynomials, *J. Algebra* 111 (1987) 483–506.
- [6] V.V. Deodhar, Duality in parabolic set up for questions in Kazhdan–Lusztig theory, *J. Algebra* 142 (1991) 201–209.
- [7] J.M. Douglass, An inversion formula for relative Kazhdan–Lusztig polynomials, *Comm. Algebra* 18 (1990) 371–387.
- [8] J. Du, IC bases and quantum linear groups, *Proc. Sympos. Pure Math.* 56 (1994) 135–148.
- [9] B. Elias, G. Williamson, The Hodge theory of Soergel bimodules, *Ann. Math.* 180 (3) (2014) 1089–1136.
- [10] B. Elias, G. Williamson, Soergel calculus, preprint, arXiv:1309.0865v1, 2013.
- [11] M. Geck, S. Kim, G. Pfeiffer, Minimal length elements in twisted conjugacy classes of finite Coxeter groups, *J. Algebra* 229 (2000) 570–600.
- [12] R.B. Howlett, Y. Yin, Inducing W -graphs, *Math. Z.* 244 (2) (2003) 415–431.
- [13] R.B. Howlett, Y. Yin, Inducing W -graphs II, *Manuscripta Math.* 115 (4) (2004) 495–511.
- [14] A. Hultman, Fixed points of involutive automorphisms of the Bruhat order, *Adv. Math.* 195 (2005) 283–296.
- [15] A. Hultman, The combinatorics of twisted involutions in Coxeter groups, *Trans. Amer. Math. Soc.* 359 (2007) 2787–2798.
- [16] A. Hultman, Twisted identities in Coxeter groups, *J. Algebraic Combin.* 28 (2008) 313–332.
- [17] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
- [18] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* 53 (1979) 165–184.
- [19] G. Lusztig, Hecke Algebras with Unequal Parameters, CRM Monogr. Ser., vol. 18, Amer. Math. Soc., 2003.
- [20] G. Lusztig, A bar operator for involutions in a Coxeter group, *Bull. Inst. Math. Acad. Sin. (N.S.)* 7 (2012) 355–404.
- [21] G. Lusztig, D. Vogan, Singularities of closures of K -orbits on flag manifold, *Invent. Math.* 71 (1983) 365–379.
- [22] G. Lusztig, D.A. Vogan, Hecke algebras and involutions in Weyl groups, *Bull. Inst. Math. Acad. Sin. (N.S.)* 7 (2012) 323–354.
- [23] G. Lusztig, D.A. Vogan, Hecke algebras and involutions in Coxeter groups, preprint, arXiv:1312.3237v2, 2014.
- [24] K. Lux, A. Schaeffer, C.R. Vinoot, Strong reality properties of normalizers of parabolic subgroups in finite Coxeter groups, *Comm. Algebra* 40 (8) (2012) 3056–3070.
- [25] E. Marberg, Comparing and characterizing some constructions of canonical bases from Coxeter systems, in: *Representations of Reductive Groups: In Honor of the 60th Birthday of David A. Vogan, Jr.*, Birkhäuser, 2015, pp. 399–436, arXiv:1403.4013.
- [26] Götz Pfeiffer, Gerhard Röhrle, Special involutions and bulky parabolic subgroups in finite Coxeter groups, *J. Aust. Math. Soc.* 79 (1) (2005) 141–147.
- [27] E.M. Rains, M.J. Vazirani, Deformations of permutation representations of Coxeter groups, *J. Algebraic Combin.* 37 (2013) 455–502.

- [28] R.W. Richardson, T.A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata* 35 (1990) 389–436.
- [29] R.W. Richardson, T.A. Springer, Complements to: the Bruhat order on symmetric varieties, *Geom. Dedicata* 49 (1994) 231–238.
- [30] W. Soergel, Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, *J. Amer. Math. Soc.* 3 (2) (1990) 421–445.
- [31] W. Soergel, Kazhdan–Lusztig polynomials and a combinatoric for tilting modules, *Represent. Theory* 1 (1997) 83–114.
- [32] W. Soergel, Kazhdan–Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen, *J. Inst. Math. Jussieu* 6 (3) (2007) 501–525.
- [33] T.A. Springer, Some results on algebraic groups with involutions, in: *Adv. Stud. Pure Math.*, vol. 6, Kinokuniya/North-Holland, 1985, pp. 525–543.
- [34] J.R. Stembridge, Admissible W -graphs, *Represent. Theory* 12 (2008) 346–368.
- [35] J.R. Stembridge, Admissible W -graphs and commuting Cartan matrices, *Adv. in Appl. Math.* 44 (2010) 203–224.
- [36] D. Vogan, Irreducible characters of semisimple Lie groups III: proof of the Kazhdan–Lusztig conjectures in the integral case, *Invent. Math.* 71 (1983) 381–417.
- [37] B. Webster, Canonical bases and higher representation theory, *Compos. Math.* 151 (2015) 121–166.