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On the structure of groups with polynomial growth III

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ABSTRACT

We show that a compactly generated locally compact group of polynomial growth having no non-trivial compact normal subgroups can be embedded as a co-compact subgroup into a semidirect product of a connected, simply connected, nilpotent Lie group and a compact group. There is also a uniqueness statement for this extension.

0. Introduction and main results

Let G be a locally compact (l.c.), compactly generated group. λ denotes a Haar measure on G and V a compact neighbourhood of the identity e, generating G. The group G is said to be of *polynomial growth*, if there exists $d \in \mathbb{N}$ such that $\lambda(V^n) = O(n^d)$ for $n \in \mathbb{N}$. The group G is called *almost nilpotent*, if it has a nilpotent subgroup H such

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that G/H is compact. A classical result of Gromov [11] asserts that a finitely generated discrete group has polynomial growth if and only if it is almost nilpotent. Any almost nilpotent group has polynomial growth, but it is well known that the converse is no longer true in the non-discrete case (see [17] 1.4.3 for explicit examples). Nevertheless, it turns out that there are very close relations between the two classes and this will be the main object of the present paper.

If G is any compactly generated l.c. group of polynomial growth, it has a maximal compact normal subgroup C ([17] Prop. 1). Therefore, we will formulate the main theorems for groups having no non-trivial compact normal subgroups. G/C is always a Lie group ([16] Th. 2). Any compactly generated Lie group G of polynomial growth has a maximal nilpotent normal subgroup N, the (non-connected) nilradical of G, denoted by $N = \mathrm{nil}(G)$ ([17] Prop. 3). In the discrete case, this is called the Fitting subgroup ([23] p. 15).

Theorem 1. *Let G be a compactly generated l.c. group of polynomial growth having no non-trivial compact normal subgroups. $N = \mathrm{nil}(G)$ shall be its (non-connected) nilradical. Then there exists a closed subgroup L of G such that $G = NL$ and $L/\mathrm{nil}(L)$ is compact (in particular, L is almost nilpotent).*

For discrete polycyclic groups there is a similar result about nilpotent almost–supplements for the Fitting subgroup ([23] Sec. 3C; see our Remark 3.1 for further discussion).

Theorem 2. *Let G be a compactly generated l.c. group of polynomial growth having no non-trivial compact normal subgroup. Then G can be embedded as a closed subgroup into a semidirect product $\widetilde{G} = \widetilde{N} \rtimes K$ such that K is compact, $\widetilde{N}$ is a connected, simply connected nilpotent Lie group, K acts faithfully on $\widetilde{N}$ and $\widetilde{G}/G$ is compact.*

Then $\widetilde{G}$ is also a Lie group; but G need not be normal in $\widetilde{G}$ (see Example 4.12 (c), (f)). Thus, although G need not be almost nilpotent, it is always contained as a co-compact subgroup in an almost nilpotent (and almost connected) group $\widetilde{G}$. For G connected, this was shown in [2] Th. 3.6 (see also Remark 4.11 (b)).

It follows that any group G as in Theorem 2 has a faithful linear representation (Corollary 3.6), G is isomorphic to a distal linear group (as considered in [1]). $\widetilde{G}$ is isomorphic to a real-algebraic linear group which is (for $\widetilde{G}$ minimal) an algebraic hull of G in the sense of [20] Def. 4.39 (see Remark 4.11 (a) for further discussion).

It turns out that the minimal extensions $\widetilde{G}$ as above (or more specifically, with K chosen minimal) are determined uniquely up to isomorphism.

Theorem 3. *Let G, $\widetilde{G}$, $\widetilde{G}'$ be l.c. groups, $j : G \to \widetilde{G}$, $j' : G \to \widetilde{G}'$ shall be continuous, injective homomorphisms such that $j(G)$, $j'(G)$ are closed, $\widetilde{G}/j(G)$, $\widetilde{G}'/j'(G)$ compact, $\widetilde{G} = \widetilde{N} \rtimes K$, $\widetilde{G}' = \widetilde{N}' \rtimes K'$ with K, K' compact, $\widetilde{N}$, $\widetilde{N}'$ connected, simply connected*

nilpotent, $\widetilde{N}j(G)$ dense in $\widetilde{G}$ and K' acting faithfully on $\widetilde{N}'$.
Then there exists a unique continuous homomorphism $\Phi\colon \widetilde{G} \to \widetilde{G}'$ such that $\Phi \circ j = j'$. Φ is surjective iff $\widetilde{N}'j'(G)$ is dense in $\widetilde{G}'$. Φ is injective iff K acts faithfully on $\widetilde{N}$.

Thus, if $\widetilde{G}$, $\widetilde{G}'$ are given as in Theorem 2 and $\widetilde{N}j(G)$, $\widetilde{N}'j'(G)$ are both dense (which can always be attained by minimizing K, K'), then Φ is an isomorphism. Due to this uniqueness, we call a group $\widetilde{G}$ as in Theorem 2 with $\widetilde{N}G$ dense in $\widetilde{G}$, the *algebraic hull* of G and $\widetilde{N} = \mathrm{nil}(\widetilde{G})$ the *connected nil-shadow* of G.

Theorem 1 and 2 are based on the splitting techniques introduced by Malcev and developed further by Wang, Mostow and Auslander (see also [3]). We build upon [26] and extend it in Section 2 for our purpose (see 2.1, Remark 3.7 and Remarks 4.11 for further discussion). Section 3 contains the proofs of Theorem 1 and 2. In Section 4 the proof of Theorem 3 is given, based on various structural properties of subgroups of semidirect products like those appearing in Theorem 2. Examples 4.12 contains various examples for the algebraic hull and related objects.

1. Notations and auxiliary results

1.1. If $\mathcal{B}$ is a group acting on G by automorphisms, then G is said to be an $FC^-_{\mathcal{B}}$-group if the orbits $\{\alpha(x)\colon \alpha \in \mathcal{B}\}$ are relatively compact in G for all $x \in G$. For $\mathcal{B}$ the inner automorphisms, G is called an FC^--group. G is called a *generalized $\overline{FC}$-group* if there exists a series $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = (e)$ of closed normal subgroups of G such that G_i/G_{i+1} is an FC^--group and compactly generated for $i = 0, \ldots, n-1$ (see [17] 1.2.1 for further discussion and references). Any compactly generated group of polynomial growth is a generalized $\overline{FC}$-group. As worked out in [17], generalized $\overline{FC}$-groups have some nice algebraic properties, the class contains all discrete polycyclic groups, connected solvable groups and compact groups (thus it should allow unified formulations for some results of [20] Ch. III that are developed there separately for the discrete and the connected case). Conversely, every generalized $\overline{FC}$-group can be built up from members of these subclasses.

1.2. We refer to [4] and [27] for basic results on the algebraic theory of nilpotent groups. If G is a connected nilpotent Lie group, then G is simply connected iff it is torsion free ([25] Th. 3.6.1). If $\mathfrak{g}$ denotes the Lie algebra of G, then the exponential function $\exp\colon \mathfrak{g} \to G$ is always surjective and if G is simply connected, then exp defines a homeomorphism.

If N is any compactly generated, torsion free nilpotent group, it can always be embedded as a closed subgroup into a connected, simply connected nilpotent Lie group $N_{\mathbb{R}}$ such that $N_{\mathbb{R}}/N$ is compact (but N need not be normal). $N_{\mathbb{R}}$ is called the *(real) Malcev-completion* of G (see [4] Ch. 4, [27] Sec. 11, 12, [3] Ch. II and [20] Ch. II). $N_{\mathbb{R}}$ is determined uniquely up to isomorphism. If $\varphi\colon N \to G$ is any continuous homomorphism into a connected, simply connected nilpotent Lie group G, it has a unique extension $\varphi_{\mathbb{R}} : N_{\mathbb{R}} \to G$.

1.3. *Semidirect products:* $G = H \rtimes K$ means that H, K are closed subgroups of the l.c. group G, H normal, $G = HK$, $H \cap K = \{e\}$ ("internal product"). In most cases we will follow [15] (2.6) (i.e., the left factor is normal). The restrictions of the inner automorphisms define a continuous action of K on H, we write $k \circ h = k\,h\,k^{-1}$. Conversely, if l.c. groups H, K are given and a continuous action of K on H (i.e., a continuous homomorphism $K \to \mathrm{Aut}(H)$ - compare [13] III.3) one can define the ("external") semidirect product $H \rtimes K$ by considering the cartesian product $H \times K$ of the topological spaces with group multiplication $(h_1, k_1)\,(h_2, k_2) = (h_1(k_1 \circ h_2), k_1k_2)$. Then H is isomorphic to the closed normal subgroup $\{(h, e) : h \in H\}$, similarly for K. For σ-compact l.c. groups both viewpoints are equivalent (respectively, they lead to isomorphic groups) and we will not distinguish further on.

Next, we give a result on combining two group extensions ("pasting of two groups along a common subgroup"). This is probably known, but we could not find a reference.

Proposition 1.4. *Let G, H_1 be l.c. groups such that $H = G \cap H_1$ (with induced topology from G) is a closed normal subgroup of G and a subgroup of H_1 for which the inclusion $H \to H_1$ is continuous. Assume that G acts continuously on H_1 by automorphisms (see 1.3) such that $x \circ h = x\,h\,x^{-1}$ whenever $(x, h) \in (H \times H_1) \cup (G \times H)$.*
Then there exists a l.c. group G_1 and continuous homomorphisms $j : G \to G_1$, $j_1 : H_1 \to G_1$ such that $j_1(H_1)$ is a closed normal subgroup of G_1, $j = j_1$ on H, $G_1 = j_1(H_1)\,j(G)$, $j(H) = j_1(H_1) \cap j(G)$, $j_1(x \circ h) = j(x)\,j_1(h)\,j(x)^{-1}$ for all $x \in G$, $h \in H_1$. If H is closed in H_1, then $j(G)$ is closed in G_1.

If H is closed in H_1, and the topologies of G and H_1 coincide on H, it will result from the proof that j defines a topological isomorphism of G and $j(G)$, similarly for j_1. For this reason, we will skip j, j_1 in general and consider G, H_1 as subgroups of G_1. Then the properties amount to $G_1 = H_1\,G$, $H_1 \triangleleft G_1$, $x \circ h = x\,h\,x^{-1}$ for all $x \in G$, $h \in H_1$.

Proof. Put $G^* = H_1 \rtimes G$ (with respect to the given action, see 1.3) and $H^* = \{\,(x^{-1}, x) : x \in H\}$. Then by easy computations, it follows from the properties of the action that H^* is a closed normal subgroup of G^* (e.g., the subgroup property follows from $(x^{-1}, x)\,(y^{-1}, y) = (x^{-1}(x \circ y^{-1}), xy) = (y^{-1}x^{-1}, xy)$ for $x, y \in H$, since by assumption, $x \circ y^{-1} = x\,y^{-1}x^{-1}$ for $x \in H$). Put $G_1 = G^*/H^*$, $j(x) = (e, x)H^*$ for $x \in G$, $j_1(h) = (h, e)H^*$ for $h \in H_1$. This satisfies the properties stated above. $\square$

1.5. See [15] for basic properties of l.c. groups and [25] for Lie groups. e will always denote the unit element of a group G. $Z(G)$ stands for the centre of G, $\mathrm{Aut}(G)$ will denote the group of topological automorphisms of G with its standard topology (see [15] (26.3)). G^0 denotes the connected component of the identity.

2. The splitting technique

2.1. In this section, we use the following setting. G shall be a (not necessarily connected) Lie group whose topological commutator group $[G,G]^-$ is compactly generated, nilpotent, torsion free and such that G/G^0 is nilpotent (^{0} denoting the connected component of the identity). In particular, G is an extension of a nilpotent group by an abelian group and therefore solvable.

In addition, we consider a fixed closed subgroup $H \supseteq [G,G]^-$ such that H is compactly generated, nilpotent and torsion free. Then G/H^0 is nilpotent (observe that $G^0/[G,G]^{-0}$ is central in $G/[G,G]^{-0}$, hence $G/[G,G]^{-0}$ is nilpotent). In principle, the proofs could also be done without specifying such an H, but this approach makes it easier to use the results of [26]. If G is compactly generated and has no non-trivial compact normal subgroups, one can always take $H = N = \mathrm{nil}(G)$ (the nilradical - see 2.8), then $\mathrm{Aut}_H(G)$, defined below, coincides with $\mathrm{Aut}(G)$. In the terminology of [3], H is a CN-group. If H is connected and open in G and G/H is finitely generated, this coincides with the class of solvable groups G considered in [26] sec. 6 (contained in the class of $\mathcal{S}$-groups defined in [26] sec. 10). More generally, if H is connected and G/H compactly generated, one gets the ϵ-category of [24]. The main results will be Proposition 2.15 and Corollary 2.16 on existence of the splitting (containing [26] (10.2)) and Proposition 2.22 on uniqueness up to conjugacy.

$\mathrm{Aut}(G)$ will denote the group of topological automorphisms of G (with its standard topology, [15] (26.3)). For $x \in G$, $\iota_x(y) = xyx^{-1}$ denotes the corresponding inner automorphism of G, $\iota\colon G \to \mathrm{Aut}(G)$ is a homomorphism, $\iota_{\theta(x)} = \theta \circ \iota_x \circ \theta^{-1}$ for $\theta \in \mathrm{Aut}(G)$. As in [26] p. 2, we say that $\theta \in \mathrm{Aut}(G)$ is *unipotent*, if there exists an integer $n > 0$ such that $(\mathrm{ad}\,\theta)^n$ is the identity on G, where $(\mathrm{ad}\,\theta)(x) = \theta(x)\,x^{-1}$ (if G is connected, this is equivalent to the statement that $d\theta - \mathrm{id}$ is nilpotent on the Lie algebra of G – recall that G is solvable). H/H^0 is finitely generated, nilpotent and torsion free, G/H abelian. By well known results (compare [27] 9.3, 9.5) nilpotency of the group G/H^0 is equivalent to unipotency of the automorphisms of H/H^0 induced by ι_x ($x \in G$). We put $\mathrm{Aut}_H(G) = \{\theta \in \mathrm{Aut}(G) : H \text{ is } \theta\text{-invariant}\}$ ($\theta|H$ will denote the restriction of the mapping) and (extending [26] p. 8)

$$\mathrm{Aut}_1(G) = \{\theta \in \mathrm{Aut}_H(G) :\ \theta \text{ induces the identity on } G/H \\ \text{and a unipotent automorphism of } H/H^0\}\ .$$

Clearly, this depends on H, so we will sometimes write more precisely $\mathrm{Aut}_{1,H}(G)$. Note that if H is not connected, $\mathrm{Aut}_1(G)$ need not be a subgroup, but it is always $\mathrm{Aut}_H(G)$-invariant. The assumptions on G, H imply that $\iota_x \in \mathrm{Aut}_1(G)$ for all $x \in G$. For $\theta \in \mathrm{Aut}(G)$, we write $G_\theta = \{x \in G : \theta(x) = x\}$. If G is connected, we call $\theta \in \mathrm{Aut}(G)$ *semisimple* if the corresponding linear transformation $d\theta$ of the Lie algebra $\mathfrak{g}$ of G is semisimple (i.e., it diagonalizes after suitable extension of the base field). Recall that any $\theta \in \mathrm{Aut}(\mathbb{R}^n)$ has a unique decomposition $\theta = \theta_s \circ \theta_u = \theta_u \circ \theta_s$, where θ_s is semisimple, θ_u unipotent (multiplicative Jordan decomposition – see [5] VII, Th. 1, p. 42). θ_s is a polynomial of θ, hence any θ-invariant subspace is also θ_s-invariant. If θ is an automorphism of a Lie algebra, the same is true for θ_s, θ_u (an easy consequence of [6] VII, Prop. 12, p. 16). This carries over to automorphisms of connected, simply connected Lie groups.

If G, H are given as above, we can consider the Malcev completion $H_{\mathbb{R}}$ of H and by 1.2 and Proposition 1.4, we can consider G and $H_{\mathbb{R}}$ as closed subgroups of a (uniquely determined) Lie group $G_{\mathbb{R}}$ such that $H_{\mathbb{R}}$ is normal in $G_{\mathbb{R}}$, $G \cap H_{\mathbb{R}} = H$ and $G_{\mathbb{R}} = H_{\mathbb{R}} G$. Then $G_{\mathbb{R}}/H_{\mathbb{R}} \cong G/H$, in particular, the pair $G_{\mathbb{R}}, H_{\mathbb{R}}$ satisfies again our general requirements and $G_{\mathbb{R}}/G$ (being homeomorphic to $H_{\mathbb{R}}/H$) is compact (more generally, $G_{\mathbb{R}}$ can be defined as above whenever H is a closed normal subgroup of a l.c. group G such that H is compactly generated, nilpotent and torsion free; but be aware that $G_{\mathbb{R}}$ may have non-trivial compact normal subgroups, even if G does not, see also Corollary 3.5; furthermore, $G_{\mathbb{R}}$ depends on H and it need not be connected). Any $\theta \in \mathrm{Aut}_H(G)$ has a unique extension $\theta_{\mathbb{R}} \in \mathrm{Aut}_{H_{\mathbb{R}}}(G_{\mathbb{R}})$, $\theta \in \mathrm{Aut}_{1,H}(G)$ implies $\theta_{\mathbb{R}} \in \mathrm{Aut}_{1,H_{\mathbb{R}}}(G_{\mathbb{R}})$. If θ is unipotent, the same is true for $\theta_{\mathbb{R}}$.

Lemma 2.2. *For $\theta \in \mathrm{Aut}_1(G)$, the following statements are equivalent:*

(i) $\theta|\,G^0$ *is semisimple,* $G = G^0\,G_\theta$.
(ii) $\theta|\,H^0$ *is semisimple,* $G = H^0\,G_\theta$.
(iii) $\theta_{\mathbb{R}}|H_{\mathbb{R}}$ *is semisimple,* $G_{\mathbb{R}} = H\,(G_{\mathbb{R}})_{\theta_{\mathbb{R}}}$.

Proof. (i) $\Rightarrow$ (ii): it will be enough to show that $G^0 \subseteq H^0 G_\theta$. The assumption $\theta \in \mathrm{Aut}_1(G)$ implies $(\mathrm{ad}\,\theta)(G^0) \subseteq H^0$, consequently $d\theta$ induces the identity on $\mathfrak{g}/\mathfrak{h}$ (where $\mathfrak{h}$ denotes the Lie algebra of H). $d\theta$ being semisimple, it follows that $\mathfrak{g} = \mathfrak{g}_\theta + \mathfrak{h}$ (where $\mathfrak{g}_\theta = \{X \in \mathfrak{g} : d\theta(X) = X\}$). Clearly, $\mathfrak{g}_\theta$ is the Lie algebra of G_θ and it follows (as in [25] L. 3.18.4) that $H^0 G_\theta^0$ is open in G.
(ii) $\Rightarrow$ (iii): We have $H = H^0 H_\theta$. It is easy to see that $(H_\theta)_{\mathbb{R}} \subseteq (H_{\mathbb{R}})_{\theta_{\mathbb{R}}}$, thus $H_{\mathbb{R}} = H^0 (H_{\mathbb{R}})_{\theta_{\mathbb{R}}}$. Then $G_{\mathbb{R}} = H_{\mathbb{R}} G$ implies $G_{\mathbb{R}} = H^0 (G_{\mathbb{R}})_{\theta_{\mathbb{R}}}$. In addition, we get a decomposition of the Lie algebra $\mathfrak{h}_{\mathbb{R}}$ of $H_{\mathbb{R}}$ into a sum (similar as above) and then semisimplicity of $d\theta|\,\mathfrak{h}$ ($= d\theta_{\mathbb{R}}|\,\mathfrak{h}$) implies semisimplicity of $d\theta_{\mathbb{R}}$.
(iii) $\Rightarrow$ (i): We have $\mathrm{ad}\,\theta_{\mathbb{R}}(H_{\mathbb{R}}) \subseteq H^0$ and this implies that $d\theta_{\mathbb{R}}$ induces the identity on $\mathfrak{h}_{\mathbb{R}}/\mathfrak{h}$. As in the first step, we get that $H_{\mathbb{R}} = H^0 (H_{\mathbb{R}})_{\theta_{\mathbb{R}}}$, hence $H = H^0 H_\theta$ and $G_{\mathbb{R}} = H^0 (G_{\mathbb{R}})_{\theta_{\mathbb{R}}}$. Since $(G_{\mathbb{R}})_{\theta_{\mathbb{R}}} \cap G = G_\theta$, this gives $G = H^0 G_\theta \subseteq G^0 G_\theta$. We get a surjective homomorphism from $G^0 \cap G_\theta$ to G^0/H^0 and since $G^0 \cap G_\theta$ is σ-compact, this is an open mapping ([15] Th. 5.29). Hence the mapping has to remain surjective on $(G^0 \cap G_\theta)^0 = G_\theta^0$ and it follows that $G^0 = H^0 G_\theta^0$. As in the second step, this gives a decomposition of $\mathfrak{g}$ and semisimplicity of $d\theta$. $\square$

Definition 2.3. $\theta \in \mathrm{Aut}(G)$ is called *semisimple* if it satisfies condition (i) of Lemma 2.2.

Note that if H is any subgroup of G as in 2.1 such that $\theta \in \mathrm{Aut}_{1,H}(G)$ holds, then θ satisfies Lemma 2.2 (ii), (iii) as well. In particular (by (iii)), $\theta_{\mathbb{R}} \in \mathrm{Aut}(G_{\mathbb{R}})$ is again semisimple. But the converse is not true in general (take e.g., $G = \mathbb{Z}$, $\theta(n) = -n$, then $\theta_{\mathbb{R}}$ is semisimple but θ is not). If H is connected and open, G/H finitely generated (G, H as in 2.1), $\theta \in \mathrm{Aut}_1(G)$, our Definition is equivalent to that of [26] sec. 6. If (for general G, H as in 2.1) $\theta', \theta'' \in \mathrm{Aut}_1(G)$ are commuting semisimple automorphisms, it follows as in the proof of [26] (8.8) (see also Corollary 2.6 below with $L = G_\theta$) that $\theta' \circ \theta''$ is again semisimple and $\theta' \circ \theta'' \in \mathrm{Aut}_1(G)$.

Lemma 2.4. *Assume that G, H are given as in 2.1, $\theta \in \mathrm{Aut}_1(G)$ and let ρ be the semisimple part of $\theta | H^0$. Then there exists a unique semisimple automorphism $\theta_s \in \mathrm{Aut}_1(G)$ which extends ρ and commutes with θ.*
If H is connected, we have $G_{\theta_s} = \{x \in G : \theta(x)\, x^{-1} \in H_\rho\}$. For general H, we have $G_{\theta_s} = \{x \in G : (\mathrm{ad}\,\theta)^n(x) = e$ for some $n > 0\}$ and $\theta_s = (\theta_{\mathbb{R}})_s | G$. In particular $G_\theta \subseteq G_{\theta_s}$.

Proof. First, assume that H is connected. To prove uniqueness, it will be enough (by Lemma 2.2 (ii)) to verify the first formula for G_{θ_s} (then $G_\theta \subseteq G_{\theta_s}$ follows as well in this case). If $x \in G_{\theta_s}$, then (using $\theta \circ \theta_s = \theta_s \circ \theta$ and $\theta \in \mathrm{Aut}_1(G)$) we get $\theta(x)x^{-1} \in G_{\theta_s} \cap H = H_\rho$. For the converse, take $x \in G$, then we have (Lemma 2.2 (ii)) $x = yz$ with $y \in H,\ z \in G_{\theta_s}$. If $\theta(x)x^{-1} \in H_\rho$, we can (since $\theta(x)x^{-1} = \theta(y)\theta(z)z^{-1}y^{-1}$) apply [26] (5.1) with $w = y,\ v = \theta(z)z^{-1}$ (note a misprint in [26]: it should read $\theta(w)vw^{-1}$ instead of $\rho(w)vw^{-1}$) and conclude that $y \in H_\rho \subseteq G_{\theta_s}$. This gives $x \in G_{\theta_s}$. Furthermore, concerning the second formula for G_{θ_s}, [26] (5.1) shows that for $w \in H,\ (\mathrm{ad}\,\theta)(w) \in H_\rho$ implies $w \in H_\rho$. Thus (by induction) $(\mathrm{ad}\,\theta)^n(x) = e$ for some $n > 0$ implies $(\mathrm{ad}\,\theta)(x) \in H_\rho$, i.e., $x \in G_{\theta_s}$. The other inclusion follows from the fact that $\theta | H_\rho$ is unipotent.
Concerning existence of θ_s, this follows from [26] (8.1) if H is open (and still connected): he assumes that G/H is finitely generated, but (recall that G/H is abelian) any finite subset of G is contained in an open subgroup G_1 of G such that G_1/H is finitely generated; since uniqueness has already been proved, this allows to define the automorphism θ_s unambiguously on all of G; since H is open and θ_s-invariant, continuity holds automatically. If H is not open, we can refine the topology to make it open (observe that on H the two topologies coincide). Then the argument above produces a unique extension θ_s of ρ for the refined topology. To prove continuity of θ_s for the original topology, we may assume that G is σ-compact (since $\theta \in \mathrm{Aut}_1(G)$, any subgroup G_1 containing H is automatically θ-invariant, the same for θ_s). The description of G_{θ_s} that was demonstrated above shows that G_{θ_s} is closed in the original topology. By Lemma 2.2 (ii) and [15] Th. 5.29, G is topologically isomorphic to a quotient of the semidirect product $H \rtimes G_{\theta_s}$. On $H \rtimes G_{\theta_s}$, the mapping $\theta(y, z) = (\rho(y), z)$ is a group automorphism (since θ_s is known, to be a group automorphism) and θ is clearly continuous, hence the same is true for the induced mapping θ_s on the quotient group. This finishes the proof when H is connected.

If H is not connected, we consider the extension $\theta_{\mathbb{R}}$ to $G_{\mathbb{R}}$ (see 2.1). Put $\rho_{\mathbb{R}} = (\theta_{\mathbb{R}}|H_{\mathbb{R}})_s$. Since H^0 is $\theta_{\mathbb{R}}$-invariant, it is also $\rho_{\mathbb{R}}$-invariant. Hence, by uniqueness of Jordan decomposition, $\rho = \rho_{\mathbb{R}}|H^0$. Since $\theta \in \mathrm{Aut}_1(G)$, it induces a unipotent transformation on H/H^0. Its extension to $H_{\mathbb{R}}/H^0$ coincides (by uniqueness) with the transformation induced by $\theta_{\mathbb{R}}|H_{\mathbb{R}}$. Thus $\theta_{\mathbb{R}}|H_{\mathbb{R}}$ induces a unipotent transformation on $H_{\mathbb{R}}/H^0$, hence $\rho_{\mathbb{R}}$ induces the identity on $H_{\mathbb{R}}/H^0$ which implies $H_{\mathbb{R}} \subseteq H^0\,(H_{\mathbb{R}})_{\rho_{\mathbb{R}}}$. Let θ_s be any extension of ρ as in the Lemma. From $\theta_s \in \mathrm{Aut}_1(G)$, it follows as above that $(\theta_s)_{\mathbb{R}}|H_{\mathbb{R}}$ induces the identity on $H_{\mathbb{R}}/H^0$. As an easy consequence, $\theta_{\mathbb{R}} \circ (\theta_s)_{\mathbb{R}}^{-1}$ is unipotent on $H_{\mathbb{R}}$, thus uniqueness of the Jordan decomposition implies $(\theta_s)_{\mathbb{R}}|H_{\mathbb{R}} = \rho_{\mathbb{R}}$. As observed after Definition 2.3, $(\theta_s)_{\mathbb{R}} \in \mathrm{Aut}_{1,H_{\mathbb{R}}}(G_{\mathbb{R}})$ is semisimple, hence uniqueness in the connected case implies $(\theta_s)_{\mathbb{R}} = (\theta_{\mathbb{R}})_s$, thus $\theta_s = (\theta_{\mathbb{R}})_s|\,G$. This proves uniqueness in the general case. For existence, it suffices to show that G is $(\theta_{\mathbb{R}})_s$-invariant. But semisimplicity implies $G_{\mathbb{R}} = H_{\mathbb{R}}\,(G_{\mathbb{R}})_{(\theta_{\mathbb{R}})_s}$ and we already know that $H_{\mathbb{R}} = H^0\,(H_{\mathbb{R}})_{\rho_{\mathbb{R}}}$, thus $G_{\mathbb{R}} = H^0\,(G_{\mathbb{R}})_{(\theta_{\mathbb{R}})_s}$ which implies invariance of any subgroup containing H^0 (and also that $(\theta_{\mathbb{R}})_s|\,G \in \mathrm{Aut}_1(G)$). Finally, since $G_{\theta_s} = G \cap (G_{\mathbb{R}})_{(\theta_{\mathbb{R}})_s}$ and $\mathrm{ad}\,\theta = (\mathrm{ad}\,\theta_{\mathbb{R}})|\,G$, the formula for G_θ follows from the connected case. □

2.5. For $\theta \in \mathrm{Aut}_1(G)$, we write $s(\theta) = \theta_s$ (defined by Lemma 2.4), $\theta_u = \theta \circ \theta_s^{-1}$. It follows easily that $\theta_u \in \mathrm{Aut}_1(G)$ is unipotent, $\theta = \theta_s \circ \theta_u = \theta_u \circ \theta_s$, and (by the corresponding result for operators on vector spaces and Lemma 2.4) this is the only such decomposition in $\mathrm{Aut}_1(G)$ for which the factors commute with θ. Lemma 2.2 (i) implies that $d\theta_s$ is the semisimple part of $d\theta$ (on the Lie algebra $\mathfrak{g}$). Combined with the formula for G_{θ_s}, it follows that θ_s (and hence θ_u as well) does not depend on the choice of H, as long as there exists *some* H for which $\theta \in \mathrm{Aut}_{1,H}(G)$ (note that $\theta \in \mathrm{Aut}_{1,H}(G)$ holds iff for $G_1 = G \rtimes \mathbb{Z}$ with the action defined by θ, the pair G_1, H satisfies the assumptions of 2.1; in particular, by 2.1, existence of such an H can be characterized by the conditions that the closed subgroup generated by $[G,G]$ and $(\mathrm{ad}\,\theta)(G)$ should be compactly generated, nilpotent and torsion free and θ should induce a unipotent transformation on G/G^0; if G is compactly generated and has no non-trivial compact normal subgroup, one can always take $H = N$, as defined in Remark 2.8). By uniqueness, we have $s(\psi \circ \theta \circ \psi^{-1}) = \psi \circ s(\theta) \circ \psi^{-1}$ for $\psi \in \mathrm{Aut}(G)$, $\theta \in \mathrm{Aut}_1(G)$ (note that $\psi \circ \theta \circ \psi^{-1} \in \mathrm{Aut}_{1,\psi(H)}(G)$). In particular, if $\psi \in \mathrm{Aut}(G)$ commutes with θ, it commutes also with θ_s, θ_u (see also [26] (8.6)). For $\theta = \iota_x$ $(x \in G)$ we just write $s(x)$ $(= s(\iota_x))$. Note that in this case the inclusion $G_\theta \subseteq G_{\theta_s}$ implies that $\sigma = s(x)$ satisfies $\sigma(x) = x$. Furthermore, we put

$$\mathcal{S} = \{s(x) : x \in G\}\,.$$

The example after Definition 2.3 shows (for $G = \mathbb{R}^n, L = \mathbb{Z}^n$) that if θ is semisimple on G and L is a general θ-invariant subgroup, then $\theta|L$ need not be semisimple in the sense of Definition 2.3. Furthermore, if θ is given by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, then $L = \mathbb{Z}^3$ is not invariant under θ_s.

Observe that if G, H are as in 2.1 and L is a closed subgroup of G such that $L \cap H^0$ is connected, then $L, L \cap H$ satisfy again the assumptions of 2.1 (using that $(L \cap H)^0 = L \cap H^0$ and algebraically $L/(L \cap H^0) \cong LH^0/H^0 \subseteq G/H^0$ holds; furthermore, since H is nilpotent, any closed subgroup of H is compactly generated - compare [17] Prop. 2). If $\theta \in \mathrm{Aut}_1(G)$, L is as above and θ-invariant, then $\theta|L \in \mathrm{Aut}_1(L)$.

Corollary 2.6. *Assume that L is a closed subgroup of G such that $L \cap H^0$ is connected, $\theta \in \mathrm{Aut}_1(G)$ and L is θ-invariant. Then L is θ_s-invariant and $\theta_s|\,L = (\theta|L)_s$, $\theta_u|\,L = (\theta|L)_u$.*
In particular, if θ is semisimple, then $\theta|\,L$ is semisimple.

Proof. Put $\theta_L = \theta|L$, $H_L = L \cap H$. $(\theta_L)_s$ is defined by Lemma 2.4 (see above) and we have $L_{(\theta_L)_s} = L \cap G_{\theta_s}$. L^0 is θ-invariant and it follows from the properties of Jordan decomposition (2.1) that L^0 is θ_s-invariant and that $(d\theta_L)_s$ is the restriction of $(d\theta)_s = d\theta_s$. Thus $(\theta_L)_s = \theta_s$ on L^0 and Lemma 2.2 (i) (for θ_L) gives the result. $\square$

Lemma 2.7. *Assume that $\mathcal{C}$ is a subgroup of $\mathrm{Aut}(G)$, $\mathcal{C}_1$ is a normal subgroup of $\mathcal{C}$ such that $\mathcal{C}_1 \subseteq \mathrm{Aut}_1(G)$, $\mathcal{C}_1$ is nilpotent and $[\mathcal{C}, \mathcal{C}_1]$ consists of unipotent transformations. Then s is a group homomorphism on $\mathcal{C}_1$, $s(\mathcal{C}_1)$ is commutative and centralizes $\mathcal{C}$.*

Proof. We use the notation for commutators $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1}$ as in [18] p. 129. We consider the ascending central series $(e) = \mathcal{C}^{(0)} \subseteq \cdots \subseteq \mathcal{C}^{(k)} = \mathcal{C}_1$ for $\mathcal{C}_1$ (i.e., $\mathcal{C}^{(i+1)}/\mathcal{C}^{(i)}$ is the centre of $\mathcal{C}_1/\mathcal{C}^{(i)}$) and put $\mathcal{C}^{(k+1)} = \mathcal{C}$. Take $\sigma \in \mathcal{C}_1$. By induction, we want to show that if $\tau \in \mathcal{C}^{(i)}$, then σ_s commutes with τ. This is trivial for $i = 0$, so we assume that the statement holds for $i - 1$ (where $i \geq 1$). We have $\tau' = [\sigma^{-1}, \tau] \in \mathcal{C}^{(i-1)}$ (for $i = k{+}1$ use that $\sigma \in \mathcal{C}_1$), hence τ' commutes with σ_s and by assumption, τ' is unipotent. Observe that $\tau\sigma\tau^{-1} = \sigma\tau' = \sigma_s\sigma_u\tau'$. Let $\mathfrak{h}$ be the Lie algebra of H. By [26] (2.2), $\mathcal{C}_1$ induces a triangular group of transformations on $\mathfrak{h}$ and then the same is true for the unipotent parts of these transformations and the group generated by them. It follows that the group generated by $\{\xi_u : \xi \in \mathcal{C}_1\}$ contains just unipotent transformations on H^0, in particular, $\sigma_u\tau'$ is unipotent on H^0. Consequently, $\sigma_s|H^0$ is the semisimple part of $(\tau\sigma\tau^{-1})|H^0$ and then uniqueness in Lemma 2.4 implies $\sigma_s = s(\tau\sigma\tau^{-1}) = \tau\sigma_s\tau^{-1}$, providing the induction step.
In particular, σ_s commutes with $\mathcal{C}_1$, hence (see 2.5), for any $\tau \in \mathcal{C}_1$, it commutes also with τ_s and τ_u. Thus $\sigma\tau = \sigma_s\tau_s\sigma_u\tau_u$ and (recall that by 2.3 $\sigma_s\tau_s$ is semisimple) as above, we get $s(\sigma\tau) = s(\sigma)\, s(\tau)$. $\square$

Remark 2.8. As in the previous proof (using the Lie algebra $\mathfrak{h}_{\mathbb{R}}$ of $H_{\mathbb{R}}$ instead of $\mathfrak{h}$), it follows from [26] (2.3) that the group $\mathcal{N}$ generated by $\{(\iota_x)_u : x \in G\}$ is nilpotent, contained in $\mathrm{Aut}_1(G)$ and consists of unipotent transformations. In particular, $N = \mathrm{nil}(G) = \{x \in G : \iota_x \text{ unipotent}\}$ is a nilpotent characteristic subgroup of G containing H. It is the biggest nilpotent normal subgroup of G (in particular closed). We call it the (non-connected) *nilradical* of G. If G is compactly generated, existence of N follows also from [17] Prop. 3, see also [26] (9.1).

Take $x \in G$ and put $\sigma = s(x)$. It is an easy consequence that for $y \in G$, $s(y) = \sigma$ holds iff $y \in (N \cap G_\sigma)\, x$.

If $\mathcal{C} \subseteq \mathrm{Aut}_H(G)$, we put $G_{\mathcal{C}} = \bigcap_{\theta \in \mathcal{C}} G_\theta$. Note that $G_{\mathcal{C}} \cap H^0$ is connected (since the fixed points correspond to a linear subspace of the Lie algebra $\mathfrak{h}$ of H). Hence (see 2.5), $G_{\mathcal{C}}$, $G_{\mathcal{C}} \cap H$ satisfy the conditions of 2.1. We put $L_{\mathcal{C}} = \mathrm{nil}(G_{\mathcal{C}})$. Then $L_{\mathcal{C}} \cap H^0 = G_{\mathcal{C}} \cap H^0$. It is easy to see that if $\langle \mathcal{C} \rangle$ denotes the subgroup generated by $\mathcal{C}$, then $G_{\mathcal{C}} = G_{\langle \mathcal{C} \rangle}$.

Corollary 2.9. *Let L be a nilpotent subgroup of G, $\sigma \in \mathrm{Aut}(G)$ such that L is σ-invariant and $\sigma(x)\, x^{-1} \in N$ for all $x \in L$. Then σ commutes with $s(L)$.*

Observe that $\sigma(x)\, x^{-1} \in N$ holds for every x, whenever σ induces the identity on G/N, in particular if $\sigma \in \mathrm{Aut}_1(G)$.

Proof. Put $\mathcal{C}_1 = \{\iota_x : x \in L\}$ and let $\mathcal{C}$ be the group generated by $\mathcal{C}_1$ and σ. Then $\mathcal{C}_1$ is nilpotent, contained in $\mathrm{Aut}_1(G)$ and normal in $\mathcal{C}$ (since L is σ-invariant). $[G, G] \subseteq H \subseteq N$ and $\sigma(x)x^{-1} \in N$ for $x \in L$ imply that $[\mathcal{C}, \mathcal{C}_1] \subseteq \{\iota_y : y \in N\}$, hence $[\mathcal{C}, \mathcal{C}_1]$ consists of unipotent transformations. Now Lemma 2.7 shows that $\mathcal{C}$ centralizes $s(\mathcal{C}_1) = s(L)$. $\square$

Lemma 2.10. *Assume that $\mathcal{C}$ is a subgroup of $\mathrm{Aut}_1(G)$ containing only semisimple transformations. If $\mathcal{C}_0$ is a normal subgroup of $\mathcal{C}$ such that $G_{\mathcal{C}_0} \cap H^0 = G_{\mathcal{C}} \cap H^0$, then $G_{\mathcal{C}_0} = G_{\mathcal{C}}$.*

Proof. Take $\theta \in \mathcal{C}$. Normality of $\mathcal{C}_0$ implies that $G_{\mathcal{C}_0}$ is θ-invariant. By Corollary 2.6 and Lemma 2.2, $\theta|\, G_{\mathcal{C}_0} \cap H^0 = \mathrm{id}$ implies $\theta|G_{\mathcal{C}_0} = \mathrm{id}$. $\square$

Corollary 2.11. *Let $\mathcal{C}$ be a commuting subset of $\mathrm{Aut}_1(G)$ consisting of semisimple transformations. Then we have $G = H^0 G_{\mathcal{C}} = N G_{\mathcal{C}}$ and there exists a finite subset $\mathcal{C}_0$ of $\mathcal{C}$ such that $G_{\mathcal{C}_0} = G_{\mathcal{C}}$.*

Proof. Choose a finite subset $\mathcal{C}_0$ of $\mathcal{C}$ so that $\dim(G_{\mathcal{C}_0} \cap H^0)$ is minimal. Then Lemma 2.10 (applied to the groups generated by $\mathcal{C}_0$ and $\mathcal{C}$) implies $G_{\mathcal{C}_0} = G_{\mathcal{C}}$. The equation $G = H^0 G_{\mathcal{C}_0}$ follows from Lemma 2.2 (ii) by induction on the cardinality of $\mathcal{C}_0$ (recall that G_θ, $G_\theta \cap H$ also satisfy the assumptions of 2.1 and for $\theta \in \mathcal{C}_0$, G_θ is $\mathcal{C}_0$-invariant and the restrictions are semisimple by Corollary 2.6) – compare [26] (8.8). $\square$

Lemma 2.12. *Let $\mathcal{C}$ be a subset of $\mathrm{Aut}(G)$ satisfying $G = N G_{\mathcal{C}}$. Then the following statements are equivalent:*

(i) *$\sigma \in s(G_{\mathcal{C}})$.*
(ii) *$\sigma \in \mathcal{S}$ and it commutes with $\mathcal{C}$.*
(iii) *$\sigma \in \mathcal{S}$ and $G_{\mathcal{C}}$ is σ-invariant.*

In particular, by Corollary 2.11, this applies to any commuting subset $\mathcal{C}$ of $\mathrm{Aut}_1(G)$ consisting of semisimple transformations.

Proof. (i) $\Rightarrow$ (ii) follows from 2.5. (ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i): We put $H = [G, G]^-$, then $\mathcal{C} \subseteq \mathrm{Aut}_H(G)$, hence by Remark 2.8, $G_\mathcal{C}$, $G_\mathcal{C} \cap H$ satisfy the assumptions of 2.1. If $G_\mathcal{C}$ is σ-invariant, then by Corollary 2.6, $\sigma|G_\mathcal{C}$ is semisimple and then by Lemma 2.2 (ii) $G_\mathcal{C} = (G_\mathcal{C} \cap H^0)(G_\mathcal{C} \cap G_\sigma)$. This implies $G = N(G_\mathcal{C} \cap G_\sigma)$. Take $x \in G$ such that $\sigma = s(x)$. Then $x \in G_\sigma$ and $x = zy$ with $y \in G_\mathcal{C} \cap G_\sigma,\ z \in N$. It follows that $z \in N \cap G_\sigma$, hence by Remark 2.8, $s(x) = s(y)$. $\square$

Lemma 2.13. *Let $\mathcal{C}$ be a subset of* $\mathrm{Aut}(G)$ *satisfying $G = NG_\mathcal{C}$. Then the following statements are equivalent:*

(i) *$\sigma \in s(L_\mathcal{C})$.*
(ii) *$\sigma \in \mathcal{S},\ \sigma(t) = t$ for all $t \in G_\mathcal{C}$.*
(iii) *$\sigma \in \mathcal{S}$, it commutes with $\mathcal{C}$ and $\sigma(t) = t$ for all $t \in G_\mathcal{C} \cap H^0$.*

Proof. Again we put $H = [G, G]^-$.
(i) $\Rightarrow$ (ii), (iii): By Lemma 2.12, σ commutes with $\mathcal{C}$. Take $x \in L_\mathcal{C}$ such that $\sigma = s(x)$. Then ι_x is unipotent on $G_\mathcal{C}$ (see 2.8) and Corollary 2.6 (for $L = G_\mathcal{C}$) implies that $\sigma = s(x)$ is the identity on $G_\mathcal{C}$.
(iii) $\Rightarrow$ (ii): If σ commutes with $\mathcal{C}$, then $G_\mathcal{C}$ is σ-invariant. By Corollary 2.6, $\sigma|G_\mathcal{C}$ is semisimple and by Lemma 2.2 (ii) (with $G_\mathcal{C} \cap H^0$ instead of H^0), σ is the identity on $G_\mathcal{C}$.
(ii) $\Rightarrow$ (i): $G_\mathcal{C}$ is σ-invariant, hence by Lemma 2.12 $\sigma \in s(G_\mathcal{C})$. Take $x \in G_\mathcal{C}$ such that $\sigma = s(x)$. Then $G_\mathcal{C}$ is ι_x-invariant and by assumption, $s(x) = \sigma$ is the identity on $G_\mathcal{C}$. Hence Corollary 2.6 implies that ι_x is unipotent on $G_\mathcal{C}$, i.e., $x \in L_\mathcal{C}$. $\square$

Lemma 2.14. (i) *Let L be a nilpotent subgroup of G and put $\mathcal{C} = s(L)$. Then $\mathcal{C}$ is an abelian group contained in $\mathcal{S}$ and we have $L \subseteq L_\mathcal{C}$.*
(ii) *Let $\mathcal{C}$ be any subset of* $\mathrm{Aut}(G)$ *satisfying $G = NG_\mathcal{C}$ and put $\mathcal{C}_1 = s(L_\mathcal{C})$. Then $s(L_{\mathcal{C}_1}) = \mathcal{C}_1$.*
If in addition $\mathcal{C} \subseteq \mathcal{S}$ holds, then $\mathcal{C} \subseteq \mathcal{C}_1$ (in particular, the elements of $\mathcal{C}$ commute and the generated subgroup is contained in $\mathcal{S}$), $G_\mathcal{C} = G_{\mathcal{C}_1},\ L_\mathcal{C} = L_{\mathcal{C}_1}$.
(iii) *If $\mathcal{C}$ is a commuting subset of $\mathcal{S}$ then $L_\mathcal{C}$ is a maximal nilpotent subgroup of G.*

Thus the maximal nilpotent subgroups of G are all of the form $L_\mathcal{C}$, where $\mathcal{C}$ is an abelian group contained in $\mathcal{S}$. In particular (for $\mathcal{C} = \{\mathrm{id}\}$), $N = \mathrm{nil}(G)$ is a maximal nilpotent subgroup of G (but this follows also directly from the definition in Remark 2.8).

Proof. (i): H^0 being nilpotent and torsion free, we can identify $(L \cap H^0)_\mathbb{R}$ with a subgroup of H^0 which is L-invariant. Thus $L' = (L \cap H^0)_\mathbb{R} L$ is still nilpotent. By Lemma 2.7 (with $\mathcal{C}_1 = \{\iota_x : x \in L'\}$, $\mathcal{C}' = \mathcal{C}_1$ in place of $\mathcal{C}$), s is a homomorphism on L'. Hence $s(L') = s(L)$

is always a group and (replacing L by L') we see that it is no restriction to assume that $L \cap H^0$ is connected. Furthermore (again by Lemma 2.7) $\mathcal{C} = s(L)$ is commutative.
If $x \in L$, then ι_x is unipotent on L, hence by Corollary 2.6, $\sigma = s(x)$ is the identity on L. This implies $L \subseteq G_{\mathcal{C}}$. $\sigma \in \mathcal{C}$ implies that σ is the identity on $G_{\mathcal{C}}$, hence by Lemma 2.13 (ii), $\sigma \in s(L_{\mathcal{C}})$. Take $y \in L_{\mathcal{C}}$ such that $\sigma = s(x) = s(y)$. By Remark 2.8, $y\,x^{-1} \in N \cap G_{\mathcal{C}} \subseteq L_{\mathcal{C}}$ and it follows that $x \in L_{\mathcal{C}}$.
(ii): By Lemma 2.13 (ii), $G_{\mathcal{C}} \subseteq G_{\mathcal{C}_1}$. Put $\mathcal{C}_2 = s(L_{\mathcal{C}_1})$. If $\sigma \in \mathcal{C}_2$, then by Lemma 2.13 (ii), σ is the identity on $G_{\mathcal{C}_1} \supseteq G_{\mathcal{C}}$. Thus $\sigma \in s(L_{\mathcal{C}}) = \mathcal{C}_1$, proving that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. By (i) (with $L_{\mathcal{C}}$ in place of L), we have $L_{\mathcal{C}} \subseteq L_{\mathcal{C}_1}$, hence $\mathcal{C}_1 = s(L_{\mathcal{C}}) \subseteq s(L_{\mathcal{C}_1}) = \mathcal{C}_2$ which gives $\mathcal{C}_2 = \mathcal{C}_1$.
With the additional assumption $\mathcal{C} \subseteq \mathcal{S}$, Lemma 2.13 (ii) again implies that any $\sigma \in \mathcal{C}$ belongs to $s(L_{\mathcal{C}}) = \mathcal{C}_1$, i.e., $\mathcal{C} \subseteq \mathcal{C}_1$. In particular, by (i), $\mathcal{C}$ is contained in an abelian subgroup of $\mathcal{S}$. $\mathcal{C} \subseteq \mathcal{C}_1$ implies that $G_{\mathcal{C}_1} \subseteq G_{\mathcal{C}}$ and by Lemma 2.13 (ii), $G_{\mathcal{C}} \subseteq G_{\mathcal{C}_1}$. Thus $G_{\mathcal{C}} = G_{\mathcal{C}_1}$ and then $L_{\mathcal{C}} = L_{\mathcal{C}_1}$.
(iii): Assume that $L_{\mathcal{C}} \subseteq L$, where L is a nilpotent subgroup of G and put $\mathcal{C}' = s(L)$. Then by (ii) (using $\mathcal{C} \subseteq \mathcal{S}$ and Corollary 2.11), $\mathcal{C} \subseteq \mathcal{C}_1 = s(L_{\mathcal{C}}) \subseteq s(L) = \mathcal{C}'$. Consequently $L_{\mathcal{C}'} \subseteq G_{\mathcal{C}'} \subseteq G_{\mathcal{C}}$ and by (i), $L_{\mathcal{C}} \subseteq L \subseteq L_{\mathcal{C}'}$. Since $L_{\mathcal{C}}$ is normal in $G_{\mathcal{C}}$ (Remark 2.8), it follows that $L_{\mathcal{C}}$ is L-invariant. Take $x \in L$, then (L being nilpotent) ι_x is unipotent on $L_{\mathcal{C}}$. Recall that $L_{\mathcal{C}} \cap H^0 = G_{\mathcal{C}} \cap H^0$ and that ι_x induces a unipotent transformation on G/H^0. Combined, we see that ι_x is unipotent on $G_{\mathcal{C}}$. Thus $x \in L_{\mathcal{C}}$. This proves that $L = L_{\mathcal{C}}$. □

Proposition 2.15. *Let $\mathcal{C}_0$ be a commuting subset of $\mathcal{S}$ such that the dimension of $G_{\mathcal{C}_0} \cap H^0$ is minimal (among all such subsets). Put $L = G_{\mathcal{C}_0}$, $\mathcal{C} = s(L)$. Then the following properties hold.*
(i) *L is a maximal nilpotent subgroup of G (in particular, L is closed), $L \cap H^0$ is connected, $G = H^0L$, $L = L_{\mathcal{C}} = L_{\mathcal{C}_0} = G_{\mathcal{C}}$. The dimension of $L \cap H^0$ is minimal among the maximal nilpotent subgroups of G.*
(ii) *$\mathcal{C}$ is a subgroup of $\mathrm{Aut}_1(G)$ consisting of semisimple transformations. It is a maximal commuting subset of $\mathcal{S}$, $\mathcal{C}_0 \subseteq \mathcal{C}$.*
(iii) *$\beta(xy) = s(y)$ (where $x \in H^0$, $y \in L$) defines a continuous surjective group homomorphism $\beta\colon G \to \mathcal{C}$, $\ker\beta = N$, $\beta(x)\,\iota_x^{-1}$ is unipotent for all $x \in G$.*

Proof. (i): If $\mathcal{C}_1$ is any commuting set with $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{S}$, then minimality of $\dim(G_{\mathcal{C}_0} \cap H^0)$ implies $G_{\mathcal{C}_1} \cap H^0 = G_{\mathcal{C}_0} \cap H^0$ (recall that $G_{\mathcal{C}_1} \cap H^0$ is always connected). Take $x \in G_{\mathcal{C}_0}$ and put $\sigma = s(x)$. By Lemma 2.12, σ commutes with $\mathcal{C}_0$. Put $\mathcal{C}_1 = \mathcal{C}_0 \cup \{\sigma\}$. Then σ is the identity on $G_{\mathcal{C}_1} \supseteq G_{\mathcal{C}_0} \cap H^0$. $G_{\mathcal{C}_0}$ being invariant under σ and ι_x, Corollary 2.6 and Lemma 2.2 (ii) imply that σ is the identity on $G_{\mathcal{C}_0}$ and then that ι_x is unipotent on $G_{\mathcal{C}_0}$. Thus $x \in L_{\mathcal{C}_0}$. This proves that $G_{\mathcal{C}_0} = L_{\mathcal{C}_0}$. By Lemma 2.14 (iii), L is a maximal nilpotent subgroup of G, Corollary 2.11 shows that $G = H^0L$. Lemma 2.14 (ii) implies that $G_{\mathcal{C}_0} = G_{\mathcal{C}}$, $L_{\mathcal{C}_0} = L_{\mathcal{C}}$. The minimality statement about $\dim(L \cap H^0)$ results also from Lemma 2.14.

(ii): By Lemma 2.14 (i), $\mathcal{C}$ is an abelian group contained in $\mathcal{S}$, $\mathcal{C}_0 \subseteq \mathcal{C}$ by Lemma 2.14 (ii). If $\mathcal{C}_1$ is as at the beginning, the reasoning as above gives $L_{\mathcal{C}_1} = G_{\mathcal{C}_1} \subseteq G_{\mathcal{C}_0} = L_{\mathcal{C}_0}$. Then maximality of $L_{\mathcal{C}_1}$ (Lemma 2.14 (iii)) implies $L_{\mathcal{C}_1} = L_{\mathcal{C}_0}$, hence $\mathcal{C}_1 \subseteq s(L_{\mathcal{C}_1}) = s(L_{\mathcal{C}_0}) = \mathcal{C}$. This proves maximality of $\mathcal{C}$.
(iii): By Lemma 2.7, s is a homomorphism on L and clearly $s(y)$ is the identity for $y \in L \cap H^0 \subseteq N$. Since H^0 is normal in G and by (i), $G = H^0 L$, it follows easily that β is well defined on G and a surjective group homomorphism.
If $\mathfrak{h}$ denotes the Lie algebra of H, then by 2.5 and Lemma 2.4, $d(s(y)|H) = (d(\iota_y|H))_s$. If $y = \exp(Y)$ (where $Y \in \mathfrak{g}$), then by [25] (2.13.6) and Th. 2.13.2, $d(\iota_y|H) = \exp(\mathrm{ad}_{\mathfrak{h}} Y)$ (where $\mathrm{ad}_{\mathfrak{h}}$ denotes the adjoint representation of $\mathfrak{g}$ on $\mathfrak{h}$). Furthermore, uniqueness of the Jordan decomposition implies $\big(\exp(\mathrm{ad}_{\mathfrak{h}} Y)\big)_s = \exp\big((\mathrm{ad}_{\mathfrak{h}} Y)_s\big)$ (recall that $(\mathrm{ad}_{\mathfrak{h}} Y)_s$ is also the semisimple part in the additive Jordan decomposition of the operator $\mathrm{ad}_{\mathfrak{h}} Y$ on $\mathfrak{h}$. It follows (using also [13] Th. IX.1.2) that the mapping $t \mapsto s\big(\exp(tY)\big)|\,H^0$ (from $\mathbb{R}$ to $\mathrm{Aut}(H^0)$) is continuous, hence by [25] Th. 2.11.2, the mapping $y \mapsto s(y)|H^0$ from G^0 to $\mathrm{Aut}(H^0)$ is continuous. Since by (i) $L = L_{\mathcal{C}}$, we know that $s(y)$ is the identity on L for $y \in L$. As in the proof of Lemma 2.4, G is isomorphic to a quotient of a semidirect product $H^0 \rtimes L$ and it follows easily that $y \mapsto s(y)$ from L^0 to $\mathrm{Aut}(G)$ in continuous (hence the same is true on L) and then that β is continuous.
By definition (see also Remark 2.8) $z = xy \in \ker\beta$ (where $x \in H^0,\ y \in L$) iff $y \in N$ and this is equivalent to $z \in N$. Again by Remark 2.8, $\beta(x)\,\iota_x^{-1}$ is unipotent for all $x \in G$. $\square$

Corollary 2.16. *Let $\mathcal{C}, \beta$ be as in Proposition 2.15. Assume that $\mathcal{C}'$ is a subgroup of $\mathrm{Aut}_H(G)$ with $\mathcal{C} \subseteq Z(\mathcal{C}')$. Put $G' = G \rtimes \mathcal{C}',\ N' = \{(x, \beta(x^{-1})) : x \in G\}$.*
Then N' is nilpotent, closed and normal in G', $G' = N'\mathcal{C}'$, $N' \cap \mathcal{C}' = (e)$. Thus $G' = N' \rtimes \mathcal{C}'$ holds algebraically and in fact topologically as well. If $\mathcal{C}'$ consists of semisimple transformations, then $N' = \mathrm{nil}(G')$.

Proof. Take $\sigma \in \mathcal{C}'$. Since it commutes with $\mathcal{C}$, the group $G_{\mathcal{C}} = L_{\mathcal{C}}$ is σ-invariant. By 2.1 and 2.5, $s(\sigma(x)) = \sigma \circ s(x) \circ \sigma^{-1}$, hence $s(\sigma(x)) = s(x)$ for $x \in L$. This implies $\beta \circ \sigma = \beta$ and then a short computation shows that N' is a normal subgroup of G' (evidently closed).
Commutativity of $\mathcal{C}$ implies $[N', N'] \subseteq \ker\beta = N$. For $x \in G$, the restriction of $\iota_{(x,\beta(x^{-1}))}$ to G equals $\iota_x \circ \beta(x^{-1})$ which is unipotent and belongs to $\mathrm{Aut}_1(G)$ (in fact, even to the nilpotent group $\mathcal{N}$ of Remark 2.8). Hence N' is nilpotent by [27] (9.3). It follows easily from continuity of β that the isomorphism of $G \rtimes \mathcal{C}'$ and $N \rtimes \mathcal{C}'$ is a homeomorphism. Assume that $N'' \supsetneqq N'$ is a nilpotent subgroup of G'. Then there exists $\sigma \in N'' \cap \mathcal{C}'$ with $\sigma \neq \mathrm{id}$. Since $N \subseteq N'$, σ should be unipotent on N. If σ is semisimple, this would imply $\sigma|N = \mathrm{id}$ and Lemma 2.2 (ii) would give $\sigma = \mathrm{id}$. The remaining properties are clear. $\square$

Remark 2.17. It is easy to see that $[G', N'] \subseteq H$ (for $\mathcal{C}' = \mathcal{C}$ even $[G', G'] = [G, G]$ holds). If P' denotes the group of compact elements of N', then one gets $P' = \{(x, \beta(x^{-1})) : x \in P\}$, where P denotes the group of compact elements of $L_{\mathcal{C}}$. One has $P' \subseteq Z(G')$. P' is

non-trivial (hence N' is not torsion free) whenever G has non-trivial compact (necessarily abelian) subgroups (even when G has no non-trivial compact normal subgroups). If H is connected, one can show (similarly as in [26] (9.2)) that there exists a closed torsion free subgroup N'' of N' with $N'' \supseteq H$ and $N' = N''P'$. Then $G' = N'' \rtimes (P' \times \mathcal{C}')$ (the group P' acts trivially on N''; but in general the complementary group N'' is not unique).
Take for example, $G = \mathbb{C} \rtimes \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \circ z = e^{2\pi i t} z$ for $t \in \mathbb{T}$, $z \in \mathbb{C}$, $\mathcal{C} = \mathcal{C}' = \iota(\mathbb{T}) \cong \mathbb{T}$. Then $N = \mathbb{C}$, $L_{\mathcal{C}} = \mathbb{T} = P \cong P'$, $N' \cong \mathbb{C} \times \mathbb{T}$.
If $\mathcal{C}' \subseteq \mathrm{Aut}_1(G)$ is locally compact, abelian, $\mathcal{C}' \supseteq \mathcal{C}$, then G', H satisfy again the assumptions of 2.1 and any semisimple $\sigma \in \mathcal{C}'$ defines a semisimple automorphism of G'.

Lemma 2.18. *If $\sigma \in \mathrm{Aut}_1(G)$ is semisimple, $x \in G$ and $\sigma(x)\,x^{-1} \in Z(G)$ holds, then $x \in (Z(G) \cap H^0)\,G_\sigma$. Under the additional assumption $\sigma \in \mathcal{S}$, we get $x \in G_\sigma$, i.e., $\sigma(x) = x$.*

$Z(G)$ denotes the centre of G. Thus for $\sigma \in \mathcal{S}$, $x \in G_\sigma$ is equivalent to $[\iota_x, \sigma] = \mathrm{id}$.

Proof. By Lemma 2.2 (ii), $x = vy$ with $v \in H^0$, $y \in G_\sigma$. Then $\sigma(x)x^{-1} = \sigma(v)v^{-1}$ and the first statement follows from [26] (5.6) (with $N_1 = Z(G) \cap H^0$). If $\sigma \in \mathcal{S}$, then by Corollary 2.6, σ is the identity on $Z(G)$, i.e., $Z(G) \subseteq G_\sigma$. $\square$

Lemma 2.19. *Assume that $\sigma \in \mathrm{Aut}(H^0)$ is semisimple, $\theta \in \mathrm{Aut}(H^0)$, $\theta \circ \sigma^{-1}$ unipotent, $[\theta, \sigma] \in \iota(H^0)$. Then $H^0 = (\mathrm{ad}\,\theta)^2(H^0)\,H^0_\sigma$.*

Proof (compare [26] (5.2)). First we treat the special case $w \in Z(H^0)$. We have $Z(H^0) \cong \mathbb{R}^n$ (written additively). $\theta|\,Z(H^0)$ is given by a matrix A, $\sigma|\,Z(H^0)$ by a semisimple matrix A_1, where $A, A_1 \in \mathrm{GL}(\mathbb{R}^n)$. Then $H^0_\sigma \cap Z(H^0) \cong \ker(A_1 - I)$, $\mathrm{ad}\,\theta\,(Z(H^0)) \cong \mathrm{im}(A - I)$. Since $[\theta, \sigma]$ is the identity on $Z(H^0)$, it follows that A, A_1 commute, hence A_1 is the semisimple part of A. In particular, $\mathbb{R}^n = \mathrm{im}(A_1 - I) \oplus \ker(A_1 - I)$, $\mathrm{im}(A_1 - I)$ is A-invariant and $A - I$ is invertible on $\mathrm{im}(A_1 - I)$. Thus $\mathrm{im}(A_1 - I) \subseteq \mathrm{im}((A - I)^2)$, proving $w \in (\mathrm{ad}\,\theta)^2(H^0)\,H^0_\sigma$.
In the general case, we use induction on $\dim H^0$. The special case covers $\dim H^0 = 1$. For the induction step, we apply the hypothesis to $H^0/Z(H^0)$. Thus, given $w \in H^0$, there exist $u_0 \in (\mathrm{ad}\,\theta)^2(H^0)$, $v_0 \in H^0$ such that $\sigma(v_0)v_0^{-1} \in Z(H^0)$ and $w = u_0 v_0 z_0$ for some $z_0 \in Z(H^0)$. By Lemma 2.18, $v_0 = z_1 v_1$ for some $z_1 \in Z(H^0)$, $v_1 \in H^0_\sigma$. From the special case above, we get $u_2 \in (\mathrm{ad}\,\theta)^2(Z(H^0))$, $v_2 \in H^0_\sigma \cap Z(H^0)$ such that $z_1 z_0 = u_2 v_2$. Then $u = u_0 u_2$, $v = v_1 v_2$ will satisfy our requirements. $\square$

Lemma 2.20. *Let $\sigma \in \mathrm{Aut}_1(G)$ be semisimple and $x \in G$ such that $\sigma \circ \iota_x^{-1}$ is unipotent. Then there exists $u \in [G, H^0]$ such that $uxu^{-1} \in G_\sigma$.*

Proof. By Lemma 2.2 (ii), $G = H^0 G_\sigma$. Write $x = yz$ with $y \in H^0, z \in G_\sigma$. Put $\theta = \iota_x$. It follows that $[\sigma, \theta] = \iota_{\sigma(x)x^{-1}} = \iota_{\sigma(y)y^{-1}} \in \iota(H^0)$. By Lemma 2.19, there exist $u \in$

$\operatorname{ad}\theta(H^0) \subseteq [G, H^0]$, $v \in G_\sigma \cap H^0 = H^0_\sigma$ such that $y = \operatorname{ad}\theta(u)\, v = [u, x]^{-1}v$. Then $uxu^{-1} = [u, x]x = vz \in G_\sigma$. $\square$

Lemma 2.21. *For $\beta, \mathcal{C}$ as in Proposition 2.15, $\sigma \in \mathcal{C}$, $x \in G$, the following statements are equivalent:*

(i) $s(x) = \sigma$.
(ii) $x \in G_\sigma$, $\beta(x) = \sigma$.

Proof. By Proposition 2.15, $\sigma = s(y)$ for some $y \in L \subseteq G_\sigma$. By Remark 2.8, $s(x) = \sigma$ iff $x \in (N \cap G_\sigma)\, y$. Since by Proposition 2.15 (iii) $\ker \beta = N$, this proves our claim. $\square$

For $\alpha \in \operatorname{Aut}(G)$, $u \in G$, we write $u\,\alpha\,u^{-1} = \iota_u \circ \alpha \circ \iota_{u^{-1}}$.

Proposition 2.22. *Let $\mathcal{C}$ be the subgroup of $\mathcal{S}$ constructed in Proposition 2.15. If $\mathcal{C}_1$ is any commuting subset of $\mathcal{S}$, there exists $u \in [G, H^0]$ such that $u\,\mathcal{C}_1 u^{-1} \subseteq \mathcal{C}$. Putting $\mathcal{C}_2 = \mathcal{C} \cap \mathcal{C}_1$, one can take $u \in [G_{\mathcal{C}_2}, H^0 \cap G_{\mathcal{C}_2}]$.*
In particular, $\mathcal{S} = \bigcup \{\, u\mathcal{C}u^{-1} : u \in [G, H^0]\,\}$.

Proof. First, we assume that $\mathcal{C}_1$ is finite. If $\mathcal{C}_2 = \mathcal{C}_1$, there is nothing to prove. So, take $\sigma_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $x_1 \in G$ with $\sigma_1 = s(x_1)$ (which entails $x_1 \in G_{\sigma_1}$). Since σ_1 commutes with $\mathcal{C}_2$, G_{σ_1} is invariant under $\mathcal{C}_2$ and the restrictions of the transformations are semisimple by Corollary 2.6. Hence by Corollary 2.11, $G_{\sigma_1} = (H^0 \cap G_{\sigma_1})\,(G_{\mathcal{C}_2} \cap G_{\sigma_1})$ (if $\mathcal{C}_2 = \emptyset$, we put $G_{\mathcal{C}_2} = G$). By Remark 2.8, it follows that we may assume that $x_1 \in G_{\mathcal{C}_2}$. Consider β as in Proposition 2.15 and put $\sigma = \beta(x_1)$. Then $\sigma \in \mathcal{C}$, hence $G_{\mathcal{C}_2}$ is σ-invariant. Applying Lemma 2.20 to $G_{\mathcal{C}_2}$ and the restriction of σ (use also Proposition 2.15 (iii)), there exists $u \in [G_{\mathcal{C}_2}, H^0 \cap G_{\mathcal{C}_2}]$ such that $ux_1u^{-1} \in G_\sigma$. Then $u \in [G, H^0]$ and by Lemma 2.21, $u\sigma_1u^{-1} = s(ux_1u^{-1}) = \sigma$. Since $u \in G_{\mathcal{C}_2}$, we have $u\sigma'u^{-1} = \sigma'$ for $\sigma' \in \mathcal{C}_2$. Thus $u\mathcal{C}_1u^{-1} \cap \mathcal{C}$ strictly contains $\mathcal{C}_2$ and, repeating this argument, we can reach our goal after finitely many steps.
In the general case, there exists by Corollary 2.11 a finite subset $\mathcal{C}_1'$ of $\mathcal{C}_1$ such that $G_{\mathcal{C}_1'} = G_{\mathcal{C}_1}$. By the special case treated above, we may assume that $\mathcal{C}_1' \subseteq \mathcal{C}$. Then $G_{\mathcal{C}} \subseteq G_{\mathcal{C}_1}$. Take $\sigma \in \mathcal{C}_1$, then σ is the identity on $G_{\mathcal{C}_1}$, hence by Lemma 2.13, $\sigma \in s(L_{\mathcal{C}}) = \mathcal{C}$ (Proposition 2.15 (i)). Thus $\mathcal{C}_1 \subseteq \mathcal{C}$. $\square$

Remarks 2.23.
(a) In Lemma 2.12, the implication (i) $\Rightarrow$ (ii) holds for general subsets $\mathcal{C}$ of $\operatorname{Aut}(G)$ (same proof). Furthermore, by Corollary 2.9, the elements of $s(G_{\mathcal{C}}) \cup \mathcal{C}$ always commute with those of $s(L_{\mathcal{C}})$.
(b) With some further arguments the following conditions give other characterizations of the elements $\sigma \in s(L_{\mathcal{C}})$ ($\mathcal{C}$ as in Lemma 2.13):

(iv) $\sigma \in \mathcal{S}$, it commutes with $s(L_{\mathcal{C}})$ and $\sigma(t) = t$ for all $t \in G_{\mathcal{C}} \cap H^0$.
(v) $\sigma \in \mathcal{S}$, $\sigma(t) = t$ for all $t \in L_{\mathcal{C}}$.

In particular, it follows that $s(L_{\mathcal{C}})$ is a maximal commuting subset (group) in the set $\{\sigma \in \mathcal{S} :\ \sigma| G_{\mathcal{C}} \cap H^0 = \mathrm{id}\}$.
(c) In Lemma 2.13, the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) hold for arbitrary subsets $\mathcal{C}$ of $\mathrm{Aut}_H(G)$ (same proof).
(d) $s(G_{\mathcal{C}})$ is not a group, unless $G_{\mathcal{C}} = L_{\mathcal{C}}$, i.e. $G_{\mathcal{C}}$ is nilpotent ($\mathcal{C}$ any subset of $\mathrm{Aut}_H(G)$). Indeed, assume that $s(G_{\mathcal{C}})$ is a group, take $\sigma \in s(G_{\mathcal{C}})$. If $x \in G_{\mathcal{C}}$, then by 2.5, $\iota_x \circ \sigma \circ \iota_{x^{-1}} \in s(G_{\mathcal{C}})$, hence $\iota_{\sigma(x)x^{-1}} = [\iota_x, \sigma] \in s(G_{\mathcal{C}})$. Since $\sigma(x)x^{-1} \in H^0$, $\iota_{\sigma(x)x^{-1}}$ is unipotent, hence it must be the identity. Consequently $\sigma(x)x^{-1} \in Z(G)$ and by Lemma 2.18, we get $\sigma(x) = x$ for all $x \in G_{\mathcal{C}}$. Then Lemma 2.13 (ii) implies $\sigma \in s(L_{\mathcal{C}})$, i.e., $s(G_{\mathcal{C}}) = s(L_{\mathcal{C}})$. Now take $x \in G_{\mathcal{C}}$, then by Remark 2.8, there exists $y \in L_{\mathcal{C}}$ such that $yx^{-1} \in N \cap G_{\mathcal{C}} \subseteq L_{\mathcal{C}}$, hence $x \in L_{\mathcal{C}}$.

As a special case, if $\mathcal{C}_0$ is a commuting subset of $\mathcal{S}$ and $\dim(G_{\mathcal{C}_0} \cap H^0)$ is not minimal (compare Proposition 2.15), then $G_{\mathcal{C}_0} \neq L_{\mathcal{C}_0}$, in particular: $s(G_{\mathcal{C}_0})$ is not a group (if $\mathcal{C}$ is a maximal commuting set with $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{S}$, then by Proposition 2.22, $G_{\mathcal{C}_0} \cap H^0 \neq G_{\mathcal{C}} \cap H^0$ and by Proposition 2.15, $L_{\mathcal{C}} = G_{\mathcal{C}} \subseteq G_{\mathcal{C}_0}$, but by Lemma 2.14 (iii), $L_{\mathcal{C}}$ is not strictly contained in $L_{\mathcal{C}_0}$).
(e) In Lemma 2.19, the assumption $[\theta, \sigma] \in \iota(H^0)$ can be replaced by assuming the existence of a normal series $(e) = H_0 \lhd H_1 \cdots \lhd H_k = H^0$, where H_i are closed, connected and invariant under θ, σ and $[\theta, \sigma]$ induces the identity on H_i/H_{i-1} ($i = 1, \dots, k$). The same argument shows that $H^0 = (\mathrm{ad}\ \sigma)(H^0)\, H^0_\sigma$ (for any semisimple automorphism σ of H^0), compare [18] L. 5.4.
(f) For $G = \mathbb{R} \times \mathbb{T}$, $H = \mathbb{R} \times (0)$, $\sigma(x, y) = (x, y + x)$, one has $G_\sigma = \mathbb{Z} \times \mathbb{T}$, thus $G = H^0 G_\sigma$ but $\sigma \notin \mathrm{Aut}_H(G)$ and $G_\sigma \cap H^0$ is not connected (compare Remark 2.8). Alternatively, one could take $H_1 = (e)$. Then $\sigma \in \mathrm{Aut}_{H_1}(G)$ but $G \neq H_1^0 G_\sigma$.
(g) For $L = L_{\mathcal{C}}$ with $\mathcal{C}$ as in Proposition 2.15 one can show that $N_G(L) = L$, i.e. L is "self-normalizing". If G is a connected solvable Lie group with Lie algebra $\mathfrak{g}$, then L is connected. If $\mathfrak{l}$ denotes the Lie algebra of L then $\mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{g}$ and conversely. [7] 3.1 gives an explicit construction of the nil-shadow of G (based on [8] Sec. 3). His mapping T coincides with β of our Proposition 2.15 and his multiplication $*$ corresponds to the multiplication on N' arising in our Corollary 2.16.
(h) The Example before Corollary 2.6 shows that Jordan decomposition does not always work for automorphisms of $\mathbb{Z}^3$ and by duality one also gets counter-examples for $\mathbb{T}^3$. This limits the possibility to weaken the assumptions of 2.1. If G is a l.c. group such that $[G, G]$ and G/G^0 are nilpotent, let K be the group of compact elements in $[G, G]^-$ (in the compactly generated case, this is just the maximal compact normal subgroup). If K^0 is central in G (or more generally, if $\iota_x|K^0$ is unipotent for all $x \in G$) one can extend most of the results of this section (extending similarly the definition of $\mathrm{Aut}_1(G)$). This applies in particular when G is any connected (but not necessarily simply connected) solvable l.c. group (then, by Iwasawa's theorem, every compact normal subgroup of G is contained in the centre, [14] Th. 9.82). We will not make use of this generalization, but see also Remark 3.7 and Section 4.

Example 2.24. Assume that G_0, H_0 satisfy the assumptions of 2.1 and consider $G = G_0 \times G_0$, $H = H_0 \times H_0$, $\sigma(x,y) = (y,x)$. Then $\sigma \in \mathrm{Aut}_H(G)$, $d\sigma$ is semisimple, but if $G_0 \neq H_0$, then $\sigma \notin \mathrm{Aut}_1(G)$. We have $G_\sigma = \{(x,x) : x \in G_0\}$, $L_\sigma = \{(x,x) : x \in N_0\}$ (where $N_0 = \mathrm{nil}(G_0)$). Hence if $G_0 \neq H_0$, then $G \neq HG_\sigma$, i.e., Corollary 2.11 does not hold in this case. L_σ is not maximal nilpotent (since it is strictly contained in $N_0 \times N_0$), i.e., Lemma 2.14 (iii) does not hold. If $\mathcal{C}_0$ is a subset of $\mathrm{Aut}_1(G_0)$ as in Proposition 2.15 and for $\tau \in \mathrm{Aut}(G_0)$, $\tilde{\tau}(x,y) = (\tau(x), y)$, $\widetilde{\mathcal{C}} = \{\sigma\} \cup \{\tilde{\tau} : \tau \in \mathcal{C}_0\}$, then $L_{\widetilde{\mathcal{C}}} = \{(x,x) : x \in L_{\mathcal{C}_0}\}$ and for $\widetilde{\mathcal{C}}_1 = s(L_{\widetilde{\mathcal{C}}})$, we get $L_{\widetilde{\mathcal{C}}_1} = \{(x,y) : x, y \in L_{\mathcal{C}_0}\}$, thus $\widetilde{\mathcal{C}}_1$ is strictly contained in $s(L_{\widetilde{\mathcal{C}}_1})$, i.e., Lemma 2.14 (ii) does not hold.

3. Proofs of Theorem 1 and 2

Proof of Theorem 1. Let R be the (non-connected) radical, N the nilradical of G ([17] Prop. 3). By [17] Prop. 5, there exists a closed subgroup G_1 of R with finite index and such that $[G_1, G_1] \subseteq N \subseteq G_1$. Since R/R^0 is a discrete, finitely generated group of polynomial growth, it has a nilpotent subgroup of finite index. Hence, we can assume in addition that G_1/R^0 is nilpotent and (by easy arguments as in [17]) that G_1 is normal in G. It follows that G_1, N satisfy the assumptions of 2.1. Choose $\mathcal{C}$ as in Proposition 2.15 and put $L_1 = L_{\mathcal{C}} = (G_1)_{\mathcal{C}}$. Then by Proposition 2.15, L_1 is nilpotent and $G_1 = N^0 L_1$. Let L be the normalizer of L_1 in G. Proposition 2.22 implies $G = [G_1, N^0]\, L \subseteq N^0 L$.
We claim that $L \cap G_1 = L_1$. Put $L' = L \cap G_1$, $L'' = L \cap N^0$. Then $L' = L_1 L''$, L_1 and L'' are nilpotent normal subgroups of L, hence L' is nilpotent ([20] L. 4.7). Since L_1 is maximal (Proposition 2.15), we conclude that $L_1 = L'$, proving our claim. L_1 being nilpotent and normal in L, it follows that $\mathrm{nil}(L) \supseteq L_1$. Since $LG_1 = G$, we have $L/L_1 = L/(L \cap G_1) \cong LG_1/G_1 = G/G_1$ and this is compact by ([17] Prop. 4), finishing our proof. □

Remark 3.1. The argument shows that in fact $G = N^0 L$. The same proof works if G is a generalized $\overline{FC}$-group without non-trivial compact normal subgroups under the additional assumption that G/G^0 has polynomial growth (by the standard properties of [12], this assumption is equivalent to R/R^0 having polynomial growth – recall that G^0/R^0 is compact). In particular, the additional assumption is satisfied, if N is connected (by [17] Prop. 5).

With some further efforts, it can be shown that Theorem 1 (with $G = N^0 L$) is valid for arbitrary compactly generated Lie groups G of polynomial growth (if P denotes the maximal compact normal subgroup of N and P^0 is central in G, things are easier, using the generalizations mentioned in Remark 2.23 (h)). However, it does not hold for arbitrary generalized $\overline{FC}$-groups (see Example 3.2 below). If G is a generalized $\overline{FC}$-group and a Lie group, one can show the existence of a closed subgroup L such that $L/\mathrm{nil}(L)$ is compact and NL is an open subgroup of finite index in G (in the discrete case, i.e., G is a finite extension of a polycyclic group, this is [23] Cor. 2, p. 48, where $\mathrm{nil}(L)$ is called an almost-supplement for $\mathrm{nil}(G)$).

Example 3.2. The conclusion of Theorem 1 does not hold in general for discrete torsion free polycyclic groups (in particular not for arbitrary generalized $\overline{FC}$-groups). Take $A = \mathbb{Z}^n$, let α, β be two commuting automorphisms of A such that $\operatorname{im}(\alpha - \mathrm{id}) + \operatorname{im}(\beta - \mathrm{id}) \neq A$ and choose $v_0 \in A$ not belonging to the left side. We consider $G = (A \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ with the first action defined by α and for the second one, the "affine" action arising from β on A and $1 \circ (0,1) = (v_0, 1)$. Altogether, $G \cong \{(v,k,l) : v \in A,\ k,l \in \mathbb{Z}\}$ and the multiplication is $(v,k,l)\,(v',k',l') = \Big(v + \alpha^k\big(\beta^l(v') + (\beta^{l-1} + \cdots + \mathrm{id}) \circ (\alpha^{k'-1} + \cdots + \mathrm{id})\,(v_0)\big), k+k', l+l'\Big)$ for $l, k' > 0$. If for $(k,l) \neq (0,0)$ $\alpha^k \circ \beta^l - \mathrm{id}$ is always injective, it is easy to see that $N = A$ and any nilpotent subgroup B of G with $B \nsubseteq A$ satisfies $B \cap A = (e)$. But the choice of v_0 implies that G cannot be written as $A \rtimes B$ for some subgroup B of G.
Explicitly, for $n = 4$, we have $A \cong \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and α, β can be found as follows: $\alpha = \alpha_0 \otimes \mathrm{id}$, $\beta = \mathrm{id} \otimes \alpha_0$, where $\alpha_0 - \mathrm{id}$ is not surjective and the eigenvalues of α_0 are not roots of unity, e.g., take α_0 given by the matrix $\left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$. Here $\operatorname{im}(\alpha_0 - \mathrm{id})$ has index 2 in $\mathbb{Z}$ and it turns out that $\{(v,k,l) : (k,l) \in 4\mathbb{Z} \times \mathbb{Z}\}$ splits, i.e., the conclusion of Theorem 1 holds for this subgroup.

We add here a further structural property, partially extending [17] Prop. 6.

Proposition 3.3. *Let G be a generalized $\overline{FC}$-group without non-trivial compact normal subgroups. Then the nilradical N is a maximal nilpotent subgroup of G.*
Thus, if $x \in G$ and the induced automorphism of N is unipotent, then $x \in N$. In particular, the centralizer $C_G(N)$ equals the centre $Z(N)$.

Proof. We start with the statement on the centralizer $Z_1 = C_G(N)$. Clearly, $Z_1 \cap N = Z(N)$ and Z_1 is normal in G. Let R be the (non-connected) radical of G. Then it follows easily from maximality of N that $Z_1 \cap R \subseteq N$ (otherwise, consider the last non-trivial term of the derived series of the solvable group $(Z_1 \cap R)N/N$). Let R_1 be the radical of Z_1. It is a characteristic subgroup, hence maximality of R implies $R_1 \subseteq R$ and it follows that $R_1 = Z(N)$. Then by [17] Prop. 4 (applied to the generalized $\overline{FC}$-group Z_1), $Z_1/Z(N)$ is compact. Thus Z_1 is a Z-group in the sense of [10], in particular an FC^--group and it is compactly generated by [17] Prop. 2. Then by [10] Th. 3.20, $[Z_1, Z_1]^-$ is compact. Since G has no non-trivial compact normal subgroups, it follows that Z_1 is abelian, consequently $Z_1 = R_1 = Z(N)$.
For the general case, assume that M is a nilpotent subgroup of G containing N. We may assume M to be closed. Then by [17] Prop. 6, M/N is compact. We use the ascending central series $(e) = N_0 \subseteq \cdots \subseteq N_k = N$, recall that N_i/N_{i-1} is torsion free. For $x \in M$, we consider the automorphism $\varphi_i(x) \in \operatorname{Aut}(N_i/N_{i-1})$ induced by ι_x $(i = 1, \ldots, k)$. φ_i is a continuous homomorphism and $N \subseteq \ker \varphi_i$, hence $\operatorname{im} \varphi_i$ is compact. Since ι_x is unipotent on N, it follows that all φ_i are trivial, thus $[M, N_i] \subseteq N_{i-1}$ for all i. Let $\mathcal{H}$ be the set of those continuous automorphisms of N that induce the identity on N_i/N_{i-1} for $i = 1, \ldots, k$. Then it is easy to see that $\mathcal{H}$ is a nilpotent normal subgroup of $\operatorname{Aut}(N)$. For $x \in G$, let $\varphi(x) \in \operatorname{Aut}(N)$ be the restriction of ι_x. Then $H = \{x \in G : \varphi(x) \in \mathcal{H}\}$ is

a normal subgroup of G with $M \subseteq H$. By the first part of the proof, $\ker\varphi = C_G(N) = Z(N)$ and by the definition of $\mathcal{H}$ (take $i = 1$) $N_1 = Z(N) \subseteq Z(H)$. It follows that H is nilpotent and then maximality gives $H = N$, hence $M \subseteq N$. $\square$

Remark 3.4. This need not be true when there are non-trivial compact normal subgroups. Take e.g. a direct product of a compact semisimple group and a nilpotent group.

Corollary 3.5. *If G is as in Proposition 3.3, then $G_{\mathbb{R}}$ has no non-trivial compact normal subgroup. $N_{\mathbb{R}}$ is the nilradical of $G_{\mathbb{R}}$.*

Proof. We have $G_{\mathbb{R}} = N_{\mathbb{R}}\, G$ (see 2.1, taking $H = N$). Assume that P is a compact normal subgroup of $G_{\mathbb{R}}$. Since $N_{\mathbb{R}}$ is normal in $G_{\mathbb{R}}$ and torsion free, it follows that $[P, N_{\mathbb{R}}] = (e)$. Take $x \in P$, then $x = uv$ with $u \in N_{\mathbb{R}},\ v \in G$. Since ι_v coincides with $\iota_{u^{-1}}$ on $N_{\mathbb{R}}$, we get that ι_v is unipotent on $N_{\mathbb{R}}$, hence also on N. Thus $v \in N$, resulting in $P \subseteq P \cap N = (e)$.
It is easy to see that $G_{\mathbb{R}}$ is again a generalized $\overline{FC}$-group. If N_1 denotes its nilradical, then $N_1 \supseteq N_{\mathbb{R}}$ and (by maximality) $N_1 \cap G = N$, giving $N_1 = N_{\mathbb{R}}$. $\square$

Proof of Theorem 2.

(a) Assume that the nilradical N is connected and that G/N is compact. Then G is an almost connected Lie-group. Let K be a maximal compact subgroup. By [13] Th. XV.3.7, we have $G = NK$. Since N is torsion free, $N \cap K$ is trivial, thus $G \cong N \rtimes K$. Furthermore, $K \cap C_G(N)$ is easily seen to be normal in G, hence it must also be trivial, proving faithfulness of the action of K. So we may take $\widetilde{G} = G$ in this case (in fact, this argument just needs that N is some connected nilpotent normal subgroup for which G/N is compact, but then it is not hard to see that necessarily $N = \mathrm{nil}(G)$ holds).

(b) For the general case, it will be enough (using (a)) to show the existence of a Lie group $\widetilde{G}$ without non-trivial compact normal subgroups, having G as a closed subgroup such that $\widetilde{G}/G$ is compact, $\widetilde{N} = \mathrm{nil}(\widetilde{G})$ is connected and $\widetilde{G}/\widetilde{N}$ is compact. We take up the notations from the Proof of Theorem 1 above. Put $K_1 = \overline{\mathcal{C}}$. Then K_1 is a compact abelian subgroup of $\mathrm{Aut}(G_1)$ (recall that $G_1 = N^0 L_1$ and each $\sigma \in \mathcal{C}$ is the identity on L_1, thus it suffices to consider the restrictions to N^0; let $\mathfrak{n}$ be the Lie algebra of N^0, then $\mathrm{Aut}(N^0) \cong \mathrm{Aut}(\mathfrak{n})$ – using [25] Th. 2.7.5 and [13] Th. IX.1.2; by [16] Th. 1, for $x \in G_1$, the eigenvalues of $d\iota_x$ have modulus 1, hence the same is true for the eigenvalues of $(d\iota_x)_s$ and this equals $d\iota_{s(x)}$ by 2.5). Put $G_2 = G_1 \rtimes K_1$ and $N_2 = \mathrm{nil}(G_2)$. By Corollary 2.16, we have $G_2 \cong N_2 \rtimes K_1$ and G_2/G_1 ($\cong K_1$) is compact.
For $x \in L$, we define $\varphi(x)\,(y, \sigma) = (xyx^{-1}, \iota_{x,1} \circ \sigma \circ \iota_{x,1}^{-1})$, where $\iota_{x,1}$ denotes the restriction of ι_x to G_1 (compare [3] p. 243). Recall that $L_1 \triangleleft L$ and $\mathcal{C} = s(L_1)$, thus $\iota_{x,1} \circ \mathcal{C} \circ \iota_{x,1}^{-1} = \mathcal{C}$. Easy computations show that $\varphi(x) \in \mathrm{Aut}(G_2)$ and that φ is a homomorphism on L. For $x \in G_1 \cap L$ ($= L_1$), we get $\varphi(x) = \iota_{x,2}$, the corresponding inner automorphism of G_2. Then for $u \in G_1,\ x \in L$, we extend the definition by $\varphi(ux) = \iota_{u,2} \circ \varphi(x)$. Again one can check that $\varphi\colon G\,(= G_1 L) \to \mathrm{Aut}(G_2)$ is well defined and a continuous homomorphism.

For $z \in G$, the restriction of $\varphi(z)$ to G_1 is $\iota_{z,1}$ and for $z \in G_1$, we have $\varphi(z) = \iota_{z,2}$. This allows to apply Proposition 1.4, there exists a locally compact group G_3 (in fact a Lie group) having G_2, G as closed subgroups with G_2 normal, $G_2 \cap G = G_1,\ G_2\, G = G_3$ (consequently, $G_3 = GK_1$). Since G/G_1 is compact, it follows that G_3/G_2 is compact and then that G_3/N_2 and G_3/G are compact. In particular, by [12] Th. I.4, G_3 has polynomial growth. Put $N_3 = \mathrm{nil}(G_3)$. Then $G_2 \lhd G_3$ implies $N_2 \subseteq N_3$. Let P_3 be the maximal compact normal subgroup of G_3 ([17] Prop. 1) and put $G_4 = G_3/P_3$. Then $G \cap P_3$ is trivial and we get an embedding of G into G_4. For $N_4 = \mathrm{nil}(G_4)$, we have $N_4 \supseteq N_3P_3/P_3$. Since N_4 is torsion free, we can finish the construction by putting $\widetilde{N} = (N_4)_{\mathbb{R}},\ \widetilde{G} = (G_4)_{\mathbb{R}} = (N_4)_{\mathbb{R}} G_4$ (see 2.1, with $H = N_4$) which has the required properties (it has no non-trivial compact normal subgroups by Corollary 3.5; alternatively we could factor once more by the maximal compact normal subgroup; see also the comment to Corollary 4.9 for further explanations). $\square$

Corollary 3.6. *Let G be a compactly generated group of polynomial growth without non-trivial compact normal subgroups. Then G has a faithful finite dimensional representation, i.e., for appropriate $n > 0$ there exists an injective continuous homomorphism $\pi : G \to \mathrm{GL}(\mathbb{R}^n)$ such that $\pi(G)$ is closed and $\pi(N)$ (where $N = \mathrm{nil}(G)$) consists of upper triangular unipotent matrices.*
Moreover, there exists such a faithful representation π having the additional property that the eigenvalues of $\pi(x)$ are of modulus 1 for all $x \in G$.

In [1] a (real or complex) linear group is called *distal*, if every eigenvalue of its elements has absolute value 1.

Proof (compare [26] Th. 3). By Theorem 2, we can assume that $G = \widetilde{N} \rtimes K$, where K is compact, $\widetilde{N}$ is connected nilpotent torsion free and the action of K on $\widetilde{N}$ is faithful. Thus K can be considered as a closed subgroup of $\mathrm{Aut}(\widetilde{N})$. Now we use the Birkhoff embedding theorem ([26] p. 16, [3] p. 239). It gives a faithful representation π of $\widetilde{N}$ by upper triangular unipotent matrices together with a representation of $\mathrm{Aut}(\widetilde{N})$ (where $\pi(\sigma)$ is semisimple for σ semisimple), combining to a representation of $\widetilde{N} \rtimes \mathrm{Aut}(\widetilde{N})$. Note that in particular $\pi(K)$ acts faithfully on $\pi(\widetilde{N})$. By compactness, the (complex) eigenvalues of $\pi(x)$ have modulus 1 for $x \in K$. Fixing $x \in K$, $\pi(\langle x \rangle \widetilde{N})$ is trigonalizable (over $\mathbb{C}$) by [26] (2.2). Hence the eigenvalues of $\pi(xy)$ have modulus 1 for $y \in \widetilde{N}$. $\square$

Remark 3.7. By some additional arguments, one can also prove an analogue of Theorem 2 for a generalized $\overline{FC}$-group G without non-trivial compact normal subgroups. One gets an embedding into some group $\widetilde{G} = \widetilde{N} \rtimes S$ such that $\widetilde{N}$ is connected, simply connected, nilpotent and S is an almost connected SIN-group (i.e., by [10] Th. 2.9, $S \cong V \rtimes K$, where $V \cong \mathbb{R}^n,\ K$ is compact and K^0 acts trivially on V) such that the action of S on $\widetilde{N}$ is semisimple. G becomes a closed subgroup, but one can no longer expect that $\widetilde{G}/G$ is compact. As in Corollary 3.6, one gets again faithful finite dimensional representations.

In [28] Th. 3 (for discrete groups) an intermediate type of embeddings is studied. By splitting only the compact part K, a generalized $\overline{FC}$-group G without non-trivial compact normal subgroups can be embedded as a closed, co-compact subgroup into a semidirect product $\widetilde{S} \rtimes K$ where $\widetilde{S}$ is a connected, simply connected and super-solvable Lie group (i.e., $\widetilde{S}$ has a faithful representation by real triangular matrices), K compact, acting faithfully on $\widetilde{S}$. [28] Cor. 4 gives for G discrete a uniqueness result, similar to our Theorem 3, for this type of embedding. This should extend to general G.

Such embeddings are related to the semisimple splittings of [3] (see also [23] Ch. 7; by considering closures in the automorphism group, we arrive at somewhat bigger groups in the non-discrete case). To point out the differences, note that in Theorem 2 we get $\widetilde{N} = \mathrm{nil}(\widetilde{G})$ to be connected which entails that GK can be a proper subset of $\widetilde{G}$ (not even a subgroup in general) and G need not be normal in $\widetilde{G}$. Since we want K to be compact, $G\widetilde{N}$ will in general be only a dense subgroup of $\widetilde{G}$. On the other hand, we do not require that G/N is torsion free. See also Remarks 4.11 for further discussion.

In [24] rather general splitting results are stated. But the handling of the definitions is not always consistent and the presentation is rather intransparent. Therefore, we have decided to rely on the earlier version of Wang [26] as a basis of our exposition. We have tried to avoid too much use of results from algebraic groups (this might also give some shorter arguments in Section 4).

With almost the same proof Theorem 2 extends to the case where only N (instead of G) has no non-trivial compact normal subgroup. But if C, the maximal compact subgroup of G, is non-trivial, then for any embedding as in Theorem 2 (with $\widetilde{N}$ simply connected), C (that embeds into K, see also Proposition 4.8 (c)) will act trivially on $\widetilde{N}$. Thus the action of K on $\widetilde{N}$ will no longer to be faithful. If G is a compactly generated l.c. group of polynomial growth satisfying the assumptions of 2.1, one can (after first passing to $G_{\mathbb{R}} = H_{\mathbb{R}}G$) use the modification sketched in Remark 2.17 to get an embedding (as a closed subgroup) into a group $\widetilde{N} \rtimes K$ where $\widetilde{N}$ is simply connected, nilpotent, K compact abelian. This contains some further examples there C is non-trivial.

More generally, using the generalizations mentioned in Remark 2.23 (h)), one can extend a large part of the proof of Theorem 2 (up to G_3) to the case where G is a compactly generated l.c. group of polynomial growth with maximal compact subgroup C and there exists a closed normal subgroup H such that G/H is compact and $\iota_x|C$ is unipotent for all $x \in H$. But there are examples of (non torsion free) compactly generated nilpotent Lie groups that cannot be embedded into a connected nilpotent group. Hence the last step of the argument will fail in general and this produces only a certain analogue of the groups G_{an} described in Proposition 3.8.

If G is any compactly generated Lie group of polynomial growth, the following properties can be shown to be equivalent:

(a) G has a faithful (continuous) finite dimensional representation.
(b) G has a closed normal subgroup H such that G/H is compact and H has no non-trivial compact normal subgroup.

(c) G has a closed normal subgroup H such that G/H is compact and $[G^0H, H]^{-\,0}$ is torsion free.
(d) R (the non-connected radical of G) has a subgroup R_1 of finite index such that $[R_1, R_1]^-$ is nilpotent and torsion free.

Then the group R_1 in (d) can be chosen to be G-invariant and the group H in (b) can be found so that $[R_1, R_1] \subseteq H \subseteq R$. There exists a faithful finite dimensional representation π such that $\pi(G)$ is closed and distal. There is also an embedding of G as a closed subgroup of some $\widetilde{G} \cong \widetilde{N} \rtimes K$ such that K is compact, $\widetilde{N}$ connected nilpotent torsion free, $\widetilde{G}/G$ compact. But the action of K on $\widetilde{N}$ need not be faithful and N need not be contained in $\widetilde{N}$ (recall that in case $\pi(N)$ consists of unipotent transformations, N must be torsion free).

A classical case where this is satisfied are finitely generated (discrete) groups of polynomial growth (and more generally, extensions of polycyclic groups by finite groups, not necessarily torsion free). Here one can even get a faithful representation by integer-valued matrices ([23] Th. 5, p. 92).

For a compactly generated Lie nilpotent group G one gets that G has a faithful (continuous) finite dimensional representation iff $[G, G]^{-\,0}$ (the identity component of the topological commutator group) is torsion free (this extends the characterization of [13] Th. XVIII.3.2 in the connected case; for connected G it follows that $[G, G]$ must already be closed, but this need not be true in the non-connected case).

The construction used in the proof of Theorem 2 gives also a smaller almost nilpotent extension.

Proposition 3.8. *Let G be a compactly generated group of polynomial growth without non-trivial compact normal subgroups.*
(i) *There exists a Lie group G_{an} containing G as a closed subgroup such that G_{an}/G, G_{an}/N_{an} are compact (where $N_{an} = \mathrm{nil}(G_{an})$) and $G_{an} = \overline{N_{an}G}$.*
Furthermore, G_{an} has no non-trivial compact normal subgroup, $G \cap N_{an} = N$, $[N_{an}, N_{an}] \subseteq N$ (in particular, N is normal in G_{an}) and there exists a compact connected abelian subgroup K_1 of G_{an} such that $G_{an} = GK_1$ and $K_1 \cap C_{G_{an}}(G) = \{e\}$. The group $N_{an}K_1 \cap G$ is G_{an}-invariant and contained, with finite index, in R (radical of G).
(ii) *If G is almost connected, then $N_{an} = \widetilde{N}$ is connected and $G_{an} = \widetilde{G}$ coincides with the group of Theorem 2.*

Proof. We take $G_{an} = G_4$, as constructed in the proof of Theorem 2. Then (i) follows (replacing G_1 by some subgroup of finite index, one can always achieve that K_1 is connected, see also Corollary 4.9 and the comment there).
If G is almost connected (since it is a Lie group, this means that G/G^0 is finite), we can take $G_1 = R^0$. Then Corollary 2.16 shows that N_2 is connected. N_2 being co-compact in G_3, the same is true for its image N_2P_3/P_3 in G_4. Since N_2P_3/P_3 is connected and

N_4 torsion free, it follows (e.g. [20] Rem. 2.6) that N_4 equals N_2P_3/P_3. Hence $\widetilde{G} = G_4$ (alternatively, one could use Corollary 4.9; then Proposition 4.4 (a) implies that $\widetilde{N} \cap G^0 K_1$ is a co-compact connected subgroup of $\widetilde{N}$, hence $\widetilde{N} \subseteq G^0 K_1$). $\square$

4. Subgroups of semidirect products

As a preparation for the proof of Theorem 3, we start with two technical lemmas and then we collect some properties of subgroups of the semidirect products that arise in Theorem 2 (in particular, consequences of the assumption "$\widetilde{G}/G$ compact"). This allows to give an explicit description of the decomposition of Theorem 1 (Corollary 4.9) and to identify (Proposition 4.8) various constituents that came up in [17].

To make the induction arguments easier, we consider now also nilpotent Lie groups N that are not torsion free. Analogously to Definition 2.3, $\sigma \in \mathrm{Aut}(N)$ is called *semisimple*, if the corresponding transformation $d\sigma$ of the Lie algebra $\mathfrak{n}$ of N is semisimple and $N = N^0 N_\sigma$. If N' is a closed normal σ-invariant subgroup of N, it is easy to see that the induced automorphism on N/N' is again semisimple.

Lemma 4.1. *Let N be a nilpotent Lie group, $\mathcal{K}$ a connected subgroup of* $\mathrm{Aut}(N)$ *consisting of semisimple transformations and let N_1 be a closed connected $\mathcal{K}$-invariant subgroup of N. Then for $\sigma \in \mathcal{K}$, we have* $(\mathrm{ad}\,\sigma)(N) \cap N_1 = (\mathrm{ad}\,\sigma)(N_1)$.

Proof. Semisimplicity implies $(\mathrm{ad}\,\sigma)(N) = (\mathrm{ad}\,\sigma)(N^0)$, thus we may assume N connected. Let $\widetilde{N}$ be the universal covering group of N, $\pi : \widetilde{N} \to N$ the canonical projection, $\Gamma = \ker \pi$, $\widetilde{N}_1 = \pi^{-1}(N_1)^0$. Denote by $\widetilde{\mathcal{K}}$ the group of lifted automorphisms $\tilde{\sigma}$. Consider $x \in N$ with $\sigma(x)\,x^{-1} \in N_1$ and take $\tilde{x} \in \widetilde{N}$ with $\pi(\tilde{x}) = x$. Then $\tilde{\sigma}(\tilde{x})^{-1}\tilde{x}^{-1} \in \pi^{-1}(N_1) = \Gamma\widetilde{N}_1$. Let $\widetilde{P}$ be the analytic subgroup of $\widetilde{N}$ generated by Γ (we have $\Gamma \subseteq Z(\widetilde{N})$; if this is written additively, $\widetilde{P}$ is just the vector subspace generated by Γ). Then $M = \widetilde{P}\widetilde{N}_1$ is an analytic subgroup of $\widetilde{N}$, $\tilde{\sigma}(\tilde{x})\,\tilde{x}^{-1} \in M$. By [26] (5.6), we have $\tilde{x} = \tilde{y}\,z$ with $\tilde{y} \in M$, $z \in \widetilde{N}_{\tilde{\sigma}}$. Since $\widetilde{\mathcal{K}}$ ($\cong \mathcal{K}$) is connected, it has to be trivial on Γ, hence also on $\widetilde{P}$, thus we can assume that $\tilde{y} \in \widetilde{N}_1$. Put $y = \pi(\tilde{y})$, then $y \in N_1$, $\sigma(y)\,y^{-1} = \sigma(x)\,x^{-1}$. $\square$

Corollary 4.2. *Let N, $\mathcal{K}$ be as above.*
(a) *If $\sigma \in \mathcal{K}$ and N_1 is any closed $\mathcal{K}$-invariant subgroup, then $\sigma|N_1$ is again semisimple.*
(b) *If $\mathcal{K}$ is commutative, then $N = N^0 N_{\mathcal{K}}$.*
(c) *If $\mathcal{K}$ is commutative and N' is any closed normal $\mathcal{K}$-invariant subgroup, then $N_{\mathcal{K}}N'/N' = (N/N')_{\mathcal{K}}$.*

Proof. (a): If $x \in N_1$, then $\sigma(x)\,x^{-1} \in N_1^0$ ($\mathcal{K}$ is connected), hence by Lemma 4.1, $x = y\,z$ with $y \in N_1^0$, $z \in N_\sigma$.
(b), (c): As in the proof of Corollary 2.11, there is a finite subset $\mathcal{K}_0$ of $\mathcal{K}$ such that $N_{\mathcal{K}} = N_{\mathcal{K}_0}$. Then an easy induction argument (see also [26] (8.8)) proves our claim. $\square$

Lemma 4.3. *Let $\mathcal{G}$ be a triangular group of automorphisms of $N = \mathbb{R}^n$ and assume that $\mathcal{G}$ is nilpotent and that the closure $\mathcal{K}$ of $s(\mathcal{G})$ (the semisimple parts) is connected. Then any closed $\mathcal{G}$-invariant subgroup H of N is $\mathcal{K}$-invariant.*

Proof. By Lemma 2.7, $\mathcal{K}$ is commutative and centralizes $\mathcal{G}$. If $\mathcal{N}$ denotes the set of unipotent transformations in $\mathcal{G}\mathcal{K}$, then $\mathcal{N}$ is a subgroup. For $\sigma \in \mathcal{G}$, $s(\sigma)$ is a polynomial in σ, hence the result follows immediately when H is connected. Thus (passing to N/H^0), we may assume that H is discrete and that it generates N as a real vector space.
First, we claim that $N_{\mathcal{K}} \cap H$ is non-trivial. For $\sigma \in \mathcal{G}$, we consider its restriction to the $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ $(= \mathbb{Q}H)$. It has a Jordan decomposition (for the base field $\mathbb{Q}$, [5] VII, Th. 1, p. 42) and by uniqueness, the real extensions of the components have to coincide with σ_s, σ_u. Thus $H_{\mathbb{Q}}$ is $s(\mathcal{G})$-invariant and $\mathcal{N}$-invariant. $\mathcal{N}$ is still triangular and it follows easily that $N_{\mathcal{N}} \cap H_{\mathbb{Q}}$ is non-trivial, hence $N_{\mathcal{N}} \cap H$ is non-trivial. Since $N_{\mathcal{N}} \cap H$ is $\mathcal{G}$-invariant and $\mathcal{N}$ acts trivially on this group, it is also $\mathcal{K}$-invariant. $\mathcal{K}$ being connected, $N_{\mathcal{N}} \cap H$ discrete, it follows that $\mathcal{K}$ acts trivially on $N_{\mathcal{N}} \cap H$, i.e., $N_{\mathcal{N}} \cap H \subseteq N_{\mathcal{K}}$.
As observed in the proof of Corollary 4.2, there is a finite subset $\mathcal{K}_0$ of $\mathcal{K}$ such that $N_{\mathcal{K}_0} = N_{\mathcal{K}}$. For $\sigma \in \mathcal{K}$, the projection to N_σ obtained from the primary decomposition is a rational polynomial in σ ([25] Th. 3.1.1). Combined, this gives a projection p to $N_{\mathcal{K}}$ which is a rational polynomial in elements of $\mathcal{K}$. Put $M = (\mathrm{id} - p)(N)$, defining a complementary subspace for $N_{\mathcal{K}}$, invariant under $\mathcal{G}$ and $\mathcal{K}$. Since $H_{\mathbb{Q}}$ is p-invariant, it follows that $(\mathrm{id} - p)(H_{\mathbb{Q}}) \subseteq H_{\mathbb{Q}}$, hence $H_1 = (\mathrm{id} - p)(H_{\mathbb{Q}}) \cap H$ generates M as a real vector space. Thus, if M would be non-trivial, the same would be true for H_1 and the argument above would imply that $N_{\mathcal{K}} \cap H_1$ is non-trivial which is impossible. It follows that $N_{\mathcal{K}} = N$, finishing the proof. $\square$

Proposition 4.4. *Let $\widetilde{N}$ be a compactly generated nilpotent Lie group, K an abelian connected locally compact group with a continuous semisimple action on $\widetilde{N}$. Let G be a subgroup of $\widetilde{G} = \widetilde{N} \rtimes K$ such that $N = G \cap \widetilde{N}$ is closed in $\widetilde{N}$, $\widetilde{G}/G$ is compact and $\widetilde{N}G$ dense in $\widetilde{G}$. Then the following properties hold.*
(a) *Put $\widetilde{M} = \{x \in \widetilde{N}\colon\ k \circ x = x \text{ for all } k \in K\}$, $\widetilde{L} = \widetilde{M} \times K$, $L = G \cap \widetilde{L}$.*
Then G and N are K-invariant, $G = N^0 L$, $\widetilde{G} = N^0 \widetilde{L}$, $\widetilde{N} = N^0 \widetilde{M}$ and $\widetilde{N}/(\widetilde{N} \cap GK)$ $(\cong \widetilde{M}/(\widetilde{M} \cap GK))$ is compact.
(b) *If $Z(\widetilde{N})GK$ is dense in $\widetilde{G}$, then $[\widetilde{G}, \widetilde{G}] \subseteq N$ (in particular, G is normal in $\widetilde{G}$, $\widetilde{M}/(\widetilde{M} \cap N)$ is abelian).*
(c) *If H is a connected, closed G-invariant subgroup of $\widetilde{N}$ and $\widetilde{N}$ is torsion free, then H is normal in $\widetilde{G}$.*
(d) *If H is any closed G-invariant subgroup of $\widetilde{N}$, then H is K-invariant (in particular, if $Z(\widetilde{N})GK$ is dense in $\widetilde{G}$, then H is normal in $\widetilde{G}$).*

By a semisimple action on $\widetilde{N}$ (denoted by $\circ$), we mean that each transformation shall be semisimple (see also Lemma 4.7). We do not require G to be closed. By [15] (5.24 b), compactness of $\widetilde{G}/G$ is equivalent to compactness of the Hausdorff space $\widetilde{G}/\overline{G}$.

Proof. (α) First, we assume that $\widetilde{N}$ is abelian and that $G \cap \widetilde{N}$ and $\widetilde{M}$ are trivial. We claim that this implies $\widetilde{N}$ to be trivial. Connectedness of K easily implies (use the dual action) that any continuous action on a compact abelian group is trivial. Hence $\widetilde{N}$ must be torsion free and, replacing $\widetilde{N}$ by $\widetilde{N}_{\mathbb{R}}$, we can assume that $\widetilde{N}$ is connected, i.e., $\widetilde{N} \cong \mathbb{R}^n$, written additively. Let $K' = \widetilde{N}G \cap K$ be the projection of G to K ($\cong \widetilde{G}/\widetilde{N}$). Triviality of $G \cap \widetilde{N}$ gives a mapping $c\colon K' \to \widetilde{N}$ such that $G = \{\, c(x)\, x : x \in K'\}$ and c is a crossed homomorphism, i.e., $c(xy) = c(x) + x \circ c(y)$ for $x, y \in K'$. Then commutativity of K leads to $x \circ c(y) - c(y) = y \circ c(x) - c(x)$. Triviality of $\widetilde{M}$ implies that for each $v \in \widetilde{N} \setminus \{0\}$ there exists $x \in K$ such that $x \circ v \neq v$. Assume that $\widetilde{N}$ is non-trivial, i.e., $n > 0$. Considering the root space decomposition for the extended action of K on $\mathbb{C}^n$, it is easy to see (using that K is connected) that there exists $x_0 \in K$ for which all roots are different from 1, i.e., $\alpha(v) = x_0 \circ v - v$ is an isomorphism on $\widetilde{N}$ (compare [6] p. 28). By assumption, K' is a dense subgroup of K, hence we can assume that $x_0 \in K'$. Putting $v_0 = -\alpha^{-1}\big(c(x_0)\big)$, it follows that $c(x) = v_0 - x \circ v_0$ for all $x \in K'$. This would imply that $G = v_0 K'(-v_0)$ is conjugate to K', contradicting compactness of $\widetilde{G}/G$.

(β) Now we assume just that $\widetilde{N} \cap G$ is trivial and claim that $\widetilde{M} = \widetilde{N}$ and that GK is abelian. Observe that if N' is a closed $\widetilde{G}$-invariant subgroup of $\widetilde{N}$, we can consider $\widetilde{G}_1 = (\widetilde{N}/N') \rtimes K$ and then the image G_1 of G in this quotient satisfies again the assumptions of the Proposition (recall that $\widetilde{N} \cap G$ is trivial). First, if $\widetilde{N}$ is abelian, we take $N' = \widetilde{M}$. Then by Corollary 4.2 (c), $(\widetilde{N}/N')_K$ is trivial, hence (α) implies $\widetilde{M} = \widetilde{N}$. In the general case, we consider $N' = [\widetilde{N}, \widetilde{N}]^-$. Then the abelian case (and again Corollary 4.2 (c)) gives $\widetilde{N} = [\widetilde{N}, \widetilde{N}]^- \widetilde{M}$. But it is well known (using induction on the nilpotency-class) that this implies $\widetilde{M} = \widetilde{N}$. Clearly $[\widetilde{G}, \widetilde{G}] \subseteq \widetilde{N}$, hence triviality of $\widetilde{N} \cap G$ implies that G is abelian, thus GK is abelian (in fact, a little further argument shows that $[\widetilde{N}, \widetilde{N}]^-$ must be compact in this case).

In steps (γ), (δ) the Proposition will be proved by induction on the nilpotency-class n of $\widetilde{N}$. The result is trivial when $\widetilde{N}$ is trivial, hence we assume now that (a) holds for $G_1, \widetilde{G}_1, \widetilde{N}_1$, when $\widetilde{N}_1$ has nilpotency-class smaller than n.

(γ) We will now prove (c) and (d). Consider $N' = Z(\widetilde{N})$, $\widetilde{N}_1 = \widetilde{N}/N'$ and $\overline{G}_1$ the closure of the image of G in $\widetilde{G}_1 = \widetilde{N}_1 \rtimes K$. The action of $\widetilde{G}$ on $\widetilde{N}$ by inner automorphisms induces an action of $\widetilde{G}_1$ on $\widetilde{N}$ and on $\tilde{\mathfrak{n}}$ (the Lie algebra of $\widetilde{N}$). The inductive assumption gives a decomposition $\overline{G}_1 = N_1^0 L_1$ with $L_1 \subseteq \widetilde{M}_1 \times K$. For $x = y\,z \in L_1$ with $y \in \widetilde{M}_1$ ($\subseteq \widetilde{N}/N'$), $z \in K$, the corresponding operators on $\tilde{\mathfrak{n}}$ implement (by uniqueness) the Jordan decomposition for the operators from L_1 (see also Corollary 4.5).

Assume that H is connected with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h}$ is L_1-invariant, hence it is also invariant under the semisimple parts. Thus $\mathfrak{h}$ is K'-invariant, where K' denotes the projection of L_1 to K. Since K' contains the projection of G to K, it follows that $\mathfrak{h}$ is K-invariant. This proves (d) when H is connected.

For (c) observe that by (a) (for $\widetilde{G}_1$), $\overline{G}_1$ is K-invariant, hence $\big(Z(\widetilde{N})GK\big)^-$ is a subgroup of $\widetilde{G}$, $\mathfrak{h}$ is invariant under this subgroup and $\widetilde{N} \cap \big(Z(\widetilde{N})GK\big)^-$ must be co-compact in $\widetilde{N}$

(since $\widetilde{G}/G$ is compact). Then $\mathfrak{h}$ is $\widetilde{N}$-invariant by [20] Th. 2.3, Cor. 2 (after extending the action of $\widetilde{N}$ on $\tilde{\mathfrak{n}}$ to $\widetilde{N}_{\mathbb{R}}$, using [20] Th. 2.11). This proves (c).
For the general case of (d) we first assume that $\widetilde{N}$ is torsion free. Then by (c), H^0 is normal. Replacing $\widetilde{N}$ by $\widetilde{N}/H^0$, we can assume that H is discrete. In addition (replacing $\widetilde{N}$ by $\widetilde{N}_{\mathbb{R}}$), we may assume that $\widetilde{N}$ is connected. Let D be the additive subgroup of $\tilde{\mathfrak{n}}$ generated by $\log H$. By [20] Th. 2.12 (see the detailed version on p. 34), D is discrete and clearly G-invariant. Considering $\widetilde{N}_1 = \widetilde{N}/Z(\widetilde{N})$ as above, it follows from Lemma 4.3 (and the inductive assumption) that D is K-invariant. K connected implies that K acts trivially on D, hence it is trivial on H.
If $\widetilde{N}$ is not torsion free, let P be the maximal compact normal subgroup. Then (passing to $\widetilde{N}/P$), it follows that PH is K-invariant. By Corollary 4.2 (a), K acts semisimply on PH and P. Since P^0 is abelian, K acts trivially on P^0. PH is isomorphic to a quotient of $P \rtimes H$, hence $(PH)^0 = P^0H^0 = H^0P^0$. For $x \in H$ we have by Corollary 4.2 (b), $x = y\,x_0$ with $y \in H^0$, $k \circ x_0 = x_0$ for all $k \in K$. Then we get $(k \circ x)x^{-1} = (k \circ y)y^{-1}$, hence $k \circ x \in H$.

(δ) Next, we prove (a) and (b). First, we assume that $Z(\widetilde{N})GK$ is dense in $\widetilde{G}$. By (d), N is normal in $\widetilde{G}$. Thus, taking $N' = N$, $\widetilde{G}_1 = \widetilde{N}/N' \rtimes K$, it follows from ($\beta$) and Corollary 4.2 (c) that $\widetilde{N} = N\widetilde{M}$ and that (G_1 denoting the image of G in $\widetilde{G}_1$) G_1K is abelian. This implies that $\widetilde{G}_1$ is abelian, thus $[\widetilde{G}, \widetilde{G}] \subseteq N$, proving (b). Furthermore, by (a) and (b) of Corollary 4.2, $N = N^0(\widetilde{M} \cap N)$, thus $\widetilde{N} = N^0\widetilde{M}$.
In the general case, it follows from the induction hypothesis (see (γ)) that $\widetilde{G}_2 = \big(Z(\widetilde{N})GK\big)^-$ is a K-invariant subgroup of $\widetilde{G}$, containing G. Put $\widetilde{N}_2 = \widetilde{G}_2 \cap \widetilde{N}$, then $\widetilde{G}_2 = \widetilde{N}_2 \rtimes K$ and from the special case above, we get that G and N are K-invariant and that $\widetilde{N}_2 \subseteq N^0\widetilde{M}$. Now, if $\widetilde{N}$ is torsion free, we can (passing to $\widetilde{N}_{\mathbb{R}}$) assume that it is connected as well. By (c), N^0 is normal in $\widetilde{N}$, hence $N^0\widetilde{M}$ is a connected subgroup of $\widetilde{N}$. $\widetilde{N}_2$ is clearly co-compact in $\widetilde{N}$, hence by [25] Th. 3.18.2, $N^0\widetilde{M} = \widetilde{N}$. If $\widetilde{N}$ is not torsion free, let P be its maximal compact normal subgroup. Then the previous argument, applied to $\widetilde{N}/P$ (and combined with Corollary 4.2 (c)), gives $\widetilde{N} = (PN)^0\widetilde{M}$. As noted in ($\gamma$), $(PN)^0 = N^0P^0$ and $P \subseteq \widetilde{M}$, leading again to $\widetilde{N} = N^0\widetilde{M}$. The remaining properties follow easily. □

Corollary 4.5. *Let $G, \widetilde{G}$ be as in Proposition 4.4 and assume that G is closed in $\widetilde{G}$, $N = G \cap \widetilde{N}$ torsion free, K a Lie group. Then G, N $(= H)$ satisfy the assumptions of 2.1. L is nilpotent, $\iota_G(GK) \subseteq \mathrm{Aut}_1(G)$. For $x = y\,z \in L$ with $y \in \widetilde{M}$, $z \in K$, we have $\iota_G(y) = \iota_G(x)_u$, $\iota_G(z) = \iota_G(x)_s$.*
Put $K' = K \cap \widetilde{N}G$. Then $\mathcal{C}_0 = \mathcal{C} = \iota_G(K')$ satisfy the properties of Proposition 2.15. One gets $L = G_{\mathcal{C}} = L_{\mathcal{C}}$, $\mathcal{C} = s(L)$ and $\overline{\mathcal{C}} = \iota_G(K)$.

Taking $G = \widetilde{G}$, similar statements hold for $\widetilde{G}$ when $\widetilde{N}$ is torsion free. As before, $\iota_G(x)$ denotes the restriction of the inner automorphism $\iota_{\widetilde{G}}(x)$ to G.

Proof. $L < \widetilde{M} \times K$ is clearly nilpotent, $[G,G] \subseteq N$ and $G/N^0 \cong L/(L \cap N^0)$ (by Proposition 4.4 (a)). The remaining properties are easy. $L = L_{\mathcal{C}} = G_{\mathcal{C}}$ follows from density of K' in K. Maximality of $\mathcal{C}$ in $s(G)$ follows from Remark 2.23 (d). □

Corollary 4.6. *Let $G, \widetilde{G}$ be as in Proposition 4.4 and assume that $\widetilde{N}$ is torsion free. Then $[\widetilde{N},\widetilde{N}] \cap [G,G]$ $(\subseteq [\widetilde{N},\widetilde{N}] \cap N)$ is a co-compact subgroup (not necessarily closed) of $[\widetilde{N},\widetilde{N}]$.*

Proof. Proposition 4.4 (b) gives $[GK, GK] \subseteq N$. If N^0 is central in G, it follows similarly as in the proof of Proposition 4.4 (c) that N^0 is central in $\widetilde{G}$. In the general case (using that $[G, N^0]$ is normal in $\widetilde{G}$ by Proposition 4.4 (c)), this implies that $[\widetilde{G}, N^0] = [G, N^0]$ holds. Thus we can factor by $[\widetilde{N}, N^0]$. Then N^0 is central in $\widetilde{N}$. By Proposition 4.4 (a), we get that $[\widetilde{N},\widetilde{N}] = [\widetilde{M},\widetilde{M}]$ and that $\widetilde{M} \cap GK = \widetilde{M} \cap LK$ is co-compact in $\widetilde{M}$. Since $L \subseteq \widetilde{M} \times K$, it follows that $[L, L] = [LK, LK] = [\widetilde{M} \cap LK, \widetilde{M} \cap LK]$ is co-compact in $[\widetilde{M},\widetilde{M}]$, giving the desired conclusion. □

Lemma 4.7. *Let $\widetilde{N}$ be a compactly generated torsion free nilpotent Lie group, K a connected compact Lie group with a continuous action on $\widetilde{N}$, put $\widetilde{G} = \widetilde{N} \rtimes K$. Then the following holds.*
(i) *The action of K is semisimple (as defined after Proposition 4.4).*
(ii) *If G is a closed subgroup of $\widetilde{G}$ such that $\widetilde{N}G$ is dense in $\widetilde{G}$, then $[K, K] \subseteq \widetilde{N}G^0$.*

Proof. (i): Connectedness of K implies that the action on $\widetilde{N}/\widetilde{N}^0$ is trivial. For $x \in K$, the restricted transformation on $\widetilde{N}^0$ is semisimple (by compactness). Then as in the step (iii) ⇒ (i) of the proof of Lemma 2.2, semisimplicity on $\widetilde{N}$ follows easily.
(ii): This is related to a theorem of Auslander, Wang and Zassenhaus (compare the proof of [21] Th. 4.3). $\widetilde{G}$ has polynomial growth, hence the same is true for G and G/G^0 ([12] Th. I.3, I.4). Thus by Gromov's theorem (passing to a nilpotent subgroup of finite index), we can assume that G/G^0 is nilpotent. Put $K' = (\widetilde{N}G) \cap K,\ K'' = (\widetilde{N}G^0) \cap K$. We have $K = [K, K]\, Z(K)^0$ and $Z_1 = [K, K] \cap Z(K)^0$ is finite ([14] Cor. 6.16). Let $\varphi\colon K \to K/Z(K) \cong [K, K]/Z_1$ be the quotient mapping. G^0 maps continuously onto K'', hence $\varphi(K'')$ is an analytic subgroup of the semisimple group $[K, K]/Z_1$ and K'-invariant. Since K' is dense in K, it follows (considering the Lie algebra) that $\varphi(K'')$ is normal and that it is closed ([25] Th. 4.11.6). K'/K'' is isomorphic to a quotient of G/G^0, hence it is nilpotent. $[K, K]$ being semisimple, it follows that $\varphi(K'')$ has finite index in $\varphi(K')$, hence $\varphi(K')$ is closed as well and dense, giving $\varphi(K') = \varphi(K'') = [K, K]/Z_1$, thus $[K, K] \subseteq K''Z_1$. K'' being an analytic subgroup of K, we arrive (considering the Lie algebra) at $[K, K] \subseteq K''$. □

Proposition 4.8. *Let $\widetilde{N}$ be a compactly generated torsion free nilpotent Lie group, K a compact Lie group with a continuous action on $\widetilde{N}$. Let G be a closed subgroup of $\widetilde{G} = \widetilde{N} \rtimes K$ such that $\widetilde{N}G$ is dense in $\widetilde{G}$ and $\widetilde{G}/G$ compact. Put $N = G \cap \widetilde{N}$. Then the following properties hold.*

(a) *If R_K denotes the (non-connected) radical of K, then $R = G \cap (\widetilde{N} \rtimes R_K)$ is the radical of G.*
(b) *For $K_1 = Z(K^0)^0$, the groups $\widetilde{G}_1 = \widetilde{N} \rtimes K_1,\ G_1 = G \cap \widetilde{G}_1$ satisfy the assumptions of Proposition 4.4. Every connected semisimple subgroup of $\widetilde{G}$ is contained in G (in particular, $[K^0, K^0]$ is a Levi subgroup of G^0). N^0 is a normal subgroup of $\widetilde{G}$ and if $\widetilde{N}$ is connected, then $N_{\mathbb{R}}$ is normal as well.*
(c) *Assume that the action of K is faithful. Then G has no non-trivial compact normal subgroups and N is the nilradical of G. If $x \in \widetilde{G}$ normalizes N and acts unipotently on N, then $x \in \widetilde{N}$. We have $C_{\widetilde{G}}(G) = Z(\widetilde{G}) \subseteq Z(\widetilde{N})$.*
(d) *Assume that K is connected. Then $K \cap G$ is a maximal compact subgroup of G. If C is any compact subgroup of $\widetilde{G}$, there exists $n \in N^0$ such that $n\,Cn^{-1} \subseteq K$, G is C-invariant, $[G^0C, \widetilde{N}] \subseteq N^0$.*
(e) *Assume that K is abelian. Then $[G, G]$ ($\subseteq N$) is co-compact in $[\widetilde{G}, \widetilde{G}]^-$. Thus if $\widetilde{N}$ is connected, $[\widetilde{N}, \widetilde{N}] \subseteq [\widetilde{G}, \widetilde{G}] = ([G, G]^-)_{\mathbb{R}} \subseteq N_{\mathbb{R}}$.*

Proof. (a) is easy.
To prove (b), we first consider $\widetilde{G}_2 = \widetilde{N} \rtimes K^0,\ G_2 = G \cap \widetilde{G}_2$. The subgroup K^0 has finite index in K, hence $\widetilde{G}/\widetilde{G}_2$, G/G_2 are finite and it follows easily that $G_2, \widetilde{G}_2$ satisfy again the assumptions of the Proposition. By Lemma 4.7 (ii) (and since $G_2^0 = G^0$), $[K^0, K^0] \subseteq \widetilde{N}G^0 \subseteq \widetilde{G}_2$. It follows (since $K^0 = [K^0, K^0]K_1$, [14] Cor. 6.16) that $\widetilde{G}_1 G_2 = \widetilde{G}_2$ and that $\widetilde{N}G_1$ is dense in $\widetilde{G}_1$. Then $G_2/G_1 \cong \widetilde{G}_2/\widetilde{G}_1$ is compact which implies that $\widetilde{G}/G_1$ and also $\widetilde{G}_1/G_1$ are compact. This gives the assumptions of Proposition 4.4. Putting $\widetilde{M} = \{x \in \widetilde{N} :\ k \circ x = x \text{ for all } k \in K_1\}$, it follows from Proposition 4.4 (a) that $\widetilde{N} = N^0\widetilde{M}$. Clearly $[G_2, G_1] \subseteq N$, thus $[G^0, G_1] \subseteq N^0$. If H is a connected, closed G-invariant subgroup of $\widetilde{N}$ then by Proposition 4.4 (c), H is normal in $\widetilde{G}_1$, thus it is $\widetilde{G}_1G$-invariant, hence H is normal in $\widetilde{G}$. This applies to N^0 and $N_{\mathbb{R}}$ (which is G-invariant by [20] Th. 2.11). By connectedness, the action of K_1 on $\widetilde{N}/N^0 \cong \widetilde{M}/(N^0 \cap \widetilde{M})$ is trivial. Thus $[\widetilde{G}_2, K_1] \subseteq N^0$ and then $[G^0, G_1K_1] \subseteq N^0$. By Proposition 4.4 (a), $\widetilde{M} \cap G_1K_1$ is a co-compact subgroup of $\widetilde{M}$. Applying [20] Th. 2.11 to the image of this subgroup in $\widetilde{M}/(N^0 \cap \widetilde{M})$, it follows that the induced action of G^0 on $\widetilde{M}/(N^0 \cap \widetilde{M})$ is trivial, thus $[G^0, \widetilde{N}] \subseteq N^0$ ($\subseteq G^0$).
Let $\varphi\colon \widetilde{G}_2 \to K^0$ ($\cong \widetilde{G}_2/\widetilde{N}$) be the quotient mapping and let C be a Levi subgroup of G^0. We have shown that $\varphi(G^0) \supseteq [K^0, K^0]$. Considering the Lie algebras of G^0 and K^0, it follows that φ maps the connected radical of G^0 to $Z(K^0)^0$, consequently $\varphi(C) = [K^0, K^0]$. Clearly, $[K^0, K^0]$ is a Levi subgroup of $\widetilde{G}_2^0$. By [25] Th. 3.18.13, there exists $x \in \widetilde{N}^0$ such that $\iota_x(C) \subseteq [K^0, K^0]$. We have shown above that G^0 is $\widetilde{N}$-invariant, hence $\iota_x(C) \subseteq G^0$ and we may assume that $C \subseteq [K^0, K^0]$. Then $C = \varphi(C) = [K^0, K^0] \subseteq G^0$ and again by [25] Th. 3.18.13, any Levi subgroup of $\widetilde{G}_2^0$ is contained in $N^0[K^0, K^0] \subseteq G^0$. Note further that $[K^0, K^0] \subseteq G^0$ implies $\big[[K^0, K^0], \widetilde{N}\big] \subseteq N^0$, hence $[K^0, \widetilde{N}] \subseteq N^0$.
For (e), we can assume that $\widetilde{N}$ is connected and then factor by $([G, G]^-)_{\mathbb{R}}$ ($\subseteq \widetilde{N}$, being normal in $\widetilde{G}$ as above). Then G is abelian. By Corollary 4.6, $\widetilde{N}$ is abelian. Easy calculations (using that K is abelian) show that $G \cap \widetilde{M} \rtimes K$ acts trivially on G_1K_1, hence

(recall that $\widetilde{M} \cap G_1K_1$ is co-compact in $\widetilde{M}$) by [20] Th. 2.11 it acts trivially on $\widetilde{M}$ and it follows that $\widetilde{G}$ is abelian.

Next, we come to (d). Here we assume $K^0 = K$, thus $\widetilde{G}_2 = \widetilde{G}$. We can (replacing $\widetilde{N}$ by $\widetilde{N}_{\mathbb{R}}$) also assume that $\widetilde{N}$ is connected and then $\widetilde{G}$ is connected as well. We have shown above that $[G^0K, \widetilde{N}] \subseteq N^0$. If C is a compact subgroup of G, it follows from [13] Th. XV.3.1 (clearly K is a maximal compact subgroup of $\widetilde{G}$) that there exists $x \in \widetilde{N}$ such that $xCx^{-1} \subseteq K$. Then $[K, \widetilde{N}] \subseteq N^0$ implies $C \subseteq N^0K$. Repeating the argument with $N^0 \rtimes K$ instead of $\widetilde{G}$, it follows that we can assume that $x \in N^0$. Then $xCx^{-1} \subseteq K \cap G$. If (as above) φ denotes the projection to K, we have $\varphi(C) = \varphi(xCx^{-1})$ and maximality of $K \cap G$ follows. $C \subseteq N^0K$ gives $[G^0C, \widetilde{N}] \subseteq N^0$. G_1 is K_1-invariant by (b) and Proposition 4.4 (a). Furthermore, since K is connected, $K = [K, K]\, K_1$, hence (b) implies $G = G_1[K, K]$ and it follows that G is K-invariant and also invariant under xKx^{-1} for any $x \in G$.

Finally, we prove (c). Again, we can assume that $\widetilde{N}$ is connected. Then by (b), $N_{\mathbb{R}}$ is normal in $\widetilde{G}$. Assume that $x \in \widetilde{G}$ normalizes N, $x = y\,z$ with $y \in \widetilde{N}$, $z \in K$. If x acts unipotently on N, then the same is true on $N_{\mathbb{R}}$ (considering the commutator series of $N_{\mathbb{R}}$ and using [20] Th. 2.3, Cor. 1, this can be reduced to the abelian case which is easy). From [26] (2.2), (2.3) (applied to the automorphisms of $N_{\mathbb{R}}$ defined by the elements of $\langle x\rangle\widetilde{N}$), we conclude that ι_z is both semisimple and unipotent on $N_{\mathbb{R}}$ and it follows that z centralizes $N_{\mathbb{R}}$. $C_{\widetilde{G}}(N_{\mathbb{R}})$ is normal in $\widetilde{G}$. By (b) and Proposition 4.4 (a), $K_1 = Z(K^0)^0$ is faithful on N, thus $K_1 \cap C_{\widetilde{G}}(N_{\mathbb{R}})$ is trivial and we get (K_1 is normal in K) that $z \in C_K(K_1)$. By (e), we have $[\widetilde{G}_1, \widetilde{G}_1] \subseteq N_{\mathbb{R}}$. Put $\widetilde{G}_3 = \widetilde{N} \rtimes C_K(K_1)$. Then $[\widetilde{G}_3 \cap G, G_1] \subseteq N$. By assumption, $\widetilde{N}G$ is dense in $\widetilde{G}$ and since $\widetilde{N}C_K(K_1)$ ($\supseteq \widetilde{N}K^0$) is open, it follows that $\widetilde{G}_3 = \widetilde{G}_1(\widetilde{G}_3 \cap G)$. This gives $[\widetilde{G}_3, G_1] \subseteq N_{\mathbb{R}}$. Then $z \in C_K(K_1)$ implies $[z, G_1K_1] \subseteq N_{\mathbb{R}}$. Now Proposition 4.4 (a) and [20] Th. 2.11 give $[z, \widetilde{N}] \subseteq N_{\mathbb{R}}$. Since ι_z is semisimple on $\widetilde{N}$, it follows that z centralizes $\widetilde{N}$, thus (by faithfulness) $z = e$. This proves that $x \in \widetilde{N}$.

It follows that $C_{\widetilde{G}}(G) \subseteq C_{\widetilde{G}}(N) \subseteq \widetilde{N}$. Take $x \in C_{\widetilde{G}}(G) \cap N^0$. Applying Corollary 2.6 to the automorphisms defined by G_1 (and using Corollary 4.5), it follows that x commutes with $K_1 \cap (G_1\widetilde{N})$ which is dense in K_1. We get that x commutes with K_1. By Proposition 4.4 (d), $C_{\widetilde{G}}(G)$ is K_1-invariant, $[\widetilde{N}, K_1] \subseteq N^0$ by Proposition 4.4 (a). Then semisimplicity (using Lemma 4.7 and Corollary 4.2 (a)) implies that $C_{\widetilde{G}}(G)$ commutes with K_1. By Proposition 4.4 (a), $\widetilde{N}/(\widetilde{N} \cap G_1K_1)$ is compact, hence by [20] Th. 2.11, $C_{\widetilde{G}}(G) \subseteq Z(\widetilde{N})$. Since $G\widetilde{N}$ is dense in $\widetilde{G}$, we arrive at $C_{\widetilde{G}}(G) \subseteq Z(\widetilde{G})$.

Next, assume that C is a compact normal subgroup of G. Clearly $C \cap \widetilde{N}$ must be trivial, thus C centralizes N. It follows that $C \subseteq \widetilde{N}$, hence C is trivial.

Let N_1 be the nilradical of G. Clearly $N_1 \supseteq N$. Take $x \in N_1$. Then x normalizes N and acts unipotently, hence $x \in \widetilde{N}$. Thus $x \in \widetilde{N} \cap G = N$, proving that $N_1 = N$. $\square$

Proof of Theorem 3.

(α) In (α)$-$(γ), we assume that $j'(G)\widetilde{N}'$ is dense in $\widetilde{G}'$. This is no restriction (reducing K') when proving existence of Φ (that the same is true concerning uniqueness will be seen in (δ)). Furthermore, we assume in the steps $(\alpha) - (\gamma)$ that K, K' are connected abelian and K acts faithfully on $\widetilde{N}$. Then by Proposition 4.4 (a), $j(G)$ is K-invariant. Since j induces a topological isomorphism between G and $j(G)$, we get a continuous homomorphism $\alpha\colon K \to \mathrm{Aut}(G)$ satisfying $j\big(\alpha(k)(x)\big) = k\, j(x)\, k^{-1}$ for all $k \in K,\ x \in G$. By Proposition 4.4 (a), the action of K on $j(G)$ is faithful (since it is faithful on $\widetilde{N}$; alternatively one can use Proposition 4.8 (c)), hence α is injective. Similarly, we get $\alpha'\colon K' \to \mathrm{Aut}(G)$. We claim that there exists $n \in N^0$ such that $\alpha'(K') = n\, \alpha(K)\, n^{-1}$ (as explained before Proposition 2.22, this notation is shorthand for $\iota_n \circ \alpha(K) \circ \iota_n^{-1}$). The claim will be proved in (β) below. Then, replacing j' by $j'' = j' \circ \iota_n^{-1}$ (which replaces $\alpha'(k)$ by $\alpha''(k) = \iota_n \circ \alpha'(k) \circ \iota_n^{-1}$ for $k \in K'$), it will be enough to show existence of Φ under the additional assumption $\alpha(K) = \alpha'(K')$. This will be done in (γ). In (δ) we will prove uniqueness of Φ for general K, K' connected and K acting faithfully, then in (ϵ) uniqueness for general K, K' will be shown. Finally, existence for general K, K' will be treated in (φ) and also the question of surjectivity and injectivity.

(β) We assume that K, K' are connected and abelian, K acts faithfully on $\widetilde{N}$. Put $\mathcal{C}_0 = \alpha(K \cap \widetilde{N} j(G))$, $\overline{\mathcal{C}} = \alpha(K)$, $\mathcal{C}_0' = \alpha'(K' \cap \widetilde{N}' j'(G))$, $\overline{\mathcal{C}'} = \alpha'(K')$. Then Corollary 4.5 takes us to the setting of Section 2 (j transfers $\overline{\mathcal{C}}$ to $\iota_{j(G)}(K)$, similarly for $\mathcal{C}_0$ and for j'). By Proposition 2.22 (note that Proposition 4.8 (c) implies $j^{-1}(\widetilde{N}) = j'^{\,-1}(\widetilde{N}') = \mathrm{nil}(G)$, i.e., both instances of Corollary 4.5 refer to the same subgroup $H = N$ of G), there exists $n \in N^0$ such that $\mathcal{C}_0' = n\, \mathcal{C}_0\, n^{-1}$. By assumption, $K \cap \widetilde{N} j(G)$ (resp. $K' \cap \widetilde{N}' j'(G)$) is dense in K (resp. K'), it follows that $\alpha(K) = n\, \alpha'(K')\, n^{-1}$ (actually, $\overline{\mathcal{C}}$ coincides with the closure of $\mathcal{C}_0$).

(γ) Now we prove existence of the extension Φ under the assumption that K, K' are connected abelian and $\alpha(K) = \alpha'(K')$. First, we want to show that this implies $j^{-1}(K) \subseteq j'^{\,-1}(K')$ and that $j' \circ j^{-1}(k) = \alpha'^{\,-1} \circ \alpha(k)$ holds for $k \in K \cap j(G)$.
Take $x \in j^{-1}(K)$. Then $\iota_x = \alpha(j(x)) \in \alpha'(K')$. Put $k' = \alpha'^{\,-1}(\iota_x)$, then $k'^{\,-1} j'(x)$ centralizes $j'(G')$, hence by Proposition 4.8 (c), $k'^{\,-1} j'(x) \in Z(\widetilde{G}')$. Thus $j'(x) \in Z(\widetilde{G}')K'$. Since $j^{-1}(K)$ is compact and $Z(\widetilde{G}') \subseteq \widetilde{N}'$ is torsion free, it follows that $j'(x) \in K'$, proving $j^{-1}(K) \subseteq j'^{\,-1}(K')$. Since $\alpha'(j'(x)) = \iota_x$, the second formula follows easily.
For $k \in K,\ x \in G$, we define $\Phi\big(j(x)\, k\big) = j'(x)\, \alpha'^{\,-1} \circ \alpha(k)$. Then it follows from the properties above that $\Phi : j(G)K \to j'(G)K'$ is well defined and (see the definition of α, α') that it is a homomorphism. Furthermore, $\Phi(K) \subseteq K'$. Closedness of $j(G)$ implies that $j(G)K$ is isomorphic to a quotient of $G \rtimes K$ ([15] Th. 5.21) and continuity of Φ follows. $(j(G)K) \cap \widetilde{N}$ is a nilpotent normal subgroup of $j(G)K$. Applying Proposition 4.8 (c) to $\Phi(j(G)K)$ (see also (δ) below), it follows that $\Phi(j(G)K \cap \widetilde{N}) \subseteq \mathrm{nil}(\Phi(j(G)K)) = \Phi(j(G)K) \cap \widetilde{N}'$. By Proposition 4.4 (a), $(j(G)K) \cap \widetilde{N}$ is a co-compact subgroup of $\widetilde{N}$. By [20] Th. 2.11, $\Phi|\big(j(G)K \cap \widetilde{N}\big)$ has a unique extension to a continuous homomorphism $\widetilde{N} \to \widetilde{N}'$ (again denoted by Φ). Uniqueness of the extension implies $\Phi(knk^{-1}) = \Phi(k)\Phi(n)\Phi(k^{-1})$ for $k \in K,\ n \in \widetilde{N}$ and then Φ extends further to a homomorphism $\widetilde{G} \to \widetilde{G}'$.

(δ) We claim that $\Phi(\widetilde{N}) \subseteq \widetilde{N}'$ holds for any $\Phi, \widetilde{G}, \widetilde{G}'$ as in Theorem 3. We have $\Phi(j(G)) = j'(G)$. Since $j(G)$ is co-compact in $\widetilde{G}$ and $j'(G)$ is closed, it follows that

$\Phi(\widetilde{G})$ is closed in $\widetilde{G}'$. Thus (reducing K' temporarily) it also satisfies the assumptions of Proposition 4.8 and Proposition 4.8 (c) gives $\mathrm{nil}\big(\Phi(\widetilde{G})\big) = \Phi(\widetilde{G}) \cap \widetilde{N}'$. Since (again by Proposition 4.8 (c)) $\widetilde{N}' = \mathrm{nil}(\widetilde{G}')$, we conclude that $\Phi(\widetilde{N}) \subseteq \mathrm{nil}\big(\Phi(\widetilde{G})\big) \subseteq \widetilde{N}'$, proving our claim. It follows that $\Phi(\widetilde{G})$ is contained in the closure of $\widetilde{N}'j'(G)$. Thus we can always assume that $\widetilde{N}'j'(G)$ is dense in $\widetilde{G}'$ when proving uniqueness.
Next, we prove uniqueness of the extension for K connected, $j'(G)\widetilde{N}'$ dense in $\widetilde{G}'$ and K acting faithfully on $\widetilde{N}$. Assume that $\Phi_1, \Phi_2 \colon \widetilde{N} \rtimes K \to \widetilde{N}' \rtimes K'$ are continuous group homomorphisms satisfying $\Phi_i \circ j = j'$ for $i = 1, 2$. If $x \in \widetilde{G}$ normalizes $j(G)$, then $\iota_{\Phi_1(x)}$ coincides with $\iota_{\Phi_2(x)}$ on $j'(G)$. Put $\Psi(x) = \Phi_2(x)^{-1}\Phi_1(x)$. Then $\Psi(x)$ commutes with $j'(G)$, hence Proposition 4.8 (c) implies, $\Psi(x) \in Z(\widetilde{G}') \subseteq \widetilde{N}'$. By Proposition 4.8 (d), $j(G)$ is K-invariant. It follows that $\Psi \colon j(G)K \to Z(\widetilde{G}')$ is a continuous group homomorphism and by assumption, $j(G) \subseteq \ker\Psi$. Since $\widetilde{N}'$ is torsion free, we get that Ψ must be trivial, hence Φ_1, Φ_2 coincide on K.
By Proposition 4.4 (a), $j(G)K$ contains a co-compact subgroup of $\widetilde{N}$, thus by [20] Th. 2.11 (recall that $\Phi_i(\widetilde{N}) \subseteq \widetilde{N}'$) Φ_1, Φ_2 coincide on $\widetilde{N}$, proving that $\Phi_1 = \Phi_2$.

(ϵ) Now, we prove uniqueness of the extension for general K, K'. Consider Φ_1, Φ_2 as in (δ). Let $K_{\widetilde{N}}$ be the kernel of the action of K on $\widetilde{N}$. This is a compact normal subgroup of $\widetilde{G}$, hence $\Phi_i(K_{\widetilde{N}})$ is a compact normal subgroup of $\Phi_i(\widetilde{G}')$. By faithfulness of the action on $\widetilde{N}'$, it follows from Proposition 4.8 (c) (applied to $G = \Phi_i(\widetilde{G}')$) that $\Phi_i(K_{\widetilde{N}})$ must be trivial, hence $K_{\widetilde{N}} \subseteq \ker\Phi_i$ holds for $i = 1, 2$.
Passing to $K/K_{\widetilde{N}}$ (and composing j with the quotient mapping), we can now assume that K acts faithfully on $\widetilde{N}$. Put $G_2 = j^{-1}(\widetilde{N} \rtimes K^0) \cap {j'}^{-1}(\widetilde{N} \rtimes {K'}^0)$. Then G_2 is a closed subgroup of G with finite index. It follows that $\widetilde{N}j(G_2)$ is dense in $\widetilde{N} \rtimes K^0$ (a connected group has no proper closed subgroups of finite index) and similarly for $j'(G_2)$. Thus, we can apply (δ) and conclude that Φ_1, Φ_2 coincide on $\widetilde{N} \rtimes K^0$. Density of $\widetilde{N}j(G)$ in $\widetilde{G}$ implies that $\widetilde{G} = (\widetilde{N} \rtimes K^0)\, j(G)$, consequently $\Phi_1 = \Phi_2$.

(φ) We show existence of the extension for general K, K'. Consider $K_{\widetilde{N}}$ as in (ϵ). Similarly we get $j'(j^{-1}(K_{\widetilde{N}}) \cap G) = \{e\}$, hence $K_{\widetilde{N}} \cap j(G) = \{e\}$. Thus we can pass to $K/K_{\widetilde{N}}$ and assume that K acts faithfully on $\widetilde{N}$. Put $K_1 = Z(K^0)^0,\ K_1' = Z({K'}^0)^0,\ G_1 = j^{-1}(\widetilde{N} \rtimes K_1),\ G_1' = {j'}^{-1}(\widetilde{N} \rtimes K_1'),\ G_2 = G_1 \cap G_1'$. Then G_1, G_1' are closed normal subgroups of G. We have $R_K^0 = K_1$ ([25] Th. 4.11.7), hence by Proposition 4.8 (a), G_1 has finite index in the radical R of G and the same is true for G_1'. It follows that G_1/G_2 is finite and from Proposition 4.8 (b), we get (similarly as in (ϵ)) that $\big(\widetilde{N}j(G_2)\big) \cap K_1$ is dense in K_1, analogously for $\big(\widetilde{N}'j'(G_2)\big) \cap K_1'$. Thus we can apply the connected abelian case ((α)-(γ)) to G_2 with K, K' replaced by K_1, K_1'. This gives a homomorphism $\Phi \colon \widetilde{N} \rtimes K_1 \to \widetilde{N}' \rtimes K_1'$ satisfying $\Phi \circ j = j'$ on G_2. Uniqueness of the extension (shown in (ϵ)) implies that $\Phi\big(j(x)\, y\, j(x)^{-1}\big) = j'(x)\, \Phi(y)\, j'(x)^{-1}$ for all $x \in G,\ y \in \widetilde{N} \rtimes K_1$. Now take $x \in G_1$, then $j(x) \in \widetilde{N} \rtimes K_1$ and it follows that $z = \Phi\big(j(x)\big)^{-1} j'(x)$ commutes with $\Phi(\widetilde{N} \rtimes K_1) \supseteq j'(G_2)$. Hence by Proposition 4.8 (c), $z \in \widetilde{N}'$. This implies $j'(x) \in \widetilde{N} \rtimes K_1'$, hence $x \in G_1'$. This shows that $G_2 = G_1 \subseteq G_1'$.
It follows from density of $\widetilde{N}j(G)$ in $\widetilde{G}$ and Proposition 4.8 (b) that $\widetilde{G} = (\widetilde{N} \rtimes K_1)\, j(G)$. On $j(G)$ we put $\Phi = j' \circ j^{-1}$. Then, by the properties above, the two definitions of Φ

agree on $j(G) \cap (\widetilde{N} \rtimes K_1)$ and they can be combined to give a continuous homomorphism $\widetilde{G}_1 \to \widetilde{G}'_1$.

By construction, we always have $K_{\widetilde{N}} \subseteq \ker \Phi$ (see also (ϵ)) and by (δ), $\Phi(\widetilde{G}) \subseteq \big(\widetilde{N}' j'(G)\big)^{-}$. If K acts faithfully and $\widetilde{N}' j'(G)$ is dense in $\widetilde{G}'$, we can interchange the rôles of $\widetilde{G}$ and $\widetilde{G}'$ and in the usual manner, it follows from uniqueness of the extension that Φ is an isomorphism. For the general case, this implies $\Phi(\widetilde{G}) = \big(\widetilde{N}' j'(G)\big)^{-}$ and $\ker \Phi = K_{\widetilde{N}}$. □

Corollary 4.9. *Let $G, \widetilde{G}, G_1, K_1$ be as in Proposition 4.8 (b) and define $\widetilde{M} = \{ x \in \widetilde{N} : k \circ x = x$ for all $k \in K_1 \}$, $L = G \cap (\widetilde{M} \rtimes K)$, $L_1 = G_1 \cap L$. Then the following properties hold:*

G_1 is normal in G, G/G_1 is compact, $[G_1, G_1] \subseteq N \subseteq G_1$, $G = N^0 L$, $G_1 = N^0 L_1$, L/L_1 is compact, L_1 is nilpotent.

Thus L satisfies the properties of Theorem 1. We will exemplify the constructions in step (b) of the proof of Theorem 2 for this choice of G_1. By Corollary 4.5, $\overline{\mathcal{C}} = \iota_{G_1}(K_1)$, thus $G_2 = G_1 \rtimes K_1 \subseteq (\widetilde{N} \rtimes K_1) \rtimes K_1$. Since K_1 is abelian, we can interchange the K_1-components and use the representation $G_2 = \{ (x, \sigma_1, \sigma_2) : (x, \sigma_2) \in G_1, \ \sigma_1 \in K_1 \} \subseteq \widetilde{N} \rtimes (K_1 \times K_1) \subseteq \widetilde{N} \rtimes (K_1 \rtimes K)$, where the action of $K_1 \rtimes K$ on $\widetilde{N}$ is given by $(\sigma_1, \sigma_2) \circ x = (\sigma_1 \sigma_2) \circ x$. Then $N_2 = \{ (x, \sigma^{-1}, \sigma) : (x, \sigma) \in G_1 \}$. Embedding G to $\{ (x, e, \sigma) : (x, \sigma) \in G \}$ this produces the action of G on G_2 defined in the proof of Theorem 2 and one can take $G_3 = \{ (x, \sigma_1, \sigma_2) : (x, \sigma_2) \in G, \ \sigma_1 \in K_1 \}$. It is not hard to see that $P_3 = \{ (e, \sigma, \sigma^{-1}) : \sigma \in K_1 \}$ (the kernel of the action of $K_1 \rtimes K$). It follows that G_4 can be identified with the subgroup $G K_1$ of $\widetilde{N} \rtimes K$, and then N_4 corresponds to $G K_1 \cap \widetilde{N}$.

Proof. $G_1, \widetilde{G}_1$ satisfy the assumptions of Proposition 4.4 (see also the proof of Proposition 4.8 (b)). By Proposition 4.4 (a), $\widetilde{N} = N^0 \widetilde{M}$ which implies $G = N^0 L$, $G_1 = N^0 L_1$. In particular, $G_1 L = G$ is closed, giving $L/L_1 \cong G/G_1$ ([15] Th. 5.33). $L_1 \subseteq \widetilde{M} \times K_1$ is nilpotent. The remaining properties are clear. □

Next, we describe some special cases of Theorem 2.

Corollary 4.10. *Let $G, \widetilde{G}$ be as in Theorem 2, with $\widetilde{N} G$ dense in $\widetilde{G}$.*

(a) *The following properties are equivalent*
  (i) *K is abelian* (ii) $[G, G] \subseteq N$ (iii) *the action of G on $(\mathfrak{n}_{\mathbb{R}})_{\mathbb{C}}$ is trigonalizable.*

(b) *The following properties are equivalent*
  (i) *G, N satisfy the assumptions of 2.1.*
  (ii) *K is abelian and acts trivially on N/N^0.*
  (iii) *G acts unipotently on N/N^0 and the action of G on $\mathfrak{n}_{\mathbb{C}}$ is trigonalizable.*
  (iv) *K is abelian and G/G^0 is nilpotent.*

(v) *K is abelian and for $\widetilde{M}$ defined as in Corollary 4.9 one has $\widetilde{M} = \{x \in \widetilde{N} : k \circ x = x$ for all $k \in K\}$.*

(c) *G/N is compact if and only if $\widetilde{N}/N$ is compact (equivalently: $\widetilde{N} = N_{\mathbb{R}}$).*

(d) *$\widetilde{N}$ is abelian if and only if N is an FC_G^--group and there exists an abelian subgroup H of G such that NH is closed and $G/(NH)$ is compact.*

As before, $\mathfrak{n}$ denotes the Lie algebra of N, $\mathfrak{n}_{\mathbb{R}}$ that of the Malcev completion $N_{\mathbb{R}}$ and $\mathfrak{n}_{\mathbb{C}}, (\mathfrak{n}_{\mathbb{R}})_{\mathbb{C}}$ denote the complexifications of $\mathfrak{n}, \mathfrak{n}_{\mathbb{R}}$. The proof will show that in (d) one can take $H = L_1$ (the group of Corollary 4.9). Furthermore, the proof of (d) shows that N is an FC_G^--group iff it is central in $\widetilde{N}$ and this is equivalent to N^0 being central in $\widetilde{N}$.

Proof. (a) (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): The action of N on $\mathfrak{n}_{\mathbb{R}}$ is clearly unipotent, thus the same is true on $(\mathfrak{n}_{\mathbb{R}})_{\mathbb{C}}$ and (iii) follows from [26] (2.2).
(iii) $\Rightarrow$ (ii): If $x \in [G, G]$, it follows from (iii) that the automorphism of $N_{\mathbb{R}}$ induced by ι_x is unipotent, hence the same is true on N. By Proposition 3.3, this implies $x \in N$ (alternatively, one could use (c) of Proposition 4.8).
(ii) $\Rightarrow$ (i): If (ii) holds, then the image of G in $\widetilde{G}/\widetilde{N}$ ($\cong K$) is abelian and by assumption, it is dense.
(b) (i) $\Rightarrow$ (iii) follows from nilpotency of G/N^0 and using again [26] (2.2).
(iii) $\Rightarrow$ (ii): The action of $[G, G]$ on N is unipotent, hence again by Proposition 3.3, $[G, G] \subseteq N$ and from (a) it follows that K is abelian. N^0 is normal by Proposition 4.4 (c) combined with Proposition 4.8 (b). By (a), the action of G on $(\mathfrak{n}_{\mathbb{R}})_{\mathbb{C}}$ is trigonalizable, hence also that on $(\mathfrak{n}_{\mathbb{R}}/\mathfrak{n})_{\mathbb{C}}$. G acts unipotently on $\mathfrak{n}_{\mathbb{R}}/\mathfrak{n}$, hence by [26] (2.3), K acts trivially on $\mathfrak{n}_{\mathbb{R}}/\mathfrak{n}$ and the same is true on $N/N^0 \subseteq N_{\mathbb{R}}/N^0$.
(ii) $\Rightarrow$ (i): By (a), we have $[G, G] \subseteq N$, thus G/N is abelian. Since the action on N/N^0 is unipotent, it follows ([27] 9.3) that G/N^0 is nilpotent (if K is connected one can also apply Corollary 4.5).
(iv), (v) are shown similarly.
(c) This follows immediately from compactness of $\widetilde{G}/G$ and $\widetilde{G}/\widetilde{N}$.
(d) If $\widetilde{N}$ is abelian, then obviously N is an FC_G^--group. The subgroup L_1 ($\subseteq \widetilde{M} \times K_1$) of Corollary 4.9 is abelian as well and it satisfies $G_1 = NL_1$ and G/G_1 is compact.
For the converse, we can (passing to a subgroup of finite index) assume that K is connected. N is K_1-invariant by Proposition 4.4 (d). If N is an FC_G^--group, then N must be abelian (there are no non-trivial unipotent inner automorphisms) and $L_1K_1 \cap \widetilde{N}$ acts trivially on N by Corollary 4.5. Thus $G_1K_1 \cap \widetilde{N}$ commutes with N. Since $\widetilde{N}$ is torsion free and nilpotent, the centralizer of N is a connected subgroup of $\widetilde{N}$ (see also Remark 2.8). By Proposition 4.4 (a), $G_1K_1 \cap \widetilde{N}$ is co-compact in $\widetilde{N}$, hence ([20] Th. 2.1(4)), N is central in $\widetilde{N}$. Put $N' = [K, N] = [K, N^0]$, $\widetilde{M}' = \{x \in \widetilde{N} : k \circ x = x$ for all $k \in K\}$, $\widetilde{L}' = \widetilde{M}' \times K$. Then N' is a closed normal subgroup of $\widetilde{G}$, $N' \cap \widetilde{M}'$ is trivial and $\widetilde{N} = N'\widetilde{M}'$ by [18] L. 5.4 and Proposition 4.8 (d) (in particular $N' = [K, \widetilde{N}]$). Thus $\widetilde{G} = N' \rtimes \widetilde{L}'$.

Let $H' = N'H \cap \widetilde{L}'$ be the projection of H to $\widetilde{L}'$. Then H' is an abelian subgroup of G and $(\widetilde{M}' \cap N)H'$ is co-compact in $\widetilde{L}'$ and abelian (observe that $\widetilde{M}' \cap N$ is central in $\widetilde{G}$). Consequently, $H'' = \big((\widetilde{M}' \cap N)H'K\big) \cap \widetilde{M}'$ (i.e., the projection of $(\widetilde{M}' \cap N)H'$ to $\widetilde{M}'$) is a co-compact abelian subgroup of $\widetilde{M}'$. As above, it follows that $\widetilde{M}'$ must be abelian and this implies that $\widetilde{N}$ is abelian. $\square$

Remarks 4.11. (a) We want to relate our results to the notions of [20]. Let G be a compactly generated group of polynomial growth without non-trivial compact normal subgroups and let π be a continuous faithful representation of G on $\mathbb{R}^n$. Denote by $\widetilde{G}$ the Zariski-closure of $\pi(G)$ in $\mathrm{GL}(n, \mathbb{R})$. Let $\widetilde{N}$ be the unipotent radical of $\widetilde{G}$. Then we have a "Levi decomposition" (in the sense of algebraic groups) $\widetilde{G} = \widetilde{N} \rtimes K$, where K is a maximal reductive subgroup of $\widetilde{G}$ (see [20] p. 11, [1] p. 296). Then (putting as before $N = \mathrm{nil}(G)$) one can show that the following properties are equivalent:

(i) $\pi(N)$ consists of unipotent matrices, $\pi(G)$ is closed (for the Euclidean topology of $\mathrm{GL}(n, \mathbb{R})$) and distal.
(ii) $\pi(N)$ consists of unipotent matrices, $\pi(G)$ is closed (Euclidean topology) and K is compact.

If this holds, it follows that $\widetilde{G}/\pi(G)$ is compact. Furthermore, if the action of K on $\widetilde{N}$ is faithful (i.e., $C_{\widetilde{G}}(\widetilde{N}) \subseteq \widetilde{N}$), then (i) and (ii) are equivalent to

(iii) $\widetilde{G}$ is an algebraic hull of G (as defined in [20] Def. 4.39).

(Be aware that in [3] p. 228 the term algebraic hull is used in a much wider sense.)

Thus, in the case of a faithful action, $\widetilde{G}$ coincides with the groups considered in Theorem 2 and 3. To be precise: [20] considers complex algebraic groups (i.e., the Zariski closure in $\mathrm{GL}(n, \mathbb{C})$), thus our $\widetilde{G}$ is the "real algebraic hull", i.e., the set of real points of the algebraic hull in the sense of [20]. In particular, it follows from Theorem 3 that all algebraic hulls (in the sense of [20]) are isomorphic (this has also been shown in [20] L. 4.41). Since we are dealing with groups of polynomial growth, one can show (similarly as in the proof of [20] L. 4.36, using a corresponding definition of the "rank" for generalized $\overline{FC}$-groups) that the condition "$\pi(G)$ is closed" of (i), (ii) is equivalent to "π is full" in the sense of [20] Def. 4.37, i.e., $\dim(\widetilde{N}) = \mathrm{rk}(G)$.

The representation (coming from the Birkhoff embedding theorem) that was used in the proof of Corollary 3.6 has the properties leading to (iii). But in general, there are also faithful finite dimensional representations of G which satisfy (i) and (ii), but K does not act faithfully on $\widetilde{N}$ (see Examples 4.12 (d)). Let $K_{\widetilde{N}}$ be the kernel of the action of K on $\widetilde{N}$. Then $K_{\widetilde{N}}$ is normal in $\widetilde{G}$ and by Theorem 3, $\widetilde{G}/K_{\widetilde{N}}$ is isomorphic (as a locally compact group) to the algebraic hull of G.

In the case of discrete generalized $\overline{FC}$-groups (i.e., finite extensions of polycyclic groups) another construction of the algebraic hull (using Hopf algebras and working on arbitrary fields of characteristic zero) has been described in [9] (see L. 4.1.2, Prop. 4.2.2, 4.3.2). A more explicit version in terms of a "basis" of the group has been given in [22].

(b) In general, there are further almost nilpotent groups lying between G and $\widetilde{G}$. The group G_{an} of Proposition 3.8 is a co-compact extension of G that is almost nilpotent and has no non-trivial compact normal subgroup. For K_1 one can take that of Proposition 4.8 (b) and for a given hull $\widetilde{G}$, the group GK_1 does not depend on the choice of K. But in general, G need not be K_1-invariant (in particular, G need not be normal in G_{an}) and G_{an} need not split into a semidirect product of a nilpotent group and a compact group. G_{an} need not be a minimal almost nilpotent extension of G (see Examples 4.12 (a), (f)).

When G is connected, simply connected and solvable, $\widetilde{G} = G_{an}$ coincides with the semisimple splitting of [3] p. 237, $\widetilde{N}$ is called the nil-shadow of G (in the notation of [3]: $\widetilde{G} = R_S,\ \widetilde{N} = M_R,\ K_1 = T_R$, where $R = G$). Hence in the general case of our Theorem 2, we call $\widetilde{N}$ the *connected nil-shadow* of G. As mentioned in Remark 2.23 (g) this coincides with the notions of [3] and [7] for connected, simply connected, solvable Lie groups.

In [2] Th. 3.6, an arbitrary connected Lie group G of polynomial growth is embedded (as a closed normal subgroup) into a connected Lie group H such that H/G is compact and H has a co-compact normal subgroup M_0 that is connected and nilpotent. But in general M_0 need not be simply connected, even when G has no non-trivial compact normal subgroups (for G solvable with $[G,G]^-$ torsion free, H coincides with the group G' of Corollary 2.16, $M_0 = N'$, see also Remark 2.17). Thus, this does not always coincide with our algebraic hull.

In the non-connected case, one can consider splittings where the nilpotent factor is not necessarily connected. This is related to the "discrete semisimple splitting" mentioned in [3] p. 253, see also [23] p. 141. Let N^K be the closed K-invariant subgroup of $\widetilde{N}$ generated by $GK \cap \widetilde{N}$. Then $G \subseteq N^K \rtimes K$ (and N^K is minimal to get such a splitting for given K). But in general, N^K depends on the choice of K. One can show that it is always possible to choose K so that $N^K \rtimes K$ is a finite extension of G_{an}. But in general there is no uniqueness result corresponding to Theorem 3 (see Examples 4.12 (c); this aspect is somehow concealed in the formulation of [3] p. 254; compare also [23] Th. 3, p. 147).

In [19] Sec. 2, another construction of the nil-shadow based on representative functions (and working for an arbitrary generalized $\overline{FC}$-group G without non-trivial compact normal subgroups) is given. If π is any continuous finite dimensional representation of G and $\widetilde{N}_\pi$ denotes the unipotent radical of the real Zariski closure of $\pi(G)$, then $\widetilde{N}_\pi$ is a quotient of the nil-shadow $\widetilde{N}$, but the reductive part can become arbitrarily large (unless $G/\mathrm{nil}(G)$ is finite), compare the Examples 4.12 (d).

(c) [28] Ex. 2.3 shows that Proposition 4.8 (c) need not hold when $\widetilde{N}$ is replaced by a general connected, simply connected and *solvable* group.

(d) In [7] Th. 1.2, it is shown that if G is a compactly generated l.c. group of polynomial growth having no non-trivial compact normal subgroup, then there exists a co-compact closed subgroup H that can be embedded (as a closed subgroup) into a connected, simply connected, solvable Lie group S. The proof (given in [7] 7.1) reduces it in several steps to a corresponding embedding theorem ([26] Th. 3) for $\mathcal{S}$-groups. He calls S a "Lie shadow" of G. It is necessarily of polynomial growth, but in general not unique (see [7] p. 671). It follows from our Theorem 3 that the algebraic hull of S contains the algebraic hull of H which is contained in the algebraic hull of G. In particular, the nil-shadow of S must coincide with the connected nil-shadow of G (fixing also the dimension of S).

Examples 4.12.
(a) We start with the examples given in [17] 1.4.3. For $G = \mathbb{C} \rtimes \mathbb{Z}$ with the action $n \circ z = \alpha^n z$, where $|\alpha| = 1$ and α is not a root of unity, we get $\widetilde{G} = (\mathbb{C} \times \mathbb{R}) \rtimes K$ with $K = \{\beta \in \mathbb{C} : |\beta| = 1\}$ $(= K_1)$, $\beta \circ (z,t) = (\beta z, t)$ and the embedding $(z,n) \mapsto (z,n,\alpha\, n)$, $\widetilde{N} = \mathbb{C} \times \mathbb{R}$, $\widetilde{M} = \mathbb{R}$, $L = \mathbb{Z}$, $G_{an} = (\mathbb{C} \times \mathbb{Z}) \rtimes K$.
For $G = \mathbb{C}^2 \rtimes \mathbb{R}$ with $t \circ (z_1, z_2) = (e^{it\beta_1} z_1, e^{it\beta_2} z_2)$, we get $\widetilde{G} = (\mathbb{C}^2 \times \mathbb{R}) \rtimes K$ $(= G_{an})$, where $K\,(= K_1)$ denotes the closure of $\{(e^{it\beta_1}, e^{it\beta_2}) : t \in \mathbb{R}\}$, $(\gamma_1, \gamma_2) \circ (z_1, z_2, t) = (\gamma_1 z_1, \gamma_2 z_2, t)$ and (writing $\mathbf{z} = (z_1, z_2)$) the embedding $(\mathbf{z}, t) \mapsto (\mathbf{z}, t, (e^{it\beta_1}, e^{it\beta_2}))$, $\widetilde{N} = \mathbb{C}^2 \times \mathbb{R}$.
Similarly, for $G = \mathbb{R}^n \rtimes \mathbb{Z}$ with the action $n \circ v = A^n v$, where $A \in \mathrm{GL}(n, \mathbb{R})$ and all eigenvalues of A have modulus 1. We consider the multiplicative Jordan decomposition $A = A_s A_u$. We get $\widetilde{G} = (\mathbb{R}^n \rtimes \mathbb{R}) \rtimes K$, where K denotes the closure of $\{A_s^n : n \in \mathbb{Z}\}$ and the actions are $t \circ v = e^{tB} v$ with $B = \log A_u$, $C \circ (v,t) = (Cv, t)$ for $C \in K$. The embedding is given by $(v,n) \mapsto (v, n, A_s^n)$, $\widetilde{N} = \mathbb{R}^n \rtimes \mathbb{R}$. If no root of unity is an eigenvalue of A, then $\widetilde{M} = \mathbb{R}$, $L = \mathbb{Z}$. Otherwise, $\widetilde{M}$ includes the eigenspaces of A_s for the roots of unity and if one of these eigenvalues is different from 1, the action of K on $\widetilde{M}$ is non-trivial. If $K^0\,(= K_1)$ is non-trivial (i.e., A has at least one eigenvalue that is not a root of unity), then $N = \mathbb{R}^n$, $G_{an} \subseteq (\mathbb{R}^n \rtimes \mathbb{Z}) \rtimes K$, but if $K^0 \neq K$ (e.g., A has also an eigenvalue that is a root of unity different from 1), the inclusion is proper and G_{an} does not split.
Similarly, for $G = \mathbb{R}^n \rtimes \mathbb{R}$. For example, in the case $G = \mathbb{C} \rtimes \mathbb{R}$ with $t \circ z = e^{it} z$, one has $\widetilde{G} = (\mathbb{C} \times \mathbb{R}) \rtimes K$ $(= G_{an})$ with $K = \{\beta \in \mathbb{C} : |\beta| = 1\}$, $\beta \circ (z,t) = (\beta z, t)$ and the embedding $(z,t) \mapsto (z, t, e^{it})$, $\widetilde{N} = \mathbb{C} \times \mathbb{R}$, $\widetilde{M} = \mathbb{R}$, $L = \mathbb{R}$, $N = \mathbb{C} \times 2\pi\mathbb{Z}$. Thus G is almost nilpotent, but $G_{an} \neq G$, i.e., G_{an} is not minimal.
(b) An example where the action of K on $\widetilde{N}$ is not faithful (notation of Theorem 3): take $G = \mathbb{R}$, $\widetilde{G} = \mathbb{R} \times K$ with $K = \mathbb{R}/\mathbb{Z}$, $j(t) = (t, t + \mathbb{Z})$. Here $\widetilde{G}/j(G)$ is compact, but $\widetilde{G}$ is not isomorphic to the algebraic hull of G (which coincides with G).
An example where $j(G)$ is not closed: take $G = \mathbb{Z}^2$, $\widetilde{G} = \mathbb{R}$, $j(n,m) = n\alpha + m\beta$ where $\alpha, \beta \in \mathbb{R}$ are $\mathbb{Q}$-linearly independent. Then $j(G)$ is dense in R, but not closed, and the

algebraic hull of G is $G_{\mathbb{R}} = \mathbb{R}^2$.
These examples can also be used to show that in Remark 4.11 (a) the assumptions $\pi(N) \subseteq \widetilde{N}$ and $\pi(G)$ closed cannot be dropped.
(c) For G almost nilpotent, one has $\widetilde{N} = N_{\mathbb{R}}$ by Corollary 4.10 (c), and conversely. To get examples for the discrete case (where G is a finite extension of a nilpotent group), put $\widetilde{N} = \mathbb{R}^2$, $\alpha_1(x_1, x_2) = (-x_1, x_2)$, $\alpha_2(x_1, x_2) = (x_1 - x_2)$, K ($\cong \mathbb{Z}_2^2$) the subgroup of $\mathrm{GL}(2, \mathbb{R})$ generated by α_1, α_2, $\widetilde{G} = \widetilde{N} \rtimes K$, $N = \mathbb{Z}^2$ and G shall be the subgroup of $\widetilde{G}$ generated by N and $((0,0), \alpha_1)$, $((\frac{1}{2}, 0), \alpha_2)$. $\widetilde{N}/N$ being compact, it follows that $\widetilde{N} \cong N_{\mathbb{R}}$, $N = G \cap \widetilde{N}$ and $\widetilde{G}$ is the algebraic hull of G. Since K is discrete, we have $G_{an} = G$. Here, N and G are K-invariant, $N^K = \frac{1}{2}\mathbb{Z} \times \mathbb{Z}$. For $\mu = (0, 1) \in N$, $K^\mu = \mu K \mu^{-1}$, one gets $N^{K^\mu} = \{(x, y) \in (\frac{1}{2}\mathbb{Z})^2 : x + y \in \mathbb{Z}\}$ and it is easy to see (N^{K^μ} does not split into cyclic K^μ-invariant subgroups) that $N^{K^\mu} \rtimes K^\mu$ is not isomorphic to $N^K \rtimes K$. Thus there are non-isomorphic discrete splittings. G has index 2 in both extensions.
Observe (using [13] Th. XV.3.1) that for every compact subgroup C of $\widetilde{G}$ there exists $\mu \in \widetilde{N}$ such that $\mu^{-1} C \mu \subseteq K$, in particular, K^μ ($\mu \in \widetilde{N}$) gives all maximal compact subgroups of $\widetilde{G}$.
For further examples, consider $\widetilde{N} = \mathsf{H} \times \mathbb{R}$, where H denotes the three-dimensional real Heisenberg group. Explicitly, $\widetilde{N} = \mathbb{R}^4$ topologically, with the multiplication $(x_1, x_2, t_1, t_2)\,(x_1', x_2', t_1', t_2') = (x_1 + x_1', x_2 + x_2', t_1 + t_1' - x_2 x_1', t_2 + t_2')$. Let N be the (discrete) subgroup generated by $(1, 0, 0, 0)$, $(0, 1, 0, \frac{1}{4})$, $(0, 0, \frac{1}{2}, \frac{1}{2})$. Writing $\mathbf{v} = (x_1, x_2, t_1, t_2)$, we get $N = \{\mathbf{v} :\ x_1, x_2, 2t_1, 4t_2 \in \mathbb{Z},\ 4t_2 - 4t_1 - x_2 \equiv 0 \pmod 4\}$. Consider $\alpha \in \mathrm{Aut}(\widetilde{N})$ defined by $\alpha(\mathbf{v}) = (x_1, -x_2, -t_1, t_2)$, $K = \langle \alpha \rangle$, $\widetilde{G} = \widetilde{N} \rtimes K$. Finally, let G be the subgroup of $\widetilde{G}$ generated by N and $((\frac{1}{2}, 0, 0, 0), \alpha)$. Since $\alpha(0, 1, 0, \frac{1}{4}) \notin N$, we get that N and G are not α-invariant. Hence they are not K-invariant and the same can be shown if K is replaced by a conjugate group $\mu K \mu^{-1}$ ($\mu \in \widetilde{N}$). In a similar way, one can construct examples where N is K-invariant but G is not K-invariant.
When K is abelian (see Corollary 4.10 (a)), one can show similar statements as in [23] Th. 1, p. 143. Put $\widetilde{M'} = \{x \in \widetilde{N} : k \circ x = x \text{ for all } k \in K\}$, $M' = N^K \cap \widetilde{M'}$. Proposition 4.8 (e) implies $\widetilde{N} = N_{\mathbb{R}} \widetilde{M'}$. One can choose K such that N and G are K-invariant and $N^K = NM'$ if and only if there exists a nilpotent subgroup L' of G such that $G = NL'$ and N is $s(L')$-invariant (where as in 2.5, $s(x) \in \mathrm{Aut}(N_{\mathbb{R}})$ denotes the semisimple part of the automorphism ι_x of $N_{\mathbb{R}}$). However, even under these stronger assumptions there is no uniqueness in general. Similarly as above, one can construct non-isomorphic splittings $N^K \rtimes K$ of this type.
As mentioned before, [23] and [3] assumed that G/N is torsion free. But it is easy to modify the examples above to meet this requirement. For example, the first one came from an action of $\mathbb{Z}_2^2$ on $\mathbb{R}^2$ (in fact on $\mathbb{Q}^2$). This gives rise to a faithful action of $\mathbb{Z}^2$ on $\mathbb{R}^2 \times \mathbb{Z}^4$ when combining with a faithful action of $\mathbb{Z}^2$ on $\mathbb{Z}^4$ by semisimple matrices (of course, this leads outside the scope of groups of polynomial growth).
(d) On faithful representations. In (b), we mentioned examples concerning the conditions in (i), (ii) of Remark 4.11 (a). Now we consider the first example of (a), $G = \mathbb{C} \rtimes \mathbb{Z}$. A

natural choice of a faithful representation would be $\pi(z,n) = \begin{pmatrix} \alpha^n & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ ($\subseteq \mathrm{GL}(4,\mathbb{R})$). But $\pi(G)$ is not closed, the (real) Zariski closure gives $\left\{ \begin{pmatrix} \beta & z \\ 0 & 1 \end{pmatrix} : \beta, z \in \mathbb{C},\ |\beta| = 1 \right\} \cong \mathbb{C} \rtimes K$ with $K = \left\{\beta \in \mathbb{C} : |\beta| = 1\right\}$. Write $\alpha = \alpha_1^2$, take $r \in \mathbb{R}$ with $|r| \neq 0, 1$ and put $\pi_r(z,n) = \begin{pmatrix} (r\alpha_1)^n & z \\ 0 & (r/\alpha_1)^n \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ ($\subseteq \mathrm{GL}(4,\mathbb{R})$). Now $\pi_r(G)$ is closed but not distal, the (real) Zariski closure gives $\left\{ \begin{pmatrix} \beta & z \\ 0 & \gamma \end{pmatrix} : \beta, \gamma, z \in \mathbb{C},\ \beta\gamma \in \mathbb{R}^* \right\} \cong \mathbb{C} \rtimes K \times \mathbb{R}^*$ (with $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ non-compact).

$$\text{Put} \qquad \pi_{alg}(z,n) = \begin{pmatrix} \alpha^n & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{C}) \quad (\subseteq \mathrm{GL}(8,\mathbb{R}))\,.$$

$\pi_{alg}(G)$ satisfies all the properties (i)-(iii) of Remark 4.11 (a). The (real) Zariski closure (which gives the algebraic hull, isomorphic to the version in (a)) is

$$\left\{ \begin{pmatrix} \beta & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} : \beta, z \in \mathbb{C},\ |\beta| = 1,\ t \in \mathbb{R} \right\}.$$

To get an example of a faithful representation of G satisfying (i), (ii), but not (iii), of Remark 4.11 (a), take (with α_1 as above) $\pi'(z,n) = \begin{pmatrix} \alpha_1^n & z & 0 & 0 \\ 0 & \alpha_1^{-n} & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}$, giving $\left\{ \begin{pmatrix} \beta & z & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} : \beta, z \in \mathbb{C},\ |\beta| = 1,\ t \in \mathbb{R} \right\}$ as (real) Zariski closure. Then the corresponding action of $K \cong \{\beta \in \mathbb{C} : |\beta| = 1\}$ on $\widetilde{N} \cong \mathbb{C} \times \mathbb{R}$ is $\beta \circ (z,t) = (\beta^2 z, t)$. Thus K does not act faithfully.

The last example excludes a possible converse in Proposition 4.8 (c): when G has no non-trivial compact normal subgroups, K need not act faithfully on $\widetilde{N}$.

(e) In Corollary 4.10 (d), the condition of the existence of an abelian almost–supplementary group H cannot be dropped (i.e., for $\widetilde{N}$ to be abelian, it is not enough that N is an FC_G^--group). Let $\alpha, \beta \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Consider ($\mathsf{H}_{\mathbb{Z}}$ denotes the discrete Heisenberg group) the group $G = \mathsf{H}_{\mathbb{Z}} \ltimes \mathbb{C}$ given by $\mathbb{Z}^3 \times \mathbb{C}$ topologically, with multiplication $(k,l,m,z)(k',l',m',z') = (k+k', l+l', m+m'+l\,k', e^{i(k'\alpha + l'\beta)}z + z')$. Then $N = \{(0,0,m,z) : m \in \mathbb{Z},\ z \in \mathbb{C}\}$ is an FC^--group (observe that the action of $\mathsf{H}_{\mathbb{Z}}$ on $\mathbb{C}$ is semisimple). But (similarly as in (a)) $\widetilde{N} = (\mathsf{H}_{\mathbb{Z}})_{\mathbb{R}} \times \mathbb{C}$ $(= \mathsf{H} \times \mathbb{C})$, $K \cong \{\gamma \in \mathbb{C} : |\gamma| = 1\}$, $\widetilde{M} = (\mathsf{H}_{\mathbb{Z}})_{\mathbb{R}}$, thus $\widetilde{N}$ is not abelian. In the notation of Corollary 4.9, one has $L = \mathsf{H}_{\mathbb{Z}}$.

(f) In the notation of Proposition 4.8 (b), G need not be K_1-invariant (but, as mentioned earlier, $G_{an} = G\,K_1$ is always a group). Let $G = \mathbb{C} \rtimes \mathbb{Z}$ be the first example of (a) and define $\sigma \in \mathrm{Aut}(\widetilde{G} \times \widetilde{G})$ by $\sigma(x,y) = (y,x)$. Let W ($\cong \mathbb{Z}_2$) be the subgroup generated by σ and put $\widetilde{G}' = (\widetilde{G} \times \widetilde{G}) \rtimes W$. Let $j : G \to \widetilde{G}$ denote the embedding and consider the subgroup G' ($\cong (G \times G) \rtimes W$) of $\widetilde{G}$ generated by $j(G) \times j(G)$ and σ. Then $K' = (K \times K) \rtimes W$ gives a compact component for $\widetilde{G}'$, $(K')^0 = K \times K$ and taking $x \in K$ such that $x^2 \notin \{\alpha^n : n \in \mathbb{Z}\}$, G' is not invariant under the inner automorphism of $\widetilde{G}'$ defined by $(x,x) \in K'$. Similarly for subgroups conjugate to K'.
Taking $\mu \in \widetilde{M} \times \widetilde{M}$ such that $\sigma(\mu)\,\mu^{-1} \notin \mathbb{Z}^2$, one gets $\mu K' \mu^{-1} \cap G' = K \times K$. It follows that $\mu K' \mu^{-1} \cap G'$ is not a maximal compact subgroup of G', hence the corresponding statement of Proposition 4.8 (d) does not extend to the non-connected case.

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