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Nilpotency in Groups with the Minimal Condition on Centralizers*

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The Sylow-2-subgroups of a periodic group with minimal condition on centralizers are locally finite and conjugate. The same holds for the Sylow-p-subgroups for any prime p, provided the subgroups generated by any two p-elements of the group are finite. In the non-periodic context, the bounded left Engel elements of a group with minimal condition on centralizers form the Fitting subgroup.

Key Words: chain condition, nilpotency, Sylow subgroup.

INTRODUCTION

Nilpotency properties in groups with the chain condition on centralizers ($\mathfrak{M}_c$) have been studied by a number of people, generalizing corresponding results for linear groups. R. Bryant [2] proved that a locally nilpotent periodic $\mathfrak{M}_c$-group is nilpotent-by-finite and that the Sylow p-subgroups in a locally finite $\mathfrak{M}_c$-group are conjugate. Bryant and Hartley [3] also showed that a periodic locally soluble $\mathfrak{M}_c$-group is nilpotent-by-abelian-by-finite. We shall extend Bryant's first result by replacing local nilpotency by binary nilpotency (Remark 2.1) and his second result by replacing local finiteness by periodicity (for $p = 2$) or binary finiteness (for arbitrary p) (Theorem 3.1). We also answer positively a question of John Wilson about local finiteness of $\mathfrak{M}_c$-2-groups (Corollary 2.4). (Recall that a group is *binary P*, where P is a property of groups, if every 2-generated subgroup is contained in a group satisfying P.)

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Non-periodic groups were first dealt with in an intermediate case, the class of *substable* groups, which lies in between the linear and the $\mathfrak{M}_c$-groups. Local nilpotency properties of substable groups are well understood ([6, 7, 8, 4], see also [9]):

(1) Uniformly locally nilpotent substable groups are nilpotent.

(2) Locally nilpotent substable groups are hypercentral.

(3) The bounded left Engel elements of a substable group form the Fitting subgroup.

(4) The unbounded left Engel elements of a substable group form the Hirsch-Plotkin radical.

The question was asked to what extent substability can be replaced by $\mathfrak{M}_c$ in the above results. Derakhshan could show that the Fitting subgroup of any $\mathfrak{M}_c$-group is nilpotent; this quickly led to a generalization of (1) to the $\mathfrak{M}_c$ case [4]. Recently, Bludov in a short, elegant paper [1] dealt with the $\mathfrak{M}_c$-analogue of (2); he also found an independent proof of Derakhshan's result. In this paper we shall generalize (3).

1. NOTATION AND PRELIMINARIES

Our commutators are left-normed, defined inductively via $[g,h] = g^{-1}h^{-1}gh$ and $[g_0, g_1, \dots, g_{n+1}] = [[g_0, g_1, \dots, g_n], g_{n+1}]$; repeated commutators are given by $[g,_0 h] = g$ and $[g,_{n+1} h] = [[g,_n h], h]$. The *descending central series* is defined by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$; the *ascending central series* is given by $Z_0(G) = \{1\}$ and $Z_{n+1}(G) = \{g \in G: [g, G] \leq Z_n(G)\}$; this can be continued into the transfinite by taking unions at the limit stages. We also write $Z(G)$ for $Z_1(G)$. The *derived series* of G is given by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The series of *iterated centralizers* of some subset X of G is defined inductively via $C_G^0(X) = \{1\}$ and $C_G^{n+1}(X) = \{g \in \bigcap_{i \leq n} N_G(C_G^i(X)): [g, X] \subseteq C_G^n(X)\}$.

If G acts on an abelian group A, we can define analogously for any subset X of G the *iterated centralizers* of X in A via $C_A^0(X) = \{1\}$ and $C_A^{n+1}(X) = \{a \in A: a^g \in aC_A^n(X)$ for all $g \in X\}$. These are obviously subgroups of A.

We shall need the following facts, which can be found in [9]:

Fact 1.1 [2]. Suppose $K \leq H \leq G$, and $C_G(\gamma_i(K)) = C_G(\gamma_i(H))$ whenever $0 < i \leq j$. Then $C_G^j(K) = C_G^j(H)$.

Fact 1.2 (T. Yen). A locally nilpotent $\mathfrak{M}_c$-group is soluble.

DEFINITION 1.1. A group G is *hypercentral* if $G = Z_\alpha(G)$ for some ordinal α. It satisfies the *normalizer condition* if no proper subgroup of G is self-normalizing.

FACT 1.3. A nilpotent-by-finite and locally nilpotent group is hypercentral and hence satisfies the normalizer condition.

Proof. If G is finite, it is nilpotent and we are done. Otherwise, let N be a normal nilpotent subgroup of G of finite index, $g_0, \dots, g_n$ a system of representatives of G/N, and $x \in Z(N)$ a non-trivial element. Then $\langle x^{g_i}: i \leq n \rangle$ is a normal subgroup of G contained in $Z(N)$, and must contain a non-trivial central element z of the nilpotent group $\langle x, g_i: i \leq n \rangle$. But then $z \in Z(G)$ and $Z(G)$ is non-trivial. Since the assumptions of the fact are preserved under quotients, G is hypercentral. But now, if H is a proper subgroup of G and α is minimal such that $Z_\alpha(G) \nleq H$, then $Z_\alpha(G) \leq N_G(H)$, whence $N_G(H) > H$.

Fact 1.4 [2]. Let G be a locally nilpotent periodic $\mathfrak{M}_c$-group. Then G is nilpotent-by-finite; if $G/Z_i(G)$ has finite exponent for some $i < \omega$, then G is nilpotent. The maximal normal nilpotent subgroup of G is the centralizer of the d!-th powers of the elements of G, for all d sufficiently large; it also is the maximal nilpotent subgroup of finite index. Furthermore, G is hypercentral, and in particular satisfies the normalizer condition.

LEMMA 1.5. *Let G be a group, X a G-invariant subset, and H a subgroup of G satisfying the normalizer condition. Suppose K is a subgroup of G such that $H \cap X \nsubseteq K$, and put $I = \langle K \cap H \cap X \rangle$. Then there is $h \in N_{H \cap X}(I) - I$.*

Proof. If the whole of $H \cap X$ normalizes I, we are done since $I \leq K$. Otherwise $N_H(I)$ is a proper subgroup of H, so $N_H(N_H(I)) > N_H(I)$ and there is $g \in H$ which normalizes $N_H(I)$, but not I. Since X is G-invariant, g must normalize $N_{H \cap X}(I)$. On the other hand, $I = \langle I \cap X \rangle$, so g cannot normalize $I \cap X$. Therefore $N_{H \cap X}(I) \supset I \cap X$; as $I \cap (H \cap X) = I \cap X$, the assertion follows. ∎

DEFINITION 1.2. Let G be a group. Two elements x and y in G satisfy the nth Engel identity if $[x,_n y] = 1$.

An element $g \in G$ is *right Engel* if $[g,_n x] = 1$ for all x in G, where n may depend on x. If n can be chosen independently of x, then g is called *right n-Engel*, or *bounded right Engel*.

An element $g \in G$ is *left Engel* if $[x,_n g] = 1$ for all x in G, where again n may depend on x. If n can be chosen independently of x, then g is called *left n-Engel*, or *bounded left Engel*.

An element is *Engel* if it is left or right Engel. It is *bounded Engel* if it is bounded left or bounded right Engel.

Remark 1.1. Note that if g^{-1} is right n-Engel and x is in G, then $[x,_{n+1} g] = [g^{-x}g,_n g] = [g^{-x},_n g]^g = [g^{-1},_n g^{x^{-1}}]^{xg} = 1$, and so g is left $(n+1)$-Engel.

For a subset X of G we say that any two elements in X satisfy some Engel identity if for any $x, y \in X$ there is $n < \omega$ such that $[x,_n y] = 1$ or $[y,_n x] = 1$.

Fact 1.6 1.6 [5, 4]. The bounded left Engel elements in a soluble $\mathfrak{M}_c$-group form the Fitting subgroup. The Fitting subgroup of an $\mathfrak{M}_c$-group is nilpotent.

LEMMA 1.7. *Let G be a soluble group, p a prime, and X a G-invariant subset of G of p-elements such that any two elements of X satisfy some Engel identity. Then X generates a locally finite p-group.*

Proof. Consider a counter-example G of minimal derived length, and $x_1, \dots, x_n \in X$ such that $F = \langle x_1, \dots, x_n \rangle$ is not a finite p-group. Let A be the last non-trivial subgroup in the derived series of F. By minimality of the derived length, F/A must be a finite p-group. But F is finitely generated, and so is any subgroup of finite index, in particular A. On the other hand, A is abelian and there is a finite k such that A/A^k is not a p-group (where A^k denotes the subgroup of kth powers of elements in A). Hence F/A^k is not a p-group, and we may assume that G is finite.

Since X does not generate a p-group, there are two distinct subgroups S and T of G which are maximal subject to being p-subgroups generated by elements in X. Choose S and T such that $S \cap T \cap X$ is of maximal cardinality, and let $I = \langle S \cap T \cap X \rangle$. By Lemma 1.5 there are $x \in N_{S \cap X}(I) - I$ and $y \in N_{T \cap X}(I) - I$; by assumption there is some n such that $[x,_n y] = 1$ or $[y,_n x] = 1$, and by symmetry we may assume $[x,_n y] = 1$. Choose $m < n$ maximal such that $[x,_m y] \notin T$; putting $u = [x,_m y]$, note that $u \in N_G(I)$. Then $[u, y] \in T$, whence $y^u \in T$. Hence $T \cap T^u$ contains I and y^u; as $y^u \in X - I$, maximality of $S \cap T \cap X$ implies $T = T^u$. But either $u = x \in X$, or yu^{-1} is a conjugate of y and hence in X. Since $\langle T, u \rangle = \langle T, yu^{-1} \rangle$, in either case this group is an extension of T by a p-element in X normalizing T, and thus a p-group generated by elements in X. As $u \notin T$, this contradicts maximality of T. ∎

LEMMA 1.8. *Let G be a group and S a nilpotent subgroup of class c. If $H \le \bigcap_{i \le c} N_G(C_G^i(S))$ and N is the group generated by all H-conjugates of S, then N is nilpotent of class c, and $C_G^i(N) = C_G^i(S)$ for all $i \le c$.*

Proof. Let $I = (x_0, x_1, \dots)$ be a sequence of elements in the union of the H-conjugates of S, and put $y_0 = x_0$ and $y_{i+1} = [y_i, x_{i+1}]$ for all $i \ge 0$. Note that $C_G^i(S^h) = C_G^i(S)^h = C_G^i(S)$ for all $i \le c$ and all $h \in H$. Since $S \le C_G^c(S)$, it follows that $y_0 \in C_G^c(S)$. Furthermore, if $y_i \in C_G^{c-i}(S)$ and $x_{i+1} \in S^h$ for some $h \in H$, then $y_i \in C_G^{c-i}(S^h)$ and hence $y_{i+1} \in$

$C_G^{c-i-1}(S^h) = C_G^{c-i-1}(S)$, for any $i \leq c-1$. Therefore $y_c = 1$. Now every commutator of length $c+1$ in N is a product of commutators of the above form of length at least $c+1$. Thus N is nilpotent of class c.

For the second assertion, we use induction on i. For $i = 0$ the assertion is trivial, so suppose inductively that $C_G^j(N) = C_G^j(S)$ for all $j < i$. Then by inductive hypothesis $\bigcap_{j<i} N_G(C_G^j(N)) = \bigcap_{j<i} N_G(C_G^j(S^h))$ for all $h \in H$, and for any $g \in \bigcap_{j<i} N_G(C_G^j(N))$ we have $[g, N] \leq C_G^{i-1}(N)$ if and only if $[g, S^h] \leq C_G^{i-1}(N)$ for all $h \in H$, since N is generated by the H-conjugates of S and normalizes $C_G^{i-1}(N)$. But this holds if and only if $[g, S^h] \leq C_G^{i-1}(S^h)$ for all $h \in H$, which is equivalent to $g \in C_G^i(S^h)$ for all $h \in H$. Since $C_G^i(S^h) = C_G^i(S)$ for all $h \in H$, this proves the assertion. ∎

Fact 1.9 [4]. Let G be an abelian group acting on an abelian group A. Suppose that there are finitely many elements $g_0, \ldots, g_k$ in G such that $C_A(G) = C_A(g_0, \ldots, g_k)$. Let a be an element in A, and suppose that for all $i = 0, \ldots, k$ there is some non-zero $m_i < \omega$ with $(g_i - 1)^{m_i} a = 0$. Then $a \in C_A^m(G)$, with $m = 1 + \sum_{i=0}^k (m_i - 1)$.

2. BINARY NILPOTENCY CONDITIONS

THEOREM 2.1. *Let G be an $\mathfrak{M}_c$-group and X a G-invariant subset of G such that*

1. *every soluble subgroup S of G generated by elements in X is locally nilpotent and nilpotent-by-finite, and*
2. (a) *any two elements in X satisfy some Engel identity, or*

   (b) *X is closed under taking powers, and any two elements in X generate a 2-group.*

Then X generates a locally nilpotent subgroup of G.

Proof. Suppose the assertion is false, and that the group G with the G-invariant subset X is a counter-example. Fix a locally nilpotent subgroup S of G which is maximal subject to being generated by elements in X. By Fact 1.2 and the first assumption, S is nilpotent-by-finite and is a maximal soluble subgroup of G subject to being generated by elements in X. In particular $N_X(S) = S \cap X$.

Since $\langle X \rangle$ is not locally nilpotent, there is another locally nilpotent subgroup T of G which is maximal subject to being generated by elements of X. Note that if every pair of elements of X generates a 2-group, then local nilpotency implies that both S and T are 2-groups. Let $\mathscr{I}$ be the set of all subgroups I of S of the form $I = \langle S \cap T \cap X \rangle$, where T ranges through all locally nilpotent subgroups of G distinct from S which are maximal subject to being generated by elements in X. We have just seen that $\mathscr{I}$ is non-empty. Note that any I in $\mathscr{I}$ is generated by $I \cap X$.

Claim. Let $(I_i: i < \alpha)$ be an ascending sequence of groups in $\mathscr{I}$, and put $I = \bigcup_{i<\alpha} I_i$. Then there is J in $\mathscr{I}$ with $I \le J$.

Proof of Claim. I is a subgroup of S and thus nilpotent-by-finite. Let K be a nilpotent subgroup of minimal finite index in I, say of nilpotency class c. Now I satisfies the normalizer condition by Fact 1.3. So if K is not normal in I, then there is $g \in I$ normalizing $N_I(K)$ but not K. Hence KK^g is a nilpotent group with $K < KK^g \le I$, contradicting the minimality of the index $|I : K|$. It follows that K is normal in I and unique.

Put $K_i = K \cap I_i$ for all $i < \alpha$, so K_i is nilpotent of class at most c. Suppose the set of $i < \alpha$ such that there is a nilpotent subgroup K_i^* of class at most $2c+1$, with $K_i < K_i^* \le I_i$, is cofinal in α. As K has finite index in I, there are only finitely many possibilities for KK_i^*; replacing the sequence $(i: i < \alpha)$ by a cofinal subsequence, we may assume that KK_i^* is constant for all $i < \alpha$. But then for $i \le j < \alpha$ we have

$$K_i^* \le \left(KK_j^*\right) \cap I_j = \left(K \cap I_j\right)K_j^* = K_jK_j^* = K_j^*.$$

Let $K^* = \bigcup_{i<\alpha} K_i^*$. Then K^* is nilpotent of class at most $2c+1$ and $K < K^*$, contradicting the maximality of K. Hence there is $i_0 < \alpha$ such that for all $i \ge i_0$ no proper extension of K_i in I_i can be nilpotent of class at most $2c+1$. But if H is an automorphic conjugate of K_i in I_i, then both K_i and H are normal nilpotent of class at most c in I_i, so NK_i is nilpotent of class at most $2c+1$. It follows that K_i is characteristic in I_i for all $i \ge i_0$.

Since $K_i \le K_j$ and $I_i \le I_j$ for $i \le j < \alpha$, after possibly increasing i_0 we may assume that first $C_G(\gamma_j(K_{i_0}))$ is minimal possible for $j = 1, 2, \dots, c$ (and hence $C_G(\gamma_j(K_i)) = C_C(\gamma_j(K_{i_0}))$ for all $i \ge i_0$), and second $I = KI_{i_0}$ (so $I = KI_i$ for all $i \ge i_0$). Let T be a locally nilpotent subgroup of G which is distinct from S and maximal subject to being generated by elements in X, with $I_{i_0} = \langle S \cap T \cap X \rangle$ (which exists, since $I_{i_0} \in \mathscr{I}$). By Lemma 1.5 there is some $y \in N_{T \cap X}(I_{i_0}) - I_{i_0}$, in particular $y \notin S$. Put $F = \langle I_{i_0}, y \rangle$; then both I_{i_0} and its characteristic subgroup K_{i_0} are normalized by F, and so is $C_G^j(K_{i_0})$ for all $j \le c$. But

$$\begin{aligned} C_G\big(\gamma_j(K)\big) &= C_G\left(\gamma_j\left(\bigcup_{i<\alpha} K_i\right)\right) = C_G\left(\bigcup_{i<\alpha} \gamma_j(K_i)\right) \\ &= \bigcap_{i<\alpha} C_G\big(\gamma_j(K_i)\big) = C_G\big(\gamma_j(K_{i_0})\big) \end{aligned}$$

for $j = 1, 2, \dots, c$, whence $C_G^j(K) = C_G^j(K_{i_0})$ for all $j \le c$ by Fact 1.1. Hence $F \le \bigcap_{j \le c} N_G(C_G^j(K))$. By Lemma 1.8 the group N generated by all F-conjugates of K is nilpotent of class c (and obviously contains K). Now $F \le T$, so F is soluble; since F normalizes N, it follows that NF is soluble. Note that NF contains KI_{i_0} and y, so $\langle I, y\rangle$ is soluble. Furthermore I is generated by $I \cap X$, and $y \in X$; by assumption $\langle I, y\rangle$ is locally nilpotent, and hence contained in a locally nilpotent group T_1 which is maximal subject to being generated by elements in X. Now $I \le S \cap T_1$, and $y \in T_1 - S$ implies $S \ne T_1$. Therefore $J = \langle S \cap T_1 \cap X\rangle$ is the required group in $\mathscr{I}$ containing I. ▮

Claim. Let $I \in \mathscr{I}$. Then there is $J > I$ with $J \in \mathscr{I}$.

Proof of Claim. Let T be a locally nilpotent subgroup of G distinct from S and maximal subject to being generated by elements in X, such that $I = \langle S \cap T \cap X\rangle$. By Lemma 1.5 there is some $y \in N_{T \cap X}(I) - I$; in Case 2.(b) we may choose y with $y^2 \in I$. Similarly, there is some $x \in N_{S \cap X}(I) - I$ (which in Case 2.(b) we choose such that $x^2 \in I$).

In Case 2.(b) the group $\langle x, y\rangle I/I$ is dihedral, so $\langle x, y, I\rangle$ is soluble and generated by elements in X, whence locally nilpotent by assumption. Let T_1 be a locally nilpotent group containing x, y, and I, maximal subject to being generated by elements in X, and put $J := \langle S \cap T_1 \cap X\rangle$. Then $y \in T_1$, so $T_1 \ne S$ and $J \in \mathscr{I}$; since $x \in J - I$ we have $I < J$ as required.

In Case 2.(a) consider $I_0 := \langle S \cap S^y \cap X\rangle$. If $S^y = S$, we could extend S to $\langle S, y\rangle$, contradicting the maximality of S. Hence $I_0 \in \mathscr{I}$; since y normalizes I, we have $I_0 \ge I$. If $I_0 > I$ we are done, so assume $I_0 = I$. By Lemma 1.5 there is some $z \in N_{S^y \cap X}(I) - I$; by assumption there is some n such that $[z,_n x] = 1$ or $[x,_n z] = 1$. In the second case we may replace S^y by $S^{y^{-1}}$, x by $z^{y^{-1}}$, and z by $x^{y^{-1}}$, thus reducing to the first case.

Choose $m < n$ maximal such that $[z,_m x] \notin S$ and put $u = [z,_m x]$; note that u normalizes I. Then $[u, x] \in S$, whence $x^u \in S \cap X$. Hence $S \cap S^u$ contains I and x^u. If $S = S^u$, then $\langle S, u\rangle$ is a soluble group properly containing S. But either $u = z \in X$, or xu^{-1} is a conjugate of x and hence in X; in either case $\langle S, u\rangle$ is generated by elements in X, contradicting maximality of S. Hence $S^u \ne S$, and we may take $J = \langle S \cap S^u \cap X\rangle$. Then $J \in \mathscr{I}$ and $J \ge I$; since $x^u \in J - I$, we are done. ▮

This shows that $\mathscr{I}$ is non-empty and chains in $\mathscr{I}$ have upper bounds in $\mathscr{I}$, but $\mathscr{I}$ does not have a maximal element, contradicting Zorn's Lemma. ▮

Note that under the assumptions of the Theorem, since a locally nilpotent $\mathfrak{M}_c$-group is soluble by Fact 1.2, the group generated by X is also nilpotent-by-finite.

COROLLARY 2.2. *Let G be an $\mathfrak{M}_c$-group and X a G-invariant subset of p-elements in G, for some prime p. If every pair of elements of X satisfies some Engel identity or generates a 2-group (for $p = 2$), then X generates a locally nilpotent p-group.*

Proof. If every pair of elements of X generates a 2-group, we can close X under taking powers. Since a soluble 2-generated 2-group is finite and thus nilpotent, in particular it satisfies some Engel identity. By Lemma 1.7 a soluble subgroup of G generated by a subset of X is locally nilpotent. By Fact 1.4 it also is nilpotent-by-finite. We may now apply Theorem 2.1 to see that X generates a locally nilpotent group, which must be a p-group by local nilpotency. ∎

COROLLARY 2.3. *Let G be an $\mathfrak{M}_c$-group and X a G-invariant subset of p-elements in G, for some prime p. If every pair of elements of X generates a finite p-group, then X generates a locally finite p-group.*

Proof. This is obvious, as any two elements in a nilpotent group satisfy some Engel identity. ∎

COROLLARY 2.4. *An $\mathfrak{M}_c$-2-group is locally finite. A periodic $\mathfrak{M}_c$-group G is locally nilpotent if and only if every pair of elements generates a nilpotent subgroup.*

Proof. The first assertion follows immediately from Corollary 2.2. For the second assertion, suppose every pair of elements generates a nilpotent subgroup. For each prime p let X_p be the set of all p-elements of G. Since a nilpotent group generated by two p-elements must be a finite p-group, X_p is a normal locally finite p-group by Corollary 2.3; this holds for all primes p. But any two elements of coprime order must commute (as they generate a nilpotent subgroup), so G is the direct product of all the X_p and locally nilpotent. The converse is trivial. ∎

Remark 2.1. In particular, a periodic binary nilpotent group is nilpotent-by-finite by Fact 1.4.

Remark 2.2. Let X be a G-invariant periodic subset of an $\mathfrak{M}_c$-group G which generates a locally nilpotent subgroup. For $s < \omega$ put $X(s) = \{x \in X\colon x^s = 1\}$. Suppose q is a power of some prime p such that the exponent, or the nilpotency class, of every group generated by two elements in $X(q)$ is bounded, say by p^n for some $n < \omega$. Note that if a group H is nilpotent of class c and generated by elements in X_q, it is easy to see that $\gamma_i(H)/\gamma_{i+1}(H)$ has a set of generators of order dividing q for all $i \geq 1$, so the exponent of H is bounded by q^c. Hence the case of bounded nilpotency class reduces to the case of bounded exponent.

Let N be the maximal (normal) nilpotent subgroup of $\langle X(q)\rangle$ of finite index, as given by Fact 1.4. Then $x^{-1}x^y = [x, y] \in N$ for every $x \in X(q)$ and $y \in N$, and this is an element of order at most p^n. Let N_0 be the subgroup of N generated by all elements of order at most p^n. Then N_0 is normal in $\langle X(q)\rangle$, and $\gamma_i(N_0)/\gamma_{i+1}(N_0)$ has a generating set of elements of order at most p^n for all $i \geq 1$; it follows that N_0 has finite exponent. Put $N_x = \langle N_0, x\rangle$; clearly N_x has finite exponent and must be nilpotent by Fact 1.4. As it is normalized by N, the group NN_x is nilpotent and equals N by maximality of N. Thus $x \in N$, and $X(q)$ generates a normal nilpotent subgroup. In particular, if this happens for all prime powers q, then X generates a subgroup of the Fitting subgroup of G, which is nilpotent by Fact 1.6.

We shall now consider non-periodic groups.

COROLLARY 2.5. *Let G be an $\mathfrak{M}_c$-group. Then the bounded left Engel elements form the Fitting subgroup of G.*

Proof. If we denote the set of bounded left Engel elements by E, then E is G-invariant, and any two elements of E satisfy some Engel condition. By Fact 1.6 a soluble subgroup generated by a subset of E is nilpotent; Theorem 2.1 implies that E generates a locally nilpotent group. But now E generates a soluble subgroup by Fact 1.2, which must nilpotent by Fact 1.6 again. It follows that every bounded left Engel element is in the Fitting subgroup.

Conversely, since the Fitting subgroup is nilpotent by Fact 1.6, say of class c, every element in the Fitting subgroup is left $(c+1)$-Engel. ∎

As a particular case, we obtain:

COROLLARY 2.6. *An $\mathfrak{M}_c$-group generated by bounded left Engel elements is nilpotent.*

3. CONJUGACY OF THE SYLOW SUBGROUPS

THEOREM 3.1. *Let G be an $\mathfrak{M}_c$-group, and p a prime. If G is periodic and $p = 2$, or if every pair of p-elements of G generates a finite subgroup, then the Sylow-p-groups of G are conjugate.*

Proof. Note first that under the assumptions of the theorem, the maximal p-subgroups of G (i.e., the Sylow-p-subgroups) are locally finite by Corollary 2.4. Assume for a contradiction that G is a counter-example to the assertion.

Claim. We may assume that G is countable.

Proof of Claim. Consider the structure $(G, 1, \cdot, \mathscr{S}, \mathscr{T})$, where $\mathscr{S}$ and $\mathscr{T}$ are unary predicates for two non-conjugate Sylow-p-subgroups S and T, respectively. By the downward Löwenheim-Skolem Theorem, G has a countable elementary substructure H. Now $\mathscr{S}^H$ is a subgroup of $\mathscr{S}^G$, which is equal to S, and hence a p-group. Furthermore every $g \in G$ normalizing $\mathscr{S}^G$ and with $g^p \in \mathscr{S}^G$ lies in $\mathscr{S}^G$ itself, since S is a Sylow-p-subgroup. But this is a first-order property and hence also true in H. By Fact 1.3 any p-Sylow subgroup of H containing $\mathscr{S}^H$ satisfies the normalizer condition, so $\mathscr{S}^H$, and similarly $\mathscr{T}^H$, are Sylow-p-subgroups of H. Now non-conjugacy of $\mathscr{S}$ and $\mathscr{T}$ is a first-order statement which is true in G, so it must also be true in H. Hence H is a countable $\mathfrak{M}_c$-group with two non-conjugate Sylow-p-subgroups $\mathscr{S}^H$ and $\mathscr{T}^H$. ▮

By the chain condition on centralizers, we may assume that the Sylow-p-subgroups of every proper centralizer in G are conjugate.

Claim. There exist finite p-subgroups A_0 and B_0 of G such that no G-conjugate of B_0 generates together with A_0 a (necessarily finite) p-group.

Proof of Claim. This is similar to the proof of Theorem B in [2], but we have to be a bit more careful. Let S and T be two non-conjugate Sylow-p-subgroups of G. By countability, S and T are the union of ascending chains $(S_i\colon i < \omega)$ and $(T_i\colon i < \omega)$ of finite subgroups. Suppose any two finite p-subgroups of G have conjugates which generate a p-group. Then there is an ascending chain $(U_i\colon i < \omega)$ of finite p-groups, such that for every i there is $g_i \in G$ and $h_i \in G$ with $S_i^{g_i} \le U_i$ and $T_i^{h_i} \le U_i$: start with $g_0 = 1$ and $h_0 \in G$ such that $\langle S_0, T_0^{h_0}\rangle$ is a finite p-group U_0; if U_{i-1} has been found, the assumption first yields $g_i \in G$ such that $\langle U_{i-1}, S_i^{g_i}\rangle$ is a finite p-group, say F_0, and then $h_i \in G$ such that $\langle F_0, T_i^{h_i}\rangle$ is a finite p-group, which we choose for U_i. Extend $\bigcup_{i<\omega} U_i$ to a maximal p-group U. Then either S and U or U and T are not conjugate. By symmetry we may assume that S and U are not conjugate.

By Fact 1.4, for some $d < \omega$ the centralizer $C_U(u^d\colon u \in U)$ is the unique maximal nilpotent subgroup N of finite index in U, say of index $|U:N| = n$. Now every S_i has only finitely many normal subgroups of index at most n, and $S_i \cap N^{g_j^{-1}}$ is such a subgroup for all $j \ge i$. So there is some normal subgroup $\overline{S}_0$ of index at most n in S_0 such that $\overline{S}_0^{g_j} = S_0^{g_j} \cap N$ for infinitely many $j < \omega$. Then for every $i < \omega$ there is some $j < \omega$ with $j \ge i$ and $\overline{S}_0^{g_j} = S_0^{g_j} \cap N$; as $S_i^{g_j} \le U$, we may replace g_i by g_j. Repeating this process (but replacing only g_j for $j \ge i$ at the ith stage), we may assume that there is a sequence $(\overline{S}_i\colon i < \omega)$ such that $\overline{S}_i$ is a normal subgroup of S_i of index at most n and $\bar{S}_i^{g_j} = S_i^{g_j} \cap N$ for all i, j with $i \leq j < \omega$. Thus, $\bar{S}_i = S_i \cap \bar{S}_j$ for all i, j with $i \leq j < \omega$, the sequence $(\bar{S}_i\colon i < \omega)$ is ascending, and $\bigcup_{i<\omega} \bar{S}_i$ is a normal subgroup $\bar{S}$ of S of index at most n.

Define $C = C_G(u^d\colon u \in U)$, and consider C and its conjugates (which are again centralizers). If $C = G$, then U is nilpotent and the proof of Lemma 3.3 of [2] applies, which yields that U and S are conjugate, contradicting our assumptions. Hence $N \leq C < G$. By the chain condition on centralizers the ascending sequence $(\bigcap_{j\geq i} C^{g_j^{-1}}\colon i < \omega)$ has a maximal element $\bar{C}$, say $\bar{C} = \bigcap_{j \geq s} C^{g_j^{-1}}$. Since

$$\bar{C} \geq \bigcap_{j\geq i} C^{g_j^{-1}} \geq \bigcap_{j \geq i} N^{g_j^{-1}} \geq \bar{S}_i$$

for all $i < \omega$, we see that $C^{g_s^{-1}} \geq \bar{C} \geq \bar{S}$; replacing U by a conjugate, we may assume $C \geq \bar{S}$. Extend $U \cap C$ (which is N) and $\bar{S}$ to Sylow-p-subgroups $\tilde{U}$ and $\tilde{S}$ of C; since $C < G$, our assumptions imply that there is $g \in C$ with $\tilde{S}^g = \tilde{U}$.

We shall show inductively that if X and Y are two Sylow-p-subgroups of G with $|Y : X \cap Y| = k < \omega$, then X and Y are conjugate in G with $|X : X \cap Y| = |Y : X \cap Y|$. This is clear for $k = 1$. By Fact 1.4, for $k > 1$ there are $x \in N_X(X \cap Y) - Y$ and $y \in N_Y(X \cap Y) - X$ of order p modulo $X \cap Y$. By assumption they generate a finite group modulo $X \cap Y$, so there is some $h \in N_G(X \cap Y)$ such that x^h, y and $X \cap Y$ are contained in a Sylow-p-subgroup Z of G. But $|Y : Z \cap Y| < k$, so Z and Y are conjugate by inductive hypothesis, with $|Z : Z \cap Y| = |Y : Z \cap Y|$. This implies that $|Z : X \cap Y| = |Y : X \cap Y| = k$, whence $|Z : Z \cap X^h| < k$. Again by inductive hypothesis, Z and X^h are conjugate with $|X^h : X^h \cap Z| = |Z : X^h \cap Z|$. Therefore $|X^h : X \cap Y| = |Z : X \cap Y|$, and finally $|X : X \cap Y| = |X^h : X \cap Y| = |Z : X \cap Y| = |Y : X \cap Y|$. This finishes the induction.

Now let $\hat{S}$ be a Sylow-p-subgroup of G extending $\tilde{S}$. Since $\tilde{S} \geq \bar{S}$, the last paragraph applied to $X = \hat{S}$ and $Y = S$ implies that S and $\hat{S}$ are conjugate. Since $\hat{S}^g \geq \tilde{S}^g = \tilde{U} \geq N$ and $|U : N|$ is finite, the last paragraph applies again with $X = \hat{S}^g$ and $Y = U$. This proves conjugacy of U and $\hat{S}$, whence of U and S, a contradiction. The claim is shown. ∎

Let $\mathscr{I}$ be the set of all triples (A, B, C), where A and B are finite p-subgroups of G such that $\langle A, B^g \rangle$ is not a p-group for any $g \in G$, and $C = A \cap B$. Then $(A_0, B_0, A_0 \cap B_0) \in \mathscr{I}$, so $\mathscr{I}$ is non-empty.

Claim. If $(A, B, C) \in \mathscr{I}$, then there is some $(A^*, B^*, C^*) \in \mathscr{I}$ such that $C \leq C^* \leq B$, $B^* = \langle C^*, b \rangle \leq B$ for some $b \in N_B(C^*)$ with $b^p \in C^*$, and $\langle A^h, B^* \rangle$ is not a p-group for any $h \in G$.

Proof of Claim. B is nilpotent and $\langle A^g, B\rangle$ is not a p-group for any $g \in G$. Therefore there is some maximal i such that for some $g \in G$ the group $\langle A^g, Z_i(B)C\rangle$ is a p-group, and a maximal C^* with $Z_i(B)C \leq C^* < Z_{i+1}(B)C$, such that for some $g' \in G$ the group $\langle A^{g'}, C^*\rangle$ is a p-group A^*. Take $b \in Z_{i+1}(B) - C^*$ with $b^p \in C^*$, and put $B^* = \langle b, C^*\rangle$. Since $Z_i(B) \leq C^* \leq B$ we get $b \in N_B(C^*)$. Now $\langle A^{*h}, B^*\rangle \geq \langle A^{g'h}, B^*\rangle$ for any $h \in G$; since $\langle A^h, B^*\rangle$ is not a p-group for any $h \in G$ by maximality of C^*, neither is $\langle A^{*h}, B^*\rangle$. Furthermore, $C^* \leq A^* \cap B^* < B^*$; as $|B^* : C^*| = p$ we get $C^* = A^* \cap B^*$, and $(A^*, B^*, C^*) \in \mathscr{I}$. ∎

Claim. There is a sequence $(A_i, B_i, C_i\colon i < \omega) \subseteq \mathscr{I}$ such that

1. $C_i \leq C_{i+1}$ for all $i < \omega$,
2. for all odd $i < \omega$ there is $a_i \in N_G(C_i)$ with $a_i^p \in C_i$ and $A_i = \langle C_i, a_i\rangle$, and
3. for all even $i > 0$ there is $b_i \in N_G(C_i)$ with $b_i^p \in C_i$ and $B_i = \langle C_i, b_i\rangle$, and $\langle A_{i-1}^g, B_i\rangle$ is not a p-group for any $g \in G$.

Proof of Claim. Start with any $(A_0, B_0, C_0) \in \mathscr{I}$. If (A_i, B_i, C_i) has been found for odd i, the above claim yields $(A_{i+1}, B_{i+1}, C_{i+1}) \in \mathscr{I}$; the even case is symmetric. ∎

Let $C = \bigcup_{i<\omega} C_i$. By Fact 1.4 there is some $d < \omega$ such that $C_C(g^d\colon g \in C)$ is the maximal (necessarily normal) nilpotent subgroup of finite index in C, say of class c. For any subgroup H of G, put $D(H) = \{g^d\colon g \in H\}$, and $E(H) = C_H(D(H))$. In particular, $E(C)$ has finite index in C and is nilpotent of class c.

Claim. There is a p-group N_0 containing C and $i_1 < \omega$, such that $A_i \leq N_G(N_0)$ for all odd $i \geq i_1$, and $B_i \leq N_G(N_0)$ for all even $i \geq i_1$.

Proof of Claim. By the chain condition on centralizers, there is some $i_0 < \omega$ such that $C_G(D(C_{i_0}))$ is minimal possible. Note that then $E(C_i) = E(C_j) \cap C_i \leq E(C_j)$ whenever $i_0 \leq i \leq j$, whence $E(C) = \bigcup_{i \geq i_0} E(C_i)$. Furthermore, there is some $i_1 > i_0$ such that first $C_G(\gamma_j(E(C_{i_1})))$ is minimal possible for $j = 1, 2, \ldots, c$ (and hence $C_G(\gamma_j(E(C_i))) = C_G(\gamma_j(E(C_{i_1})))$ for all $j = 1, 2, \ldots, c$ and $i \geq i_1$), and second $C = E(C)C_{i_1}$ (whence $C = E(C)C_i$ for all $i \geq i_1$). As $\gamma_j(E(C)) = \bigcup_{i \geq i_1} \gamma_j(E(C_i))$, this implies that $C_G(\gamma_j(E(C))) = C_G(\gamma_j(E(C_i)))$ for $j = 1, 2, \ldots, c$ and all $i \geq i_1$, whence $C_G^j(E(C)) = C_G^j(E(C_i))$ for all $i \geq i_1$ and all $j \leq c$ by Fact 1.1.

Put $F = \bigcap_{j \leq c} N_G(C_G^j(E(C)))$. For every odd $i \geq i_1$ we have $A_i \leq F$, since A_i normalizes C_i and therefore $E(C_i)$; similarly $B_i \leq F$ for all even $i \geq i_1$.

By Lemma 1.8 the group N generated by all F-conjugates of $E(C)$ is nilpotent of class c. Since $E(C)$ is a p-group, so is N; clearly N is normalized by F. Furthermore, $C_i \leq A_i \cap B_i \leq F$, so C_i normalizes N for all $i \geq i_1$. Put $N_0 = NC$; this is a p-group since C is a p-group normalizing N. Now $C = E(C)C_i$ implies $N_0 = NC_i$ for all $i \geq i_1$. As A_i normalizes C_i (and N) for odd $i \geq i_1$, it normalizes N_0; similarly B_i normalizes C_i and N, and hence N_0, for all even $i \geq i_1$.

Consider some odd $i \geq i_1$. Then $A_i = \langle C_i, a_i \rangle$, and a_i normalizes N_0; on the other hand, $B_{i+1} = \langle C_{i+1}, b_{i+1} \rangle$ and b_{i+1} normalizes N_0. By assumption, $a_i N_0$ and $b_{i+1} N_0$ generate a finite subgroup of $N_G(N_0)/N_0$ (for $p = 2$ this is a dihedral group); by Sylow's Theorem there is $g \in N_G(N_0)$ such that $\langle a_i^g, b_{i+1} \rangle$ is a p-group P normalizing N_0. Therefore A_i^g and B_{i+1} generate a p-group contained in $N_0 P$, contradicting the definition of B_{i+1}.

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