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Classification of the vertex operator algebras V_L^+ of class \mathcal{S}^4



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ABSTRACT

In this paper, we prove that an even rootless lattice L is isomorphic to $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, or BW_{16} if the lattice type vertex operator algebra V_L^+ is of class \mathcal{S}^4 . In addition, we prove that the vertex operator algebra $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the vertex operator algebras $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 .

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1. Introduction

Frenkel, Lepowsky, and Meurman constructed the moonshine vertex operator algebra (VOA) V^\natural in [10], which is one of the most important examples of VOAs. In the same work, they conjectured that a holomorphic VOA $V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$ of central charge 24 with $\dim V_0 = 1$ and $V_1 = 0$ is isomorphic to V^\natural . This conjecture, called the FLM conjecture, is an analogy of the uniqueness of the extended binary Golay code of length 24 and the Leech lattice, and it has yet to be solved.

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VOAs of class \mathcal{S}^n were introduced in [16] as follows: A VOA is said to be of class \mathcal{S}^n if the invariant subVOA under its automorphism group coincides with the subVOA V_ω generated by the Virasoro element ω up to degree n subspace. For example, V^\natural is of class \mathcal{S}^{11} (cf. [3,5]). In [16], it was shown that the pair of the central charge and the dimension of degree 2 subspace of a VOA of class \mathcal{S}^8 with minimal conformal weight 2 is (24, 196 884), those of V^\natural . Therefore, it is expected that V^\natural is unique under the condition of class \mathcal{S}^8 and minimal conformal weight 2, and that this point of view provides a new approach for the FLM conjecture.

As other related topics of VOAs of class \mathcal{S}^n , there are classification problems of VOAs of classes \mathcal{S}^4 and \mathcal{S}^6 . If a VOA is of class \mathcal{S}^n , then all its homogeneous subspaces are conformal n -designs, introduced in [12]. Note that the notion of conformal designs based on a VOA is a natural analogue of the notions of combinatorial designs or spherical designs based on binary codes or lattices, respectively. It was proved in [12] that a VOA of class \mathcal{S}^6 with minimal conformal weight 1 is isomorphic to the lattice VOA associated to the root lattice A_1 or E_8 by using conformal designs. Also, if a VOA is of class \mathcal{S}^n , then the quadratic Casimir elements up to degree n , introduced in [16], belong to V_ω . By using the elements, it was proved in [21] that a VOA of class \mathcal{S}^4 with minimal conformal weight 1 is isomorphic to the simple affine VOA with level 1 associated to one of Deligne's exceptional Lie algebras A_1 , A_2 , G_2 , D_4 , F_4 , E_6 , E_7 , and E_8 . From these results, in order to complete the classification, we should consider VOAs whose minimal conformal weights are greater than 1. The moonshine VOA V^\natural is one of such examples, and other candidates are given in [12,16,21].

Let $V = \bigoplus_{n \geq 0} V_n$ be a VOA of central charge c with $\dim V_0 = 1$ and $V_1 = 0$. Then the pair of c and $d := \dim V_2$ is restricted by the Matsuo–Norton trace formula if V is of class \mathcal{S}^4 (see [16] for details). For example,

$$(c, d) : \quad (4, 22), \quad (8, 156), \quad (16, 2296)$$

satisfy the restriction. On the other hand, the VOAs $V_{\sqrt{2}D_4}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ have these pairs, where BW_{16} is the Barnes–Wall lattice of rank 16. Due to these circumstances, it is expected that those VOAs are of class \mathcal{S}^4 , and some of them are of class \mathcal{S}^6 . This is our motivation and the reason why we focus on the lattice type VOAs V_L^+ of class \mathcal{S}^n in this paper. Our main result of this paper is the following:

Main result. (See Theorems 5.15 and 6.8.) *Let L be an even lattice without roots. Then V_L^+ is of class \mathcal{S}^4 if and only if L is isomorphic to $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, or BW_{16} . Moreover, the VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 .*

In the following, we explain the method for the classification of V_L^+ of class \mathcal{S}^4 .

Let L be an even lattice without roots. The automorphism group $\text{Aut}(V_L^+)$ of V_L^+ contains the subgroup G induced by the orthogonal group $O(L)$ of L . In [19], it was

shown that $G \subsetneq \text{Aut}(V_L^+)$ if and only if L is obtained by Construction B from some binary code. We show that V_L^+ is not of class \mathcal{S}^4 by finding an element of $(V_L^+)_4^{\text{Aut}(V_L^+)} \setminus V_\omega$ if $\text{Aut}(V_L^+) = G$. Hence, L is obtained by Construction B if V_L^+ is of class \mathcal{S}^4 . In particular, L is isomorphic to $2A_1$ if the rank of L is 1.

Next, we assume that L is obtained by Construction B from a binary code C . Then $O(L)$ has the subgroup H induced by the automorphism group of C . Moreover, if C has T -decompositions of Type B, then L has orthogonal transformations not in H (see [13] for the details). We prove that $H \subsetneq O(L)$ if and only if C has T -decompositions of Type B. Using this result, we show that V_L^+ is not of class \mathcal{S}^4 if $O(L) = H$ and the rank of L is greater than 1. Hence, C has T -decompositions of Type B if V_L^+ is of class \mathcal{S}^4 . By the definition of T -decompositions, the rank of L is a multiple of 4. Due to these results, we obtain $L \cong \sqrt{2}D_4$ if the rank of L is 4, and $L \cong \sqrt{2}E_8$ if the rank of L is 8. From now on, we assume that the rank of L is greater than 8. Let \mathcal{S}_C be the binary code generated by all T -decompositions of Type B of C . Then we prove that C is equivalent to the first order Reed–Muller code $RM(1, 4)$ of length 16 if \mathcal{S}_C is singly even. Note that the lattice obtained by Construction B from $RM(1, 4)$ is isomorphic to BW_{16} . Moreover, we prove that V_L^+ is not of class \mathcal{S}^4 if \mathcal{S}_C is doubly even by finding an element of $(V_L^+)_4^{\text{Aut}(V_L^+)} \setminus V_\omega$. Thus, we obtain the classification of V_L^+ of class \mathcal{S}^4 .

Finally, we prove that the VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 . Since the lattices $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, and BW_{16} are obtained by Construction B, the automorphism groups of the associated lattice type VOAs V_L^+ contain extra automorphisms constructed in [10]. First, we give bases of the fixed point subspaces of $\text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ and extra automorphisms in the degree 5 and 7 subspaces of V_L^+ . From this, we prove that $V_{2A_1}^+$ is of class \mathcal{S}^7 . Next, we consider the fixed points of the orthogonal groups of lattices D_4 , E_8 , and BW_{16} . The lattices belong to the series of the Barnes–Wall lattices, and their orthogonal groups are studied in [11] and [2]. By using the invariant ring theory of the index 2 subgroup discussed in [2] of the Clifford group, we prove the remaining assertion.

This paper is organized as follows. In Section 2, we recall the notion of VOAs, and the definition of VOAs of class \mathcal{S}^n introduced in [16]. Also, we recall the lattice VOA V_L and its subVOA V_L^+ associated to an even lattice L . In Section 3, we give a generator of the orthogonal group of the lattice obtained by Construction B from a binary code. In Section 4, we recall some properties of the extra automorphisms of V_L^+ constructed in [10], and study their fixed point subspaces in V_L^+ . In Section 5, we prove that an even rootless lattice L is isomorphic to $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, or BW_{16} if V_L^+ is of class \mathcal{S}^4 by using the results obtained in Sections 3 and 4. In Section 6, we prove that the VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 .

Notations

A_n, D_n, E_n	root lattices of type A, D, E of rank n .
$\text{Aut}(C)$	the group of all automorphisms of a binary code C .

$\text{Aut}(V)$	the group of all automorphisms of a VOA V .
BW_{16}	the Barnes–Wall lattice of rank 16.
C^\perp	the dual code of a binary code C .
$D(k)$	the set of all weight (resp., norm) k vectors in D if D is a subset of \mathbb{F}_2^n (resp., if D is a subset of \mathbb{R}^n).
$E(C)$	$:= \{\epsilon_x \mid x \in C^\perp\}$.
$\text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$	the group of all group homomorphisms from L to $\mathbb{Z}/2\mathbb{Z}$.
\mathfrak{h}	$:= \mathbb{C} \otimes_{\mathbb{Z}} L$.
$\mathfrak{h}(J)$	the subspace of \mathfrak{h} spanned by $\{\alpha_i \mid i \in J\}$, where $J \subset \Omega_n$.
L^*	the dual lattice of a lattice L .
\widehat{L}	the central extension of a lattice L by $\langle \kappa \rangle$ with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$.
$L_X(C)$ ($X \in \{A, B\}$)	the lattice obtained by Construction X from a binary code C .
$L_J(n)$	the $(n+1)$ -th operator of ω_J .
$M_{\mathfrak{h}}(1)$	the free bosonic VOA associated to \mathfrak{h} .
$O(L)$	the orthogonal group of L .
$O(\widehat{L})$	the automorphism group of \widehat{L} .
O_L	the subset of L^*/L defined in (3.2).
$RM(1, 4)$	the first order Reed–Muller code of length 16.
$\text{supp}(x)$	the support of $x \in \mathbb{F}_2^n$.
\mathcal{T}_C	the set of all T -decompositions of Type B of a binary code C .
V_ω	the subVOA of V generated by the Virasoro element ω .
$V^{\text{Aut}(V)}$	the fixed point subspace of $\text{Aut}(V)$ in V .
V_L	the lattice VOA associated to an even lattice L .
V_L^+	the fixed point subspace of θ in V_L .
$\text{wt}(x)$	the weight of $x \in \mathbb{F}_2^n$.
$\{\alpha_i \mid i \in \Omega_n\}$	an orthogonal basis of \mathbb{R}^n consisting of norm 2 vectors.
ϵ_x	the orthogonal transformation associated to $x \in \mathbb{F}_2^n$ defined in Section 3.2.
θ	an involution in $\nu^{-1}(-id_L)$, where id_L is the identity element of $O(L)$.
μ	the map from $\text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ to $O(\widehat{L})$ defined in (2.4).
ν	the map from $O(\widehat{L})$ to $O(L)$ defined in (2.5).
ρ_s	an orthogonal transformation of $L_B(C)$ defined in (3.6).
σ_0	an extra automorphism of V_L^+ associated to $\alpha_1 + L$ defined in Section 4.1.
ω_J	the Virasoro element of $M_{\mathfrak{h}(J)}(1)$.
Ω_n	the set of n letters $1, 2, \dots, n$.
$\langle \cdot, \cdot \rangle$	a positive-definite symmetric bilinear form of \mathbb{R}^n .

2. Preliminaries

2.1. Vertex operator algebras

In this subsection, we recall the notion of vertex operator algebras and their automorphisms (see [10,14]). Also we recall the notion of the conjugate module of a vertex operator algebra associated to an automorphism (see [8]).

A *vertex operator algebra* (VOA) V is a \mathbb{Z} -graded \mathbb{C} -vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ equipped with a linear map

$$\begin{aligned} Y(\cdot, z) : V &\rightarrow \text{End}(V)[[z, z^{-1}]] \\ v &\mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \end{aligned}$$

and two non-zero vectors $\mathbf{1}$ and ω in V , which are called the *vacuum vector* and the *Virasoro element*, respectively, satisfying certain conditions. As one of the conditions, the following *Virasoro relation* holds on V :

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m+n, 0} c$$

for $m, n \in \mathbb{Z}$, where $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, δ_{ij} is the Kronecker symbol and $c \in \mathbb{C}$; the number c is called the *central charge* of V . Throughout the paper, we assume that V is of CFT-type, i.e.,

$$V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n, \quad V_0 = \mathbb{C}\mathbf{1}.$$

An element $\sigma \in GL(V)$ is called an *automorphism* of V if

$$\sigma(u_n v) = \sigma(u)_n \sigma(v) \text{ for } u, v \in V, n \in \mathbb{Z}, \quad \sigma(\omega) = \omega.$$

Let $\text{Aut}(V)$ denote the group of all automorphisms of V .

Let $M = (M, Y_M)$ be a V -module, where $Y_M(\cdot, z) : V \rightarrow \text{End}(M)[[z, z^{-1}]]$, $v \mapsto \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$ (see [14] for the definition of modules). The isomorphism class of M is denoted by $[M]$. For $g \in \text{Aut}(V)$, the V -module $g.M$ is defined as $(M, Y_{g.M})$, where

$$Y_{g.M}(v, z) := Y_M(g^{-1}v, z) \text{ for } v \in V.$$

Remark 2.1. Let M be an irreducible V -module and $g \in \text{Aut}(V)$. Then $g.M$ is also an irreducible V -module. In particular, the map $[M] \mapsto g.[M] = [g.M]$ induces an action of $\text{Aut}(V)$ on the set of all isomorphism classes of irreducible V -modules.

2.2. Lattice VOAs

In this subsection, we first review some basic facts of lattices. Additionally, we review the construction of lattice type VOAs V_L and V_L^+ associated to an even lattice L , and their automorphisms induced from lattices.

Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the Euclidean space of dimension n . A subset $L \subset \mathbb{R}^n$ is called a *lattice* of rank n if there exists a basis $\{v_i\}_{i=1}^n$ of \mathbb{R}^n such that $L = \bigoplus_{i=1}^n \mathbb{Z}v_i$. A lattice L is said to be *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. Define the *dual lattice* L^* of a lattice L as

$$L^* := \{u \in \mathbb{R}^n \mid \langle u, v \rangle \in \mathbb{Z} \text{ for all } v \in L\}.$$

The following lemma is straightforward.

Lemma 2.2. *Let L be a lattice. For $\beta + 2L^* \in L^*/2L^*$, define*

$$\begin{aligned} f_\beta : L &\rightarrow \mathbb{Z}/2\mathbb{Z} \\ \alpha &\mapsto \langle \beta, \alpha \rangle + 2\mathbb{Z}. \end{aligned}$$

Under the correspondence $\beta + 2L^ \mapsto f_\beta$, $L^*/2L^*$ is isomorphic to $\text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ as groups.*

It is easily seen the following lemma.

Lemma 2.3. *Let L be a lattice. Then $\{\alpha \in L \mid f(\alpha) = 0 \text{ for all } f \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})\}$ is equal to $2L$.*

Proof. Set $X = \{\alpha \in L \mid f(\alpha) = 0 \text{ for all } f \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})\}$. Let $\alpha \in X$. By Lemma 2.2, $\langle \beta, \alpha \rangle \in 2\mathbb{Z}$ for all $\beta + 2L^* \in L^*/2L^*$. Therefore $\alpha \in 2(L^*)^* = 2L$ and $X \subset 2L$. By definition, $2L \subset X$. Hence, we have this lemma. \square

Let L be an even lattice of rank n . We review the construction of the lattice VOA V_L associated to L of rank n briefly (for details, see [10]). Let $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the form $\langle \cdot, \cdot \rangle$ on L to \mathfrak{h} \mathbb{C} -bilinearly. Then we regard \mathfrak{h} as an abelian Lie algebra with the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ be the corresponding affine Lie algebra. Write $u(m) := u \otimes t^m$ for $u \in \mathfrak{h}$, $m \in \mathbb{Z}$. We regard \mathbb{C} as the one-dimensional $(\mathfrak{h} \otimes \mathbb{C}[t]) \oplus \mathbb{C}\mathbf{c}$ -module such that $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially and \mathbf{c} acts identically on \mathbb{C} . Let

$$M_{\mathfrak{h}}(1) := \mathfrak{U}(\hat{\mathfrak{h}}) \otimes_{\mathfrak{U}((\mathfrak{h} \otimes \mathbb{C}[t]) \oplus \mathbb{C}\mathbf{c})} \mathbb{C},$$

where $\mathfrak{U}(X)$ denotes the universal enveloping algebra of a Lie algebra X . Then $M_{\mathfrak{h}}(1)$ is isomorphic to the symmetric algebra of $\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ as vector spaces, and it is a VOA of central charge n ; the Virasoro element is given by

$$\omega := \frac{1}{2} \sum_{h \in H} h(-1)^2 \mathbf{1}, \quad (2.1)$$

where H is an orthonormal basis of \mathfrak{h} , and the vacuum vector $\mathbf{1} := 1 \in \mathbb{C}$.

Let \widehat{L} be the central extension of L by $\langle \kappa \rangle = \langle \kappa \mid \kappa^2 = 1 \rangle$ with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$:

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \widehat{L} \twoheadrightarrow L \longrightarrow 1,$$

where $\bar{\cdot}: \widehat{L} \rightarrow L$ denotes the canonical group homomorphism. Let $e: L \rightarrow \widehat{L}, \alpha \mapsto e_\alpha$, be a section such that $e_0 = 1$. We regard \mathbb{C} as the one-dimensional $\langle \kappa \rangle$ -module on which κ acts as multiplication by -1 . Let $\mathbb{C}\{L\} := \mathbb{C}[\widehat{L}] \otimes_{\langle \kappa \rangle} \mathbb{C}$, where $\mathbb{C}[\widehat{L}]$ denotes the group algebra of \widehat{L} , and set $\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$ for $a \in \widehat{L}$. Then

$$V_L := M_{\mathfrak{h}}(1) \otimes \mathbb{C}\{L\} \quad (2.2)$$

has a VOA structure whose central charge is n (see [10]), and (2.1) is also the Virasoro element of V_L .

Since the central charge of V_L is a positive integer, the next lemma follows from the Kac determinant formula (cf. [16, Section 1.2]).

Lemma 2.4. *Let L be an even lattice, and V_ω the subVOA of V_L generated by the Virasoro element (2.1). For $2 \leq k \leq 7$, a basis of $(V_\omega)_k$ is given by $\{L(-n_1) \cdots L(-n_l) \mathbf{1} \mid n_1 \geq \cdots \geq n_l \geq 2, n_j \in \mathbb{Z}, \sum_{j=1}^l n_j = k\}$.*

Now we recall the automorphisms of V_L induced from those of \widehat{L} (for details, see [10] and [9]). Let $O(L)$ be the orthogonal group of L , and $O(\widehat{L})$ the automorphism group of \widehat{L} . By [10, Proposition 5.4.1],

$$1 \longrightarrow \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mu} O(\widehat{L}) \xrightarrow{\nu} O(L) \longrightarrow 1 \quad (2.3)$$

is exact, where the maps μ and ν are defined by

$$\begin{aligned} \mu: \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) &\rightarrow O(\widehat{L}) \\ \lambda &\mapsto \mu(\lambda): \widehat{L} \rightarrow \widehat{L}, a \mapsto (-1)^{\lambda(\bar{a})} a \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \nu: O(\widehat{L}) &\rightarrow O(L) \\ \sigma &\mapsto \nu(\sigma): L \rightarrow L, \alpha \mapsto \overline{\sigma(e_\alpha)}, \end{aligned} \quad (2.5)$$

respectively.

Every $\sigma \in O(\widehat{L})$ induces an automorphism of V_L , denoted also by σ , in the following way [10]:

$$\sigma(u_1(-n_1) \cdots u_k(-n_k) \otimes \iota(a)) := \nu(\sigma)(u_1)(-n_1) \cdots \nu(\sigma)(u_k)(-n_k) \otimes \iota(\sigma(a)) \quad (2.6)$$

for $u_i \in \mathfrak{h}$, $n_i \in \mathbb{Z}_{\geq 1}$, $k \in \mathbb{Z}_{\geq 0}$ and $a \in \widehat{L}$. We can regard $O(\widehat{L})$ as a subgroup of $\text{Aut}(V_L)$ by this action.

Let $\theta \in \nu^{-1}(-id_L)$, where id_L is the identity element of $O(L)$. Set $V_L^\pm = \{u \in V_L \mid \theta(u) = \pm u\}$; V_L^+ is a subVOA of V_L whose central charge is n . We know from [7, Corollary D. 7] that the VOA structure of V_L^+ is independent of the choice of $\theta \in \nu^{-1}(-id_L)$. Moreover, the following proposition about V_L^+ holds.

Proposition 2.5. (Cf. [17, Proposition 2.7].) *The VOA V_L^+ is completely reducible as a V_ω -module.*

Remark 2.6. In fact, it was proved in the proof of [17, Proposition 2.7] that V_L^+ has an \mathbb{R} -form equipped with a positive-definite invariant bilinear form.

Remark 2.7. It is well-known that V_L^- and $V_{\lambda+L}^\pm$ for $\lambda + L \in L^* \cap (L/2)$ are irreducible V_L^+ -modules (for details, see [1, Theorem 7.7]).

Note that $O(\widehat{L})/\langle \theta \rangle$ is a subgroup of $\text{Aut}(V_L^+)$. The following lemma will be used in Section 5.

Lemma 2.8. (See [10, (10.4.13)].) *Let L be an even lattice. Then the following sequence is exact:*

$$1 \longrightarrow \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \longrightarrow O(\widehat{L})/\langle \theta \rangle \longrightarrow O(L)/\langle -id_L \rangle \longrightarrow 1.$$

2.3. VOAs of class \mathcal{S}^n

We recall the notion of VOAs of class \mathcal{S}^n introduced in [16]. Also, we give a sufficient condition that a VOA is of class \mathcal{S}^n . Let V_ω denote the subVOA of a VOA V generated by the Virasoro element ω , and set $V^{\text{Aut}(V)} = \{v \in V \mid g(v) = v \text{ for all } g \in \text{Aut}(V)\}$.

Definition 2.9. (See [16, Definition 1.1].) A VOA V is said to be of class \mathcal{S}^n if $V^{\text{Aut}(V)}$ coincides with V_ω up to degree n subspace, i.e.,

$$V_m^{\text{Aut}(V)} = (V_\omega)_m \text{ for } 0 \leq m \leq n.$$

Remark 2.10. It is easily seen from the definition of $\text{Aut}(V)$ that $(V_\omega)_m$ is always contained in $V_m^{\text{Aut}(V)}$ for each $m \in \mathbb{Z}_{\geq 0}$.

Remark 2.11. For each $0 \leq n \leq 11$, the decomposition of the degree n subspace of the moonshine VOA V^\natural constructed in [10] as a monster module is obtained by monstrous moonshine (cf. [3, 5]). From this, one can see that V^\natural is of class \mathcal{S}^{11} .

Proposition 2.12. *Let W be a subVOA of a VOA V with the same Virasoro element ω . Assume that V is completely reducible as a V_ω -module. If $W_n = (V_\omega)_n$, then $W_{n-1} = (V_\omega)_{n-1}$.*

Proof. The assertion for $n = 1$ is obvious since $V_0 = \mathbb{C}\mathbf{1}$. Let $n > 1$. By our assumption, there exists the complement U of V_ω in V as a V_ω -module. Let $u \in W_{n-1}$ and write it as $u = u_1 + u_2$, where $u_1 \in (V_\omega)_{n-1}$, $u_2 \in U \cap V_{n-1}$. Since W is a subVOA of V , we have $L(-1)u \in W_n = (V_\omega)_n$. Therefore, we have $L(-1)u_2 \in V_\omega$ because $L(-1)u_1 \in V_\omega$. Also, $L(-1)u_2 \in U$ since U is a V_ω -module. Hence $L(-1)u_2 \in V_\omega \cap U$, and $L(-1)u_2 = 0$ because U is the complement of V_ω i.e., $V_\omega \cap U = 0$. By [14, Proposition 3.11.2], $u_2 \in \ker L(-1) \subset V_0$. Since $n > 1$ and $u_2 \in V_{n-1}$, we obtain $u_2 = 0$ and $u = u_1 \in (V_\omega)_{n-1}$. This proves the statement. \square

Since $V^{\text{Aut}(V)}$ is a subVOA of V with the same Virasoro element, the following corollary holds.

Corollary 2.13. *Let V be a VOA. Assume that V is completely reducible as a V_ω -module. If $V_n^{\text{Aut}(V)} = (V_\omega)_n$, then V is of class \mathcal{S}^n .*

3. Orthogonal group of $L_B(C)$

3.1. Lattice obtained by Construction B

In this subsection, we review fundamental facts about binary codes, and recall the constructions of lattices from binary codes, which are called the Constructions A and B. Also, we recall the definition of frames of even lattices introduced in [13].

A *binary code* C of length n is a linear subspace of \mathbb{F}_2^n . Define $(u, v) := \sum_{i=1}^n u_i v_i \pmod{2}$ for $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{F}_2^n$. The *dual code* C^\perp of C is defined by

$$C^\perp := \{u \in \mathbb{F}_2^n \mid (u, v) = 0 \text{ for all } v \in C\}.$$

The *support* $\text{supp}(x)$ and the *weight* $\text{wt}(x)$ of $x \in \mathbb{F}_2^n$ are defined by $\text{supp}(x) := \{1 \leq i \leq n \mid x_i \neq 0\}$ and $\text{wt}(x) := |\text{supp}(x)|$, respectively. A binary code C is said to be *even* (resp., *doubly even*) if $\text{wt}(c) \in 2\mathbb{Z}$ (resp., $\text{wt}(c) \in 4\mathbb{Z}$) for all $c \in C$. If C is even but is not doubly even, then C is said to be *singly even*. An element $\sigma \in S_n$ is called an *automorphism* of C if $\sigma(C) = C$. Let $\text{Aut}(C)$ denote the group of all automorphisms of C .

Define $\Omega_n := \{1, \dots, n\}$ and let $\{\alpha_i \mid i \in \Omega_n\}$ be an orthogonal basis of \mathbb{R}^n consisting of norm 2 vectors. Set $\alpha_x := \sum_{i \in \text{supp}(x)} \alpha_i$ for $x \in \mathbb{F}_2^n$. Let C be a binary code of length n . The lattices

$$L_A(C) := \sum_{c \in C} \mathbb{Z} \frac{1}{2} \alpha_c + \sum_{i \in \Omega_n} \mathbb{Z} \alpha_i, \quad L_B(C) := \sum_{c \in C} \mathbb{Z} \frac{1}{2} \alpha_c + \sum_{i, j \in \Omega_n} \mathbb{Z} (\alpha_i + \alpha_j) \quad (3.1)$$

are called the lattices obtained by *Construction A* and *Construction B* from C with respect to $\{\alpha_i \mid i \in \Omega_n\}$, respectively. In this paper, the lattices $L_A(C)$ and $L_B(C)$ are constructed by using $\{\alpha_i \mid i \in \Omega_n\}$ unless otherwise noted.

Notation 3.1. For $x_1, \dots, x_l \in \mathbb{F}_2$ and $k_1, \dots, k_l, m \in \mathbb{Z}_{\geq 1}$, let $(x_1^{k_1} x_2^{k_2} \cdots x_l^{k_l})$ and $(x_1 \cdots x_l)^m$ denote

$$\underbrace{(x_1, \dots, x_1)}_{k_1}, \dots, \underbrace{(x_l, \dots, x_l)}_{k_l} \in \mathbb{F}_2^{k_1 + \cdots + k_l}$$

and

$$\underbrace{(x_1, \dots, x_l, x_1, \dots, x_l, \dots, x_1, \dots, x_l)}_{l \cdot m} \in \mathbb{F}_2^{lm},$$

respectively.

Example 3.2. The following are examples of the lattices obtained by Construction B (see [4]).

$$\begin{aligned} L_B(\{(0^1)\}) &\cong 2A_1, & L_B(\{(0^4)\}) &\cong \sqrt{2}D_4, \\ L_B(\{(0^8), (1^8)\}) &\cong \sqrt{2}E_8, & L_B(RM(1, 4)) &\cong BW_{16}. \end{aligned}$$

Here, A_n , D_n , and E_n are root lattices of type A , D , and E of rank n , respectively, $RM(1, 4)$ is the first order Reed–Muller code of length 16, and BW_{16} is the Barnes–Wall lattice of rank 16.

The statements (1), (2) of the following lemma are easy, and (3) holds by the results in [15, p. 400].

Lemma 3.3. *The following hold:*

- (1) $\{(1^{16}), (1^8 0^8), (1^4 0^4)^2, (1^2 0^2)^4, (10)^8\}$ is a basis of $RM(1, 4)$.
- (2) The binary code $RM(1, 4)$ is a maximal doubly even code of length 16 without weight 4 vectors, i.e., there is no doubly even binary code D without weight 4 vectors such that $RM(1, 4) \subsetneq D$.
- (3) The automorphism group of $RM(1, 4)$ acts triply transitively on Ω_{16} .

Let k be a nonnegative integer. For subsets $X \subset \mathbb{R}^n$ and $D \subset \mathbb{F}_2^n$, define $X(k) := \{u \in X \mid \langle u, u \rangle = k\}$ and $D(k) := \{d \in D \mid \text{wt}(d) = k\}$. The following proposition about $L_B(C)$ is easy.

Proposition 3.4. *Let C be a binary code. Then the following hold:*

- (1) C is doubly even if and only if $L_B(C)$ is even.
- (2) Let n be the length of C . If C is doubly even, then the dual lattice $L_B(C)^*$ of $L_B(C)$ is given by $L_B(C)^* = L_A(C^\perp) + \mathbb{Z} \frac{1}{4} \alpha_{(1^n)}$.
- (3) Assume that C is doubly even. Then $C(4) = \emptyset$ if and only if $L_B(C)(2) = \emptyset$.

Let L be an even lattice of rank n with $L(2) = \emptyset$. A subset $\{\pm\beta_i \mid i \in \Omega_n\}$ of \mathbb{R}^n is called a *frame* of L if $\{\beta_i \mid i \in \Omega_n\}$ is an orthogonal basis of \mathbb{R}^n consisting of norm 2 vectors, and if $\beta_i \pm \beta_j \in L$ for $i, j \in \Omega_n$ [13, Section 1]. A frame F of L is said to be of *Type B* if $F \not\subset L$ and $F \subset L^*$. Let \mathcal{F}_L denote the set of all frames of Type B of L .

Remark 3.5. By (3.1), $L_B(C)$ has a frame $\{\pm\alpha_i \mid i \in \Omega_n\}$ of Type B if it is even. Also, if an even lattice of rank n has a frame $\{\pm\beta_i \mid i \in \Omega_n\}$ of Type B, then it is isomorphic to the lattice obtained by Construction B from some binary code with respect to $\{\beta_i \mid i \in \Omega_n\}$ [13, Lemma 2.1.3].

Define

$$O_L := \{\lambda + L \in L^*/L \mid 2\lambda \in L, \#(\lambda + L)(2) = 2n\}. \quad (3.2)$$

Since $O(L) = O(L^*)$, any element $\sigma \in O(L)$ acts on L^*/L as $\sigma(\lambda + L) := \sigma(\lambda) + L$ for $\lambda + L \in L^*/L$. Note that O_L is stable under this action of $O(L)$. It follows from [19, Proposition 1.8] that $(\lambda + L)(2)$ is a frame of Type B of L for each $\lambda + L \in O_L$. We have the following lemma.

Lemma 3.6. *The map $O_L \ni \lambda + L \mapsto (\lambda + L)(2) \in \mathcal{F}_L$ is bijective, and commute with the actions of $O(L)$.*

Proof. It is easily seen that the map is injective. We show the surjectivity. Let $F \in \mathcal{F}_L$ and $\gamma \in F$. Then $F \subset (\gamma + L)(2)$ since F is a frame of L . Suppose, to the contrary, that $F \subsetneq (\gamma + L)(2)$. Let $\gamma' \in (\gamma + L)(2) \setminus F$. Since $\gamma' - \gamma \in L$ and L is even, we have $\langle \gamma, \gamma' \rangle \in \mathbb{Z}$. Then $\langle \gamma, \gamma' \rangle = 0$ because $L(2) = \emptyset$ and the following inequality holds:

$$0 < \langle \gamma \pm \gamma', \gamma \pm \gamma' \rangle = 4 \pm 2\langle \gamma, \gamma' \rangle.$$

Since $\gamma + L = \beta + L$ for any $\beta \in F$, we also have $\langle \beta, \gamma' \rangle = 0$ by the same argument for γ . Hence $\gamma' = 0$ because F spans \mathbb{R}^n , contrary to the norm of γ' . Thus we have $F = (\gamma + L)(2)$. In particular, $|(\gamma + L)(2)| = 2n$, and hence $\gamma + L \in O_L$. Therefore the map is surjective. The last assertion obviously holds. \square

3.2. Orthogonal transformations of $L_B(C)$

Let C be a doubly even binary code of length n with $C(4) = \emptyset$, and $L = L_B(C)$. In this subsection, we give a generator of $O(L)$. Note that $L(2) = \emptyset$ by Proposition 3.4 (3).

For $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$, let ϵ_x denote the orthogonal transformation on \mathbb{R}^n defined by $\epsilon_x(\alpha_i) := (-1)^{x_i} \alpha_i$ for $i \in \Omega_n$. By the definition of $L_B(C)$, $\epsilon_x \in O(L)$ if and only if $x \in C^\perp$. Define

$$E(C) := \{\epsilon_x \mid x \in C^\perp\}.$$

Then $E(C)$ is a subgroup of $O(L)$ since $\epsilon_x \epsilon_y = \epsilon_{x+y}$ for $x, y \in C^\perp$. Every $\sigma \in S_n$ induces an orthogonal transformation of \mathbb{R}^n , denoted also by σ , by $\sigma(\alpha_i) := \alpha_{\sigma(i)}$ for $i \in \Omega_n$. Then we can regard $\text{Aut}(C)$ as a subgroup of $O(L)$ by this action. Also, $E(C)\text{Aut}(C)$ is a subgroup of $O(L)$ because $\text{Aut}(C)$ normalizes $E(C)$ in $O(L)$.

We now recall the definition of T -decompositions of binary codes.

Definition 3.7. (See [13, Section 3.3].) Let $n = 4m$ ($m \in \mathbb{Z}_{>0}$). A subset $S = \{s_1, \dots, s_m\} \subset \mathbb{F}_2^{4m}$ is called a T -decomposition of C if the following conditions are satisfied:

$$\text{wt}(s_i) = 4 \quad (1 \leq i \leq m), \quad (3.3)$$

$$s_i + s_j \in C \quad (1 \leq i, j \leq m), \quad (3.4)$$

$$\text{supp}(s_i) \cap \text{supp}(s_j) = \emptyset \quad (1 \leq i \neq j \leq m). \quad (3.5)$$

A T -decomposition S is said to be of *Type B* if it satisfies $S \not\subset C$ and $S \subset C^\perp$. Let \mathcal{T}_C denote the set of all T -decompositions of Type B of C . The following lemma is obtained from (3.3) and (3.5) immediately.

Lemma 3.8. If $S \in \mathcal{T}_C$, then the sum of all elements of S is the all-one vector in \mathbb{F}_2^n .

The following lemma is obtained in [13].

Lemma 3.9. (See [13, Lemma 3.3.2].) Let $n = 4m$ ($m \in \mathbb{Z}_{>0}$) and $T = \{t_1, \dots, t_m\} \in \mathcal{T}_C$. Then there exists $s \in C^\perp$ such that $|\text{supp}(s) \cap \text{supp}(t_k)| = 1$ for all $1 \leq k \leq m$. For such an element s , the linear transformation ρ_s associated to s is defined as follows:

$$\rho_s(\alpha_i) := \begin{cases} \frac{1}{2}\alpha_{t_k} & \text{if } \{i\} = \text{supp}(s) \cap \text{supp}(t_k), \\ \alpha_{\text{supp}(s) \cap \text{supp}(t_k)} + \alpha_i - \frac{1}{2}\alpha_{t_k} & \text{if } i \in \text{supp}(t_k) \setminus \text{supp}(s). \end{cases} \quad (3.6)$$

Then ρ_s is an orthogonal transformation of $L_B(C)$.

Define

$$\begin{aligned} O_L\left(\frac{1}{2}\right) &:= O_L \cap \{\lambda + L \mid \lambda \in L_A(C^\perp)\}, \\ O_L\left(\frac{1}{4}\right) &:= O_L \cap \left\{ \lambda + L \mid \lambda \in \frac{1}{4}\alpha_{(1^n)} + L_A(C^\perp) \right\}. \end{aligned} \quad (3.7)$$

Since C is doubly even, $(1^n) \in C^\perp$. Hence $L_B(C)^* = L_A(C^\perp) \amalg (\frac{1}{4}\alpha_{(1^n)} + L_A(C^\perp))$ by Proposition 3.4 (2). Also, we have $O_L = O_L(\frac{1}{2}) \amalg O_L(\frac{1}{4})$.

Lemma 3.10. For each $\lambda + L \in O_L(\frac{1}{4})$, there exists $c \in C^\perp$ such that $\epsilon_c(\lambda + L) = \frac{1}{4}\alpha_{(1^n)} + L$ or $\epsilon_c(\lambda + L) = \frac{1}{4}\alpha_{(1^n)} + \alpha_1 + L$.

Proof. Since

$$\frac{1}{4}\alpha_{(1^n)} + L_A(C^\perp) = \prod_{c \in C^\perp} \left(\frac{1}{4}\alpha_{(1^n)} + \frac{1}{2}\alpha_c + \sum_{i \in \Omega_n} \mathbb{Z}\alpha_i \right),$$

there exists $c \in C^\perp$ such that $\lambda + L = \frac{1}{4}\alpha_{(1^n)} - \frac{1}{2}\alpha_c + \alpha_1 + L$ or $\lambda + L = \frac{1}{4}\alpha_{(1^n)} - \frac{1}{2}\alpha_c + L$. Then the assertion holds because $\epsilon_c(\frac{1}{4}\alpha_{(1^n)} - \frac{1}{2}\alpha_c + \alpha_1 + L) = \frac{1}{4}\alpha_{(1^n)} + \alpha_1 + L$ and $\epsilon_c(\frac{1}{4}\alpha_{(1^n)} - \frac{1}{2}\alpha_c + L) = \frac{1}{4}\alpha_{(1^n)} + L$. \square

By Lemma 3.10, $\frac{1}{4}\alpha_{(1^n)} + \alpha_1 + L \in O_L(\frac{1}{4})$ or $\frac{1}{4}\alpha_{(1^n)} + L \in O_L(\frac{1}{4})$ if $O_L(\frac{1}{4}) \neq \emptyset$. If $\frac{1}{4}\alpha_{(1^n)} + \alpha_1 + L \in O_L(\frac{1}{4})$ (resp., $\frac{1}{4}\alpha_{(1^n)} + L \in O_L(\frac{1}{4})$), then $n = 8$ (resp., $n = 16$) and C is equivalent to a binary code which contains $\{(0^8), (1^8)\}$ (resp., $RM(1, 4)$) by [20, Lemma 2.6 (resp., Lemma 2.7)]. By Proposition 3.4, C is doubly even and $C(4) = \emptyset$. Hence $C \cong \{(0^8), (1^8)\}$ or $C \cong RM(1, 4)$ by Lemma 3.3 (2). We see from Example 3.2 that the following proposition holds.

Proposition 3.11. (Cf. [20, Lemmas 2.6 and 2.7].) *If $O_L(\frac{1}{4}) \neq \emptyset$, then L is isomorphic to $\sqrt{2}E_8$ or BW_{16} .*

Recall that the set $\{\pm\alpha_i \mid i \in \Omega_n\}$ is a frame of $L = L_B(C)$. Hence $(\alpha_1 + L)(2) = \{\pm\alpha_i \mid i \in \Omega_n\}$ by Lemma 3.6. From this, $\alpha_1 + L \in O_L(\frac{1}{2})$.

Lemma 3.12. *The stabilizer H_{α_1+L} of $\alpha_1 + L \in O_L$ in $O(L)$ is equal to $E(C)\text{Aut}(C)$.*

Proof. We see from the definitions of $E(C)$ and $\text{Aut}(C)$ that $E(C)\text{Aut}(C)$ is a subgroup of H_{α_1+L} . Hence, it suffices to prove that $E(C)\text{Aut}(C) \supset H_{\alpha_1+L}$. Let $g \in H_{\alpha_1+L}$. Since $(\alpha_1 + L)(2) = \{\pm\alpha_i \mid i \in \Omega_n\}$ and g preserves $(\alpha_1 + L)(2)$, we can write $g(\alpha_i) = (-1)^{x_\sigma(i)}\alpha_{\sigma(i)}$ ($i \in \Omega_n$) for some $\sigma \in S_n$ and $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$. Since $\{\alpha_i \mid i \in \Omega_n\}$ is a basis of \mathbb{R}^n , we have $g = \epsilon_x\sigma$. For $c \in C$, we have $\sigma(c) \in C$ and $|\text{supp}(\sigma(c)) \cap \text{supp}(x)| \in 2\mathbb{Z}$ since

$$g\left(\frac{1}{2}\alpha_c\right) = \frac{1}{2}\epsilon_x\sigma(\alpha_c) = \frac{1}{2}\alpha_{\sigma(c)} - \sum_{i \in \text{supp}(\sigma(c)) \cap \text{supp}(x)} \alpha_i \in L.$$

Consequently, we obtain $x \in C^\perp$ and $\sigma \in \text{Aut}(C)$. \square

Lemma 3.13. *If there exists $c \in C^\perp(4)$ such that $\frac{1}{2}\alpha_c + \alpha_1 + L \in O_L(\frac{1}{2})$, then $\frac{1}{2}\alpha_c + L \in O_L(\frac{1}{2})$.*

Proof. It follows from Proposition 3.4 (3) that $c \notin C$. Since $C = (C^\perp)^\perp$, there exists $d \in C^\perp$ such that $(c, d) = 1$. Then $\epsilon_d(\frac{1}{2}\alpha_c + \alpha_1 + L) = \frac{1}{2}\alpha_c + L$. Since $\epsilon_d \in O(L)$, the assertion holds. \square

Lemma 3.14. *If there exists $c \in C^\perp(4)$ such that $\frac{1}{2}\alpha_c + L \in O_L(\frac{1}{2})$, then there exists $X \in \mathcal{T}_C$ such that $c \in X$.*

Proof. Set $X = \{x \in C^\perp \mid \frac{1}{2}\alpha_x \in (\frac{1}{2}\alpha_c + L)(2)\}$. Let us show that X is a T -decomposition of Type B. Let $d_1, d_2 \in X$ with $d_1 \neq d_2$. Then we have

$$\text{supp}(d_1) \cap \text{supp}(d_2) = \emptyset \quad (3.8)$$

because $(\frac{1}{2}\alpha_c + L)(2)$ is a frame of L by Lemma 3.6 and $\frac{1}{2}|\text{supp}(d_1) \cap \text{supp}(d_2)| = \langle \frac{1}{2}\alpha_{d_1}, \frac{1}{2}\alpha_{d_2} \rangle$. We also have $d_1 + d_2 \in C$ since $\frac{1}{2}\alpha_{d_1+d_2} = \frac{1}{2}\alpha_{d_1} + \frac{1}{2}\alpha_{d_2} \in L$. Obviously, for any $x \in X$ $\text{wt}(x) = 4$. Set

$$S = \left\{ \frac{1}{2}\alpha_x - \alpha_y \mid x \in X, y \in \mathcal{E}_n, \text{supp}(y) \subset \text{supp}(x) \right\},$$

where \mathcal{E}_n is the set of all even weight vectors in \mathbb{F}_2^n . By the definition of X , $S \subset (\frac{1}{2}\alpha_c + L)(2)$. If we show $(\frac{1}{2}\alpha_c + L)(2) = S$, then the assertion holds because the cardinality of X is $\frac{n}{4}$ by (3.8) and the definition of O_L . By Proposition 3.4 (2), $(\frac{1}{2}\alpha_c + L)(2) \subset L_A(C^\perp)(2)$. Since $\frac{1}{2}\alpha_c + L \neq \alpha_1 + L$ and

$$L_A(C^\perp)(2) = \left\{ \frac{1}{2}\alpha_x - \alpha_y \mid x \in C^\perp(4), y \in \mathbb{F}_2^n, \text{supp}(y) \subset \text{supp}(x) \right\} \amalg \{\pm\alpha_i \mid i \in \Omega_n\}, \quad (3.9)$$

we have

$$\left(\frac{1}{2}\alpha_c + L \right) (2) \subset \left\{ \frac{1}{2}\alpha_x - \alpha_y \mid x \in C^\perp(4), y \in \mathcal{E}_n, \text{supp}(y) \subset \text{supp}(x) \right\}. \quad (3.10)$$

Let $u \in (\frac{1}{2}\alpha_c + L)(2)$. By (3.10), $u = \frac{1}{2}\alpha_x - \alpha_y$ for some $x \in C^\perp(4)$ and $y \in \mathcal{E}_n$ such that $\text{supp}(y) \subset \text{supp}(x)$. Since $\frac{1}{2}\alpha_x$ is also an element of $(\frac{1}{2}\alpha_c + L)(2)$, we have $x \in X$ and $u \in S$. This completes the proof of this lemma. \square

We assume that $|O_L(\frac{1}{2})| > 1$. Let $\lambda + L \in O_L(\frac{1}{2}) \setminus \{\alpha_1 + L\}$. Since $\lambda + L \subset L_A(C^\perp)$ and (3.9), there exists $c \in C^\perp(4)$ such that $\lambda + L = \frac{1}{2}\alpha_c + L$ or $\lambda + L = \frac{1}{2}\alpha_c + \alpha_1 + L$. By Lemma 3.13, we may assume that $\lambda + L = \frac{1}{2}\alpha_c + L$. Therefore, we have the following corollary.

Corollary 3.15. *If $|O_L(\frac{1}{2})| > 1$, then $\mathcal{T}_C \neq \emptyset$.*

Proposition 3.16. *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Then $E(C)\text{Aut}(C) \subsetneq O(L)$ if and only if $\mathcal{T}_C \neq \emptyset$.*

Proof. We first assume that $E(C)\text{Aut}(C) \subsetneq O(L)$, and prove $\mathcal{T}_C \neq \emptyset$. If $O_L(\frac{1}{4}) \neq \emptyset$, then C is equivalent to $\{(0^8), (1^8)\}$ or $RM(1, 4)$ by Proposition 3.11. It is obvious that these codes have a T -decomposition of Type B. Hence we may assume that $O_L(\frac{1}{4}) = \emptyset$. By Lemma 3.12 and our assumption, we have $H_{\alpha_1+L} \subsetneq O(L)$, where H_{α_1+L} is the stabilizer of $\alpha_1 + L$ as in Lemma 3.12. From this, the cardinality of the $O(L)$ -orbit through $\alpha_1 + L$ is greater than 1, which implies that $|O_L(\frac{1}{2})| > 1$. It follows from Corollary 3.15 that $\mathcal{T}_C \neq \emptyset$. Conversely, assume that $\mathcal{T}_C \neq \emptyset$. Since an orthogonal transformation of L defined in (3.6) does not belong to $E(C)\text{Aut}(C)$, we have $E(C)\text{Aut}(C) \subsetneq O(L)$. This proves the proposition. \square

Although the next proposition follows from Theorem 2 in [13], we give a proof for completeness.

Proposition 3.17. (Cf. [13, Theorem 2].) *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Then the subgroup generated by $E(C)\text{Aut}(C)$ and the orthogonal transformations defined in (3.6) acts on O_L transitively.*

Proof. Let K be the subgroup of $O(L)$ generated by $E(C)\text{Aut}(C)$ and the orthogonal transformations defined in (3.6), and let $\lambda + L \in O_L(\frac{1}{4})$. By Lemma 3.10 and Proposition 3.11, we may assume that $\lambda + L = \frac{1}{4}\alpha_{(1^8)} + \alpha_1 + L$ or $\lambda + L = \frac{1}{4}\alpha_{(1^{16})} + L$. For $T \in \mathcal{T}_C$ and $s \in C^\perp$ such that $|\text{supp}(s) \cap \text{supp}(t)| = 1$ for all $t \in T$, we have

$$\rho_s \left(\frac{1}{4}\alpha_{(1^8)} + \alpha_1 + L \right) = \frac{1}{2}\alpha_s + \frac{1}{2}\alpha_{t'} + L = \frac{1}{2}\alpha_{s+t'} - \alpha_1 + L \in O_L \left(\frac{1}{2} \right),$$

where $t' \in T$ such that $1 \in \text{supp}(t')$, and

$$\rho_s \left(\frac{1}{4}\alpha_{(1^{16})} + L \right) = \frac{1}{2}\alpha_s + L \in O_L \left(\frac{1}{2} \right).$$

Let $\lambda' + L \in O_L(\frac{1}{2}) \setminus \{\alpha_1 + L\}$. From the equations above, it suffices to prove that there exists $\sigma \in K$ such that $\sigma(\lambda' + L) = \alpha_1 + L$. By Lemma 3.13 and (3.9), we may assume that $\lambda' + L = \frac{1}{2}\alpha_c + L$ for some $c \in C^\perp(4)$. There exists $X \in \mathcal{T}_C$ such that $c \in X$ by Corollary 3.15. Then $\rho_{s'}(\frac{1}{2}\alpha_c + L) = \alpha_1 + L$ for $s' \in C^\perp$ such that $|\text{supp}(s') \cap \text{supp}(x)| = 1$ for all $x \in X$. Since $\rho_{s'} \in K$, the assertion holds. \square

The following lemma is a fundamental fact of group theory.

Lemma 3.18. *Let G be a finite group which acts on a finite set X , and H a subgroup of G . If there exists $x \in X$ whose stabilizer in G is contained in H , and if H acts on X transitively, then $G = H$.*

Corollary 3.19. *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Then $O(L)$ is generated by $E(C)\text{Aut}(C)$ and the orthogonal transformations defined in (3.6).*

Proof. The assertion follows from Proposition 3.17, and Lemmas 3.12, 3.18. \square

4. Extra automorphisms of V_L^+

4.1. The automorphism group of V_L^+

In this subsection, we recall extra automorphisms of V_L^+ constructed in [10, Section 11]. Also, we review the full automorphism group of the VOA V_L^+ associated with an even lattice L obtained by Construction B without norm 2 vectors (cf. [19]).

Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. In [10, (11.2.6)], an extra automorphism of V_L^+ was constructed from $\{\pm\alpha_i \mid i \in \Omega_n\}$. For each frame of Type B of L such a construction can be applied. Hence, an extra automorphism is constructed from an element of O_L by Lemma 3.6. For $\lambda + L \in O_L$, let us denote by $\sigma_{\lambda+L}$ this extra automorphism. Then the automorphism of V_L^+ constructed in [10, (11.2.6)] is equal to $\sigma_0 := \sigma_{\alpha_1+L}$. Note that $\sigma_{\lambda+L}$ is also an automorphism of $V_{\mathbb{Z}\lambda+L}$ of order 2 for $\lambda+L \in O_L$. The following propositions about some automorphisms of V_L^+ are obtained in [10, 19].

Proposition 4.1. (See [10, Proposition 12.2.7].) *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Then,*

$$\sigma_0(V_L^-) = V_{\alpha_1+L}^+, \quad \sigma_0(V_{\alpha_1+L}^-) = V_{\alpha_1+L}^-.$$

In particular,

$$\sigma_0 \cdot [V_L^-] = [V_{\alpha_1+L}^+], \quad \sigma_0 \cdot [V_{\alpha_1+L}^-] = [V_{\alpha_1+L}^-].$$

Proposition 4.2. (See [19, Proposition 2.9].) *Let L be an even lattice. For $g \in O(\widehat{L})$, $\lambda \in L^* \cap \frac{1}{2}L$,*

$$\{g \cdot [V_{\lambda+L}^\pm]\} = \{[V_{\nu(g^{-1})(\lambda)+L}^\pm]\},$$

where ν is defined in (2.5). Moreover, if g satisfies $\nu(g) = id_L$ (i.e., $g \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$), then

$$g \cdot [V_{\lambda+L}^\pm] = \begin{cases} [V_{\lambda+L}^\pm] & \text{if } g(2\lambda) = 0, \\ [V_{\lambda+L}^\mp] & \text{if } g(2\lambda) = 1. \end{cases}$$

Define \mathcal{O}_L by the orbit of $[V_L^-]$ under the action of $\text{Aut}(V_L^+)$, that is, $\mathcal{O}_L = \{g \cdot [V_L^-] \mid g \in \text{Aut}(V_L^+)\}$.

Proposition 4.3. (See [19, Theorem 3.15].) *Let L be an even lattice with $L(2) = \emptyset$. Assume that L is isomorphic to neither $\sqrt{2}E_8$ nor BW_{16} . Then $\mathcal{O}_L = \{[V_L^-], [V_{\lambda+L}^\pm] \mid \lambda + L \in O_L\}$.*

Theorem 4.4. *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Then $\text{Aut}(V_L^+) = \langle O(\widehat{L}) / \langle \theta \rangle, \sigma_0 \rangle$.*

Remark 4.5. In [19], it was proved that $\text{Aut}(V_L^+)$ is generated by $O(\widehat{L})/\langle\theta\rangle$ and extra automorphisms $\sigma_{\lambda+L}$ ($\lambda+L \in O_L$). Theorem 4.4 shows that $\text{Aut}(V_L^+)$ can be generated by $O(\widehat{L})/\langle\theta\rangle$ and σ_0 only.

Proof of Theorem 4.4. If we show that the subgroup of $\text{Aut}(V_L^+)$ generated by $O(\widehat{L})/\langle\theta\rangle$ and σ_0 acts transitively on \mathcal{O}_L , then the assertion of Theorem 4.4 follows from Lemma 3.18 since $O(\widehat{L})/\langle\theta\rangle$ is the stabilizer of $[V_L^-]$ in $\text{Aut}(V_L^+)$ by [19, Proposition 3.10]. Since the assertion of Theorem 4.4 was already shown in [19, Section 4] if $L \cong \sqrt{2}E_8$ or BW_{16} , we assume that L is isomorphic to neither $\sqrt{2}E_8$ nor BW_{16} . Then $\mathcal{O}_L = \{[V_L^-], [V_{\lambda+L}^\pm] \mid \lambda+L \in O_L\}$ by Proposition 4.3. Let $[V_{\lambda+L}^-] \in \mathcal{O}_L$. If we suppose that $f(2\lambda) = 0$ for any $f \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$, then it follows from Lemma 2.3 that $\lambda \in L$. This is a contradiction because $L \notin O_L$. Hence, there exists $g \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ such that $g(2\lambda) = 1$. From this, we obtain $g \cdot [V_{\lambda+L}^-] = [V_{\lambda+L}^+]$ by Proposition 4.2. Therefore it suffices to show that there exists $h \in \langle O(\widehat{L})/\langle\theta\rangle, \sigma_0 \rangle (\subset \text{Aut}(V_L^+))$ such that $h \cdot [V_{\lambda+L}^+] = [V_L^-]$. Now we have $O_L = O_L(\frac{1}{2})$ by Proposition 3.11. Then there exists $\varphi \in O(L)$ such that $\varphi(\lambda+L) = \alpha_1+L$ by Proposition 3.17. Also, since ν defined in (2.5) is surjective by (2.3), there exists $\varphi' \in O(\widehat{L})$ such that $\nu(\varphi') = \varphi$. Therefore, by the argument above and Proposition 4.2, there exists $\eta \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ such that $\eta\varphi' \cdot [V_{\lambda+L}^+] = [V_{\alpha_1+L}^+]$, and hence $\sigma_0\eta\varphi' \cdot [V_{\lambda+L}^+] = [V_L^-]$ by Proposition 4.1. Thus, $\langle O(\widehat{L})/\langle\theta\rangle, \sigma_0 \rangle$ acts on \mathcal{O}_L transitively, and the proof is complete. \square

4.2. A basis of $(V_{LB(C)}^+)_k^{\sigma_0} \cap M_{\mathfrak{h}}(1)_k$ for each $k \leq 7$

In this subsection, we give a basis of $(V_{LB(C)}^+)_k^{\sigma_0} \cap M_{\mathfrak{h}}(1)$ for each $k \leq 7$. For a subset $J \subset \Omega_n$, let us denote by $\mathfrak{h}(J)$ the subspace of \mathfrak{h} generated by $\{\alpha_i \mid i \in J\}$, and let $\mathfrak{h}_i := \mathfrak{h}(\{i\})$ for $i \in \Omega_n$. Let ω_J (resp., $\omega_i := \omega_{\{i\}}$) be the Virasoro element of $M_{\mathfrak{h}(J)}(1)$ (resp., $M_{\mathfrak{h}_i}(1)$) and set $L_J(n) := (\omega_J)_{n+1}$ (resp., $L_i(n) := (\omega_i)_{n+1}$) for $n \in \mathbb{Z}$. Since $\omega_i = \frac{1}{4}\alpha_i(-1)^2\mathbf{1}$, we obtain the following lemma by direct computation.

Lemma 4.6. Fix $i \in \Omega_n$. In $M_{\mathfrak{h}_i}(1)$, the following equations hold:

$$\begin{aligned} L_i(-7)\mathbf{1} &= \frac{1}{2}\alpha_i(-6)\alpha_i(-1)\mathbf{1} + \frac{1}{2}\alpha_i(-5)\alpha_i(-2)\mathbf{1} + \frac{1}{2}\alpha_i(-4)\alpha_i(-3)\mathbf{1}, \\ L_i(-5)L_i(-2)\mathbf{1} &= \frac{1}{2}\alpha_i(-6)\alpha_i(-1)\mathbf{1} + \frac{1}{8}\alpha_i(-4)\alpha_i(-1)^3\mathbf{1} + \frac{1}{8}\alpha_i(-3)\alpha_i(-2)\alpha_i(-1)^2\mathbf{1}, \\ L_i(-4)L_i(-3)\mathbf{1} &= \alpha_i(-6)\alpha_i(-1)\mathbf{1} + \frac{1}{2}\alpha_i(-5)\alpha_i(-2)\mathbf{1} \\ &\quad + \frac{1}{4}\alpha_i(-3)\alpha_i(-2)\alpha_i(-1)^2\mathbf{1} + \frac{1}{8}\alpha_i(-2)^3\alpha_i(-1)\mathbf{1}, \\ L_i(-3)L_i(-2)^2\mathbf{1} &= \frac{3}{2}\alpha_i(-6)\alpha_i(-1)\mathbf{1} + \frac{1}{2}\alpha_i(-4)\alpha_i(-3)\mathbf{1} + \frac{1}{4}\alpha_i(-4)\alpha_i(-1)^3\mathbf{1} \\ &\quad + \frac{1}{4}\alpha_i(-3)\alpha_i(-2)\alpha_i(-1)^2\mathbf{1} + \frac{1}{32}\alpha_i(-2)\alpha_i(-1)^5\mathbf{1}. \end{aligned}$$

Remark 4.7. Let L_1 and L_2 be even lattices. Then, $V_{L_1 \oplus L_2} \cong V_{L_1} \otimes V_{L_2}$ as a VOA if $\langle L_1, L_2 \rangle = 0$.

We see from the definition of extra automorphisms in [10] that σ_0 acts on $\bigotimes_{i=1}^n V_{\mathbb{Z}\alpha_i} \cong V_{L_A(\{(0^n)\})}$ diagonally, and we have the following proposition.

Proposition 4.8. Let $L = L_B(C)$ be an even lattice of rank n with $L(2) = \emptyset$. Then $(V_L^+)^{\sigma_0} \cap M_{\mathfrak{h}}(1) \cong (\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0} \cap (\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1))$.

Proof. Let $N = \bigoplus_{i=1}^n 2\mathbb{Z}\alpha_i$. Then we have

$$(V_L^+)^{\sigma_0} \cap M_{\mathfrak{h}}(1) = (V_L^+)^{\sigma_0} \cap V_N^+ \cap M_{\mathfrak{h}}(1) = \{u \in V_N^+ \cap M_{\mathfrak{h}}(1) \mid \sigma_0(u) = u\}. \quad (4.1)$$

By Remark 4.7,

$$V_N^+ \cong \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i} \right)^+. \quad (4.2)$$

Also, $M_{\mathfrak{h}}(1)$ is isomorphic to $\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1)$ via $V_N \cong \bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}$. Set $V_{2\mathbb{Z}\alpha_i}^x = \{u \in V_{2\mathbb{Z}\alpha_i} \mid \theta(u) = (-1)^x u\}$ for $i \in \Omega_n$ and $x \in \mathbb{F}_2$. Then

$$\left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i} \right)^+ = \bigoplus_{\substack{c=(c_1, \dots, c_n) \in \mathbb{F}_2^n \\ \text{wt}(c) \in 2\mathbb{Z}}} \bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^{c_i}.$$

By Proposition 4.1, $\sigma_0 \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^{c_i} \right) = \bigotimes_{i \notin \text{supp}(c)} V_{2\mathbb{Z}\alpha_i}^+ \otimes \bigotimes_{i \in \text{supp}(c)} V_{\alpha_i + 2\mathbb{Z}\alpha_i}^+$ for $c \in \mathbb{F}_2^n$ because $L_B(\{(0^1)\}) \cong 2\mathbb{Z}\alpha_i$ for $i \in \Omega_n$. Hence $\sigma_0 \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+ \right) = \bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+$ and

$$\left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i} \right)^+ \cap \left(\bigoplus_{\substack{c \in \mathbb{F}_2^n \\ \text{wt}(c) \in 2\mathbb{Z}_{>0}}} \sigma_0 \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^{c_i} \right) \right) = 0.$$

Since $\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+$ and $\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1)$ are subspaces of $\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}$, we have

$$\left\{ u \in \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i} \right)^+ \cap \left(\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1) \right) \mid \sigma_0(u) = u \right\} = \left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+ \right)^{\sigma_0} \cap \left(\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1) \right). \quad (4.3)$$

Thus we obtain the desired conclusion by (4.1), (4.2), and (4.3). \square

Lemma 4.9. $(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+)_k^{\sigma_0} = (\bigotimes_{i=1}^n (V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0})_k$ for $0 \leq k \leq 7$.

Proof. Set $(V_{2\mathbb{Z}\alpha_i}^+)^{\delta} = \{u \in V_{2\mathbb{Z}\alpha_i}^+ \mid \sigma_0(u) = (-1)^{\delta}u\}$ for $i \in \Omega_n$ and $\delta \in \mathbb{F}_2$, and set $U_c = \bigotimes_{i=1}^n (V_{2\mathbb{Z}\alpha_i}^+)^{c_i}$ for $c = (c_1, \dots, c_n) \in \mathbb{F}_2^n$. Then

$$\left(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+ \right)^{\sigma_0} = \bigotimes_{i=1}^n (V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0} \oplus \bigoplus_{c \in \mathbb{F}_2^n, \text{wt}(c) \in 2\mathbb{Z}_{>0}} U_c. \quad (4.4)$$

Let us denote by $(U_c)_m$ the degree m subspace of U_c for $c \in \mathbb{F}_2^n$. For any $c \in \mathbb{F}_2^n$ such that $\text{wt}(c) \in 2\mathbb{Z}_{>0}$, we show $k \geq 8$ if $(U_c)_k \neq 0$. There exist $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^n k_i = k$ and $\bigotimes_{i=1}^n (V_{2\mathbb{Z}\alpha_i}^+)^{c_i}_{k_i} \neq 0$ if $(U_c)_k \neq 0$. Since $\text{wt}(c) \in 2\mathbb{Z}_{>0}$, there exist $l, m \in \Omega_n$ such that $c_l = c_m = 1$. Then $k \geq k_l + k_m \geq 8$ because $s \geq 4$ if $(V_{2\mathbb{Z}\alpha_j}^+)_s^1 \neq 0$. This completes the proof of this lemma. \square

Lemma 4.10. For each $i \in \Omega_n$, $(V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0} \cap M_{\mathfrak{h}_i}(1)_k = (V_{\omega_i})_k$ for $0 \leq k \leq 7$, where V_{ω_i} is the subVOA of $V_{2\mathbb{Z}\alpha_i}^+$ generated by ω_i .

Proof. Since $(V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0} \cap M_{\mathfrak{h}_i}(1)$ is a subVOA of $V_{2\mathbb{Z}\alpha_i}^+$, we only need to show the case of $k = 7$ by Propositions 2.5 and 2.12. It follows from Lemma 4.6 that

$$(V_{2\mathbb{Z}\alpha_i}^+)_7 \cap M_{\mathfrak{h}_i}(1)_7 = (V_{\omega_i})_7 \oplus \langle \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \alpha_i(-5)\alpha_i(-2)\mathbf{1}, \alpha_i(-4)\alpha_i(-1)^3\mathbf{1} \rangle_{\mathbb{C}}.$$

By [10, Corollary 11.2.4], we have $\{u \in \langle \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \alpha_i(-5)\alpha_i(-2)\mathbf{1}, \alpha_i(-4)\alpha_i(-1)^3\mathbf{1} \rangle_{\mathbb{C}} \mid \sigma_0(u) = u\} = 0$. Hence the assertion of this lemma holds. \square

Proposition 4.11. $(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+)_k^{\sigma_0} \cap (\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1))_k = (\bigotimes_{i=1}^n V_{\omega_i})_k$ for $0 \leq k \leq 7$.

Proof. By Lemma 4.9, we have $(\bigotimes_{i=1}^n V_{2\mathbb{Z}\alpha_i}^+)_k^{\sigma_0} \cap (\bigotimes_{i=1}^n M_{\mathfrak{h}_i}(1))_k = (\bigotimes_{i=1}^n ((V_{2\mathbb{Z}\alpha_i}^+)^{\sigma_0} \cap M_{\mathfrak{h}_i}(1)))_k$ for $0 \leq k \leq 7$. Hence the assertion follows from Lemma 4.10. \square

Combining Propositions 4.8, 4.11 and Lemma 2.4, we obtain the following corollary.

Corollary 4.12. Let $L = L_B(C)$ be an even lattice of rank n with $L(2) = \emptyset$. For $0 \leq k \leq 7$, a basis of $(V_L^+)_k^{\sigma_0} \cap M_{\mathfrak{h}}(1)_k$ is given by

$$\left\{ L_{i_1}(-m_1) \cdots L_{i_l}(-m_l)\mathbf{1} \mid m_1 \geq \cdots \geq m_l \geq 2, m_j \in \mathbb{Z}, \sum_{j=1}^l m_j = k, 1 \leq i_1, \dots, i_l \leq n \right\}.$$

5. Classification of V_L^+ of class \mathcal{S}^4

In this section, we show that an even lattice L with $L(2) = \emptyset$ is isomorphic to $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, or BW_{16} if V_L^+ is of class \mathcal{S}^4 . The following proposition is obtained in [19].

Proposition 5.1. (See [19, Proposition 3.16].) Let L be an even lattice with $L(2) = \emptyset$. Then $O(\widehat{L})/\langle \theta \rangle \subsetneq \text{Aut}(V_L^+)$ if and only if L is obtained by Construction B.

Lemma 5.2. Let L be an even lattice. Then $\mathbb{C}\{L\}^{\mu(\text{Hom}(L, \mathbb{Z}/2\mathbb{Z}))} = \mathbb{C}\{2L\}$, where μ is the map defined in (2.4).

Proof. By Lemma 2.3, $\alpha \in 2L$ if and only if $\mu(f)(\iota(e_\alpha)) = \iota(e_\alpha)$ for all $f \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$. From this, we can easily deduce the assertion. \square

Remark 5.3. The orthogonal group $O(L)$ acts on $M_{\mathfrak{h}}(1)$ by (2.3), (2.5), and (2.6), and its action is obviously equal to the action of $O(\widehat{L})$. Hence we write the action of $O(\widehat{L})$ on $M_{\mathfrak{h}}(1)$ as the action of $O(L)$.

Corollary 5.4. Let L be an even lattice with $L(2) = \emptyset$. Then

$$(V_L^+)_k^{\text{Aut}(V_L^+)} = \begin{cases} ((V_L^+)_k^{\sigma_0} \cap M_{\mathfrak{h}}(1)_k)^{O(L)} & \text{if } L \text{ is obtained by Construction B,} \\ M_{\mathfrak{h}}(1)_k^{O(L)} & \text{otherwise} \end{cases}$$

for $0 \leq k \leq 7$.

Proof. Let $\lambda \in \text{Hom}(L, \mathbb{Z}/2\mathbb{Z})$. It follows from (2.3) and (2.6) that $\mu(\lambda)$ acts identically on $M_{\mathfrak{h}}(1)$. Hence $V_L^{\mu(\text{Hom}(L, \mathbb{Z}/2\mathbb{Z}))} = V_{2L}$ by (2.2) and Lemma 5.2. Also, $(V_L^+)^{\mu(\text{Hom}(L, \mathbb{Z}/2\mathbb{Z}))} = V_{2L}^+$ since $\mu(\text{Hom}(L, \mathbb{Z}/2\mathbb{Z}))$ and θ are commutative. It follows that $(V_L^+)_k^{\mu(\text{Hom}(L, \mathbb{Z}/2\mathbb{Z}))} = M_{\mathfrak{h}}(1)_k^+$ for $0 \leq k \leq 7$ because $L(2) = \emptyset$. Hence the assertion of this corollary follows from Lemma 2.8, Theorem 4.4, and Proposition 5.1. \square

Note that $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an $O(L)$ -module by extending the action of $O(L)$ on L to \mathfrak{h} \mathbb{C} -linearly, and $O(L)$ acts diagonally on the tensor algebra and the symmetric algebra of \mathfrak{h} . The following lemma is easy.

Lemma 5.5. Let L be an even lattice. Let $m, n \in \mathbb{Z}_{>0}$. If $m \neq n$, then the subspace of $M_{\mathfrak{h}}(1)$ spanned by $\{u_1(-m)u_2(-n)\mathbf{1} \mid u_i \in \mathfrak{h}\}$ is isomorphic to $\mathfrak{h} \otimes \mathfrak{h}$ as $O(L)$ -modules, otherwise, it is isomorphic to the degree 2 subspace of the symmetric algebra of \mathfrak{h} as $O(L)$ -modules.

Proposition 5.6. Let L be an even lattice with $L(2) = \emptyset$. If V_L^+ is of class \mathcal{S}^4 , then L is obtained by Construction B.

Proof. Let n be the rank of L . Assume that L is not obtained by Construction B. We show that V_L^+ is not of class \mathcal{S}^4 . Since $\{\alpha_i \mid i \in \Omega_n\}$ is an orthogonal basis of \mathbb{R}^n consisting of norm 2 vectors, $\sum_{i=1}^n \alpha_i(-3)\alpha_i(-1)\mathbf{1} \in (V_L^+)_4$ is fixed by all orthogonal transformations of L by Lemma 5.5. Hence, this element belongs to $(V_L^+)_4^{\text{Aut}(V_L^+)}$ by Corollary 5.4. By direct computation, we have

$$\begin{aligned}
L(-4)\mathbf{1} &= \frac{1}{2} \sum_{i=1}^n \alpha_i(-3)\alpha_i(-1)\mathbf{1} + \frac{1}{4} \sum_{i=1}^n \alpha_i(-2)^2\mathbf{1}, \\
L(-2)^2\mathbf{1} &= \frac{1}{16} \sum_{1 \leq i, j \leq n} \alpha_i(-1)^2\alpha_j(-1)^2\mathbf{1} + \frac{1}{2} \sum_{i=1}^n \alpha_i(-3)\alpha_i(-1)\mathbf{1}.
\end{aligned}$$

From this, $\sum_{i=1}^n \alpha_i(-3)\alpha_i(-1)\mathbf{1}$ cannot belong to $(V_\omega)_4$ by Lemma 2.4. Thus we obtain $(V_L^+)_4^{\text{Aut}(V_L^+)} \supsetneq (V_\omega)_4$. \square

Corollary 5.7. *Let L be an even lattice of rank 1 with $L(2) = \emptyset$. If V_L^+ is of class \mathcal{S}^4 , then L is isomorphic to $2A_1$.*

Proof. By Proposition 5.6, there exists a binary code C such that $L \cong L_B(C)$. We see from our assumptions and Proposition 3.4 that C is a doubly even code of length 1. Hence we have $C = \{(0^1)\}$. Therefore $L \cong 2A_1$ by Example 3.2. \square

Proposition 5.8. *Let $L = L_B(C)$ be an even lattice of rank $n > 1$ with $L(2) = \emptyset$. If V_L^+ is of class \mathcal{S}^4 , then $\mathcal{T}_C \neq \emptyset$.*

Proof. Assume that $\mathcal{T}_C = \emptyset$, and we show that V_L^+ is not of class \mathcal{S}^4 . By Proposition 3.16 and Corollary 5.4,

$$(V_L^+)_4^{\text{Aut}(V_L^+)} = ((V_L^+)_4^{\sigma_0} \cap M_{\mathfrak{h}}(1)_4)^{E(C)\text{Aut}(C)}. \quad (5.1)$$

It is easy to check that $\sum_{i=1}^n L_i(-2)^2\mathbf{1}$ is fixed by $E(C)\text{Aut}(C)$. Hence by (5.1) and Corollary 4.12, $\sum_{i=1}^n L_i(-2)^2\mathbf{1} \in (V_L^+)_4^{\text{Aut}(V_L^+)}$. However, since $n > 1$ and $L(-2)^2\mathbf{1} = \sum_{i=1}^n L_i(-2)^2\mathbf{1} + 2 \sum_{1 \leq i < j \leq n} L_i(-2)L_j(-2)\mathbf{1}$, $(V_\omega)_4$ does not contain $\sum_{i=1}^n L_i(-2)^2\mathbf{1}$ by Lemma 2.4. Hence we have $(V_L^+)_4^{\text{Aut}(V_L^+)} \supsetneq (V_\omega)_4$. \square

From Proposition 5.8, we also see that the rank of L is a multiple of 4 if V_L^+ is of class \mathcal{S}^4 and the rank of L is greater than 1.

Corollary 5.9. *Let L be an even lattice with $L(2) = \emptyset$. Assume that V_L^+ is of class \mathcal{S}^4 . Then the following hold:*

- (1) *If the rank of L is 4, then $L \cong \sqrt{2}D_4$.*
- (2) *If the rank of L is 8, then $L \cong \sqrt{2}E_8$.*

Proof. First, we see from Proposition 3.4 that C is a doubly even binary code with $C(4) = \emptyset$. Since the length of C is equal to the rank of $L = L_B(C)$, we obtain $C = \{(0^4)\}$ if the rank of L is 4. Hence, $L \cong \sqrt{2}D_4$ by Example 3.2. Assume that the rank of L is equal to 8. By Proposition 5.8, $\mathcal{T}_C \neq \emptyset$. Then $(1^8) \in C$ by Lemma 3.8 and (3.4). Therefore, $C = \{(0^8), (1^8)\}$, and $L \cong \sqrt{2}E_8$ by Example 3.2. \square

For a binary code C , let \mathcal{S}_C be the binary code generated by all the elements of all T -decompositions of Type B of C , i.e., $\mathcal{S}_C := \langle \bigcup_{T \in \mathcal{T}_C} T \rangle_{\mathbb{F}_2} (\subset C^\perp)$.

Lemma 5.10. *Let C be a doubly even binary code of length $4m$ with $C(4) = \emptyset$, where $m \in \mathbb{Z}_{>2}$. If \mathcal{S}_C is singly even, then $C \cong RM(1, 4)$.*

Proof. Since \mathcal{S}_C is singly even, there exist $f, f' \in \bigcup_{S \in \mathcal{T}_C} S$ such that $(f, f') = 1$. Also, f and f' belong to different T -decompositions by (3.5). Hence we may assume that $f \in T$ and $f' \in T'$ for some $T, T' \in \mathcal{T}_C$ with $T \neq T'$. Since $(f + g, f') = 0$ for $g \in T$, we have $(g, f') = 1$. In particular, $|\text{supp}(g) \cap \text{supp}(f')| \in \{1, 3\}$ for all $g \in T$. Hence, $m = 4$ by $\sum_{g \in T} |\text{supp}(g) \cap \text{supp}(f')| = \text{wt}(f') = 4$ and $|T| = m > 2$. Also, $(g, g') = 1$ for any $g \in T$ and $g' \in T'$ because $(g, f' + g') = 0$. Since the weights of elements of T' are 4, we obtain $|\text{supp}(g) \cap \text{supp}(g')| = 1$ for any $g \in T$ and $g' \in T'$. Therefore, we may assume that $T = \{(1^4 0^{12}), (0^4 1^4 0^8), (0^8 1^4 0^4), (0^{12} 1^4)\}$ and $T' = \{(1000)^4, (0100)^4, (0010)^4, (0001)^4\}$, up to equivalence. Then C contains $RM(1, 4)$ by Lemma 3.3 (1) and (3.4). Since the length of C is 16, the assertion of the lemma follows from Lemma 3.3 (2). \square

The direct sum of codes $C_1 \subset \mathbb{F}_2^{n_1}$ and $C_2 \subset \mathbb{F}_2^{n_2}$ is the binary code

$$C_1 \oplus C_2 := \{(u_1, \dots, u_{n_1}, v_1, \dots, v_{n_2}) \mid (u_1, \dots, u_{n_1}) \in C_1, (v_1, \dots, v_{n_2}) \in C_2\} \subset \mathbb{F}_2^{n_1+n_2}.$$

A binary code is said to be *decomposable* if it is a direct sum of codes, otherwise it is said to be *indecomposable*. For a binary code C , we can write $C = \bigoplus_{i=1}^r C_i$ for some $r \in \mathbb{Z}_{>0}$, where C_i is an indecomposable component for $i \in \Omega_r$. Note that this decomposition is unique. The following lemma about the automorphism group of a decomposable code holds.

Lemma 5.11. (See [18, Lemma 2.4].) *Let $C = C_1 \oplus \dots \oplus C_r$, where each C_i is a direct sum of equivalent codes, and for $i \neq j$ no summand of C_i is equivalent to a summand of C_j . Then $\text{Aut}(C) = \text{Aut}(C_1) \cdots \text{Aut}(C_r)$.*

Let $\mathcal{S}_C = \bigoplus_{i=1}^r D_i$, where D_1, \dots, D_r are indecomposable components of \mathcal{S}_C . Define $K_i := \bigcup_{c \in D_i} \text{supp}(c)$ for $i \in \Omega_r$.

Lemma 5.12. *Let C be a doubly even binary code with $C(4) = \emptyset$ and $\mathcal{T}_C \neq \emptyset$, and let $T \in \mathcal{T}_C$. Assume that $\mathcal{S}_C = \bigoplus_{i=1}^r D_i$ is doubly even. Then the orthogonal transformations defined in (3.6) fix the Virasoro element $\omega_{K_i} (= \sum_{j \in K_i} \omega_j)$ of $M_{\mathfrak{h}(K_i)}(1)$, for all $i \in \Omega_r$.*

Proof. Set $T_i = T \cap D_i$ for $i \in \Omega_r$. Then $\bigsqcup_{f \in T_i} \text{supp}(f) \subset K_i$. Note that $\mathcal{S}_C(4) = \bigsqcup_{i=1}^r D_i(4)$ since \mathcal{S}_C is doubly even. Hence, $T = \bigsqcup_{i=1}^r T_i$ and $\Omega_n = \bigsqcup_{i=1}^r \bigsqcup_{f \in T_i} \text{supp}(f)$. From this, $K_i = \bigsqcup_{f \in T_i} \text{supp}(f)$ for any $i \in \Omega_r$. Thus we have

$$\omega_{K_i} = \sum_{f \in T_i} \omega_{\text{supp}(f)}. \quad (5.2)$$

Now, for $t \in T$ and $s \in C^\perp$ such that $|\text{supp}(s) \cap \text{supp}(t')| = 1$ for all $t' \in T$, $\rho_s|_{\mathfrak{h}(t)}$ is an orthogonal transformation of $\mathfrak{h}(t) := \mathfrak{h}(\text{supp}(t))$ by (3.6), and hence it is an automorphism of $M_{\mathfrak{h}(t)}(1)$. From this,

$$\rho_s(\omega_{\text{supp}(t)}) = \omega_{\text{supp}(t)}. \quad (5.3)$$

It follows from (5.2) that the assertion of this lemma holds. \square

Let d_{4s} , e_7 , and e_8 be binary codes of length $4s$, 7 , and 8 generated by the rows of the following matrices:

$$d_{4s} : \begin{pmatrix} 1111 & 0000 & \cdots & 0000 \\ 0011 & 1100 & \cdots & 0000 \\ \vdots & \ddots & \ddots & \vdots \\ 0000 & \cdots & 0011 & 1100 \\ 0000 & \cdots & 0000 & 1111 \end{pmatrix}, \quad e_7 : \begin{pmatrix} 1111000 \\ 0011110 \\ 1010101 \end{pmatrix}, \quad e_8 : \begin{pmatrix} 11110000 \\ 00111100 \\ 00001111 \\ 10101010 \end{pmatrix}. \quad (5.4)$$

Proposition 5.13. (Cf. [18, Theorem 6.5].) *A doubly even binary code generated by weight 4 vectors is equivalent to a direct sum of some of the e_7 , e_8 , and d_{4s} 's.*

Proof. Let $C = \bigoplus_{i=1}^r C_i$ be a doubly even binary code generated by $S \subset \mathbb{F}_2^n(4)$, where C_1, \dots, C_r are indecomposable components of C . Obviously, $C_i \supset \langle S \cap C_i \rangle_{\mathbb{F}_2}$ for $i \in \Omega_r$. Since C is doubly even, $C(4) = \prod_{i=1}^r C_i(4)$. Hence, we have $S = \prod_{i=1}^r (C_i \cap S)$. We obtain $C_i = \langle S \cap C_i \rangle_{\mathbb{F}_2}$ for $i \in \Omega_r$ because S generates C . It follows from [18, Theorem 6.5] that C_i is equivalent to e_7 , e_8 , or d_{4s} ($s \geq 1$). This proves the assertion of this proposition. \square

Proposition 5.14. *Let $L = L_B(C)$ be an even lattice with $L(2) = \emptyset$. Assume that the rank of L is greater than 8. If V_L^+ is of class \mathcal{S}^4 , then L is isomorphic to BW_{16} .*

Proof. If \mathcal{S}_C is singly even, then the assertion follows from Example 3.2 and Lemma 5.10. Hence, it suffices to show that V_L^+ is not of class \mathcal{S}^4 if \mathcal{S}_C is doubly even. Let G be the subgroup of the orthogonal group of \mathbb{R}^n generated by $E(C)$, $\text{Aut}(\mathcal{S}_C)$, and the orthogonal transformations defined in (3.6). Since $\text{Aut}(C)$ preserves \mathcal{T}_C , we have $\text{Aut}(C) \leq \text{Aut}(\mathcal{S}_C)$. Hence $O(L) \subset G$ by Corollary 3.19, and $((V_L^+)_4^{\sigma_0} \cap M_{\mathfrak{h}}(1)_4)^G \subset (V_L^+)_4^{\text{Aut}(V_L^+)} \cap M_{\mathfrak{h}}(1)_4$ by Corollary 5.4. From this, if we find an element of $((V_L^+)_4^{\sigma_0} \cap M_{\mathfrak{h}}(1)_4)^G \setminus V_\omega$, then the assertion holds.

Let $\mathcal{S}_C = \bigoplus_{i=1}^r D_i$, where D_1, \dots, D_r are indecomposable components of \mathcal{S}_C . Note that $\mathcal{S}_C \neq 0$ by Proposition 5.8. First, we assume $r > 1$. Set $R = \{i \in \Omega_r \mid D_1 \cong D_i\}$. By Lemma 5.11, $\text{Aut}(\mathcal{S}_C) = \text{Aut}(\bigoplus_{i \in R} D_i) \text{Aut}(\bigoplus_{i \in \Omega_r \setminus R} D_i)$. Then $\text{Aut}(\mathcal{S}_C)$ fixes $\sum_{i \in R} L_{K_i}(-2)^2 \mathbf{1}$ since $\text{Aut}(\bigoplus_{i \in R} D_i)$ is isomorphic to the wreath product of $\text{Aut}(D_1)$ and the symmetric group of degree $|R|$, that is, $\text{Aut}(\bigoplus_{i \in R} D_i) \cong \text{Aut}(D_1) \wr S_{|R|}$. It follows from Lemma 5.12 that $\sum_{i \in R} L_{K_i}(-2)^2 \mathbf{1} \in ((V_L^+)_4^{\sigma_0} \cap M_{\mathfrak{h}}(1)_4)^G$. However, since

$$L(-2)^2 \mathbf{1} = \sum_{i=1}^r L_{K_i}(-2)^2 \mathbf{1} + 2 \sum_{1 \leq i < j \leq r} L_{K_i}(-2) L_{K_j}(-2) \mathbf{1}$$

and $r > 1$, $\sum_{i \in R} L_{K_i}(-2)^2 \mathbf{1}$ does not belong to $(V_\omega)_4$ by Lemma 2.4.

Next, we assume that $r = 1$, i.e., \mathcal{S}_C is indecomposable. Then we may assume that $\mathcal{S}_C = d_{4s}$ ($s > 2$) by Proposition 5.13. Set $l_i = \{2i-1, 2i\}$ for $1 \leq i \leq 2s$. Since $\text{Aut}(d_{4s})$ preserves $d_{4s}^\perp(2) = \{(0^{2j-2} 1^2 0^{4s-2j}) \mid 1 \leq j \leq 2s\}$, we have

$$\text{Aut}(d_{4s}) = \{\sigma \in S_{4s} \mid \sigma(\{l_i \mid 1 \leq i \leq 2s\}) = \{l_i \mid 1 \leq i \leq 2s\}\}. \quad (5.5)$$

Set $T_0 = \{t_i := (0^{4(i-1)} 1^4 0^{4(s-i)}) \mid 1 \leq i \leq s\}$ and set $u = (0001)^s \in \mathbb{F}_2^{4s}$. Let ρ_u denote the orthogonal transformation on \mathbb{R}^{4s} defined in (3.6) with respect to T_0 and u , and let $T \in \mathcal{T}_C$. Now we have $T \subset d_{4s}(4)$. Due to $d_{4s}(4) = \{(0^{2i-2} 1^2 0^{2j-2i-2} 1^2 0^{4s-2j}) \mid 1 \leq i < j \leq 2s\}$ and the action of $\text{Aut}(d_{4s})$, there exists $\sigma \in \text{Aut}(d_{4s})$ such that $\sigma(T_0) = T$. Let $x \in C^\perp$ such that $|\text{supp}(x) \cap \text{supp}(t)| = 1$ for all $t \in T$. Since $|\text{supp}(\sigma^{-1}(x)) \cap \text{supp}(t_i)| = 1$ for all $1 \leq i \leq s$, there exists $\tau \in \text{Aut}(d_{4s})$ such that $x = \tau\sigma(u)$, and we have $\rho_x = (\tau\sigma)\rho_u(\tau\sigma)^{-1}$. Therefore $G = \langle E(C), \text{Aut}(d_{4s}), \rho_u \rangle$. By (5.5), the element

$$2 \sum_{i=1}^{2s} L_{2i-1}(-2) L_{2i}(-2) \mathbf{1} - \sum_{1 \leq i < j \leq 2s} L_{l_i}(-2) L_{l_j}(-2) \mathbf{1} \in (V_L^+)^{\sigma_0}_4 \cap M_{\mathfrak{h}}(1)_4 \quad (5.6)$$

is fixed by $\text{Aut}(d_{4s})$. By Lemma 2.4, the element above does not belong to $(V_\omega)_4$. If we show that the element (5.6) is fixed by ρ_u , then it belongs to $((V_L^+)^{\sigma_0}_4 \cap M_{\mathfrak{h}}(1)_4)^G \setminus V_\omega$, and hence the assertion of this proposition holds. By direct computation, we have

$$\begin{aligned} & 2 \sum_{i=1}^{2s} L_{2i-1}(-2) L_{2i}(-2) \mathbf{1} - \sum_{1 \leq i < j \leq 2s} L_{l_i}(-2) L_{l_j}(-2) \mathbf{1} \\ &= \sum_{i=1}^s (L_{4i-3}(-2) - L_{4i}(-2))(L_{4i-2}(-2) - L_{4i-1}(-2)) \mathbf{1} \\ & \quad + \sum_{i=1}^s (L_{4i-3}(-2) - L_{4i-1}(-2))(L_{4i-2}(-2) - L_{4i}(-2)) \mathbf{1} - \sum_{1 \leq i < j \leq s} L_{t_i}(-2) L_{t_j}(-2) \mathbf{1}, \end{aligned}$$

where $L_{t_i}(n) = L_{\text{supp}(t_i)}(n)$. By (5.3), ρ_u fixes $L_{t_i}(-2) L_{t_j}(-2) \mathbf{1}$ for $i, j \in \Omega_s$. Fix $i \in \Omega_s$, $(L_{4i-3}(-2) - L_{4i}(-2))(L_{4i-2}(-2) - L_{4i-1}(-2)) \mathbf{1}$ is equal to

$$\frac{1}{16}(\alpha_{4i-3} - \alpha_{4i})(-1)(\alpha_{4i-3} + \alpha_{4i})(-1)(\alpha_{4i-2} - \alpha_{4i-1})(-1)(\alpha_{4i-2} + \alpha_{4i-1})(-1) \mathbf{1}.$$

Since $\rho_u(\alpha_{4i-3} - \alpha_{4i}) = \alpha_{4i-3} + \alpha_{4i}$ and $\rho_u(\alpha_{4i-2} - \alpha_{4i-1}) = \alpha_{4i-2} + \alpha_{4i-1}$, the element above is also fixed by ρ_u . By the same argument, ρ_u fixes $(L_{4i-3}(-2) - L_{4i-1}(-2))(L_{4i-2}(-2) - L_{4i}(-2)) \mathbf{1}$. Therefore, ρ_u fixes the element (5.6). This completes the proof of this proposition. \square

From [Corollaries 5.7 and 5.9](#) and [Proposition 5.14](#), we obtain the following theorem.

Theorem 5.15. *Let L be an even lattice with $L(2) = \emptyset$. If V_L^+ is of class \mathcal{S}^4 , then L is isomorphic to $2A_1$, $\sqrt{2}D_4$, $\sqrt{2}E_8$, or BW_{16} .*

6. Examples of V_L^+ of classes \mathcal{S}^5 and \mathcal{S}^7

In this section, we prove that the VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 . By [Proposition 2.5](#), we can apply [Corollary 2.13](#) to the VOA V_L^+ . Hence, if $(V_L^+)_n^{\text{Aut}(V_L^+)} = (V_\omega)_n$, then V_L^+ is of class \mathcal{S}^n .

6.1. The orthogonal groups of Barnes–Wall lattices

In this subsection, we review the orthogonal groups of $\sqrt{2}D_4$, $\sqrt{2}E_8$, and BW_{16} . It is known that D_4 and E_8 are the Barnes–Wall lattices of rank 4 and 8, respectively (cf. [\[2\]](#)).

Let \mathcal{G}_m be the subgroup of the Clifford group of genus m discussed in [\[2, p. 35\]](#). Then $O(BW_{2^m})$ is equal to \mathcal{G}_m if $m \neq 3$, and $\mathcal{G}_3 \subsetneq O(BW_8) = O(E_8)$ (cf. [\[2, 11\]](#)). Let us denote by $S^k(U)$ the degree k subspace of the symmetric algebra $S(U)$ of a vector space U . Let $\{\alpha_i \mid i \in \Omega_{2^m}\}$ be an orthogonal basis of \mathbb{R}^{2^m} consisting of norm 2 vectors. Set $Z_1, Z_2, Z_3 \in (\mathbb{R}^{2^m})^{\otimes 4}$, and $Z_4 \in (\mathbb{R}^{2^m})^{\otimes 6}$ as follows:

$$\begin{aligned} Z_1 &= \sum_{1 \leq i, j \leq 2^m} \alpha_i \otimes \alpha_i \otimes \alpha_j \otimes \alpha_j, & Z_2 &= \sum_{1 \leq i, j \leq 2^m} \alpha_i \otimes \alpha_j \otimes \alpha_i \otimes \alpha_j, \\ Z_3 &= \sum_{1 \leq i, j \leq 2^m} \alpha_i \otimes \alpha_j \otimes \alpha_j \otimes \alpha_i, & Z_4 &= \sum_{1 \leq i, j, k \leq 2^m} \alpha_i \otimes \alpha_i \otimes \alpha_j \otimes \alpha_j \otimes \alpha_k \otimes \alpha_k. \end{aligned}$$

Note that the symmetric group S_n acts on $(\mathbb{R}^{2^m})^{\otimes n}$ as $\sigma_*(u^1 \otimes \cdots \otimes u^n) := u^{\sigma^{-1}(1)} \otimes \cdots \otimes u^{\sigma^{-1}(n)}$ for $\sigma \in S_n$ and $u^1, \dots, u^n \in \mathbb{R}^{2^m}$.

A binary code is said to be *self-dual* if it is equal to its dual code. In the proof of [\[2, Theorem 5.1\]](#), it was shown that there is a basis of $((\mathbb{R}^{2^m})^{\otimes 6})^{\mathcal{G}_m}$ (resp., $((\mathbb{R}^{2^m})^{\otimes 4})^{\mathcal{G}_m}$) which corresponds bijectively to the set of all self-dual codes of length 6 (resp., length 4) if $m \geq 3$ (resp., $m \geq 2$) by using the argument of [\[11, Theorem 4.9\]](#). In fact, there are 3 self-dual codes of length 4. Also, there is only one self-dual code of length 6, up to equivalence. The following proposition holds.

Proposition 6.1. (Cf. [\[2, Theorem 5.1\]](#) and [\[11, Theorem 4.9\]](#).)

- (1) The set $\{Z_1, Z_2, Z_3\}$ is a basis of $((\mathbb{R}^{2^m})^{\otimes 4})^{\mathcal{G}_m}$ if $m \geq 2$.
- (2) The set $\{\sigma_*(Z_4) \mid \sigma \in S_6\}$ is a basis of $((\mathbb{R}^{2^m})^{\otimes 6})^{\mathcal{G}_m}$ if $m \geq 3$.

Let $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{2^m}$. Define the linear maps ψ_1 , ψ_2 , and ψ_3 by

$$\begin{aligned}
\psi_1 : \mathfrak{h}^{\otimes 4} &\rightarrow \mathfrak{h} \otimes S^3(\mathfrak{h}), & a \otimes b \otimes c \otimes d &\mapsto a \otimes bcd, \\
\psi_2 : \mathfrak{h}^{\otimes 4} &\rightarrow \mathfrak{h} \otimes \mathfrak{h} \otimes S^2(\mathfrak{h}), & a \otimes b \otimes c \otimes d &\mapsto a \otimes b \otimes cd, \\
\psi_3 : \mathfrak{h}^{\otimes 6} &\rightarrow \mathfrak{h} \otimes S^5(\mathfrak{h}), & a \otimes b \otimes c \otimes d \otimes e \otimes f &\mapsto a \otimes bcdef.
\end{aligned}$$

Clearly, these maps are surjective and $O(2^m, \mathbb{R})$ -module homomorphisms, where $O(2^m, \mathbb{R})$ is the orthogonal group of \mathbb{R}^{2^m} .

Corollary 6.2. *The following hold:*

- (1) $\{\psi_1(Z_1)\}$ is a basis of $(\mathfrak{h} \otimes S^3(\mathfrak{h}))^{\mathcal{G}_m}$ if $m \geq 2$.
- (2) $\{\psi_2(Z_1), \psi_2(Z_2)\}$ is a basis of $(\mathfrak{h} \otimes \mathfrak{h} \otimes S^2(\mathfrak{h}))^{\mathcal{G}_m}$ if $m \geq 2$.
- (3) $\{\psi_3(Z_4)\}$ is a basis of $(\mathfrak{h} \otimes S^5(\mathfrak{h}))^{\mathcal{G}_m}$ if $m \geq 3$.

Proof. Note that for $k \geq 0$, \mathcal{G}_m acts on $\mathfrak{h}^{\otimes k}$ \mathbb{C} -linearly. Hence $\{Z_1, Z_2, Z_3\}$ (resp., $\{\sigma_*(Z_4) \mid \sigma \in S_6\}$) is a basis of $(\mathfrak{h}^{\otimes 4})^{\mathcal{G}_m}$ (resp., $(\mathfrak{h}^{\otimes 6})^{\mathcal{G}_m}$) if $m \geq 2$ (resp., $m \geq 3$) by Proposition 6.1. Since ψ_1 (resp., ψ_2, ψ_3) is surjective, the image of ψ_1 (resp., ψ_2, ψ_3) of a basis is a generator of $(\mathfrak{h} \otimes S^3(\mathfrak{h}))^{\mathcal{G}_m}$ (resp., $(\mathfrak{h} \otimes \mathfrak{h} \otimes S^2(\mathfrak{h}))^{\mathcal{G}_m}, (\mathfrak{h} \otimes S^5(\mathfrak{h}))^{\mathcal{G}_m}$). Clearly, $\psi_i(Z_j)$ ($i = 1, 2, j = 1, 2, 3$) and $\psi_3(Z_4)$ are nonzero elements. Since $\psi_1(Z_1) = \psi_1(Z_2) = \psi_1(Z_3)$, the first assertion holds. We see from the definition of ψ_2 that $\psi_2(Z_1) \neq \psi_2(Z_2)$ and $\psi_2(Z_2) = \psi_2(Z_3)$. It is easy to check that $\{\psi_2(Z_1), \psi_2(Z_2)\}$ is linearly independent. Hence, the second assertion holds. Also, the last assertion holds because for any $\sigma \in S_6$ $\psi_3(Z_4) = \psi_3(\sigma_*(Z_4))$ by the definition of ψ_3 . \square

6.2. Examples

First, we show that the VOA $V_{2A_1}^+$ is of class \mathcal{S}^7 .

Proposition 6.3. *The VOA $V_{2A_1}^+$ is of class \mathcal{S}^7 .*

Proof. Since $2A_1$ is obtained by Construction B from $\{(0^1)\}$, the assertion follows from Lemma 4.10. \square

Remark 6.4. By Lemma 5.2, $(V_{2A_1}^+)^{\text{Aut}(V_{2A_1}^+)} \subset V_{4A_1}^+$. In particular, $(V_{2A_1}^+)^{\text{Aut}(V_{2A_1}^+)}_{15} \subset M_{\mathfrak{h}}(1)_{15}^+$. Also, by the argument in [6, Section 4] $\text{Aut}(V_{2A_1}^+)$ does not fix a Virasoro highest weight vector in $M_{\mathfrak{h}}(1)^+$ with highest weight 4. Due to these circumstances, one can show that $V_{2A_1}^+$ is of class \mathcal{S}^{15} because $M_{\mathfrak{h}}(1)^+$ is isomorphic to $\bigoplus_{m \geq 0} L(1, 4m^2)$ as $V_{\omega} \cong L(1, 0)$ -modules by [6, Theorem 2.7 (1)], where $L(1, 0)$ is the simple Virasoro VOA with central charge 1 and $L(1, k)$ is the Virasoro highest weight irreducible $L(1, 0)$ -module with highest weight k .

Next, we show that the VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{\sqrt{2}E_8}^+$ and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 by using Corollary 6.2. The following lemma holds by seeing subspaces of $M_{\mathfrak{h}}(1)$.

Lemma 6.5. *Let L be an even lattice. In $M_{\mathfrak{h}}(1)$, set*

- $W_1 := \text{Span}_{\mathbb{C}}\{u_1(-2)u_2(-1)u_3(-1)u_4(-1)\mathbf{1} \mid u_i \in \mathfrak{h}\}.$
- $W_2 := \text{Span}_{\mathbb{C}}\{u_1(-4)u_2(-1)u_3(-1)u_4(-1)\mathbf{1} \mid u_i \in \mathfrak{h}\}.$
- $W_3 := \text{Span}_{\mathbb{C}}\{u_1(-3)u_2(-2)u_3(-1)u_4(-1)\mathbf{1} \mid u_i \in \mathfrak{h}\}.$
- $W_4 := \text{Span}_{\mathbb{C}}\{u_1(-2)u_2(-2)u_3(-2)u_4(-1)\mathbf{1} \mid u_i \in \mathfrak{h}\}.$
- $W_5 := \text{Span}_{\mathbb{C}}\{u_1(-2)u_2(-1)u_3(-1)u_4(-1)u_5(-1)u_6(-1)\mathbf{1} \mid u_i \in \mathfrak{h}\}.$

Then W_1, W_2, W_3, W_4 , and W_5 are isomorphic to $\mathfrak{h} \otimes S^3(\mathfrak{h})$, $\mathfrak{h} \otimes S^3(\mathfrak{h})$, $\mathfrak{h} \otimes \mathfrak{h} \otimes S^2(\mathfrak{h})$, $\mathfrak{h} \otimes S^3(\mathfrak{h})$, and $\mathfrak{h} \otimes S^5(\mathfrak{h})$ as $O(L)$ -modules, respectively.

Proposition 6.6. *The VOA $V_{\sqrt{2}D_4}^+$ is of class S^5 .*

Proof. Set $G = \text{Aut}(V_{\sqrt{2}D_4}^+)$. Recall that $\sqrt{2}D_4$ is obtained by Construction B from $C = \{(0^4)\}$. Since $\text{Aut}(C) (\cong S_4)$ acts on Ω_4 doubly transitively,

$$(V_{\sqrt{2}D_4}^+)_5^G \subset ((V_{\sqrt{2}D_4}^+)_{5^0}^{\sigma_0} \cap M_{\mathfrak{h}}(1)_5)^{E(C)\text{Aut}(C)} = (V_{\omega})_5 \oplus \left\langle \sum_{i=1}^4 L_i(-3)L_i(-2)\mathbf{1} \right\rangle_{\mathbb{C}} \quad (6.1)$$

by Corollary 4.12 and Corollary 5.4. Note that $O(\sqrt{2}D_4) = O(D_4) = \mathcal{G}_2$. It follows from Corollary 6.2 (1) and Lemma 6.5 that a basis of $W_1^{O(\sqrt{2}D_4)}$ is given by $\{\sum_{1 \leq i, j \leq 4} \alpha_i(-2)\alpha_i(-1)\alpha_j(-1)^2\mathbf{1}\}$. Hence $\sum_{i=1}^4 \alpha_i(-2)\alpha_i(-1)^3\mathbf{1} \notin W_1^{O(\sqrt{2}D_4)}$. From this, $\sum_{i=1}^4 L_i(-3)L_i(-2)\mathbf{1}$ is not fixed by $O(\sqrt{2}D_4)$ because $\sum_{i=1}^4 \alpha_i(-4)\alpha_i(-1)\mathbf{1}$ is fixed by $O(\sqrt{2}D_4)$ by Lemma 5.5, and

$$\sum_{i=1}^4 L_i(-3)L_i(-2)\mathbf{1} = \frac{1}{8} \sum_{i=1}^4 \alpha_i(-2)\alpha_i(-1)^3\mathbf{1} + \frac{1}{2} \sum_{i=1}^4 \alpha_i(-4)\alpha_i(-1)\mathbf{1}.$$

Therefore $\sum_{i=1}^4 L_i(-3)L_i(-2)\mathbf{1} \notin (V_{\sqrt{2}D_4}^+)_5^G$. By (6.1), we obtain $(V_{\sqrt{2}D_4}^+)_5^G = (V_{\omega})_5$. Hence the assertion of this proposition holds. \square

Proposition 6.7. *The VOAs $V_{\sqrt{2}E_8}^+$ and $V_{BW_{16}}^+$ are of class S^7 .*

Proof. The lattices $\sqrt{2}E_8$ and BW_{16} are obtained by Construction B from $\{(0^8), (1^8)\}$ and $RM(1, 4)$, respectively. The automorphism groups $\text{Aut}(\{(0^8), (1^8)\}) \cong S_8$ and $\text{Aut}(RM(1, 4))$ act triply transitively on Ω_8 and Ω_{16} , respectively (see Lemma 3.3 (3)).

Here, we consider a more general case. Let C be a doubly even binary code of length n with $C(4) = \emptyset$ such that $\text{Aut}(C)$ acts triply transitively on Ω_n . Set

$$\begin{aligned}
X_1 &= \sum_{i=1}^n L_i(-5)L_i(-2)\mathbf{1}, & X_2 &= \sum_{i=1}^n L_i(-4)L_i(-3)\mathbf{1}, \\
X_3 &= \sum_{i=1}^n L_i(-3)L_i(-2)^2\mathbf{1}, & X_4 &= \sum_{1 \leq i \neq j \leq n} L_i(-3)L_j(-2)^2\mathbf{1}, \\
X_5 &= \sum_{1 \leq i \neq j \leq n} L_i(-3)L_i(-2)L_j(-2)\mathbf{1}.
\end{aligned}$$

Let W be the subspace of $(V_L^+)^{\sigma_0} \cap M_{\mathfrak{h}}(1)_7$ spanned by $\{X_i \mid 1 \leq i \leq 5\}$. Since $\text{Aut}(C)$ acts triply transitively on Ω_n , we have

$$((V_L^+)^{\sigma_0} \cap M_{\mathfrak{h}}(1)_7)^{E(C)\text{Aut}(C)} = (V_\omega)_7 \oplus W \quad (6.2)$$

by [Corollary 4.12](#). Hence, if we prove that

$$\{w \in W \mid \sigma(w) = w \text{ for all } \sigma \in O(L_B(C))\} = 0, \quad (6.3)$$

then V_L^+ is of class \mathcal{S}^7 by [Corollary 5.4](#). Set

$$\begin{aligned}
Y_1 &= \sum_{i=1}^n \alpha_i(-4)\alpha_i(-1)^3\mathbf{1}, & Y_2 &= \sum_{i=1}^n \alpha_i(-3)\alpha_i(-2)\alpha_i(-1)^2\mathbf{1}, \\
Y_3 &= \sum_{i=1}^n \alpha_i(-2)^3\alpha_i(-1)\mathbf{1}, & Y_4 &= \sum_{i=1}^n \alpha_i(-2)\alpha_i(-1)^5\mathbf{1}, \\
Y_5 &= \sum_{1 \leq i \neq j \leq n} \alpha_i(-2)\alpha_i(-1)\alpha_j(-1)^4\mathbf{1}, & Y_6 &= \sum_{1 \leq i \neq j \leq n} \alpha_i(-2)\alpha_i(-1)^3\alpha_j(-1)^2\mathbf{1}.
\end{aligned}$$

By direct computation, we have the following:

$$\begin{aligned}
X_1 &= \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{2} \sum_{i=1}^n \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \\
X_2 &= \frac{1}{4}Y_2 + \frac{1}{8}Y_3 + \frac{1}{2} \sum_{i=1}^n \alpha_i(-5)\alpha_i(-2)\mathbf{1} + \sum_{i=1}^n \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \\
X_3 &= \frac{1}{4}Y_1 + \frac{1}{4}Y_2 + \frac{1}{32}Y_4 + \frac{1}{2} \sum_{i=1}^n \alpha_i(-4)\alpha_i(-3)\mathbf{1} + \frac{3}{2} \sum_{i=1}^n \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \\
X_4 &= \frac{1}{32}Y_5 + \frac{1}{4} \sum_{1 \leq i \neq j \leq n} \alpha_i(-3)\alpha_i(-1)\alpha_j(-2)\alpha_j(-1)\mathbf{1}, \\
X_5 &= \frac{1}{32}Y_6 + \frac{1}{8} \sum_{1 \leq i \neq j \leq n} \alpha_i(-4)\alpha_i(-1)\alpha_j(-1)^2\mathbf{1}.
\end{aligned}$$

By [Lemmas 5.5](#) and [6.5](#),

$$\begin{aligned}
&\sum_{i=1}^n \alpha_i(-6)\alpha_i(-1)\mathbf{1}, \quad \sum_{i=1}^n \alpha_i(-5)\alpha_i(-2)\mathbf{1}, \quad \sum_{i=1}^n \alpha_i(-4)\alpha_i(-3)\mathbf{1}, \\
&\sum_{1 \leq i, j \leq n} \alpha_i(-3)\alpha_i(-1)\alpha_j(-2)\alpha_j(-1)\mathbf{1}, \quad \sum_{1 \leq i, j \leq n} \alpha_i(-4)\alpha_i(-1)\alpha_j(-1)^2\mathbf{1}
\end{aligned}$$

are fixed by $O(L)$. Hence, if $\sum_{i=1}^5 \lambda_i X_i \in \{w \in W \mid \sigma(w) = w \text{ for all } \sigma \in O(L)\}$, then we have

$$\begin{aligned} (\lambda_1 + 2\lambda_3 - \lambda_5) Y_1 &\in W_2^{O(L)}, & (\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_4) Y_2 &\in W_3^{O(L)}, \\ \lambda_2 Y_3 &\in W_4^{O(L)}, & \lambda_3 Y_4 + \lambda_4 Y_5 + \lambda_5 Y_6 &\in W_5^{O(L)}, \end{aligned} \quad (6.4)$$

where W_l for $l = 2, 3, 4$, and 5 are defined in [Lemma 6.5](#).

Here, we consider the case where L is equal to $\sqrt{2}E_8$ or BW_{16} . Applying [Corollary 6.2](#) for $m = 3, 4$ to [Lemma 6.5](#), we obtain the following:

- $\{\sum_{1 \leq i, j \leq 2^m} \alpha_i(-4)\alpha_i(-1)\alpha_j(-1)^2 \mathbf{1}\}$ is a basis of $W_2^{\mathcal{G}_m}$.
- $\{\sum_{1 \leq i, j \leq 2^m} \alpha_i(-3)\alpha_i(-2)\alpha_j(-1)^2 \mathbf{1}, \sum_{1 \leq i, j \leq 2^m} \alpha_i(-3)\alpha_j(-2)\alpha_i(-1)\alpha_j(-1) \mathbf{1}\}$ is a basis of $W_3^{\mathcal{G}_m}$.
- $\{\sum_{1 \leq i, j \leq 2^m} \alpha_i(-2)^2 \alpha_i(-2)\alpha_j(-1) \mathbf{1}\}$ is a basis of $W_4^{\mathcal{G}_m}$.
- $\{\sum_{1 \leq i, j, k \leq 2^m} \alpha_i(-2)\alpha_i(-1)\alpha_j(-1)^2 \alpha_k(-1)^2 \mathbf{1}\}$ is a basis of $W_5^{\mathcal{G}_m}$.

By direct computation, $Y_1 \notin W_2^{\mathcal{G}_m}$, $Y_2 \notin W_3^{\mathcal{G}_m}$, $Y_3 \notin W_4^{\mathcal{G}_m}$, and $\langle Y_4, Y_5, Y_6 \rangle_{\mathbb{C}} \cap W_5^{\mathcal{G}_m} = 0$. Since $\mathcal{G}_m \subset O(L)$, by [\(6.4\)](#)

$$\begin{cases} \lambda_1 + 2\lambda_3 - \lambda_5 = 0, \\ \lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_4 = 0, \\ \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0. \end{cases}$$

Clearly, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Therefore [\(6.3\)](#) holds. This completes the proof of this proposition. \square

By combining [Propositions 6.3, 6.6, and 6.7](#), we finally conclude the following.

Theorem 6.8. *The VOA $V_{\sqrt{2}D_4}^+$ is of class \mathcal{S}^5 , and the VOAs $V_{2A_1}^+$, $V_{\sqrt{2}E_8}^+$, and $V_{BW_{16}}^+$ are of class \mathcal{S}^7 .*

Remark 6.9. Although $\text{Aut}(\{(0^4)\}) \cong S_4$ acts triply transitively on Ω_4 , the VOA $V_{\sqrt{2}D_4}^+$ is not of class \mathcal{S}^6 . One of this reason is the dimension of $S^6(\mathfrak{h})^{\mathcal{G}_2}$ is 2 (see [\[2, Section 5, Remark\]](#)). Hence the dimension of the fixed point subspace of $G = \text{Aut}(V_{\sqrt{2}D_4}^+)$ in $(V_{\sqrt{2}D_4}^+)_6$ is greater than the cases of $V_{\sqrt{2}E_8}^+$ and $V_{BW_{16}}^+$. Indeed, we find the element of $(V_{\sqrt{2}D_4}^+)_6^G \setminus V_\omega$ as follows:

$$4 \sum_{i=1}^4 L_i(-4)L_i(-2) \mathbf{1} - \sum_{i=1}^4 L_i(-3)^2 \mathbf{1} + 2 \sum_{i=1}^4 L_i(-2)^3 \mathbf{1} + 10 \sum_{1 \leq i \neq j \leq 4} L_i(-2)^2 L_j(-2) \mathbf{1}.$$

Remark 6.10. One can show that the \mathbb{Z}_2 -orbifold $\tilde{V}_{BW_{32}}$ of the lattice VOA associated to the Barnes–Wall lattice BW_{32} of rank 32 is of class \mathcal{S}^7 (for the details of \mathbb{Z}_2 -orbifold, see [10]). Note that BW_{32} is isomorphic to the lattice $L_B(RM(2, 4)) + \mathbb{Z}\frac{1}{4}\alpha_{(1^{32})}$, where $RM(2, 5)$ is the second order Reed–Muller code of length 32. It is known that $\tilde{V}_{BW_{32}}$ has an automorphism whose fixed point subspace in $\tilde{V}_{BW_{32}}$ is $V_{BW_{32}}^+$, and the automorphism group of $RM(2, 5)$ acts on Ω_{32} triply transitively [15, p. 400]. By [10], there exists an automorphism of $\tilde{V}_{BW_{32}}$ which acts as σ_0 on $V_{BW_{32}}^+$. Due to these circumstances, we can apply to a similar method to the proof of Proposition 6.7.

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