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Torsion-free Aluffi algebras[☆]

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ABSTRACT

A pair of ideals $J \subseteq I \subseteq R$ has been called Aluffi torsion-free if the Aluffi algebra of I/J is isomorphic to the corresponding Rees algebra. We give necessary and sufficient conditions for the Aluffi torsion-free property in terms of the first syzygy module of the form ideal J^* in the associated graded ring of I . For two pairs of ideals $J_1, J_2 \subseteq I$ such that $J_1 - J_2 \in I^2$, we prove that if one pair is Aluffi torsion-free the other one is so if and only if the first syzygy modules of J_1 and J_2 have the same form ideals. We introduce the notion of strongly Aluffi torsion-free ideals and present some results on these ideals.

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Introduction

P. Aluffi in [1] to describe characteristic cycles of a hypersurface parallel to well-known conormal cycle in intersection theory introduces an intermediate graded algebra between the symmetric algebra of an ideal and the corresponding Rees algebra. The first author and A. Simis in [12] (see also [11]) called such an algebra, the *Aluffi algebra*. Given a Noetherian ring R and ideals $J \subset I$ of R , the Aluffi algebra of I/J is defined by

$$\mathcal{A}_{R/J}(I/J) := \operatorname{Sym}_{R/J}(I/J) \otimes_{\operatorname{Sym}_R(I)} \mathcal{R}_R(I).$$

The Aluffi algebra is squeezed as $\operatorname{Sym}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ and moreover it is a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$. The kernel of the right hand surjection so called the module of *Valabrega–Valla* as defined in [17], is the torsion of the Aluffi algebra. Thus the Rees algebra of I/J is the Aluffi algebra modulo its torsion provided that I has a regular element modulo J .

It is reasonable to ask when the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism? Geometrically, this question is important from two points of view. More precisely, the blowup of $X = \operatorname{Spec}(R/J)$ along the closed subscheme Y defined by the ideal I/J is equal to $\operatorname{Proj}(\mathcal{A}_{R/J}(I/J))$. Hence to find the equations of the blowup of X along Y , we just need to find the equations of the blowup of ambient space R along Y . For the other one, let $x \in X = \operatorname{Spec}(R/J)$ be a point. The tangent cone of X at x is the cone $\operatorname{Spec}(\operatorname{gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}))$, where \mathfrak{m}_x is the maximal ideal of the local ring $\mathcal{O}_{X,x} \simeq R_{\mathfrak{p}_x}/JR_{\mathfrak{p}_x}$. So that we may assume $\mathcal{O}_{X,x}$ is the quotient of a regular local ring (R, \mathfrak{m}) with respect of an ideal J . Then the associated graded ring $\operatorname{gr}_{\mathfrak{m}/J}(R/J)$ of \mathfrak{m}/J is isomorphic to $\operatorname{gr}_{\mathfrak{m}}(R)/J^*$, where J^* is the form ideal. The problem of determining elements f_1, \dots, f_t such that their initial forms f_1^*, \dots, f_t^* generate J^* is an essential problem in resolution of singularities. Also the torsion-free Aluffi algebras are crucial in intersection theory of regular and linear embedding (see, e.g., [4], [10]). The outline of this paper is as the follow.

In section 1, we give necessary and sufficient conditions for torsion-free Aluffi algebra, involving the standard base (in the sense of Hironaka [7]) and the first syzygy module of the form ideal in the associated graded ring.

Let $J \subseteq I$ be ideals in the ring R . We say that the pair $J \subseteq I$ is *Aluffi torsion-free* if $J \cap I^n = JI^{n-1}$ for all $n \geq 1$. In section 2, we study the behavior of the Aluffi torsion-free property with respect to contraction and extension. We prove that the sum of two Aluffi torsion-free ideals is Aluffi torsion-free if and only if one of them modulo the other one is Aluffi torsion-free. As the main result of this section, we prove that if $J_1, J_2 \subseteq I$ such that $J_1 \equiv J_2$ modulo I^2 and $J_1 \subseteq I$ is Aluffi torsion-free, then $J_2 \subseteq I$ is Aluffi torsion-free if and only if the first syzygy modules of J_1 and J_2 have the same form ideals in the associated graded module $\operatorname{gr}_I(R^m)$, where m is the number of generators of J_1 and J_2 (Theorem 2.6). In the sequel, we introduce the notion of strongly Aluffi torsion-free ideals. A pair $J = (f_1, \dots, f_t) \subseteq I \subseteq R$ is called *strongly Aluffi torsion-free*

if $J_i = (f_1, \dots, f_i)$ is Aluffi torsion-free for $i = 1, \dots, t$. We give an example of Aluffi torsion-free pair of ideals which is not strongly Aluffi torsion-free. In the case that, $J \subseteq I$ is Aluffi torsion-free, we give a criterion for strongly Aluffi torsion-freeness. We close the section with this result: let $J_1, J_2 \subseteq I$ be ideals in the ring R such that the extension of J_2 and I in the ring R/J_1 is Aluffi torsion-free. If there exists a minimal generating set f_1, \dots, f_t of J_1 such that $J_1 \subseteq I$ is strongly Aluffi torsion-free and extension of the sequence f_1, \dots, f_t in R/J_2 is regular, then $J_2 \subseteq I$ is Aluffi torsion-free.

In section 3, we focus on the case that J is an ideal in the polynomial ring $R = k[x_0, \dots, x_n]$ over a field k of characteristic zero and the ideal I stands for the Jacobian ideal of J which describe the singular subscheme of $\text{Spec}(R/J)$. We prove that if J is the ideal of a monomial curve with some special parametrization or J is the square-free Veronese ideal of degree r , then $J \subseteq I$ is Aluffi torsion-free. We close the paper with a question related to Aluffi torsion-freeness of free line arrangements.

1. The Aluffi algebra and its torsion

Throughout this section, R will be a Noetherian ring. Let $J \subseteq I \subseteq R$ be ideals. There are two important algebras related to these data. The first one is the symmetric algebra $\text{Sym}_R(I)$ and the second one is the Rees algebra, $\mathcal{R}_R(I) = \bigoplus_{n \geq 0} I^n t \subset R[t]$. It is well-known that there is a natural surjective R -algebra homomorphism $\text{Sym}_R(I) \rightarrow \mathcal{R}_R(I)$. By the functorial property of the symmetric algebra there is an other surjection $\text{Sym}_R(I) \rightarrow \text{Sym}_{R/J}(I/J)$. The (embedded) Aluffi algebra is defined by

$$\mathcal{A}_{R/J}(I/J) = \text{Sym}_{R/J}(I/J) \otimes_{\text{Sym}_R(I)} \mathcal{R}_R(I).$$

By [12, Lemma 1.2], there are R -algebra isomorphisms

$$\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_R(I)/(J, \tilde{J})\mathcal{R}_R(I) \simeq \bigoplus_{n \geq 0} I^n / JI^{n-1},$$

where J is in degree zero and \tilde{J} is in degree 1. The Rees algebra of I/J is

$$\mathcal{R}_{R/J}(I/J) \simeq \bigoplus_{n \geq 0} I^n / J \cap I^n.$$

Then there is a surjective R -algebra homomorphism $\mathcal{A}_{R/J}(I/J) \rightarrow \mathcal{R}_{R/J}(I/J)$. The kernel of the above surjection is the homogeneous ideal

$$\mathcal{W}_{J \subset I} := \bigoplus_{n \geq 2} J \cap I^n / JI^{n-1},$$

which is called the *module of Valabrega–Valla*. If J has a regular element modulo I , then the Valabrega–Valla’s module is the R/J -torsion of the Aluffi algebra [12, Proposition 2.5]. The module of Valabrega–Valla has close relation to the theory of standard

base [7]. To make a further development on this relation, we recall some facts about the filtered rings and modules.

A filtration on the ring R is a decreasing sequence of ideals $\{\mathbf{F}_n R\}_{n \geq 0}$ satisfying $(\mathbf{F}_n R)(\mathbf{F}_m R) \subseteq \mathbf{F}_{n+m} R$ for all $n, m \geq 0$. The pair $(R, \mathbf{F}_n R)$ is called a *filtered ring*. For an ideal I of a ring R there is the I -adic filtration $\mathbf{F}_n R = I^n$. A morphism of filtered rings $\varphi : (R, \mathbf{F}_n R) \rightarrow (S, \mathbf{F}_n S)$ is a homomorphism of rings $\varphi : R \rightarrow S$ such that $\varphi(\mathbf{F}_n R) \subseteq \mathbf{F}_n S$ for all $n \geq 0$.

Let $(R, \mathbf{F}_n R)$ be a filtered ring and M a R -module. A filtration on the module M is a decreasing sequence $\{\mathbf{F}_n M\}_{n \geq 0}$ of submodules of M such that $(\mathbf{F}_n R)(\mathbf{F}_m M) \subseteq \mathbf{F}_{n+m} M$ for all $m, n \geq 0$. The pair $(M, \mathbf{F}_n M)$ is called a *filtered $(R, \mathbf{F}_n R)$ -module*. A morphism of filtered $(R, \mathbf{F}_n R)$ -modules $\varphi : (M, \mathbf{F}_n M) \rightarrow (N, \mathbf{F}_n N)$ is a R -module homomorphism $\varphi : M \rightarrow N$ such that $\varphi(\mathbf{F}_n M) \subseteq \mathbf{F}_n N$ for all $n \geq 0$. This implies $\varphi(\mathbf{F}_n M) \subseteq \varphi(M) \cap \mathbf{F}_n N$. The morphism φ is called *strict* if $\varphi(\mathbf{F}_n M) = \varphi(M) \cap \mathbf{F}_n N$. If M is a R -module then $(M, \mathbf{F}_n M)$ with $\mathbf{F}_n M := (\mathbf{F}_n R)M$ is a filtered $(R, \mathbf{F}_n R)$ -module. A sequence of filtered $(R, \mathbf{F}_n R)$ -modules is called *exact* if the sequence of underlying R -modules is exact. It is called *strict* if all morphisms are strict.

Remark 1. Let $(M, \mathbf{F}_n M)$ be a filtered $(R, \mathbf{F}_n R)$ module.

- (a) Let $\varphi : L \rightarrow M$ be an injective homomorphism of R -modules. For all $n \geq 0$, put $\mathbf{F}_n L := \varphi^{-1}(\mathbf{F}_n M)$. This makes $(L, \mathbf{F}_n L)$ into a filtered module and $\varphi : (L, \mathbf{F}_n L) \rightarrow (M, \mathbf{F}_n M)$ is a strict morphism.
- (b) Let $\varphi : M \rightarrow N$ be a surjective homomorphism of R -modules. For all $n \geq 0$ put $\mathbf{F}_n N := \varphi(\mathbf{F}_n M)$. This makes $(N, \mathbf{F}_n N)$ into a filtered module and $\varphi : (M, \mathbf{F}_n M) \rightarrow (N, \mathbf{F}_n N)$ is a strict morphism.

The *associated graded ring* of a filtration $(R, \mathbf{F}_n R)$ is $\text{gr}(R) = \bigoplus_{n \geq 0} \mathbf{F}_n R / \mathbf{F}_{n+1} R$. We denote $\text{gr}_I(R)$ for the I -adic filtration. If $(M, \mathbf{F}_n M)$ is a filtered module, then *associated graded module* $\text{gr}(M) = \bigoplus_{n \geq 0} \mathbf{F}_n M / \mathbf{F}_{n+1} M$ is the graded $\text{gr}(R)$ -module. In the case of I -adic filtration we write $\text{gr}_I(M)$. It is clear that $\text{gr}(-)$ is a functor from the category of filtered modules to the category of graded modules.

Proposition 1.1 ([5], I Proposition 2.1). Let $(R, \mathbf{F}_n R)$ be a filtered ring and

$$(L, \mathbf{F}_n L) \xrightarrow{\varphi} (M, \mathbf{F}_n M) \xrightarrow{\phi} (N, \mathbf{F}_n N),$$

a strict exact sequence of filtered $(R, \mathbf{F}_n R)$ -modules. Then the induced sequence $\text{gr}(L) \xrightarrow{\text{gr}(\varphi)} \text{gr}(M) \xrightarrow{\text{gr}(\phi)} \text{gr}(N)$ is an exact sequence of $\text{gr}(R)$ -modules.

For $m \in M$ we denote by $\nu_{\mathbf{F}}(m)$ the largest integer n such that $m \in \mathbf{F}_n M$. If such n does not exist we say $\nu_{\mathbf{F}}(m) = \infty$ and if $\nu_{\mathbf{F}}(m) < \infty$, we denote by m^* the residue class of m in $\mathbf{F}_{\nu_{\mathbf{F}}(m)} M / \mathbf{F}_{\nu_{\mathbf{F}}(m)+1} M$, which is called the *initial form* of m . If $\nu_{\mathbf{F}}(m) = \infty$,

then we set $m^* = 0$. For $m_1, m_2 \in M$ if $m_1^* + m_2^* \neq 0$, then $m_1^* + m_2^* = (m_1 + m_2)^*$. If the filtration $\mathbf{F}_n M$ is multiplicative then $\text{gr}(M)$ is a ring and if $m_1^* m_2^* \neq 0$, then $(m_1 m_2)^* = m_1^* m_2^*$.

Let R be a ring and $J \subseteq I$ ideals of R . Given an element $f \in R$, we denote by $\nu = \nu_I(f)$ the number $\nu_{\mathbf{F}}(f)$ with $\mathbf{F}_n R = I^n$. We denote by J^* the homogeneous ideal of $\text{gr}_I(R)$ generated by the initial forms of the elements of J . A set of generators $\{f_1, \dots, f_t\}$ of J is called an I -standard base if $J^* = (f_1^*, \dots, f_t^*)$. If R is local, then an I -standard base of J is a generating set [7, Lemma 6].

The following remark gives necessary and sufficient conditions for the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ to be an isomorphism.

Remark 2. Let $J \subseteq I \subseteq R$ be ideals of the local ring R . By [16, Theorem 1.1], the following are equivalent.

- (a) $\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_{R/J}(I/J)$.
- (b) $J \cap I^n = JI^{n-1}$ for any $n \geq 1$.
- (c) $I^{n+1} \cap JI^{n-1} = JI^n$ for any $n \geq 1$.
- (d) There exists a minimal set of generators f_1, \dots, f_t of J such that $\{f_1, \dots, f_t\}$ is an I -standard base of J and $\nu_I(f_i) = 1$ for $i = 1, \dots, t$.

Let now $J = (f_1, \dots, f_t)$ and $\nu_I(f_i) = 1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{Z} \xrightarrow{i} R^t \xrightarrow{f} R \xrightarrow{\pi} R/J \longrightarrow 0, \quad (1)$$

where $f(a_1, \dots, a_t) = \sum_{i=1}^t a_i f_i$ and $\mathcal{Z} = \text{Syz}(J)$ is the first syzygy module of J . By Remark 1, we consider the following filtrations

$$\mathbf{F}_n R^t = \bigoplus_{i=1}^t I^{n-1}, \quad \mathbf{F}_n \mathcal{Z} = \mathbf{F}_n R^t \cap \mathcal{Z}, \quad \mathbf{F}_n R/J = (I/J)^n,$$

which make i, f and π the morphisms of filtered $(R, \mathbf{F}_n R = I^n)$ -modules. Note that i and π are strict. By Proposition 1.1, we get the corresponding complex of graded modules

$$0 \longrightarrow \text{gr}_I(\mathcal{Z}) \xrightarrow{\text{gr}(i)} \text{gr}_I(R^t) \xrightarrow{\text{gr}(f)} \text{gr}_I(R) \xrightarrow{\text{gr}(\pi)} \text{gr}_{I/J}(R/J) \longrightarrow 0. \quad (*)$$

Note that $\text{gr}_I(R^t) = \bigoplus_{i=1}^t \text{gr}_I(R)(-1)$. The map $\text{gr}(f)$ is defined by $e_i \mapsto f_i^*$, the map $\text{gr}(i)$ is inclusion and $\text{gr}(\pi)$ is surjective. We have

$$\ker(\text{gr}(\pi)) = \bigoplus_{n \geq 0} (I^{n+1} + J \cap I^n) / I^{n+1} = J^*, \quad \ker(\text{gr}(f)) = \text{Syz}(J^*).$$

Given an element $(a_1, \dots, a_t) \in \mathcal{Z}$, if $(a_1, \dots, a_t) \neq (0, \dots, 0)$, then there exist $m \geq 0$ such that $\nu_{\mathbf{F}_n \mathcal{Z}}(a_1, \dots, a_t) = m$, this means that $\nu_I(a_j) \geq m - 1$ for every i and there exists

$j \in \{1, \dots, t\}$ such that $\nu_I(a_j) = m - 1$. Hence $\psi : \mathcal{Z} \rightarrow \text{gr}_I(\mathcal{Z})$ is the canonical map which associates to every element of \mathcal{Z} its initial form in $\text{gr}_I(\mathcal{Z})$, that is, $\psi(a_1, \dots, a_t) = \overline{(a_1, \dots, a_t)} \in \mathbf{F}_m \mathcal{Z} / \mathbf{F}_{m+1} \mathcal{Z}$. On the other hand, since the sequence $(*)$ is a complex, there is a canonical embedding $\varphi : \text{gr}_I(\mathcal{Z}) \hookrightarrow \text{Syz}(J^*)$, which sends every element $\overline{(a_1, \dots, a_t)} \in \mathcal{Z} \cap \mathbf{F}_m R^t / \mathcal{Z} \cap \mathbf{F}_{m+1} R^t$ to $(\overline{a_1}, \dots, \overline{a_t})$, where $\overline{a_i}$ is the residue class of a_i in I^{m-1} / I^m . Therefore, we get a map

$$\varphi \circ \psi : \mathcal{Z} \rightarrow \text{Syz}(J^*) \quad , \quad \varphi \circ \psi((a_1, \dots, a_t)) = (\overline{a_1}, \dots, \overline{a_t}).$$

Note that $\overline{a_i} = a_i^*$ if $\nu_I(a_i) + 1 = \min_j \{\nu_I(a_j) + 1\}$ and $\overline{a_i} = 0$ if $\nu(a_i) + 1 > \min_j \{\nu(a_j) + 1\}$. The following theorem relates the torsion of the Aluffi Algebra to the first syzygy module of the form ideal $J^* \subseteq \text{gr}_I(R)$.

Theorem 1.2. *Let $J = (f_1, \dots, f_t) \subseteq I$ be ideals in the local ring R . The following are equivalent.*

- (a) $\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_{R/J}(I/J)$.
- (b) *The complex $(*)$ is exact.*
- (c) *There exists a homogeneous system of generators of $\text{Syz}(J^*)$, whose elements can be lifted to elements of \mathcal{Z} via $\varphi \circ \psi$.*

Proof. First note that the Aluffi algebra is torsion-free if and only if the map \mathfrak{f} in the sequence (1) is strict. Thus (a) implies (b) by Proposition 1.1. Assume that the complex $(*)$ is exact. Then by above $\text{gr}_I(\mathcal{Z}) = \text{Syz}(J^*)$ which yields (c). Finally, we prove that (c) implies (a). The map $\Theta : \text{Syz}(J^*) \rightarrow \text{gr}_I(\mathcal{Z})$ is inverse of φ , which is defined by sending an element $s \in \text{Syz}(J^*)$ to $\psi(a_1, \dots, a_t)$, where $\varphi \circ \psi((a_1, \dots, a_t)) = s$. Hence $\text{gr}_I(\mathcal{Z}) \simeq \text{Syz}(J^*)$. The latter implies that $J \cap I^n = JI^{n-1}$ for all $n \geq 1$. In fact, let $b \in J \cap I^n = \mathfrak{f}(R^t) \cap \mathbf{F}_n R$. Then $b = \mathfrak{f}((a_1, \dots, a_t))$ with (a_1, \dots, a_t) belonging to some $\mathbf{F}_m R^t$. If $m \geq n$, we get the assertion. If $m < n$, then by the exactness of $0 \rightarrow \text{gr}_I(\mathcal{Z}) \xrightarrow{\text{gr}(i)} \text{gr}_I(R^t)$, we get

$$(a_1, \dots, a_n) \in \mathfrak{f}^{-1}(\mathbf{F}_{m+1} R) \cap \mathbf{F}_m R^t = \mathbf{F}_m \mathcal{Z} + \mathbf{F}_{m+1} R^t.$$

Hence $b = \mathfrak{f}((c_1, \dots, c_t))$ with $(c_1, \dots, c_t) \in \mathbf{F}_{m+1} R^t$. Repeating this argument finitely many times, finally we get $b \in \mathfrak{f}(\mathbf{F}_n R^t) = JI^{n-1}$ which complete the proof. \square

2. Aluffi torsion-free ideals

In this section, we assume that all rings are Noetherian. Let $J \subseteq I$ be ideals in the ring R . If $J \subseteq I$ satisfy in one of the equivalent conditions in Remark 2 or Theorem 1.2, then the Aluffi algebra is torsion-free. Therefore we have the following definition.

Definition 2.1. A pair of ideals $J \subseteq I$ in the ring R is called *Aluffi torsion-free* if $J \cap I^n = JI^{n-1}$ for all $n \geq 1$.

Example 2.2. There are well-known examples of Aluffi torsion-free ideals.

1. If I/J in R/J is of linear type (e.g., if I is generated by regular or, more generally by a d -sequence modulo J in the sense of Huneke [8]), then $J \subseteq I$ is Aluffi torsion-free.
2. If J is generated by superficial sequence in I , then the pair $J \subseteq I$ is Aluffi torsion-free [9, Lemma 8.5.11].

The following result indicates to the behavior of Aluffi torsion-free property with respect to extension and contraction. In particular, it shows that the Aluffi torsion-free property is local.

Proposition 2.3. Let $J \subseteq I$ be ideals in the ring R . The following statements hold:

- (a) Let $\mathfrak{a} \subseteq J$ be another ideal. If $J \subseteq I$ is Aluffi torsion-free, then $\overline{J} \subseteq \overline{I}$ is Aluffi torsion-free in $\overline{R} = R/\mathfrak{a}$.
- (b) Let $R \rightarrow S$ be a flat homomorphism of rings. If $J \subseteq I$ is Aluffi torsion-free, then $JS \subseteq IS$ is Aluffi torsion-free in S .
- (c) Let $R \rightarrow S$ be a faithfully flat homomorphism of rings. If the extension of ideals $J \subseteq I$ in S is Aluffi torsion-free, then $J \subseteq I$ is Aluffi torsion-free in R . In particular, assume that (R, \mathfrak{m}) is local. If the extension of J and I in the \mathfrak{m} -adic completion \hat{R} is Aluffi torsion-free, then so does $J \subseteq I$.
- (d) The ideal $J \subseteq I$ is Aluffi torsion-free if and only if $JR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}}$ is Aluffi torsion-free for every maximal ideal \mathfrak{m} of R .

Proof. We prove (a), (b) and (c) by straightforward computations. We have

$$\overline{J} \cap \overline{I}^n = \overline{J} \cap \overline{I}^n = \overline{J \cap I^n} = \overline{JI^{n-1}} = \overline{J} \overline{I}^{n-1},$$

which proves (a). For (b), we have

$$\begin{aligned} JS \cap (IS)^n &= JS \cap I^n S = (J \otimes_R S) \cap (I^n \otimes_R S) \\ &= (J \cap I^n) \otimes_R S = (JI^{n-1}) \otimes_R S \\ &= (JI^{n-1})S = (JS)(I^{n-1}S). \end{aligned}$$

(c). As S is faithfully flat over R , $IS \cap R = I$ for all ideals I of R . We have

$$J \cap I^n = (J \cap I^n)S \cap R \subseteq (JS \cap I^n S) \cap R = (JI^{n-1})S \cap R = JI^{n-1}.$$

The second assertion yields from the fact that $R \rightarrow \hat{R}$ is faithfully flat. The part (d) follows from part (b) and local-global property. \square

Remark 3. *There is a natural question. What is the behavior of Aluffi torsion-free property with respect to operation of ideals? Here are some easy facts about this question.*

1. *The sum of two Aluffi torsion-free ideals is not Aluffi torsion-free (see Proposition 2.4).*
2. *The product and intersection of two Aluffi torsion-free need not to be Aluffi torsion-free. In $k[x, y, z]$, consider the ideals $J_1 = (xy)$ and $J_2 = (yz) \subseteq I = (x, y, z)^2$, which are Aluffi torsion-free, but $J_1 J_2 = (xy^2z)$, $J_1 \cap J_2 = (xyz) \subseteq I$ are not Aluffi torsion-free.*

Proposition 2.4. *Let $J_1, J_2 \subseteq I$ be Aluffi torsion-free ideals in the ring R . Then $J_1 + J_2 \subseteq I$ is Aluffi torsion-free if and only if $\overline{J_1} \subseteq \overline{I} \subseteq \overline{R} = R/J_2$ is Aluffi torsion-free.*

Proof. Assume that $J_1 + J_2 \subseteq I$ is Aluffi torsion-free. For all $n \geq 1$ we have

$$\overline{J_1} \cap \overline{I}^n = \frac{(J_1 + J_2) \cap I^n + J_2}{J_2} = \frac{(J_1 + J_2)I^{n-1} + J_2}{J_2} = \frac{J_1 I^{n-1} + J_2}{J_2} = \overline{J_1} \overline{I}^{n-1}.$$

For the converse, let $J_1 = (f_1, \dots, f_t)$ and $y \in (J_1 + J_2) \cap I^n$. Then $y = a + c$ with $a \in J_1$ and $c \in J_2$. If $\overline{a} = 0$ we are done, if not we get $a \in \overline{J_1} \cap \overline{I}^n = \overline{J_1} \overline{I}^{n-1}$, then $a = \sum_{i=1}^t g_i f_i + d$ with $g_i \in I^{n-1}$ and $d \in J_2$. It follows that $y = c + d + \sum_{i=1}^t g_i f_i$, where $\sum_{i=1}^s g_i f_i \in J_1 \cap I^n$ and $c + d \in J_2 \cap I^n$. Hence $(J_1 + J_2) \cap I^n \subseteq (J_1 \cap I^n) + (J_2 \cap I^n)$. Since $J_1, J_2 \subseteq I$ are Aluffi torsion-free, one has

$$(J_1 + J_2) \cap I^n \subseteq (J_1 \cap I^n) + (J_2 \cap I^n) = J_1 I^{n-1} + J_2 I^{n-1} = (J_1 + J_2) I^{n-1}. \quad \square$$

Proposition 2.5. *Let R be a local ring, $J_1, J_2 \subset R$ two ideals and $I = J_1 + J_2$. Assume that J_1 is generated by elements not in I^2 . The following are equivalent.*

- (a) *The pair $J_1 \subset I$ is Aluffi torsion-free.*
- (b) *$J_1 \cap J_2^n = J_1 I^{n-1}$ for all $n \geq 1$.*
- (c) *$\text{gr}_I R/J_1^* \simeq \text{gr}_{I/J_1}(R/J_1)$.*

Proof. Let n be a positive integer, we have

$$\begin{aligned} J_1 \cap I^n &= J_1 \cap (J_1 + J_2)^n = J_1 \cap (J_1^n, J_1^{n-1} J_2, \dots, J_1 J_2^{n-1}, J_2^n) \\ &= J_1 \cap (J_1^n, J_1^{n-1} J_2, \dots, J_1 J_2^{n-1}) + J_1 \cap J_2^n \\ &= J_1 (J_1 + J_2)^{n-1} + J_1 \cap J_2^n \\ &= J_1 I^{n-1} + J_1 \cap J_2^n, \end{aligned}$$

which prove the equivalence of (a) and (b).

The associated graded ring $\text{gr}_{I/J_1}(R/J_1)$ is isomorphic to

$$R/I \oplus I/(I^2, J_1) \oplus I^2/(I^3 + I^2 \cap J_1) \oplus \dots$$

Since J_1^* is generated by homogeneous elements in degree 1, $\text{gr}_I R/J_1^*$ is isomorphic to

$$R/I \oplus I/(I^2, J_1) \oplus I^2/(I^3 + IJ_1) \oplus \dots$$

Now assume that (c) holds. Hence for every $n \geq 1$

$$I^n/(I^{n+1} + J_1 I^{n-1}) \simeq I^n/(I^{n+1} + I^n \cap J_1). \quad (2)$$

This shows that $J_1 \cap I^n \subseteq I^{n+1} + J_1 I^{n-1}$ for every $n \geq 1$. We have

$$J_1 \cap I^n \subseteq I^{n+1} \cap J_1 + J_1 I^{n+1} \subseteq I^{n+2} + J_1 I^n + J_1 I^{n-1} = I^{n+2} + J_1 I^{n-1}.$$

By induction, for all $m \geq 1$

$$J_1 \cap I^n \subseteq J_1 I^{n-1} + I^{n+m}.$$

Since R is local, we obtain $J_1 \cap I^n = J_1 I^{n-1}$, which prove (a). The converse is clear by (2). \square

Theorem 2.6. Let $J_1 = (f_1, \dots, f_m)$ and $J_2 = (g_1, \dots, g_m) \subset I$ be ideals in the ring R and let $J_1 \subset I$ be Aluffi torsion-free. Suppose that $f_i - g_i \in I^2$ for $i = 1, \dots, m$. Let \mathcal{Z}_1 and \mathcal{Z}_2 stand for the first syzygy modules of J_1 and J_2 , respectively. Then $J_2 \subset I$ is Aluffi torsion-free if and only if $\mathcal{Z}_1 \cap I^n R^m \subseteq \mathcal{Z}_2 + I^{n+1} R^m$ for all $n \geq 0$.

Proof. (\Rightarrow) If $(a_1, \dots, a_m) \in \mathcal{Z}_1 \cap I^n R^m$, then $\sum_{i=1}^m a_i f_i = 0$ and $a_i \in I^n$. Hence $\sum_{i=1}^m a_i g_i = \sum_{i=1}^m a_i (g_i - f_i) \in J_2 \cap I^{n+2} = J_2 I^{n+1}$ by assumption. Then $\sum_{i=1}^m g_i (a_i - b_i) = 0$, where $b_i \in I^{n+1}$ and $(a_1, \dots, a_n) = ((a_1 - b_1) + b_1, \dots, (a_n - b_n) + b_n) \in \mathcal{Z}_2 + I^{n+1} R^m$.

(\Leftarrow) We only need to show that $J_2 \cap I^{n+1} \subseteq J_2 I^n$ for all $n \geq 0$. Make induction on n , the case $n = 0$ is trivial. Pick an element $\sum_{i=1}^m a_i g_i \in J_2 \cap I^{n+1}$. We may assume that $a_i \in I^{n-1}$ for $i = 1, \dots, m$, in fact we have $\sum_{i=1}^m a_i g_i \in J_2 \cap I^{n+1} \subseteq J_2 \cap I^n = J_2 I^{n-1}$ by induction hypothesis, so that $\sum_{i=1}^m a_i g_i = \sum_{i=1}^m a'_i g_i$ with $a'_i \in I^{n-1}$.

We may assume that $(a_1, \dots, a_m) \in \mathcal{Z}_1 \cap I^{n-1} R^m$. Namely we have $\sum_{i=1}^m a_i (f_i - g_i) \in I^{n-1} I^2 = I^{n+1}$. Hence $\sum_{i=1}^m a_i f_i \in J_1 \cap I^{n+1} = J_1 I^n$ and $\sum_{i=1}^m a_i f_i = \sum_{i=1}^m b_i f_i$ with $b_i \in I^n$. Now we see that $\sum_{i=1}^m (a_i - b_i) f_i = 0$, so that we may replace a_i with $(a_i - b_i)$ and this gives also $\sum_{i=1}^m a_i f_i = 0$ as required. We have now

$$\sum_{i=1}^m a_i g_i \in I^{n+1}; \quad (a_1, \dots, a_m) \in \mathcal{Z}_1 \cap I^{n-1} R^m.$$

Now by assumption, we have $(a_1, \dots, a_m) \in \mathcal{Z}_2 + I^n R^m$. Then there exists $b_i \in I^n$ and $(e_1, \dots, e_m) \in \mathcal{Z}_2$ such that $(a_1, \dots, a_m) = (e_1 + b_1, \dots, e_m + b_m)$. Then $\sum_{i=1}^m a_i g_i = \sum_{i=1}^m b_i g_i$ and replacing the a_i 's with b_i 's we may suppose that $a_i \in I^n$. Repeating the first argument above with $a_i \in I^n$ we get

$$\sum_{i=1}^m a_i g_i \in I^{n+2} ; (a_1, \dots, a_m) \in \mathcal{Z}_1 \cap I^n R^m.$$

Therefore, we have an element $\sum_{i=1}^m a_i g_i$ such that $a_i \in I^n$ and it is clear that such element belongs to $J_2 I^n$. \square

Corollary 2.7. *Let $J_1, J_2 \subseteq I$ be ideals in the ring R such that $J_1 \equiv J_2$ modulo I^2 and $J_1 \subseteq I$ is Aluffi torsion-free. Then $J_2 \subseteq I$ is Aluffi torsion-free if and only if the first syzygy modules of J_1, J_2 have the same form ideals in $\text{gr}_I(R^m)$.*

Proof. The proof is based on the symmetry of Theorem 2.6 and the fact that the condition $\mathcal{Z}_1 \cap I^n R^m \subseteq \mathcal{Z}_2 + I^{n+1} R^m$ is equivalent with $(\mathcal{Z}_1)^* \subseteq (\mathcal{Z}_2)^*$ in $\text{gr}_I(R^m) = \bigoplus_{n \geq 0} I^n R^m / I^{n+1} R^m$. More precisely, if $a \in \mathcal{Z}_1 \cap I^n R^m \setminus I^{n+1} R^m$ for some n , then $a \in \mathcal{Z}_2 + I^{n+1} R^m$. Hence $a = b + c$ with $b \in \mathcal{Z}_2$ and $c \in I^{n+1} R^m$. Thus $b^* = a^*$ which proves that $(\mathcal{Z}_1)^* \subseteq (\mathcal{Z}_2)^*$. Conversely, if $a \in \mathcal{Z}_1 \cap I^n R^m$, we choose an element $b \in \mathcal{Z}_2$ such that $a^* = b^*$ hence $a - b \in I^{n+1} R^m$ and we are done. \square

Proposition 2.8. *Let $J_1 \subseteq J_2 \subseteq I$ be ideals in the ring R . Assume that $J_1 \cap J_2^{n-1} = J_1 J_2^{n-1}$ and $I^n \subseteq J_2^n + J_1$ for all $n \geq 1$. Then $J_1 \subseteq I$ is Aluffi torsion-free.*

Proof. We show by induction on n that

$$I^n \subseteq J_2^n + \sum_{j=0}^{n-1} I^j (I J_2^{n-j-1} \cap J_1), \quad \text{for all } n \geq 1. \quad (3)$$

The case $n = 1$ is clear by second assumption. Suppose that (3) holds for some n . Multiplying (3) by I yields $I^{n+1} \subseteq I J_2^n + \sum_{j=0}^{n-1} I^{j+1} (I J_2^{n-j-1} \cap J_1)$. Again by second assumption, we also have that $I^{n+1} \subseteq J_2^{n+1} + J_1$, so that I^{n+1} contained in

$$\left[I J_2^n + \sum_{j=0}^{n-1} I^{j+1} (I J_2^{n-j-1} \cap J_1) \right] \cap (J_2^{n+1} + J_1).$$

Let a be an element of I^{n+1} . Write $a = b + c = d + e$ where

$$b \in I J_2^n, \quad c \in \sum_{j=0}^{n-1} I^{j+1} (I J_2^{n-j-1} \cap J_1), \quad d \in J_2^{n+1} \subseteq I J_2^n, \quad e \in J_1.$$

Then $e - c = d - b$, and so $d - b \in J_1 \cap I J_2^n$. Therefore, $a = d + e = d + c + (e - c)$ is in

$$J_2^{n+1} + \sum_{j=0}^{n-1} I^{j+1}(IJ_2^{n-j-1} \cap J_1) + (J_1 \cap IJ_2^n) = J_2^{n+1} + \sum_{j=0}^n I^j(IJ_2^{n-j} \cap J_1),$$

which proves (3). Now using (3) we obtain that

$$\begin{aligned} J_1 \cap I^n &\subseteq (J_1 \cap J_2^n) + (J_1 \cap IJ_2^{n-1}) + \sum_{j=1}^n I^j(IJ_2^{n-j} \cap J_1) \\ &\subseteq (J_1 \cap J_2^{n-1}) + \sum_{j=1}^n I^j(IJ_2^{n-j} \cap J_1) \\ &\subseteq J_1 J_2^{n-1} + \sum_{j=1}^n I^j(J_2^{n-j} \cap J_1) \subseteq J_1 J_2^{n-1} + \sum_{j=1}^n I^j(J_1 J_2^{n-j}) \\ &\subseteq J_1 I^{n-1} + \sum_{j=1}^n I^j(J_1 I^{n-j}) = J_1 I^{n-1} \quad \square \end{aligned}$$

2.1. Strongly Aluffi torsion-free ideals

Let R be a local ring and $I = (f_1, \dots, f_t)$ an ideal such that f_1, \dots, f_t is a regular sequence. Then for any $n \geq 1$ the pair $J_i = (f_1^n, \dots, f_i^n) \subset I^n$ is Aluffi torsion-free for $i = 1, \dots, t$ [14, Example 1.3]. We have the following definition.

Definition 2.9. The pair $J = (f_1, \dots, f_t) \subseteq I$ is called strongly Aluffi torsion-free if $J_i = (f_1, \dots, f_i) \subseteq I$ is Aluffi torsion-free for $i = 1, \dots, t$.

In general, the following example shows that Aluffi torsion-free property does not imply strongly Aluffi torsion-free property.

Example 2.10. Let $J \subseteq k[x, y, z]$ be an ideal of 5 projective points in general linear position in \mathbb{P}_k^2 , which are columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $J = (xy + 3xz - 4yz, zx^2 - 2yz^2 + xz^2, zy^2 + 6xz^2 - 7yz^2)$ which is codimension 2 perfect ideal. Let I stand for the Jacobian ideal $I = (J, I_2(\Theta))$ where Θ is the Jacobian matrix of J and $I_2(\Theta)$ is the ideal generated by 2-minors of Θ . A calculation in [3] shows that $J \subseteq I$ is Aluffi torsion-free but is not strongly Aluffi torsion-free.

The proposition below gives a criterion for strongly Aluffi torsion-free ideals.

Proposition 2.11. Let $J = (f_1, \dots, f_t) \subseteq I$ be Aluffi torsion-free ideals in the ring R . If $(J_{t-s} :_R f_{t-s+1}) = J_{t-s}$ for $1 \leq s \leq t-1$, then $J \subseteq I$ is strongly Aluffi torsion-free.

Proof. It is enough to show that $J_{t-1} \cap I^n = J_{t-1}I^{n-1}$ for any $n \geq 1$. Let a be an element of $J_{t-1} \cap I^n$. Since $J \subseteq I$ is Aluffi torsion-free and $J_{t-1} \cap I^n \subseteq J \cap I^n$ hence $a \in JI^{n-1}$. Write $a = \sum_{i=1}^{t-1} a_i f_i = \sum_{i=1}^t b_i f_i$ with $a_i \in I^n$ and $b_i \in I^{n-1}$. One has

$$b_t f_t = (a_1 - b_1)f_1 + \dots + (a_{t-1} - b_{t-1})f_{t-1} \subseteq J_{t-1}.$$

Thus $b_r \in (J_{t-1} : f_t)$. We get

$$a = (a_1 - b_1)f_1 + \dots + (a_{t-1} - b_{t-1})f_{t-1} + b_t f_t \in J_{t-1}I^{n-1} + f_t((J_{t-1} : f_t) \cap I^{n-1}).$$

Then for all $n \geq 1$ we get

$$J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t((J_{t-1} : f_t) \cap I^{n-1}).$$

Now by assumption we have

$$J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t(J_{t-1} \cap I^{n-1}).$$

Making induction on n we get

$$J_{t-1} \cap I^n \subseteq J_{t-1}I^{n-1} + f_t(J_{t-1}I^{n-2}) = J_{t-1}I^{n-1},$$

as required. \square

Remark 4. Let $J = (f_1, \dots, f_t) \subseteq R$ be an ideal such that f_1, \dots, f_t is a regular sequence. If $J \subseteq I$ is Aluffi torsion-free then by Proposition 2.11 it is strongly Aluffi torsion-free. Also by the proof of Proposition 2.11, if for all $n \geq 1$ and $1 \leq s \leq t-1$ we have

$$f_{t-s+1}((J_{t-s} : f_{t-s+1}) \cap I^n) \subseteq J_{t-s}I^n,$$

then strongly Aluffi torsion-free property holds.

Example 2.12. Let $R = k[x_1, \dots, x_n]$ and $J = (x_i x_j : 1 \leq i < j \leq n)$. By [14, Proposition 2.1], $J \subseteq I = (J, x_1^{n-1}, \dots, x_n^{n-1})$ is Aluffi torsion-free. Note that the number of generators of J is $t = n(n-1)/2$. We show that $J \subseteq I$ is strongly Aluffi torsion-free. By above remark and symmetry we just prove that $x_{n-1}x_n((J_{t-1} :_R (x_{n-1}x_n) \cap I^m)$ contained in $J_{t-1}I^m$ for all $m \geq 1$. An easy calculation show that $Q := (J_{t-1} :_R (x_{n-1}x_n)) = (x_1, \dots, x_{n-2})$. Setting $\Delta = (\hat{J}, x_1^{n-1}, \dots, x_{n-2}^{n-1})$ and $\Gamma = (x_{n-1}x_n, x_{n-1}^{n-1}, x_n^{n-1})$, where by \hat{J} we mean J without the generator $x_{n-1}x_n$. Write $I = (\Gamma, \Delta)$. We have

$$\begin{aligned} x_{n-1}x_n(Q \cap I^m) &= x_{n-1}x_n [Q \cap (\Gamma^m, \Gamma^{m-1}\Delta, \dots, \Gamma\Delta^{m-1}, \Delta^m)] \\ &= x_{n-1}x_n\Delta(I^{m-1}) + (x_{n-1}x_n)Q\Gamma^m \subseteq J_{t-1}I^m. \end{aligned}$$

Theorem 2.13. Let $J_1, J_2 \subseteq I$ be ideals in the ring R . Assume that $\overline{J_2} \subseteq \overline{I}$ is Aluffi torsion-free in $\overline{R} = R/J_1$. If there exists minimal generators f_1, \dots, f_s of J_1 such that

1. $J_1 = (f_1, \dots, f_s) \subseteq I$ is strongly Aluffi torsion-free,
2. $\{\tilde{f}_1, \dots, \tilde{f}_s\}$ is a regular sequence in $\tilde{R} = R/J_2$,

then $J_2 \subseteq I$ is Aluffi torsion-free.

Proof. We use induction on s . Assume that $s = 1$. Since $\overline{J_2} \subseteq \overline{I}$ is Aluffi torsion-free then for all $n \geq 1$ we have $(J_2 \cap I^n) + J_1 = (J_2 I^{n-1}) + J_1$. Intersecting the latter with $J_2 \cap I^n$ we get

$$J_2 \cap I^n = J_2 I^{n-1} + (J_1 \cap I^n \cap J_2).$$

But $J_1 \cap I^n \cap J_2 = (f_1) \cap I^n \cap J_2$. Hence by (1) we obtain that

$$(f_1) \cap I^n \cap J_2 = (f_1) I^{n-1} \cap J_2 = (f_1) (I^{n-1} \cap (J_2 : f_1)).$$

By (2) \tilde{f}_1 is regular in \tilde{R} , then $(J_2 : f_1) = J_2$. Hence

$$(f_1) \cap I^n \cap J_2 = f_1 (I^{n-1} \cap J_2),$$

and we obtain

$$J_2 \cap I^n = J_2 I^{n-1} + f_1 (I^{n-1} \cap J_2).$$

Now making induction on n , we get

$$J_2 \cap I^n = J_2 I^{n-1} + f_1 (J_2 I^{n-2}) \subseteq J_2 I^{n-1},$$

which prove the assertion in this case. Now assume that $s > 1$. Let $N = (f_1, \dots, f_{s-1})$ and denote by “ $\hat{}$ ” reduction modulo N . Then in the ring \hat{R} we have ideals \hat{J}_1, \hat{J}_2 and \hat{I} . Furthermore, $\hat{J}_2 \subseteq \hat{I} \subseteq \hat{R}$ and by the minimality of $\{f_1, \dots, f_s\}$ and Proposition 2.4, $\hat{J}_1 = (\hat{f}_s) \subseteq \hat{I}$ is Aluffi torsion-free. Also \hat{f}_s is regular in \hat{R} . Thus by the first step of the induction we get that $\hat{J}_2 \subseteq \hat{I}$ is Aluffi torsion-free. Since the ideal N has the same property as the ideal J_1 , the inductive assumption completes the proof. \square

3. Application and examples

In intersection theory Aluffi algebra is used for closed embedding of schemes $Y \hookrightarrow X \hookrightarrow M$ where M is a regular and Y is the singular subscheme of X . In this section, we follow this direction.

Let $R = k[x_0, \dots, x_n]$ be a polynomial ring over a field k of characteristic zero. Let $J = (f_1, \dots, f_t)$ be an ideal of height r . Denote by $\Theta = (\partial f_{ij} / \partial x_j)$ the Jacobian matrix of J and by $I_r(\Theta)$ the ideals generated by r -minors of Θ . The ideal $I = (J, I_r(\Theta))$ is called the Jacobian ideal of J which describes the singular subscheme of $\text{Spec}(R/J)$. See [14] and [13] for examples of Aluffi torsion-free ideals in this situation.

Example 3.1. Let $J \subset R = k[x, y, z]$ be the defining ideal of the monomial space curve with parametric equations $x = u^{n_1}$, $y = u^{n_2}$, $z = u^{n_3}$, where $\gcd(n_1, n_2, n_3) = 1$. Suppose that $n_1 = 2q + 1$, $n_2 = 2q + p + 1$, $n_3 = 2q + 2p + 1$, for non-negative integers p, q . If I is the Jacobian ideal of J , then the pair $J \subset I$ is Aluffi torsion-free.

Proof. Grading R by the exponents of the parameter u in the parametric equations, one knows [6] that J is a perfect codimension 2 ideal generated by the homogeneous polynomials

$$F_1 = x^{c_1} - y^{r_{12}} z^{r_{13}}, \quad F_2 = x^{r_{21}} z^{r_{23}} - y^{c_2}, \quad F_3 = x^{r_{31}} y^{r_{32}} - z^{c_3}$$

where $0 < r_{ij} < c_i$ ($i = 1, 2, 3, j \neq i$). Note the relations

$$c_1 = r_{21} + r_{31}, \quad c_2 = r_{12} + r_{32}, \quad c_3 = r_{13} + r_{23}.$$

The Jacobian matrix of J is

$$\Theta = \begin{pmatrix} c_1 x^{c_1-1} & -r_{12} y^{r_{12}-1} z^{r_{13}} & -r_{13} y^{r_{12}} z^{r_{13}-1} \\ r_{21} x^{r_{21}-1} z^{r_{23}} & -c_2 y^{c_2-1} & r_{23} x^{r_{21}} z^{r_{23}-1} \\ r_{31} x^{r_{31}-1} y^{r_{32}} & r_{32} x^{r_{31}} y^{r_{32}-1} & -c_3 z^{c_3-1} \end{pmatrix}.$$

The 2-minors of Θ are

$$\begin{aligned} f_1 &= -c_1 c_2 x^{c_1-1} y^{c_2-1} + r_{21} r_{12} x^{r_{21}-1} y^{r_{12}-1} z^{c_3}, \\ f_2 &= c_1 r_{23} x^{c_1+r_{21}-1} z^{r_{23}-1} + r_{13} r_{21} x^{r_{21}-1} y^{r_{12}} z^{c_3-1}, \\ f_3 &= -r_{12} r_{23} x^{r_{21}} y^{r_{12}-1} z^{c_3-1} + c_2 r_{13} y^{c_2+r_{12}-1} z^{r_{13}-1}, \\ f_4 &= c_1 r_{32} x^{c_1+r_{31}-1} y^{r_{32}-1} + r_{31} r_{12} x^{r_{31}-1} y^{c_2-1} z^{r_{13}}, \\ f_5 &= -c_1 c_3 x^{c_1-1} z^{c_3-1} + r_{31} r_{13} x^{r_{31}-1} y^{c_2} z^{r_{13}-1}, \\ f_6 &= r_{32} r_{13} x^{r_{31}} y^{c_2-1} z^{r_{13}-1} + c_3 r_{12} y^{r_{12}-1} z^{c_3+r_{13}-1}, \\ f_7 &= r_{21} r_{32} x^{c_1-1} y^{r_{32}-1} z^{r_{23}} + c_2 r_{31} x^{r_{31}-1} y^{c_2+r_{32}-1}, \\ f_8 &= -r_{21} r_{31} x^{c_1-1} y^{r_{32}} z^{r_{23}-1} - c_3 r_{21} x^{r_{21}-1} z^{c_3+r_{23}-1}, \\ f_9 &= -r_{32} r_{23} x^{c_1} y^{r_{32}-1} z^{r_{23}-1} + c_2 c_3 y^{c_2-1} z^{c_3-1}. \end{aligned}$$

Write \mathfrak{D} for the ideal generated by the following monomials

$$\begin{array}{lll}
M_1 = x^{r_{21}-1}y^{r_{12}-1}z^{c_3} & M_2 = x^{r_{21}-1}y^{r_{12}}z^{c_3-1} & M_3 = y^{c_2+r_{12}-1}z^{r_{13}-1} \\
M_4 = x^{r_{31}-1}y^{c_2-1}z^{r_{13}} & M_5 = x^{r_{31}-1}y^{c_2}z^{r_{13}-1} & M_6 = y^{r_{12}-1}z^{c_3+r_{13}-1} \\
M_7 = x^{r_{31}-1}y^{c_2+r_{32}-1} & M_8 = x^{r_{21}-1}z^{c_3+r_{23}-1} & M_9 = y^{c_2-1}z^{c_3-1}.
\end{array}$$

The following relations come out

$$\begin{aligned}
f_1 &= -c_1c_2x^{r_{21}-1}y^{r_{12}-1}F_3 + (r_{21}r_{12} - c_1c_2)M_1, \\
f_2 &= c_1r_{23}x^{r_{21}-1}z^{r_{23}-1}F_1 + (r_{31}r_{21} - c_1r_{23})M_2, \\
f_3 &= -r_{12}r_{23}y^{r_{12}-1}z^{r_{13}-1}F_2 - (r_{12}r_{23} + c_2r_{13})M_3, \\
f_4 &= c_1r_{32}x^{r_{31}-1}y^{r_{32}-1}F_1 + (r_{31}r_{12} + c_1r_{32})M_4, \\
f_5 &= -c_1c_3x^{r_{31}-1}z^{r_{13}-1}F_2 + (r_{31}r_{13} - c_1c_3)M_5, \\
f_6 &= r_{32}r_{13}y^{r_{12}-1}z^{r_{13}-1}F_3 + (r_{21}c_3 + r_{32}r_{13})M_6, \\
f_7 &= r_{21}r_{32}x^{r_{31}-1}y^{r_{32}-1}F_2 + (r_{31}c_2 + r_{21}r_{32})M_7, \\
f_8 &= -r_{32}r_{23}x^{r_{21}-1}z^{r_{23}-1}F_3 - (r_{21}c_3 + r_{23}r_{31})M_8, \\
f_9 &= r_{32}r_{23}y^{r_{32}-1}z^{r_{23}-1}F_1 + (c_2c_3 - r_{32}r_{23})M_9.
\end{aligned}$$

By above, J is generated by

$$F_1 = x^{p+q+1} - yz^q, \quad F_2 = xz - y^2, \quad F_3 = x^{p+q}y - z^{q+1}$$

and the Jacobian ideal is

$$I = (J, \mathfrak{D}) = (xz - y^2, x^{p+q+1}, x^{p+q}y, x^{p+q-1}y^2, yz^q, y^2z^{q-1}, z^{q+1})$$

Set $\Delta = (x^{p+q+1}, x^{p+q}y, x^{p+q-1}y^2, yz^q, y^2z^{q-1}, z^{q+1})$. By a slight adaptation of Proposition 2.5 it suffices to show that $J \cap \Delta^n \subseteq JI^{n-1}$ for every $n \geq 1$. Since J is binomial prime ideal and Δ is monomial then [2, Corollary 1.5] implies that $J \cap \Delta^n$ is generated by binomials $(u - v)h$ where $u - v \in J$ and $h \in R$ is a monomial or $h = (u - v)^c$ for some positive integer $c \geq 1$. As elements uh, vh belong to Δ^n , an easy calculation show that $(u - v)h \in JI^{n-1}$ as required. \square

Question 3.2. Let $J \subseteq k[x_1, \dots, x_n]$ be defining ideal of affine monomial curve with parametric equation $x_1 = u^{n_1}, \dots, x_m = u^{n_m}$. Let I be the Jacobian ideal of J . For which types of parametrization, the pair $J \subseteq I$ is Aluffi torsion-free.

Example 3.3. Let J be an ideal in the ring $R = k[x_0, \dots, x_n]$ with $n \geq 2$ generated with all square free monomial ideal in degree r . Let I stand for the Jacobian ideal of J . Then the pair $J \subset I$ is Aluffi torsion-free.

Proof. Let $J = (x_{i_1}x_{i_2} \cdots x_{i_r} : 0 \leq i_1 < \dots < i_r \leq n)$. It is well known that $\text{ht } J = r - 1$. The transpose Jacobian matrix of J is

$$\Theta(J) = \left[\begin{array}{c|c} x_{i_1}x_{i_2}\cdots x_{i_{r-1}} & 1 \leq i_1 < \dots < i_{r-1} \leq n \\ \hline * & \end{array} \middle| \begin{array}{c} 0 \\ \hline \Theta' \end{array} \right],$$

where Θ' is the Jacobian matrix of the ideal J' generated by all square free monomial ideal in degree r in $k[x_1, \dots, x_n]$. By induction on n and elementary columns operation we get that the Jacobian ideal I of J is $I = (J, x_i^r x_j^r : 0 \leq i < j \leq n)$. By Proposition 2.5, it is enough to show that for all $t \geq 1$

$$J \cap (x_i^r x_j^r : 0 \leq i < j \leq n)^t \subseteq JI^{t-1}.$$

The proof of the latter inclusion is based on the usual algorithmic procedure to find generators of the intersection of monomial ideal. \square

Example 3.4.

1. Let M be a $2 \times n$ generic matrix in the polynomial ring $R = k[x_i ; 1 \leq i \leq 2n]$ with $n \geq 3$. Let $J = I_2(M)$. It is well-known that $\text{ht } J = n - 1$. Let $I = (J, I_{n-1}(\Theta))$ stand for the Jacobian ideal of J . Since M is the concatenation of n scroll blocks of length 1, then by [14, Theorem 2.3], $I_{n-1}(\Theta) = (x_i ; 1 \leq i \leq 2n)^{n-1}$. In particular, the pair $J \subseteq I$ is Aluffi torsion-free.
2. Let $R = k[x_1, \dots, x_9]$. Consider the 3×3 generic matrix M in R

$$M = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}.$$

The ideal $J = I_2(M) = (\Delta_1, \Delta_2, \Delta_3)$ has codimension 4 and the Jacobian matrix of J is of the form

$$\Theta(J) = \left[\begin{array}{c|c|c} x_5 & & \\ x_6 & & \\ x_8 & * & * \\ x_9 & & \\ \hline 0 & x_6 & \\ 0 & x_9 & * \\ \hline 0 & 0 & \\ 0 & 0 & \Theta' \\ 0 & 0 & \end{array} \right],$$

where the first block of $\Theta(J)$ is the Jacobian matrix of $\Delta_1 = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_1x_8 - x_2x_7, x_1x_9 - x_3x_7)$, the second block is the Jacobian matrix of $\Delta_2 = (x_2x_6 - x_3x_5, x_2x_9 - x_3x_8)$ and in the last block Θ' is the Jacobian matrix of $\Delta_3 = I_2\left(\begin{bmatrix} x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}\right)$. Note that $\text{ht } (\Delta_2, \Delta_3) = 3$ and $\text{ht } \Delta_3 = 2$. We claim that

$I_4(\Theta(J)) = (x_1, \dots, x_9)^4$ which proves that the pair $J \subseteq I$ is Aluffi torsion-free. By using the first part of the example with $n = 3$, the 3-minors of second and third blocks of $\Theta(J)$ is generated by $(x_6, x_9)(x_4, x_5, x_6, x_7, x_8, x_9)^2$. One has

$$(x_5, x_6, x_8, x_9)(x_6, x_9)(x_4, x_5, x_6, x_7, x_8, x_9)^2 \in I_4(\Theta(J)).$$

Therefore by changing the role of x_1 and x_2 and using above argument the assertion hold.

Question 3.5. Let M be a $n \times m$ generic matrix in the polynomial ring $R = k[x_{ij} ; 1 \leq i \leq n, 1 \leq j \leq m]$. Let $J = I_2(M)$ be the ideal generated by 2-minors of M . Let I be the Jacobian ideal of J . Is the pair $J \subseteq I$ Aluffi torsion-free?

Example 3.6. Let $J(f) = (\partial f / \partial x, \partial f / \partial y, \partial f / \partial z) \subseteq R = k[x, y, z]$ denote the gradient ideal of a reduced free divisor line arrangement $X = V(f)$ of degree 3 in \mathbb{P}_k^2 . By [15, Proposition 3.7], $J(f)$ is codimension 2 perfect ideal. Then by Hilbert–Burch theorem $J(f)$ is generated by 2-minors of the 2×3 matrix of linear forms in R . If $Y = V(J(f)) \subseteq \mathbb{P}_k^2$ is non-singular then the Jacobian ideal I of $J(f)$ is (x, y, z) -primary. Therefore, by [14, Corollary 2.7] the pair $J(f) \subseteq I$ is Aluffi torsion-free.

We warm up with a conjecture.

Conjecture 3.7. Let $X = V(f)$ be a reduced free divisor of line arrangement in \mathbb{P}_k^2 . Let $J(f)$ denote the gradient ideal of f and I stands for the Jacobian ideal of $J(f)$. Then $J(f) \subseteq I$ is Aluffi torsion-free.

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