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Triangulated factorization systems and t-structures ☆

Fosco Loregian [a], Simone Virili [b,*]

[a] Tallinn University of Technology, Akadeemia tee 15/2, 12618, Tallinn, Estonia
[b] Facultad de Matemáticas, Universidad de Murcia, Campus Espinardo, 30100, Murcia, Spain

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ABSTRACT

We define *triangulated factorization systems* on triangulated categories, and prove that a suitable subclass thereof (the *normal triangulated torsion theories*) corresponds bijectively to t-structures on the same category. This result can be regarded as the triangulated analog of the theorem that says that 't-structures are normal torsion theories' in the setting of stable $(\infty, 1)$-categories, showing how the result remains true whatever the chosen model for stable homotopy theory is.

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* Corresponding author.
E-mail addresses: fosco.loregian@taltech.ee, fosco.loregian@gmail.com (F. Loregian), s.virili@um.es, virili.simone@gmail.com (S. Virili).

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Introduction

Factorization systems surely form a conspicuous part of modern category theory; this is especially because they provide the category where they live in with a rather rich structure, and they are commonly found (although very few of them can be easily built): for example, a trace of what we would today call a factorization system on the category of groups appears in the pioneering [18], published in 1948; more interestingly, as acknowledged by [22], any "synthetic" approach to homotopy theory inevitably relies on the notion of a –weak– factorization system.

Soon after having reached a consensus on the definition for these gadgets [7], category theorists wanted to make explicit the evident tight relation between (weak) factorization systems and (weakly) reflective subcategories on a same ambient category $\mathbf{C}$: this culminated with the proof, given in [4], that under mild assumptions the reflective subcategories of $\mathbf{C}$ are in bijection with the so-called *reflective pre-factorization systems* on $\mathbf{C}$.

Let us briefly recall this notion: a morphism f in $\mathbf{C}$ is *left orthogonal* to another morphism g (or g is *right orthogonal* to f), in symbols $f \perp g$, if for any commutative square of solid arrows

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, dotted, "d"] & \cdot \end{tikzcd}$$

there is a unique morphism d that makes the two above triangles commute (this defines *strong* orthogonality; in case *at least one* such d exist, we speak of weak orthogonality). Then,

- for a class $\mathcal{X} \subseteq \mathbf{C}^{\mathbf{2}}$ (where $\mathbf{C}^{\mathbf{2}}$ is the arrow category) we let ${}^{\perp}\mathcal{X}$ (resp., $\mathcal{X}^{\perp}$) be the class of morphisms which are left (resp., right) orthogonal to each element in $\mathcal{X}$;
- a *pre-factorization system* (PFS for short) on $\mathbf{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of sub-classes of $\mathbf{C}^{\mathbf{2}}$ such that $\mathcal{E} = {}^{\perp}\mathcal{M}$ and $\mathcal{E}^{\perp} = \mathcal{M}$;
- a pre-factorization system $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ on $\mathbf{C}$ such that every map $f \in \mathbf{C}^{\mathbf{2}}$ can be factored as a composition $f = m_f \circ e_f$, for $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$ is called a *factorization system* (FS for short; we call a morphism that can be factored by a PFS an $\mathfrak{F}$-*crumbled* arrow: then, a factorization system is a PFS in which every arrow is $\mathfrak{F}$-crumbled);
- a class $\mathcal{X}$ of morphisms of $\mathbf{C}$ is said to have the *3-for-2 property* if, given two composable morphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ in $\mathcal{X}$, if two elements of the set $\{f, g, g \circ f\}$ belong to $\mathcal{X}$, so does the third.

A PFS $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ is said to be *reflective* if $\mathcal{M}$ has the 3-for-2 property and if any map of the form $\begin{bmatrix} X \\ \downarrow \\ 0 \end{bmatrix}$ is $\mathfrak{F}$-crumbled. For such a PFS, the associated reflective subcategory of $\mathbf{C}$ is

$$\mathcal{M}/0 := \left\{ X \in \mathbf{C} \mid \begin{bmatrix} X \\ \downarrow \\ 0 \end{bmatrix} \in \mathcal{M} \right\} \subseteq \mathbf{C}$$

(uniqueness of lifts ensures that there is a functorial choice of an object in $\mathcal{M}/0$ for each $X \in \mathbf{C}$, precisely the object such that $X \xrightarrow{e_X} RX \xrightarrow{m_X} 0$). It is a remarkable result that *all* the reflective subcategories of $\mathbf{C}$ arise in fact in this way: given such a subcategory $\mathbf{S}$, there is a reflective PFS *generated* by all morphisms of $\mathbf{S}$.

The authors of [4] then specialize this result attempting to describe the tight relation between factorization systems and *torsion theories*, under similarly mild assumptions on $\mathbf{C}$. This approach has been extended sensibly in [21].

A factorization system $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ on $\mathbf{C}$ is said to be a *torsion theory* (TTH for short) if *both* $\mathcal{E}$ and $\mathcal{M}$ have the 3-for-2 property. This gives (thanks to the above result and its dual) a *pair* of subcategories $\mathcal{M}/0$ and $0/\mathcal{E}$ whose inclusions in $\mathbf{C}$ admit respectively a left and a right adjoint: these two subcategories form the classes of so-called *torsion* and *torsion-free* objects respectively, and relate to the classical notion of a *torsion theory* given in [5].

Suppose indeed that $\mathbf{C}$ is an abelian category. A TTH $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ on $\mathbf{C}$ is said to be *normal* if taking the $\mathfrak{F}$-factorization

$$X \xrightarrow{e} RX \xrightarrow{m} 0$$

of the final map $X \to 0$ for a given object $X \in \mathbf{C}$, and then taking the pullback

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "e"] \\ 0 \arrow[r] & RX \end{tikzcd} \tag{0.1}$$

we have $\begin{bmatrix} T \\ \downarrow \\ 0 \end{bmatrix} \in \mathcal{E}$.

Applying the definitions, one can show that the pair $(0/\mathcal{E}, \mathcal{M}/0)$ is a *classical* torsion theory (i.e. a torsion theory as defined in [5]). In fact, it is also true that *every* torsion theory arises this way (see [21]); this gives a bijection between classical torsion theories and normal TTHs.

Switching to the triangulated context, the rôle played by classical TTHs in abelian categories is now played by t-structures ([1,12]). The analogy between these two concepts was made completely formal by Beligiannis and Reiten [3] where they introduced *torsion pairs* in pre-triangulated categories. In fact, if the pre-triangulated structure is inherited from the abelian-ness of the ambient category, then torsion pairs correspond bijectively to classical TTHs, while if the pre-triangulated structure is triangulated, then torsion pairs correspond bijectively to t-structures.

The strong analogies between classical TTHs and t-structures suggests that there should be a way to describe them in terms of some kind of factorization systems, just like for TTHs in abelian categories. In fact, pursuing a similar characterization in the non-abelian setting is acknowledged in [21] as one of the most natural generalization of this technology. Nevertheless, the authors are not able to show a correspondence between t-structures on triangulated categories and factorization systems.

Somehow, this result has been prevented by a certain number of awkward properties of triangulated categories (see the introduction of [17] for a good account on this). In this respect, it is remarkable that such a theorem can be stated and proved quite naturally by getting rid of all these unwieldy features, ascending to the realm of stable $(\infty, 1)$-categories: the proof that t-structures on (the homotopy category of) a stable quasicategory correspond bijectively to normal torsion theories, regarded as particular ∞-categorical factorization systems, has been the central result of the first author's PhD thesis [13].[1]

Our first point in this paper is that the reason for the absence of this theorem from the setting of triangulated categories $\mathbf{D}$ is that there is no notion of triangulated orthogonality $\,\underline{\Join}\,$ for a pair of morphisms in $\mathbf{D}$, with formal properties comparable to those of the orthogonality relation $\perp$ but *mindful of the triangulated structure*.

The present work aims to fill this gap and solve the problem of finding a class of suitably defined *triangulated factorization systems* on $\mathbf{D}$ in bijection with the class of t-structures on $\mathbf{D}$.

We start Section 1 describing the homotopy orthogonality relation $f \,\underline{\Join}\, g$ for two morphisms in a triangulated category $\mathbf{D}$ (see Def. 1.1). After proving some natural properties, we mimic the classical theory showing that this definition is sound, in that it recovers basically all the formal properties enjoyed by the $\perp$-orthogonality relation (see 1.6–1.9). We introduce triangulated PFSs via triangulated orthogonality, triangulated FSs, triangulated TTHs and, finally, normal triangulated TTHs as the corresponding of each of the classical definitions.

We believe that this is the correct path to follow, as Def. 1.1 is exactly an orthogonality condition that keeps track of the triangulated structure of $\mathbf{D}$: as an example of this flexibility, normality for a triangulated TTH can be introduced exactly as normality for a TTH but taking a *homotopy* cartesian square (see 1.8 for the definition) in (0.1) instead of a pullback square. So apparently the definition really captures the best of both worlds.

With the theory of triangulated FSs at hand, in 2.11 we prove the following

[1] The fact that few triangulated categories generate an interesting poset of factorization systems is probably due to the fact that a nice factorization system on a category $\mathbf{A}$ interacts with co/limits on $\mathbf{A}$, and it is somehow generated by them: few triangulated categories have interesting co/limits, hence the fact that (for example) every *proper* factorization system, where the left class is contained in the class of epimorphisms, although really natural in a generic category must be trivial in a triangulated one.

Main Theorem: For a triangulated category $\mathbf{D}$, the following map is bijective:

$$\begin{array}{ccc} \left\{ \begin{smallmatrix} \text{normal triangulated} \\ \text{TTHs on } \mathbf{D} \end{smallmatrix} \right\} & \longrightarrow & \left\{ \begin{smallmatrix} t\text{-structures} \\ \text{on } \mathbf{D} \end{smallmatrix} \right\} \\ (\mathcal{E}, \mathcal{M}) & \longmapsto & \Big(0/\mathcal{E}, \Sigma(\mathcal{M}/0)\Big) \end{array}$$

As mentioned above, [8] proved a ∞-categorical version of the above theorem in the setting of stable quasicategories. In fact, quasicategories support a fairly natural theory of FSs, as rich as the classical one; we refer to [10] and [14].

Once quasicategorical FSs are defined, one can mimic the definition of normal TTH in this setting. The main results contained in [13] tells us that, for a stable quasicategory $\mathbf{C}$, the normal TTHs on $\mathbf{C}$ are in bijection with t-structures on the triangulated category $\mathsf{Ho}(\mathbf{C})$. An exercise in translation between models shows how the same result remains true

- in the setting of stable model categories, where one can speak about *homotopy factorization systems* following [2,11]; this leads to the definition of *homotopy t-structures* on stable model categories $\mathbf{M}$ as suitable analogues of normal torsion theories in the set HFS($\mathbf{M}$) of homotopy factorization systems on a model category $\mathbf{M}$;
- in the setting of DG-categories, where we speak about factorization systems (enriched in the sense of [6,16]); this leads to the definition of DG-*t-structures* as enriched analogues of normal torsion theories in the set of enriched factorization systems on a DG-category $\mathcal{D}$.

In both these settings, it is possible to recover a theorem that characterizes what, from time to time, you would like to call t-structures as a class in bijection with normal torsion theories defined in that specific model.

In subsequent work [15] we will frame our Main Theorem in a different model of a stable homotopy theory, namely *stable derivators*: this has to be regarded as the nontrivial step towards a model-independence proof saying that t-structures *are* indeed normal torsion theories whatever our preferred model for stable homotopy theory is.

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1. Triangulated factorization systems

Throughout this section we let $\mathbf{D}$ be a (fixed but arbitrary) triangulated category, with shift functor $\Sigma\colon \mathbf{D} \xrightarrow{\simeq} \mathbf{D}$. For a general background and notation on triangulated categories we refer to [20] and [9, Appendix A].

1.1. Homotopy orthogonality of morphisms

Our first task is to build a notion of orthogonality of morphisms mindful of the triangulated structure on $\mathbf{D}$.

Definition 1.1 *(Homotopy orthogonality).* Let $E_0 \xrightarrow{e} E_1$ and $M_0 \xrightarrow{m} M_1$ be two maps in $\mathbf{D}$, and complete them to triangles

$$E_0 \xrightarrow{e} E_1 \xrightarrow{\alpha_e} C_e \xrightarrow{\beta_e} \Sigma E_0 \qquad \text{and} \qquad M_0 \xrightarrow{m} M_1 \xrightarrow{\alpha_m} C_m \xrightarrow{\beta_m} \Sigma M_0. \tag{1.1}$$

We say that e is *left homotopy orthogonal* to m (while m is *right homotopy orthogonal* to e), in symbols $e \mathrel{\underline{\perp}} m$, if the following conditions are satisfied:

HO1. the following map is trivial:

$$\begin{aligned} \mathbf{D}(C_e, \Sigma^{-1}C_m) &\longrightarrow \mathbf{D}(E_1, M_0) \\ (C_e \xrightarrow{\varphi} \Sigma^{-1}C_m) &\longmapsto (E_1 \xrightarrow{\alpha_e} C_e \xrightarrow{\varphi} \Sigma^{-1}C_m \xrightarrow{\Sigma^{-1}\beta_m} M_0)\,; \end{aligned}$$

HO2. the following map is injective:

$$\begin{aligned} \mathbf{D}(C_e, C_m) &\longrightarrow \mathbf{D}(E_1, \Sigma M_0) \\ (C_e \xrightarrow{\varphi} C_m) &\longmapsto (E_1 \xrightarrow{\alpha_e} C_e \xrightarrow{\varphi} C_m \xrightarrow{\beta_m} \Sigma M_0)\,. \end{aligned}$$

The concept of homotopy orthogonality seems quite artificial, but this notion arises naturally in the setting of stable derivators: we investigate the matter in a subsequent work [15]. Notice also that one can prove by standard arguments that homotopy orthogonality does not depend on the choice of triangles in (1.1).

Remark 1.2. Condition 1.1.HO2 can be substituted by the following one:

HO2'. The unique morphism φ completing a morphism $(a, b)\colon e \to m$ in $\mathbf{D}^{\mathbf{2}}$ to a morphism of triangles, as in the following diagram, is $\varphi = 0$:

$$\begin{tikzcd} E_0 \arrow[r, "e"] \arrow[d, "a"'] & E_1 \arrow[r, "\alpha_e"] \arrow[d, "b"] & C_e \arrow[r, "\beta_e"] \arrow[d, dotted, "\varphi"] & \Sigma E_0 \arrow[d] \\ M_0 \arrow[r, "m"] & M_1 \arrow[r, "\alpha_m"] & C_m \arrow[r, "\beta_m"] & \Sigma M_0. \end{tikzcd}$$

To see this equivalence, suppose that condition 1.1.HO2 is satisfied. Then, the map $\mathbf{D}(C_e, C_m) \to \mathbf{D}(E_1, \Sigma M_0)$ sends φ to $\beta_m\varphi\alpha_e = \beta_m\alpha_m b = 0$; so by the injectivity of this map, we deduce that $\varphi = 0$. On the other hand, suppose HO2' is satisfied and consider a morphism $\psi \in \mathbf{D}(C_e, C_m)$ such that $\beta_m \psi \alpha_e = 0$; we have to show that $\psi = 0$. Indeed, since $\beta_m \psi \alpha_e = 0$ we can construct a morphism of triangles as follows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, dotted, "\exists b"] & E_1 \arrow[r] \arrow[d, "\psi\alpha_e"] & 0 \arrow[d] \\
M_0 \arrow[r, "m"] & M_1 \arrow[r, "\alpha_m"] & C_m \arrow[r, "\beta_m"] & \Sigma M_0 .
\end{tikzcd}$$

Now one can complete the central square in the following diagram to a morphism of triangles:

$$\begin{tikzcd}
E_0 \arrow[r, "e"] \arrow[d, dotted, "\exists a"'] & E_1 \arrow[r, "\alpha_e"] \arrow[d, "b"] & C_e \arrow[r, "\beta_e"] \arrow[d, "\psi"] & \Sigma E_0 \arrow[d, dotted] \\
M_0 \arrow[r, "m"] & M_1 \arrow[r, "\alpha_m"] & C_m \arrow[r, "\beta_m"] & \Sigma M_0 .
\end{tikzcd}$$

Then, by HO2', $\psi = 0$ as desired.

In what follows we verify some properties that one should expect from any well-behaved notion of orthogonality. Let us start with the following property, whose proof is an easy exercise:

Lemma 1.3. *The following are equivalent for $f \in \mathbf{D^2}$*

(i) *f is an isomorphism;*
(ii) *$f \mathrel{\overset{\perp}{\approx}} \mathbf{D^2}$;*
(iii) *$\mathbf{D^2} \mathrel{\overset{\perp}{\approx}} f$;*
(iv) *$f \mathrel{\overset{\perp}{\approx}} f$.*

The above proposition adopted an harmless abuse of notation, that is, it denoted $\mathcal{H} \mathrel{\overset{\perp}{\approx}} \mathcal{K}$ the fact that each $h \in \mathcal{H}$ is left $\overset{\perp}{\approx}$-orthogonal to every morphism of $\mathcal{K}$. To make this statement precise we introduce the following definitions.

Notation 1.4 *($\overset{\perp}{\approx}$-Orthogonal of a class).* We denote ${}^{\overset{\perp}{\approx}}(\_) \dashv (\_)^{\overset{\perp}{\approx}}$ the (antitone) Galois connection induced by the relation $\overset{\perp}{\approx}$ on full subcategories of $\mathbf{D^2}$; more explicitly, we denote

$$\begin{aligned}
\mathcal{X}^{\overset{\perp}{\approx}} &:= \{ f \in \mathbf{D^2} \mid x \mathrel{\overset{\perp}{\approx}} f, \ \forall x \in \mathcal{X} \} \\
{}^{\overset{\perp}{\approx}}\mathcal{X} &:= \{ f \in \mathbf{D^2} \mid f \mathrel{\overset{\perp}{\approx}} x, \ \forall x \in \mathcal{X} \}.
\end{aligned} \tag{1.2}$$

Remark 1.5 *($\overset{\perp}{\approx}$-Locality).* There is a related notion of orthogonality between an object X and a morphism $f \in \mathbf{D^2}$, based on the fact that we can blur the distinction between objects and their initial or terminal arrows; given these data, we say that X is *right-orthogonal* to f (or that X is an f-*local* object) if the hom functor $\mathbf{D}(-,X)$ inverts f; in fact, the map $\mathbf{D}(f,X)$ is injective if and only if the pair $(f, \begin{bmatrix} X \\ \downarrow \\ 0 \end{bmatrix})$ satisfies condition 1.1.HO1, while it is surjective if and only if $(f, \begin{bmatrix} X \\ \downarrow \\ 0 \end{bmatrix})$ satisfies condition 1.1.HO2. (Obviously, there is a dual notion of left orthogonality between f and $B \in \mathbf{D}$, or a notion of a f-*colocal* object B which reduces to left orthogonality with respect to $0 \to B$).

By the above remark, it is natural to say that two objects B and X are homotopy orthogonal if $\begin{bmatrix} 0 \\ \downarrow \\ B \end{bmatrix} \precapprox \begin{bmatrix} X \\ \downarrow \\ 0 \end{bmatrix}$. In fact, it is not difficult to show that this happens if and only if $\mathbf{D}(B,X) = 0$, that is, $B \perp X$ in the usual sense.

The following lemma can be easily verified by hand:

Lemma 1.6. *Let $\{f_i\}_{i \in I}$, $g \in \mathbf{D}^2$. If $f_i \precapprox g$ for all $i \in I$, then $\coprod_i f_i \precapprox g$. On the other hand, if $g \precapprox f_i$ for all $i \in I$, then $g \precapprox \prod_i f_i$.*

Lemma 1.7. *Let $f, g \in \mathbf{D}^2$ and let f' be a retract of f, that is, there is a commutative diagram*

$$\begin{tikzcd}
F_0' \arrow[r, "i_0"] \arrow[d, "f'"'] \arrow[rr, bend left, "\mathrm{id}"] & F_0 \arrow[r, "p_0"] \arrow[d, "f"] & F_0' \arrow[d, "f'"] \\
F_1' \arrow[r, "i_1"] \arrow[rr, bend right, "\mathrm{id}"'] & F_1 \arrow[r, "p_1"] & F_1'
\end{tikzcd}$$

If $f \precapprox g$, then $f' \precapprox g$.

Proof. Let $(a,b) \colon f' \to g$ be a morphism in $\mathbf{D}^2$ and consider the following commutative diagram, whose columns are triangles:

$$\begin{tikzcd}
F_0' \arrow[r, "i_0"] \arrow[d, "f'"'] & F_0 \arrow[r, "p_0"] \arrow[d, "f"'] & F_0' \arrow[r, "a"] \arrow[d, "f'"'] & G_0 \arrow[d, "g"] \\
F_1' \arrow[r, "i_1"] \arrow[d, "\alpha_{f'}"'] & F_1 \arrow[r, "p_1"] \arrow[d, "\alpha_f"'] & F_1' \arrow[r, "b"] \arrow[d, "\alpha_{f'}"'] & G_1 \arrow[d, "\alpha_g"] \\
C_{f'} \arrow[r, "i"] \arrow[d, "\beta_{f'}"'] & C_f \arrow[r, "p"] \arrow[d, "\beta_f"'] & C_{f'} \arrow[r, "\varphi"] \arrow[d, "\beta_{f'}"'] & C_g \arrow[d, "\beta_g"'] \\
\Sigma F_0' \arrow[r] & \Sigma F_0 \arrow[r] & \Sigma F_0' \arrow[r, "\Sigma a"] & \Sigma G_0
\end{tikzcd}$$

and notice that the composition $p \circ i$ is an isomorphism. To verify 1.1.HO2 we should prove that $\varphi = 0$, but in fact, $\varphi p = 0$ for the same condition applied to the pair (f, g), so that $\varphi \cong \varphi p\, i = 0$. On the other hand, to verify 1.1.HO1, consider a morphism $\psi \colon C_{f'} \to \Sigma^{-1} C_g$, then $\Sigma^{-1}(\beta_g)\psi\alpha_{f'} \cong \Sigma^{-1}(\beta_g)\psi p\, i\alpha_{f'} = \Sigma^{-1}(\beta_g)\psi p\alpha_f i_1 = 0 \circ i_1 = 0$, where $\Sigma^{-1}(\beta_g)\psi p\alpha_f = 0$ by the same condition applied to the pair (f, g). $\square$

Remark 1.8. To simplify the formulation of some of our forthcoming observations, let us recall that a *homotopy cartesian square* in $\mathbf{D}$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "\phi'"'] & Y' \end{tikzcd} \tag{1.3}$$

such that there exists a distinguished triangle $X \to X' \oplus Y \to Y' \to \Sigma X$, where the map $X \to X' \oplus Y$ is $\binom{\alpha}{-\phi}$, while the map $X' \oplus Y \to Y'$ is (ϕ', β). We call β the *homotopy pushout* of α, and α the *homotopy pullback* of β. We refer to [20, Ch. 1] for more details on this construction.

Lemma 1.9. *Let $(\psi \colon Y_0 \to Y_1) \in \mathbf{D^2}$ and consider a homotopy cartesian square:*

$$\begin{tikzcd} X_0 \arrow[r, "s"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & X_0' \arrow[d, "\phi'"] \\ X_1 \arrow[r, "t"'] & X_1' \end{tikzcd}$$

Then the following statements hold true:

(1) *if the pair (ϕ, ψ) satisfies 1.1.*HO2*, so does the pair (ϕ', ψ);*
(2) *if the pair (ϕ, ψ) satisfies 1.1.*HO1 *and $(\phi, \Sigma^{-1}\psi)$ satisfies 1.1.*HO2*, then $\mathbf{D}(C_{\phi'}, \Sigma^{-1} C_\psi) = 0$;*
(3) *if $\phi \mathrel{\underline{\pitchfork}} \psi$ and $\phi \mathrel{\underline{\pitchfork}} \Sigma^{-1}\psi$, then $\phi' \mathrel{\underline{\pitchfork}} \psi$.*

Proof. (1) Given a morphism $(a, b)\colon \phi' \to \psi$, we get a commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "s"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & X_0' \arrow[r, "a"] \arrow[d, "\phi'"] & Y_0 \arrow[d, "\psi"] \\
X_1 \arrow[r, "t"] \arrow[d, "\alpha"'] & X_1' \arrow[r, "b"] \arrow[d, "\alpha'"] & Y_1 \arrow[d, "\alpha_\psi"] \\
C_\phi \arrow[r, "\varphi", "\cong"'] \arrow[d, "\beta"'] & C_{\phi'} \arrow[r, "\psi"] \arrow[d, "\beta'"] & C_\psi \arrow[d, "\beta_\psi"] \\
\Sigma X_0 \arrow[r] & \Sigma X_0' \arrow[r] & \Sigma Y_0
\end{tikzcd}$$

we should prove that $\psi = 0$. By 1.1.HO2 applied to (ϕ, ψ) we get $\psi\varphi = 0$, but since φ is an isomorphism this allows us to conclude.

(2) Our two assumptions tell us that the map $\mathbf{D}(C_\phi, \Sigma^{-1}C_\psi) \to \mathbf{D}(X_1, Y_0)$ is both trivial and injective, so that $\mathbf{D}(C_{\phi'}, \Sigma^{-1}C_\psi) \cong \mathbf{D}(C_\phi, \Sigma^{-1}C_\psi) = 0$.

(3) By part (1) and $\phi \doteqdot \psi$, the pair (ϕ', ψ) satisfies 1.1.HO2. Furthermore, by part (2) and our assumptions, $\mathbf{D}(C_{\phi'}, \Sigma^{-1}C_\psi) = 0$, so the map $\mathbf{D}(C_{\phi'}, \Sigma^{-1}C_\psi) \to \mathbf{D}(\phi_1', Y_0)$ is clearly trivial. $\square$

Let $x\colon X_0 \to X_1$ and $y\colon Y_0 \to Y_1$ be morphisms in $\mathbf{D}$, and consider x and y as objects in $\mathbf{D^2}$. A morphism $f = (f_0, f_1)\colon x \to y$ in $\mathbf{D^2}$ is a commutative square of the form

$$\begin{tikzcd}
X_0 \arrow[r, "x"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\
Y_0 \arrow[r, "y"] & Y_1
\end{tikzcd}$$

Now, given a choice of completions of x and y to triangles,

$$X_0 \xrightarrow{x} X_1 \longrightarrow c(x) \longrightarrow \Sigma X_0 \qquad \text{and} \qquad Y_0 \xrightarrow{y} Y_1 \longrightarrow c(y) \longrightarrow \Sigma Y_0,$$

it seems legit to call a *cone* for f any map $c\colon c(x) \to c(y)$ that makes the following diagram commute, that is, such that (f_0, f_1, c) is a morphism of triangles:

$$\begin{tikzcd}
X_0 \arrow[r, "x"] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & c(x) \arrow[r] \arrow[d, "c"] & \Sigma X_0 \arrow[d, "\Sigma f_0"] \\
Y_0 \arrow[r, "y"] & Y_1 \arrow[r] & c(y) \arrow[r] & \Sigma Y_0
\end{tikzcd}$$

It is now natural to ask whether or not the classes $\{c\}^{\perp\!\!\!\!\perp}$ and ${}^{\perp\!\!\!\!\perp}\{c\}$ are determined by f or if they depend on the choices made. In fact, as shown by the following example, it turns out that a different choice of c can produce very different orthogonal classes:

Example 1.10. Let $X \in \mathbf{D}$ be non-trivial and consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "0"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & \Sigma X \arrow[r, "\mathrm{id}_{\Sigma X}"] \arrow[d, dotted] & \Sigma X \arrow[d, "0"] \\ X \arrow[r, "(\mathrm{id}_X,0)^t"] & X \oplus \Sigma X \arrow[r, "(0,\mathrm{id}_{\Sigma X})"] & \Sigma X \arrow[r, "0"] & \Sigma X \end{tikzcd}$$

where both rows are triangles and the square on the left-hand side trivially commutes. Now, any map $\Sigma X \to \Sigma X$ can be used to complete the above solid diagram to a morphism of triangles. In particular, both the trivial map $0_{\Sigma X} : \Sigma X \to \Sigma X$ and the identity $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ are valid choices. Since X is non-trivial, $\mathrm{id}_{\Sigma X}$ is an isomorphism while $0_{\Sigma X}$ is not and so, by Lemma 1.3, $\{\mathrm{id}_{\Sigma X}\}^{\perp\!\!\!\!\perp} = \mathbf{D^2} \neq \{0_{\Sigma X}\}^{\perp\!\!\!\!\perp}$.

In order to avoid pathologies like the one in the above example, we now recall the notion of middling good morphism of triangles; this notion will allow us to describe properties like "closure under homotopy colimits" in the correct way (see Lemma 1.12). Indeed, recall from [19] that a morphism of triangles

$$\begin{tikzcd} A_0 \arrow[r, "\phi_0"] \arrow[d, "a"'] & B_0 \arrow[r, "\psi_0"] \arrow[d, "b"'] & C_0 \arrow[r] \arrow[d, "c"] & \Sigma A_0 \arrow[d, "\Sigma a"] \\ A_1 \arrow[r, "\phi_1"] & B_1 \arrow[r, "\psi_1"] & C_1 \arrow[r] & \Sigma A_1 \end{tikzcd} \tag{1.4}$$

is said to be *middling good* if it can be completed to a 3×3 diagram whose rows and columns are triangles and where everything commutes but the lower right square, which anti-commutes:

$$\begin{tikzcd} A_0 \arrow[r, "\phi_0"] \arrow[d, "a"'] & B_0 \arrow[r, "\psi_0"] \arrow[d, "b"'] & C_0 \arrow[r] \arrow[d, "c"] & \Sigma A_0 \arrow[d, "\Sigma a"] \\ A_1 \arrow[r, "\phi_1"] \arrow[d, "\alpha_a"'] & B_1 \arrow[r, "\psi_1"] \arrow[d, "\alpha_b"'] & C_1 \arrow[r] \arrow[d, "\alpha_c"] & \Sigma A_1 \arrow[d] \\ C_a \arrow[r, "\varphi_a"] \arrow[d, "\beta_a"'] & C_b \arrow[r, "\varphi_b"] \arrow[d, "\beta_b"'] & C_c \arrow[r] \arrow[d, "\beta_c"] & \Sigma C_a \arrow[d] \\ \Sigma A_0 \arrow[r] & \Sigma B_0 \arrow[r] & \Sigma C_0 \arrow[r] & \Sigma^2 C_0 \end{tikzcd} \tag{1.5}$$

Let us recall that, given a morphism $(a, b)\colon \phi_0 \to \phi_1$ in $\mathbf{D^2}$, one can always choose a morphism $c\colon C_0 \to C_1$ such that (a, b, c) is a middling good morphism of triangles. Let us remark that it is not clear to us whether or not, given two middling good morphisms of triangles (a, b, c) and (a, b, c'), the classes $\{c\}^{\overset{\perp}{\approx}}$ and $\{c'\}^{\overset{\perp}{\approx}}$ (resp., ${}^{\overset{\perp}{\approx}}\{c\}$ and ${}^{\overset{\perp}{\approx}}\{c'\}$) are equal.

Lemma 1.11. *Let $(\chi\colon Y_0 \to Y_1) \in \mathbf{D^2}$ and consider a middling good morphism of triangles as in (1.4). If a, Σa, c, Σc, $\Sigma^{-1}c \mathrel{\overset{\perp}{\approx}} \chi$, then $b \mathrel{\overset{\perp}{\approx}} \chi$.*

Proof. By Lemma 1.9, a, $\Sigma a \mathrel{\overset{\perp}{\approx}} \chi$ implies $\mathbf{D}(C_a, \Sigma^{-1}C_\chi) = 0$, while c, $\Sigma c \mathrel{\overset{\perp}{\approx}} \chi$ implies $\mathbf{D}(C_c, \Sigma^{-1}C_\chi) = 0$. Hence, $\mathbf{D}(C_b, \Sigma^{-1}C_\chi) = 0$. On the other hand, for a morphism $(d, e)\colon b \to \chi$, we get a commutative diagram whose columns are triangles:

$$\begin{tikzcd}
A_0 \arrow[r, "\phi_0"] \arrow[d, "a"'] & B_0 \arrow[r, "d"] \arrow[d, "b"'] & Y_0 \arrow[d, "\chi"] \\
A_1 \arrow[r, "\phi_1"] \arrow[d, "\alpha_a"'] & B_1 \arrow[r, "e"] \arrow[d, "\alpha_b"'] & Y_1 \arrow[d, "\alpha_\chi"] \\
C_a \arrow[r, "\varphi_a"] \arrow[d, "\beta_a"'] & C_b \arrow[r, "\varphi"] \arrow[d, "\beta_b"'] & C_\chi \arrow[d, "\beta_\chi"] \\
\Sigma A_0 \arrow[r] & \Sigma B_0 \arrow[r] & \Sigma Y_0
\end{tikzcd}$$

Since $a \mathrel{\overset{\perp}{\approx}} \psi$, then $\varphi\varphi_a = 0$, which implies that there exists $f\colon C_c \to C_\chi$ such that $f \circ \varphi_b = \varphi$. By c, $\Sigma^{-1}c \mathrel{\overset{\perp}{\approx}} \chi$ we get $\mathbf{D}(C_c, C_\chi) = 0$, so $f = 0$, which implies $\varphi = 0$. $\square$

Lemma 1.12. *Let $\psi \in \mathbf{D^2}$ and consider two countable chains of morphisms $A_\bullet = \{A_0 \xrightarrow{j_0} A_1 \xrightarrow{j_1} A_2 \xrightarrow{j_2} \dots\}$ and $B_\bullet = \{B_0 \xrightarrow{k_0} B_1 \xrightarrow{k_1} B_2 \xrightarrow{k_2} \dots\}$. If there is a natural transformation $\alpha\colon A_\bullet \Rightarrow B_\bullet$ such that α_i, $\Sigma\alpha_i$, $\Sigma^2\alpha_i \mathrel{\overset{\perp}{\approx}} \psi$ for all $i \in \mathbb{N}$, then any map φ: hocolim $A_\bullet \to$ hocolim $B_\bullet$ completing the following diagram to a middling good map of triangles is such that $\varphi \mathrel{\overset{\perp}{\approx}} \psi$*

$$\begin{tikzcd}
\coprod_{i\in\mathbb{N}} A_i \arrow[r] \arrow[d] & \coprod_{i\in\mathbb{N}} A_i \arrow[r] \arrow[d] & \mathit{hocolim}\, A_\bullet \arrow[r] \arrow[d, dotted, "\varphi"] & + \\
\coprod_{i\in\mathbb{N}} B_i \arrow[r] & \coprod_{i\in\mathbb{N}} B_i \arrow[r] & \mathit{hocolim}\, B_\bullet \arrow[r] & +
\end{tikzcd}$$

Proof. By Lemma 1.6, $\coprod_{i\in\mathbb{N}} \alpha_i$, $\Sigma \coprod_{i\in\mathbb{N}} \alpha_i$, $\Sigma^2 \coprod_{i\in\mathbb{N}} \alpha_i \mathrel{\overset{\perp}{\approx}} \psi$, so it is enough to apply Lemma 1.11. $\square$

1.2. Triangulated factorization systems

Using the notion of homotopy orthogonality we can define triangulated factorization systems as follows:

Definition 1.13. Let $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ be a pair of classes of morphisms in $\mathbf{D}$.

(1) $\mathfrak{F}$ is a *triangulated pre-factorization system* ($\triangle$PFS for short) if
   - $\mathcal{E}^{\pitchfork} = \mathcal{M}$ and ${}^{\pitchfork}\mathcal{M} = \mathcal{E}$;
   - $\phi \in \mathcal{E}$ implies $\Sigma\phi \in \mathcal{E}$.

(2) $\mathfrak{F}$ is a *triangulated factorization system* ($\triangle$FS for short) if it is a $\triangle$PFS, and if any morphism in $\mathbf{D}$ is $\mathfrak{F}$-*crumbled*, i.e. it can be factored as a composition $\phi = m \circ e$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$.

Notice that in the second condition defining a $\triangle$PFS we could have equivalently asked that $\phi \in \mathcal{M}$ implies $\Sigma^{-1}\phi \in \mathcal{M}$.

Remark 1.14 *(Left- and right-generated $\triangle$PFS).* It is evident that a class of morphism $\mathcal{X} \subseteq \mathbf{D^2}$ which is closed under negative shifts (i.e., $x \in \mathcal{X}$ implies $\Sigma^{-1}x \in \mathcal{X}$) induces a $\triangle$PFS $({}^{\pitchfork}\mathcal{X}, ({}^{\pitchfork}\mathcal{X})^{\pitchfork})$ on $\mathbf{D}$. Dually, if $\mathcal{X}$ is closed under shifts, then $({}^{\pitchfork}(\mathcal{X}^{\pitchfork}), \mathcal{X}^{\pitchfork})$ is a $\triangle$PFS.

By the properties proved in Section 1.1 we obtain the following closure properties for the classes composing a $\triangle$PFS:

Proposition 1.15. *Let $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ be a $\triangle$PFS. Then*

(1) *$\mathcal{E}$ and $\mathcal{M}$ are closed under isomorphisms in $\mathbf{D^2}$;*
(2) *$\mathcal{E} \cap \mathcal{M}$ is the class of all isomorphisms;*
(3) *$\mathcal{E}$ is closed under arbitrary coproducts and $\mathcal{M}$ is closed under arbitrary products;*
(4) *$\mathcal{E}$ and $\mathcal{M}$ are closed under retracts;*
(5) *$\mathcal{E}$ is closed under homotopy pushouts and $\mathcal{M}$ is closed under homotopy pullbacks;*
(6) *$\mathcal{E}$ is closed under homotopy colimits in the sense that, in the same setting of Lemma 1.12, if $\alpha_i \in \mathcal{E}$ for any $i \in \mathbb{N}$, then $\varphi \in \mathcal{E}$. A dual property regarding homotopy limits holds for $\mathcal{M}$.*

The following two definitions are of capital importance for us, as they determine the class of factorization systems we are interested in:

Definition 1.16 *(Triangulated torsion theory).* A $\triangle$FS $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ is said to be a *triangulated torsion theory* (for short, $\triangle$TTH) if both $\mathcal{E}$ and $\mathcal{M}$ are 3-for-2 classes.

Definition 1.17 *(Normal* $\triangle$FS*).* Let $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ be a $\triangle$FS in $\mathbf{D}$. We say that $\mathfrak{F}$ is *normal* if, whenever we have a factorization of a final map $X \to 0$ as follows

$$X \xrightarrow{e} T \xrightarrow{m} 0 \quad \text{with} \quad e \in \mathcal{E},\ m \in \mathcal{M}\,,$$

and a triangle of the form $R \to X \xrightarrow{e} T \to \Sigma R$, the map $(R \to 0)$ belongs to $\mathcal{E}$.

2. The triangulated Rosetta stone

As in Section 1, let us fix throughout this section a triangulated category $\mathbf{D}$ with shift functor $\Sigma\colon \mathbf{D} \xrightarrow{\simeq} \mathbf{D}$.

Definition 2.1. Recall that a *t-structure* in $\mathbf{D}$ is a pair $\mathfrak{t} = (\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ of full subcategories of $\mathbf{D}$ that satisfy the following properties, where $\mathbf{D}^{\le n} := \Sigma^{-n}\mathbf{D}^{\le 0}$ and $\mathbf{D}^{\ge n} := \Sigma^{-n}\mathbf{D}^{\ge 0}$, for any $n \in \mathbb{Z}$:

t1) $\mathbf{D}(X, Y) = 0$ for any $X \in \mathbf{D}^{\le 0}$ and $Y \in \mathbf{D}^{\ge 1}$;
t2) $\mathbf{D}^{\le -1} \subseteq \mathbf{D}^{\le 0}$ and $\mathbf{D}^{\ge 1} \subseteq \mathbf{D}^{\ge 0}$;
t3) for any $X \in \mathbf{D}$ there is a distinguished triangle

$$X^{\le 0} \to X \to X^{\ge 1} \to \Sigma X^{\le 0},$$

with $X^{\le 0} \in \mathbf{D}^{\le 0}$ and $X^{\ge 1} \in \mathbf{D}^{\ge 1}$.

Given a t-structure $\mathfrak{t} = (\mathbf{D}^{\le 0}, \mathbf{D}^{\ge 0})$ in $\mathbf{D}$, one obtains two functors

$$\tau^{\le 0}\colon \mathbf{D} \to \mathbf{D}^{\le 0} \quad \text{and} \quad \tau^{\ge 1}\colon \mathbf{D} \to \mathbf{D}^{\ge 1},$$

that are respectively the right adjoint to the inclusion $\mathbf{D}^{\le 0} \to \mathbf{D}$ and the left adjoint to the inclusion $\mathbf{D}^{\ge 1} \to \mathbf{D}$.

Notation 2.2. For an object $X \in \mathbf{D}$ we will generally write $X^{\le 0}$ for $\tau^{\le 0}X$ and $X^{\ge 1}$ for $\tau^{\ge 1}X$. Furthermore, we will generally denote the unit of the co-reflection $\tau^{\le 0}$ and the co-unit of the reflection $\tau^{\ge 1}$ by the following symbols:

$$X^{\le 0} \xrightarrow{\sigma_X} X \xrightarrow{\rho_X} X^{\ge 1}.$$

For any $n \in \mathbb{Z}$, we let $\tau^{\le n} := \Sigma^{-n}\tau^{\le 0}\Sigma^n$ and $\tau^{\ge n} := \Sigma^{-n}\tau^{\ge 0}\Sigma^n$. We adopt similar notational conventions for these shifted functors.

Remark 2.3. We can equally define a t-structure as a single full additive subcategory $\mathfrak{t} \subseteq \mathbf{D}$ such that

- $\Sigma\mathfrak{t} \subseteq \mathfrak{t}$;
- each object $X \in \mathbf{D}$ fits into a distinguished triangle $X_\mathfrak{t} \to X \to X_{\mathfrak{t}^\perp} \to \Sigma X_\mathfrak{t}$ such that $X_\mathfrak{t} \in \mathfrak{t}$, $X_{\mathfrak{t}^\perp} \in \mathfrak{t}^\perp = \{Y \mid \mathbf{D}(X,Y) = 0,\ \forall X \in \mathfrak{t}\}$.

This equivalent description of t-structures calls $\mathfrak{t}$ an *aisle* and $\mathfrak{t}^\perp$ a *coaisle*. We will usually blur the distinction between a t-structure and its aisle, since the correspondence between the two is obviously bijective under $\mathbf{D}^{\leq 0} \leftrightarrows$ aisle.

2.1. The induced $\triangle$FS of a t-structure

Fix a t-structure $\mathfrak{t} = (\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ in $\mathbf{D}$, and consider the following two classes of morphism

$$\begin{aligned} \mathcal{E}_\mathfrak{t} &:= \{\phi \in \mathbf{D}^\mathbf{2} \mid \tau^{\geq 1}\phi \text{ is an iso}\} \\ \mathcal{M}_\mathfrak{t} &:= \{\psi \in \mathbf{D}^\mathbf{2} \mid \tau^{\leq 0}\psi \text{ is an iso}\}. \end{aligned} \tag{2.1}$$

This subsection is devoted to the proof of the fact that $\mathfrak{F}_\mathfrak{t} := (\mathcal{E}_\mathfrak{t}, \mathcal{M}_\mathfrak{t})$ is a $\triangle$FS.

Lemma 2.4 *(Cartesian characterization of $\mathfrak{F}_\mathfrak{t}$). In the above setting, a morphism $(\phi\colon X \to Y) \in \mathbf{D}^\mathbf{2}$ belongs to $\mathcal{E}_\mathfrak{t}$ if and only if the square*

$$\begin{tikzcd} X^{\leq 0} \arrow[r] \arrow[d, "\phi^{\leq 0}"'] & X \arrow[d, "\phi"] \\ Y^{\leq 0} \arrow[r] & Y \end{tikzcd} \tag{2.2}$$

is homotopy cartesian. Thus, if $\phi \in \mathcal{E}_\mathfrak{t}$, the cone of ϕ belongs to $\mathbf{D}^{\leq 0}$. Dually, $(\psi\colon X \to Y) \in \mathbf{D}^\mathbf{2}$ belongs to $\mathcal{M}_\mathfrak{t}$ if and only if the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi"'] & X^{\geq 1} \arrow[d, "\psi^{\geq 1}"] \\ Y \arrow[r] & Y^{\geq 1} \end{tikzcd}$$

is homotopy cartesian. Thus, if $\psi \in \mathcal{M}_\mathfrak{t}$, the cone of ψ belongs to $\mathbf{D}^{\geq 0}$.

Proof. Suppose first that $\phi \in \mathcal{E}_\mathfrak{t}$. By [20, Remark 1.3.15], the square in (2.2) can be completed to a good morphism of triangles

$$\begin{tikzcd} X^{\leq 0} \arrow[r] \arrow[d, "\phi^{\leq 0}"'] & X \arrow[r] \arrow[d, "\phi"] & X^{\geq 1} \arrow[r] \arrow[d, dotted] & \Sigma X \arrow[d] \\ Y^{\leq 0} \arrow[r] & Y \arrow[r] & Y^{\geq 1} \arrow[r] & \Sigma Y \end{tikzcd}$$

while by [1, Prop. 1.1.9], the unique map completing the above square to a morphism of triangles is $\tau^{\geq 1}\phi$. Thus, we get that the following candidate triangle is in fact a triangle

$$X^{\leq 0} \oplus \Sigma^{-1}Y^{\geq 1} \to X \oplus Y^{\leq 0} \to X^{\geq 1} \oplus Y \to \Sigma X \oplus Y^{\geq 1}.$$

The above triangle is the direct sum of the following candidate triangles (see [20, Lemma 1.2.4])

$$\Sigma^{-1}Y^{\geq 1} \to 0 \to X^{\geq 1} \tilde{\to} Y^{\geq 1} \quad \text{and} \quad X^{\leq 0} \to X \oplus Y^{\leq 0} \to Y \to \Sigma X,$$

showing that the candidate triangle on the right-hand-side is a distinguished triangle (as it is a summand of a distinguished triangle). The existence of such a triangle means exactly that the square in (2.2) is homotopy cartesian.

On the other hand, suppose the square in (2.2) is homotopy cartesian. By [20, Remark 1.4.5], this can be completed to a good morphism of triangles

$$\begin{tikzcd}
X^{\leq 0} \arrow[r] \arrow[d, "\phi^{\leq 0}"'] & X \arrow[r] \arrow[d, "\phi"] & X^{\geq 1} \arrow[r] \arrow[d, dotted, "\cong"] & \Sigma X \arrow[d] \\
Y^{\leq 0} \arrow[r] & Y \arrow[r] & Y^{\geq 1} \arrow[r] & \Sigma Y
\end{tikzcd}$$

Invoking again [1, Prop. 1.1.9], we obtain that $\tau^{\geq 1}\phi$ is an iso. $\square$

Lemma 2.5. *Consider a homotopy cartesian square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\psi"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

If $\phi^{\geq 0}$ is an isomorphism, then $\psi^{\geq 0}$ is an isomorphism. Dually, if $\psi^{\leq 0}$ is an isomorphism, then $\phi^{\leq 0}$ is an isomorphism. In other words, $\mathcal{E}_{\mathrm{t}}$ is closed under homotopy pushouts and $\mathcal{M}_{\mathrm{t}}$ is closed under homotopy pullbacks.

Proof. Suppose first that $\phi^{\geq 0}$ is an isomorphism. This means that $\mathbf{D}^{\geq 0}(\phi^{\geq 0}, B)$ is an isomorphism for any $B \in \mathbf{D}^{\geq 0}$ or, equivalently, $\mathbf{D}(\phi, B)$ is an isomorphism for any $B \in \mathbf{D}^{\geq 0}$. We have to show the same property holds for ψ. Consider the following morphism of triangles:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d, "\psi"] & \Sigma Z \arrow[d, equal] \\
Z \arrow[r] & X' \arrow[r] & Y' \arrow[r] & \Sigma Z
\end{tikzcd}$$

For any given $B \in \mathbf{D}^{\geq 0}$, we obtain a morphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{D}(\Sigma X', B) \arrow[r] \arrow[d, "\cong"] & \mathbf{D}(\Sigma Z, B) \arrow[r] \arrow[d, equal] & \mathbf{D}(Y', B) \arrow[r] \arrow[d] & \mathbf{D}(X', B) \arrow[r] \arrow[d, "\cong"] & \mathbf{D}(Z, B) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathbf{D}(\Sigma X, B) \arrow[r] & \mathbf{D}(\Sigma Z, B) \arrow[r] & \mathbf{D}(Y, B) \arrow[r] & \mathbf{D}(X, B) \arrow[r] & \mathbf{D}(Z, B) \arrow[r] & \cdots
\end{tikzcd}$$

where $\mathbf{D}(\Sigma\phi, B)$ is an isomorphism because $\mathbf{D}(\phi, \Sigma^{-1}B)$ is an isomorphism, since $\Sigma^{-1}B \in \mathbf{D}^{\geq 1} \subseteq \mathbf{D}^{\geq 0}$. Now, by the Five Lemma we obtain that $\mathbf{D}(\psi, B)$ is an isomorphism for any $B \in \mathbf{D}^{\geq 0}$, that is, $\psi^{\geq 0}$ is an isomorphism. The proof of the second part of the statement is dual. $\square$

Lemma 2.6. *Any morphism in* $\mathbf{D}$ *is* $\mathfrak{F}_{\mathfrak{t}}$*-crumbled.*

Proof. Take a map $\phi\colon X \to Y$ in $\mathbf{D}$, and let us prove that ϕ is $\mathfrak{F}_{\mathfrak{t}}$-crumbled. Let us start taking a homotopy pullback of the maps $\phi^{\geq 1}$ and ρ_Y:

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[d, dotted, "\phi_m"'] \arrow[dr, phantom, "\square"] & X^{\geq 1} \arrow[d, "\phi^{\geq 1}"] \\
Y \arrow[r, "\rho_Y"'] & Y^{\geq 1}
\end{tikzcd}$$

By Lemma 2.4, $\phi_m \in \mathcal{M}_{\mathfrak{t}}$. Consider also the following commutative solid diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\exists\phi_e"] \arrow[drr, bend left, "\rho_X"] \arrow[ddr, bend right, "\phi"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\geq 1} \arrow[d, "\phi^{\geq 1}"] \\
& Y \arrow[r, "\rho_Y"'] & Y^{\geq 1}
\end{tikzcd}$$

Then there exists a (non-unique, see [20, p. 54]) map $\phi_e\colon X \to P$ that makes the diagram commute. Finally consider the following diagram, where the dotted arrow is obtained completing to a good map of triangles:

$$\begin{tikzcd}
X^{\leq 0} \arrow[r, "\sigma_X"] \arrow[d, dotted] & X \arrow[r, "\rho_X"] \arrow[d, "\phi_e"] & X^{\geq 1} \arrow[r] \arrow[d, equal] & \Sigma X^{\leq 0} \arrow[d] \\
Y^{\leq 0} \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\phi_m"] \arrow[dr, phantom, "\square"] & X^{\geq 1} \arrow[r] \arrow[d, "\phi^{\geq 1}"] & \Sigma Y^{\leq 0} \arrow[d, equal] \\
Y^{\leq 0} \arrow[r, "\sigma_Y"'] & Y \arrow[r, "\rho_Y"'] & Y^{\geq 1} \arrow[r] & \Sigma Y^{\leq 0}
\end{tikzcd}$$

By construction $\phi = \phi_m \phi_e$. It remains to show that $\phi_e \in \mathcal{E}_\mathfrak{t}$. By Lemma 2.4, we have to verify that the top left square is homotopy cartesian. Indeed, take the following mapping cone, which is distinguished since we took a good morphism of triangles in our construction:

$$X \oplus Y^{\leq 0} \to P \oplus X^{\geq 1} \to X^{\geq 1} \oplus \Sigma X^{\leq 0} \to \Sigma X \oplus \Sigma Y^{\leq 0}.$$

This triangle is the direct sum of the following two candidate triangles (see [20, Lemma 1.2.4]):

$$0 \to X^{\geq 1} \to X^{\geq 1} \to 0,$$
$$X \oplus Y^{\leq 0} \to P \to \Sigma X^{\leq 0} \to \Sigma X \oplus \Sigma Y^{\leq 0},$$

showing that $X^{\leq 0} \to X \oplus Y^{\leq 0} \to P \to \Sigma X^{\leq 0}$ is distinguished. □

Lemma 2.7. *Given $e \in \mathcal{E}_\mathfrak{t}$ and $m \in \mathcal{M}_\mathfrak{t}$, we have $e \pitchfork m$.*

Proof. Complete e and m to triangles as follows:

$$E_0 \xrightarrow{e} E_1 \xrightarrow{\alpha_e} C_e \xrightarrow{\beta_e} \Sigma E_0 \quad M_0 \xrightarrow{m} M_1 \xrightarrow{\alpha_m} C_m \xrightarrow{\beta_m} \Sigma M_0,$$

By Lemma 2.4, there are morphisms of triangles, with $\phi = e^{\leq 0}$ and $\psi = m^{\geq 1}$,

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & E_0 \arrow[d, "e"] \\
X_1 \arrow[r] \arrow[d, "\alpha'_e"'] & E_1 \arrow[d, "\alpha_e"] \\
C_e \arrow[r, equal] \arrow[d, "\beta'_e"'] & C_e \arrow[d, "\beta_e"] \\
\Sigma X_0 \arrow[r] & \Sigma E_0
\end{tikzcd}
\qquad
\begin{tikzcd}
M_0 \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\square"] & Y_0 \arrow[d, "\psi"] \\
M_1 \arrow[r] \arrow[d, "\alpha_m"'] & Y_1 \arrow[d, "\alpha'_m"] \\
C_m \arrow[r, equal] \arrow[d, "\beta_m"'] & C_m \arrow[d, "\beta'_m"] \\
\Sigma M_0 \arrow[r] & \Sigma Y_0
\end{tikzcd}$$

where $X_0, X_1 \in \mathbf{D}^{\leq 0}$ and $Y_0, Y_1 \in \mathbf{D}^{\geq 1}$. Using the closure properties of $\mathbf{D}^{\leq 0}$ and $\mathbf{D}^{\geq 1}$, one can show that $C_e \in \mathbf{D}^{\leq 0}$ and $\Sigma^{-1}C_m \in \mathbf{D}^{\geq 1}$. Thus, $\mathbf{D}(C_e, \Sigma^{-1}C_m) = 0$ by condition 2.1.t1), giving us 1.1.HO1. It remains to verify condition 1.1.HO2, that is, suppose we have a map $f\colon C_e \to C_m$ whose image in $\mathbf{D}(E_1, \Sigma M_0)$ is trivial and let us prove that $f = 0$. Indeed, we know that $\beta_m f \alpha_e = 0$, so also $\beta'_m f \alpha'_e = 0$ and thus we can find a morphism of triangles as follows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X_1 \arrow[r, equal] \arrow[d, dotted, "f_1"] & X_1 \arrow[r] \arrow[d, "f\alpha'_e"] & 0 \arrow[d] \\
Y_0 \arrow[r] & Y_1 \arrow[r, "\alpha'_m"] & C_m \arrow[r, "\beta'_m"] & \Sigma Y_0
\end{tikzcd}$$

showing that $f\alpha'_e = \alpha'_m f_1$ for some $f_1\colon X_1 \to Y_1$. But $\mathbf{D}(X_1, Y_1) = 0$ by 2.1.t1), so $f_1 = 0$, showing that $f\alpha'_e = 0$. Hence, we can find a morphism of triangles as follows

$$\begin{tikzcd}
X_0 \arrow[r, "e"] \arrow[d] & X_1 \arrow[r, "\alpha'_e"] \arrow[d] & C_e \arrow[r, "\beta'_e"] \arrow[d, "f"] & \Sigma X_0 \arrow[d, dotted, "f_2"] \\
\Sigma^{-1}C_m \arrow[r] & 0 \arrow[r] & C_m \arrow[r, equal] & C_m
\end{tikzcd}$$

showing that $f = f_2\beta'_e$, for some $f_2\colon \Sigma X_0 \to C_m$. Now, since $\Sigma X_0 \in \mathbf{D}^{\leq -1}$ and $C_m \in \mathbf{D}^{\geq 0}$, $f_2 = 0$ and so also $f = 0$, as desired. $\square$

Proposition 2.8. *The pair of sub categories $\mathfrak{F}_{\mathfrak{t}} = (\mathcal{E}_{\mathfrak{t}}, \mathcal{M}_{\mathfrak{t}})$ defines a* $\triangle$FS.

Proof. We have already seen that any morphism is $\mathfrak{F}_{\mathfrak{t}}$-crumbled and that $\mathcal{E}_{\mathfrak{t}} \subseteq {}^{\not\perp}\mathcal{M}_{\mathfrak{t}}$. Let us show the converse inclusion. Indeed, let $(\phi\colon X \to Y) \in {}^{\not\perp}\mathcal{M}_{\mathfrak{t}}$ and choose a factorization $\phi = \phi_m\phi_e$ with $\phi_e \in \mathcal{E}_{\mathfrak{t}}$ and $\phi_m \in \mathcal{M}_{\mathfrak{t}}$. By the usual 3×3-lemma in triangulated categories, we can complete the commutative square

$$\begin{tikzcd}
X \arrow[r, "\phi_e"] \arrow[d, "\phi"'] & p \arrow[d, "\phi_m"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

to a diagram where all the rows and columns are distinguished triangles, and where everything commutes but the top left square, that anti-commutes:

$$\begin{tikzcd}
\Sigma^{-1}C_e \arrow[r] \arrow[d, equal] & \Sigma^{-1}C_\phi \arrow[r] \arrow[d] & \Sigma^{-1}C_m \arrow[r] \arrow[d] & C_e \arrow[d, equal] \\
\Sigma^{-1}C_e \arrow[r] \arrow[d] & X \arrow[r, "\phi_e"] \arrow[d, "\phi"'] & P \arrow[r] \arrow[d, "\phi_m"] & C_e \arrow[d] \\
0 \arrow[r] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
C_e \arrow[r] & C_\phi \arrow[r] & C_m \arrow[r] & \Sigma C_e
\end{tikzcd}$$

Now, since $\phi \in {}^{\perp}\mathcal{M}_{\mathfrak{t}}$, it follows by 1.1.HO2' that the map $C_\phi \to C_m$ in the above diagram is the trivial map. Thus, $\Sigma C_e \cong C_m \oplus \Sigma C_\phi$, in particular C_m is a summand of $\Sigma C_e \in \Sigma \mathbf{D}^{\leq 0} = \mathbf{D}^{\leq -1}$. Hence, $C_m \in \mathbf{D}^{\leq -1} \cap \mathbf{D}^{\geq 0} = 0$, showing that ϕ_m is an isomorphism, so that $\phi \cong \phi_e \in \mathcal{E}_{\mathfrak{t}}$. □

2.2. t-structures are normal $\triangle$TTH

We now concentrate on showing how each t-structure on $\mathbf{D}$ naturally induces a $\triangle$TTH and vice-versa; the basic idea is to mimic the proof of [13, Thm. 3.1.1] tailoring the argument to the triangulated setting.

Lemma 2.9. *$\mathfrak{F}_{\mathfrak{t}} = (\mathcal{E}_{\mathfrak{t}}, \mathcal{M}_{\mathfrak{t}})$ is a normal* $\triangle$TTH.

Proof. We have already proved that $\mathfrak{F}_{\mathfrak{t}}$ is a $\triangle$FS, while the fact that $\mathcal{E}_{\mathfrak{t}}$ and $\mathcal{M}_{\mathfrak{t}}$ are 3-for-2 classes is a trivial consequence of their definition, as they are the pre-image (under $\tau^{\geq 1}$ and $\tau^{\leq 0}$, respectively) of the class of all isomorphisms, which is a 3-for-2 class. It remains to show that $\mathfrak{F}_{\mathfrak{t}}$ is normal. Consider a factorization of a final map $X \to 0$ as follows

$$X \xrightarrow{e} T \xrightarrow{m} 0 \quad \text{with} \quad e \in \mathcal{E}_{\mathfrak{t}},\ m \in \mathcal{M}_{\mathfrak{t}},$$

and a triangle of the form $R \to X \xrightarrow{e} T \to \Sigma R$. We should prove that the map $(R \to 0)$ belongs to $\mathcal{E}_{\mathfrak{t}}$, that is, that $R \in \mathbf{D}^{\leq 0}$. By Lemma 2.4, $T \in \mathbf{D}^{\geq 1}$. Since $e \in \mathcal{E}_{\mathfrak{t}}$ and using Lemma 2.4, we can construct a commutative diagram as follows:

$$\begin{tikzcd}
X^{\leq 0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d, "e"] & X^{\geq 1} \arrow[r] \arrow[d, "\cong"] & \Sigma X^{\leq 0} \arrow[d] \\
T^{\leq 0} \arrow[r] & T \arrow[r] & T^{\geq 1} \arrow[r] & \Sigma T^{\leq 0}
\end{tikzcd}$$

Since $T \in \mathbf{D}^{\geq 1}$, we get $T^{\leq 0} = 0$ and $T \cong T^{\geq 1} \cong X^{\geq 1}$, so the fact that the square on the left-hand-side in the above diagram is homotopy cartesian provides us with a distinguished triangle of the form

$$X^{\leq 0} \to X \to T \to \Sigma X^{\leq 0}.$$

In particular, $R \cong X^{\leq 0} \in \mathbf{D}^{\leq 0}$ as desired. □

Lemma 2.10. *For a normal* $\triangle$TTH $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ *in* $\mathbf{D}$, $\mathrm{t}_{\mathfrak{F}} := (0/\mathcal{E}, \Sigma(\mathcal{M}/0))$ *is a t-structure. Furthermore, given* $X \in \mathbf{D}$ *and taking an* $\mathfrak{F}$*-factorization* $X \xrightarrow{e} T \xrightarrow{m} 0$ *of the final map* $X \to 0$*, we have that* $T \in \mathcal{M}/0$ *and* $e \colon X \to T$ *is the reflection of* X *into* $\mathcal{M}/0$ *(the coreflection of an object into* $0/\mathcal{E}$ *is constructed dually).*

Proof. We verify the three axioms of a t-structure:

- Let $X \in 0/\mathcal{E}$ and $Y \in \mathcal{M}/0$, we have to show that $\mathbf{D}(X, Y) = 0$. Indeed, let $\varphi \colon X \to Y$ and consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r, "\varphi"] \arrow[d, equal] & Y \arrow[d, equal] \\ X \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

  Notice that $(0 \to X) \in \mathcal{E}$. Furthermore, $0 \to 0$ is an isomorphism so it belongs to $\mathcal{M}$, as well as $Y \to 0$; since $\mathcal{M}$ is a 2-for-3 class, this means that also $0 \to Y$ belongs to $\mathcal{M}$. By condition 1.1.HO2, we get $\varphi = 0$.
- Let $X \in 0/\mathcal{E}$. Reasoning as in verifying 2.1.t1) above, one can show that the 2-for-3 property of $\mathcal{E}$ implies that $X \to 0$ belongs to $\mathcal{E}$. Consider now the following homotopy cartesian square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  By Proposition 1.15, the map $0 \to \Sigma X$ belongs to $\mathcal{E}$, that is $\Sigma(0/\mathcal{E}) \subseteq 0/\mathcal{E}$. One verifies similarly that $\mathcal{M}/0 \subseteq \Sigma(\mathcal{M}/0)$.
- Let $X \in \mathbf{D}$, consider a factorization of the map $X \to 0$ as follows:

$$X \xrightarrow{e} T \xrightarrow{m} 0 \quad \text{with} \quad e \in \mathcal{E},\ m \in \mathcal{M}.$$

  Now we can complete the map e to a triangle to get

$$R \to X \xrightarrow{e} T \to \Sigma R.$$

By the normality of $\mathfrak{F}$, $R \in 0/\mathcal{E}$ and $T \in \mathcal{M}/0$. $\square$

Theorem 2.11 *(The triangulated Rosetta stone). Let* $\mathbf{D}$ *be a triangulated category, then there is a bijective correspondence*

$$\begin{array}{rcl} \Phi : \left\{ \begin{smallmatrix} \textit{normal triangulated} \\ \text{TTHs } \textit{on } \mathbf{D} \end{smallmatrix} \right\} & \longleftrightarrow & \left\{ \begin{smallmatrix} \textit{t-structures} \\ \textit{on } \mathbf{D} \end{smallmatrix} \right\} : \Psi \\ (\mathcal{E}, \mathcal{M}) & \longmapsto & \Big(0/\mathcal{E}, \Sigma(\mathcal{M}/0)\Big) \\ (\mathcal{E}_{\mathfrak{t}}, \mathcal{M}_{\mathfrak{t}}) & \longleftarrow\!\shortmid & \mathfrak{t}. \end{array}$$

Proof. We have already verified in the previous subsections that Φ and Ψ are well-defined. Consider now a t-structure $\mathfrak{t}$ and let us show that $\mathfrak{t} = \Phi\Psi\mathfrak{t}$, that is, we should verify that $\mathbf{D}^{\leq 0} = 0/\mathcal{E}_{\mathfrak{t}}$. But this is true since clearly $X \in \mathbf{D}^{\leq 0}$ if and only if $0 \to X$ belongs to $\mathcal{E}_{\mathfrak{t}}$, that is, $X \in 0/\mathcal{E}_{\mathfrak{t}}$.

On the other hand, let $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ and let us show that $\mathfrak{F} = \Psi\Phi\mathfrak{F}$. Let $\phi \in \mathcal{E}_{\mathfrak{t}_{\mathfrak{F}}}$, that is, $\phi^{\geq 1}$ is an isomorphism and consider the following commutative square:

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "\phi"'] & X^{\geq 1} \arrow[d, "\phi^{\geq 1}"] \\ Y \arrow[r, "\rho_Y"] & Y^{\geq 1} \end{tikzcd}$$

Notice that ρ_X and ρ_Y belong to $\mathcal{E}$ by Lemma 2.10. The composition $\rho_Y \phi = \phi^{\geq 1}\rho_X$ belongs to $\mathcal{E}$ since $\phi^{\geq 1} \in \mathcal{E}$ (as $\mathcal{E}$ contains any isomorphism) and we have already observed that $\rho_X \in \mathcal{E}$. For the 3-for-2 property this means that $\phi \in \mathcal{E}$. This shows that $\mathcal{E}_{\mathfrak{t}_{\mathfrak{F}}} \subseteq \mathcal{E}$. One proves in the exact same way that $\mathcal{M}_{\mathfrak{t}_{\mathfrak{F}}} \subseteq \mathcal{M}$, but these two conditions together mean that $\mathfrak{F} = \mathfrak{F}_{\mathfrak{t}_{\mathfrak{F}}}$, as desired. $\square$

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