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The Ext-Algebra of a Representation-Finite Biserial Algebra

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Let Λ be a basic representation-finite biserial finite-dimensional k-algebra. We describe a method for constructing a multiplicative basis and the bound quiver of the Ext-algebra $E(\Lambda) = \coprod_{m\geq 0} \mathrm{Ext}^m_\Lambda(\Lambda/\underline{\mathbf{r}}, \Lambda/\underline{\mathbf{r}})$ of Λ using the Auslander-Reiten quiver of Λ. © 1999 Academic Press

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The purpose of this article is to describe a method for constructing a multiplicative basis and the bound quiver for the Ext-algebra of a basic representation-finite biserial finite-dimensional k-algebra Λ, where k is an algebraically closed field. The Ext-algebra of Λ is the k-vector space $E(\Lambda) = \coprod_{m\geq 0} \mathrm{Ext}^m_\Lambda(\Lambda/\underline{\mathbf{r}}, \Lambda/\underline{\mathbf{r}})$, where $\underline{\mathbf{r}}$ is the Jacobson radical of Λ, with multiplication given by the Yoneda product of exact sequences. To determine a basis for $E(\Lambda)$, we determine a basis for $\mathrm{Ext}^m_\Lambda(N, S)$ for each $m > 0$ and for each pair of indecomposable Λ-modules S, N, where S is simple. The basis for $E(\Lambda)$ will then be given by basis elements for $\mathrm{Ext}^m_\Lambda(S, T)$, where $m > 0$ and S, T is a pair of simple Λ-modules.

The Ext-algebra, which is also known as the cohomology ring of an algebra, has been of interest in the study of group algebras [9], commutative rings [18], and polynomial rings [20]. The first description of the ring structure of the Ext-algebra for a class of finite-dimensional algebras was given

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by Green and Zacharia in [17], in which a multiplicative basis for the Ext-algebra of a monomial algebra was described using minimal projective resolutions given in terms of certain paths in the quiver of the algebra. Conditions under which an algebra and its Ext-algebra have the same k-species, and under which quasi-heredity of the algebra (respectively, Ext-algebra) implies quasi-heredity of the Ext-algebra (respectively, algebra) have been studied recently by Agoston *et al.* in [1].

A finite-dimensional algebra Λ is *biserial* if for each indecomposable projective left or right Λ-module P, its radical $\underline{\mathbf{r}}P$ is the sum of at most two uniserial modules whose intersection is either simple or zero. Biserial rings were introduced by Fuller [13]. Examples of finite-dimensional biserial algebras include Nakayama algebras, algebras given by Brauer trees [2], [16], and iterated tilted algebras of types $\mathbb{A}_n$ [3] and $\tilde{\mathbb{A}}_n$ [4]. If Λ is a representation-finite finite-dimensional algebra over an algebraically closed field, then by results of Auslander and Reiten [7, Theorems 4.6 and 4.7] and Skowroński and Waschbüsch [21, Theorem 1, Corollary], Λ is biserial if and only if $\beta(\Lambda) \leq 2$, where $\beta(\Lambda)$ is the maximum number of non-projective, non-injective indecomposable summands of the middle term of any almost split sequence in $\operatorname{mod}\Lambda$, the category of finitely generated Λ-modules.

If Q is a projective–injective Λ-module, then by [6], there is an almost split sequence $0 \to \underline{\mathbf{r}}Q \to \underline{\mathbf{r}}Q/\operatorname{soc} Q \amalg Q \to Q/\operatorname{soc} Q \to 0$. If Q is uniserial, $\underline{\mathbf{r}}Q/\operatorname{soc} Q$ is indecomposable, but if Q is non-uniserial, $\underline{\mathbf{r}}Q/\operatorname{soc} Q$ is the direct sum of two indecomposable Λ-modules. Therefore, each non-uniserial projective–injective module corresponds to an almost split sequence whose middle term decomposes into three summands.

By Lemma 2 in [21], if Λ is representation-finite, then Λ is biserial if and only if Λ is isomorphic to a bound quiver algebra kQ/I satisfying the conditions

(a) for each vertex i in Q, at most two arrows end at i, and at most two arrows begin at i;

(b) for each arrow α in Q, there is at most one arrow β and at most one arrow γ such that $\beta\alpha$ and $\alpha\gamma$ are not in I.

Since Λ is a basic finite-dimensional algebra over an algebraically closed field, we may assume that $\Lambda = kQ/I$, where Q and I satisfy conditions (a) and (b) above (such an algebra is called *special biserial*).

In the first part of the paper, we determine a basis for $\operatorname{Ext}^m_\Lambda(y, x)$, for each pair of indecomposable Λ-modules x, y and each $m > 0$, where Λ is a locally representation-finite, locally finite-dimensional, simply connected k-algebra with $\beta(\Lambda) \leq 2$. These results will then apply to simply connected representation-finite biserial algebras, as well as to the universal covers of arbitrary representation-finite biserial algebras. Using covering techniques, we then obtain a basis for $\operatorname{Ext}^m_\Lambda(N, S)$, where Λ is representation-finite biserial, S is a simple Λ-module, N is an indecomposable Λ-module, and $m > 0$.

We will use concepts of quivers and relations [14], almost-split sequences [6], Auslander–Reiten quivers [14], and their universal covers [11, 15]. We refer to [19] for the description of $\operatorname{Ext}^m_\Lambda(N, M)$ as congruence classes of m-fold extensions $0 \to M \to Y_{m-1} \to \cdots \to Y_0 \to N \to 0$.

1. EXTENSIONS OVER LOCALLY REPRESENTATION-FINITE BISERIAL ALGEBRAS

In this section, Λ will be a locally representation-finite, biserial k-algebra. Let $\operatorname{mod} \Lambda$ be the category of finitely generated Λ-modules, and let $\operatorname{ind} \Lambda$ be the full subcategory of $\operatorname{mod} \Lambda$ whose objects form a complete collection of non-isomorphic indecomposable Λ-modules. Recall that a k-algebra Λ is locally representation-finite if $\sum_{y \in \operatorname{ind} \Lambda} \dim_k \operatorname{Hom}_\Lambda(x, y)$ and $\sum_{y \in \operatorname{ind} \Lambda} \dim_k \operatorname{Hom}_\Lambda(y, x)$ are finite for each indecomposable Λ-module x (this is equivalent to saying that $\operatorname{ind} \Lambda$ is *locally bounded*). The class of algebras considered in this section includes simply connected, representation-finite biserial algebras, as well as universal covers of representation-finite biserial algebras. We will construct basis elements for $\operatorname{Ext}^m_\Lambda(x, y)$ for each $m > 0$, where x, y are indecomposable Λ-modules. To construct short exact sequences, the following easily verified lemma will be needed (see also [5, Lemma 2.1]).

LEMMA 1.1. *Let R be a ring.*

(a) *If $0 \to A \overset{f_1 \amalg f_2}{\to} B_1 \amalg B_2 \overset{(g_1, g_2)}{\to} C \to 0$ and $0 \to X \overset{h_1 \amalg h_2}{\to} Y_1 \amalg Y_2 \overset{(k_1, k_2)}{\to} B_2 \to 0$ are exact sequences in* mod R *such that there is a morphism $A \overset{f}{\to} Y_1$ with $f_2 = k_1 f$. Then the sequence*

$$0 \to A \amalg X \xrightarrow{\begin{bmatrix} f_1 & 0 \\ f & h_1 \\ 0 & h_2 \end{bmatrix}} B_1 \amalg Y_1 \amalg Y_2 \xrightarrow{(g_1, g_2 k_1, g_2 k_2)} C \to 0$$

is exact.

(b) *If $0 \to A_1 \xrightarrow{f_1 \amalg g_1} B_1 \amalg A_2 \xrightarrow{(h_1, f_2)} B_2 \to 0$ and $0 \to A_2 \xrightarrow{f_2 \amalg g_2} B_2 \amalg A_3 \xrightarrow{(h_2, f_3)} B_3 \to 0$ are exact in* mod R. *Then the sequence $0 \to A_1 \xrightarrow{f_1 \amalg g_2 g_1} B_1 \amalg A_3 \xrightarrow{(h_2 h_1, -f_3)} B_3 \to 0$ is exact.*

Let (Γ_Λ, τ) be the Auslander–Reiten quiver of Λ. Since Λ is simply connected as an algebra, Γ_Λ is simply connected as a translation quiver. For a pair of vertices x and y in Γ_Λ, we define $\mathscr{R}(x, y)$ to be the full translation subquiver of Γ_Λ whose vertices lie on a path from x to y. A subquiver $\mathscr{R}(x, y)$ of Γ_Λ will be called a *rectangle* if the subquiver $\mathscr{R}(x, \tau y)$

(a) is non-empty,

(b) contains no triangular mesh (i.e., a mesh with indecomposable middle term),

(c) contains no injective vertex, except possibly non-uniserial projective–injective vertices.

Rectangles were used by Assem and Skowroński [5] in the study of subcategories of coil algebras.

EXAMPLE. Let Λ be the k-algebra given by the bound quiver

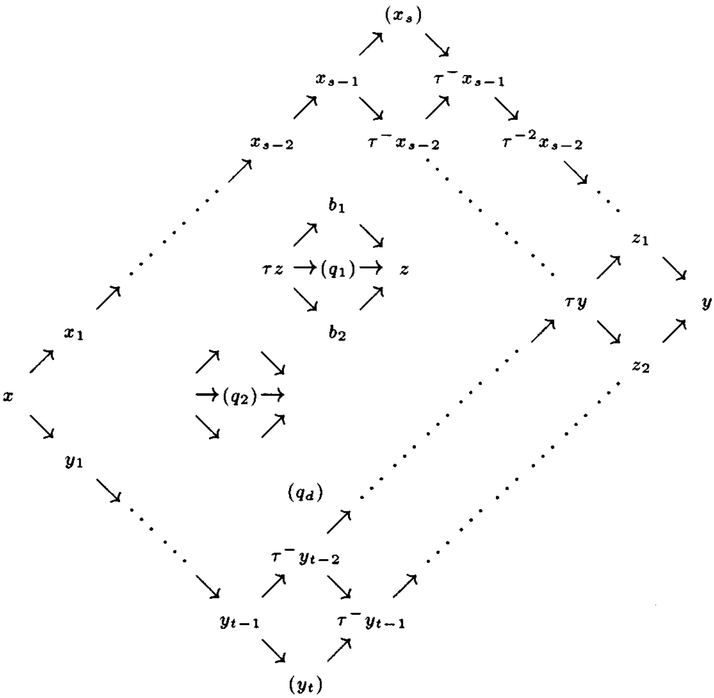
$$\begin{tikzcd}
 & 6 \arrow[dr, "\beta_1"] & & 3 \arrow[dr, "\delta_1"] & \\
7 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & 4 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_2"'] & & 1 \\
 & 5 \arrow[ur, "\beta_2"'] & & 2 \arrow[ur, "\delta_2"'] &
\end{tikzcd}$$

with commutativity relations $\beta_1\alpha_1 = \beta_2\alpha_2$, $\delta_1\gamma_1 = \delta_2\gamma_2$, and zero relations $\gamma_1\beta_1 = \gamma_2\beta_2 = 0$. Then Λ is clearly representation-finite biserial, and simply connected. The Auslander–Reiten quiver Γ_Λ of Λ is

$$\begin{tikzcd}[column sep=small]
 & & & & & & P_6 \arrow[dr] & & & & & & \\
 & P_2 \arrow[dr] & & S_3 \arrow[dr] & & \begin{smallmatrix}4\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}6\\4\end{smallmatrix} \arrow[dr] & & S_5 \arrow[dr] & & I_6 \arrow[dr] & \\
P_1 \arrow[ur] \arrow[dr] & & \underline{\mathbf{r}}P_4 \arrow[ur] \arrow[r] \arrow[dr] & P_4 \arrow[r] & \begin{smallmatrix}4\\2\,3\end{smallmatrix} \arrow[ur] \arrow[dr] & & S_4 \arrow[ur] \arrow[dr] & & \underline{\mathbf{r}}P_7 \arrow[ur] \arrow[r] \arrow[dr] & P_7 \arrow[r] & \begin{smallmatrix}7\\6\,5\end{smallmatrix} \arrow[ur] \arrow[dr] & & I_7 \\
 & P_3 \arrow[ur] & & S_2 \arrow[ur] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[ur] & & S_6 \arrow[ur] & & I_5 \arrow[ur] & \\
 & & & & & & P_5 \arrow[ur] & & & & & &
\end{tikzcd}$$

The subquiver $\mathscr{R}(\begin{smallmatrix}4\\2\,3\end{smallmatrix}, \underline{\mathbf{r}}P_7)$ is a rectangle, and in fact resembles a geometric rectangle. The subquiver $\mathscr{R}(\underline{\mathbf{r}}P_7, I_7)$ is also a rectangle since the only injective vertex in $\mathscr{R}(\underline{\mathbf{r}}P_7, \tau I_7)$ is the non-uniserial projective–injective vertex $P_7 = I_4$. The subquiver $\mathscr{R}(S_2, S_5)$ is not a rectangle since $\mathscr{R}(S_2, \tau S_5)$ contains the triangular mesh $\mathscr{R}(S_2, \begin{smallmatrix}4\\3\end{smallmatrix})$.

In general, a rectangle $\mathcal{R}(x, y)$ is of the form



By definition, a rectangle $\mathcal{R}(x, y)$ contains no triangular mesh, except possibly meshes starting with either x_{s-1} or y_{t-1}. The vertices labeled x_s and y_t are in parentheses to indicate this possibility. We will refer to these vertices as *corners* of $\mathcal{R}(x, y)$. The vertices $q_1, \ldots, q_d$ represent non-uniserial projective–injective vertices. The interior of $\mathcal{R}(x, y)$ contains no other projective or injective vertices. In the following result [5], Corollary 2.2], we obtain a short exact sequence associated with a rectangle.

PROPOSITION 1.2. *Let $\mathcal{R}(x, y)$ be a rectangle. Then there is a short exact sequence* $0 \to x \to y_0 \to y \to 0$ *in* $\operatorname{mod} \Lambda$, *where the summands of* y_0 *are the corners of* $\mathcal{R}(x, y)$, *and the non-uniserial projective–injective vertices in* $\mathcal{R}(x, y)$.

Proof. Apply Lemma 1.1(b) to the short exact sequences represented by the meshes in $\mathcal{R}(x, y)$. ∎

By the Auslander–Reiten formula (see [6]), $\dim_k \operatorname{Ext}^1_\Lambda(y, x) = \dim_k \overline{\operatorname{Hom}}_\Lambda(x, \tau y)$, so we compute the dimension of $\overline{\operatorname{Hom}}_\Lambda(x, y)$ for all indecomposable Λ-modules x, y.

LEMMA 1.3. *Let* $x = x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_w} x_w = y$, *and* $x = y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_w} y_w = y$ *be two chains of irreducible morphisms in* $\operatorname{mod}\Lambda$ *containing indecomposable modules which are not non-uniserial projective–injective. Then there exists some* $\alpha \in k$ *such that* $\overline{f_w \cdots f_0} = \alpha \overline{g_w \cdots g_0}$ *in* $\overline{\operatorname{Hom}}_\Lambda(x, y)$.

Proof. Since Λ is locally representation-finite, any morphism $x \to y$ is the sum of compositions of irreducible morphisms (see [8]). Since Λ is standard, it suffices to consider only the irreducible maps corresponding to arrows in Γ_Λ, where the maps are chosen such that their compositions satisfy the mesh relations in Γ_Λ. Since Λ is simply connected, the paths $f_w \cdots f_1$ and $g_w \cdots g_1$ are homotopic. Thus, there is a sequence of paths $\rho_0, \rho_1, \cdots, \rho_h$ such that $\rho_0 = f_w \cdots f_1$, $\rho_h = g_w \cdots g_1$, and for each i, ρ_{i+1} is obtained from ρ_i by replacing a subpath $\tau z \xrightarrow{k_1} w_1 \xrightarrow{h_1} z$ in ρ_1 by a path $\tau z \xrightarrow{k_2} w_2 \xrightarrow{h_2} z$, where we may assume that w_2 is not a non-uniserial projective–injective. Since we have the mesh relation $h_1k_1 + h_2k_2 + h_3k_3 = 0$, where h_3k_3 either is zero or factors through an injective, we have that $\overline{h_1k_1} = -\overline{h_2k_2}$. Therefore, for each i, $\overline{\rho}_{i+1} = -\overline{\rho}_i$, so the result follows. ∎

Now we may describe the dimensions of $\overline{\operatorname{Hom}}_\Lambda(x, y)$ and $\operatorname{Ext}^1_\Lambda(y, x)$ for all x, y in $\operatorname{ind}\Lambda$.

PROPOSITION 1.4. *Let* x, y *be in* $\operatorname{ind}\Lambda$.

(a) $\dim_k \overline{\operatorname{Hom}}_\Lambda(x, y) \leq 1$, *and* $\dim_k \operatorname{Ext}^1_\Lambda(y, x) \leq 1$.

(b) $\dim_k \overline{\operatorname{Hom}}_\Lambda(x, y) = 1$ *if and only if* y *is not injective and* $\mathscr{R}(x, \tau^- y)$ *is a rectangle*.

(c) $\dim_k \operatorname{Ext}^1_\Lambda(y, x) = 1$ *if and only if* $\mathscr{R}(x, y)$ *is a rectangle*.

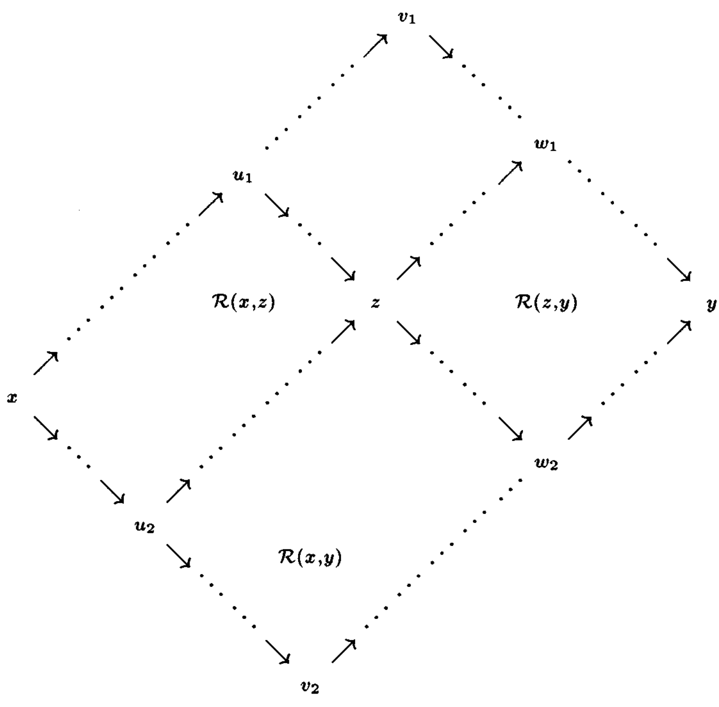
Proof. Part (a) follows from Lemma 1.3 and the Auslander–Reiten formula. If $\mathscr{R}(x, \tau^- y)$ is a rectangle, then $\operatorname{Ext}^1_\Lambda(\tau^- y, x) \neq 0$, so $\dim_k \overline{\operatorname{Hom}}_\Lambda(x, y) = 1$ by (a) and the Auslander–Reiten formula. If $\overline{\operatorname{Hom}}_\Lambda(x, y)$ is non-zero, then by Lemma 1.3, $\mathscr{R}(x, y)$ can contain no injective vertices, except possibly non-uniserial projective–injective vertices, and can contain no triangular mesh. It follows from the definition that $\mathscr{R}(x, \tau^- y)$ is a rectangle, so (b) holds, and (c) follows from (b). ∎

If $\mathscr{R}(x, y)$ is a rectangle, $\eta(x, y)$ will denote the associated non-split short exact sequence given in Proposition 1.2. Otherwise, $\eta(x, y)$ will denote the split short exact sequence $0 \to x \to x \amalg y \to y \to 0$. In the following statement, we describe pushouts and pullbacks involving short exact sequences given by rectangles.

PROPOSITION 1.5. *Suppose $\mathcal{R}(x, y)$ is a rectangle, and z is a vertex in $\mathcal{R}(x, y)$ which is not projective–injective. Then we have a commutative diagram with exact rows*

$$\begin{array}{lccccccccc}
\eta(x,z): & 0 & \longrightarrow & x & \longrightarrow & u & \longrightarrow & z & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
\eta(x,y): & 0 & \longrightarrow & x & \longrightarrow & v & \longrightarrow & y & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \| & & \\
\eta(z,y): & 0 & \longrightarrow & z & \longrightarrow & w & \longrightarrow & y & \longrightarrow & 0
\end{array}$$

Proof. The embedding of $\mathcal{R}(x, z)$ and $\mathcal{R}(z, y)$ in the rectangle $\mathcal{R}(x, y)$ is shown in the diagram



By Lemma 1.1(a), we have a short exact sequence $0 \to u_1 \amalg u_2 \to z \amalg v_1 \amalg v_2 \amalg q \to y \to 0$, where q is the direct sum of the non-uniserial projective–injective vertices in the image of $\mathcal{R}(x, y) \backslash \mathcal{R}(x, z)$. By viewing this exact sequence as a pullback diagram, we obtain the given pullback. The pushout diagram is obtained similarly. ∎

We now turn our attention to determining a basis for $\operatorname{Ext}^m_\Lambda(y, x)$ for $m > 0$. For this we define $\mathcal{R}(x_m, \ldots, x_0)$ to be the full subquiver of

Γ_Λ whose vertices lie on a path from x_m to x_0 passing consecutively through the vertices $x_m, x_{m-1}, \dots, x_1, x_0$. If $\mathcal{R}(x_i, x_{i-1})$ is a rectangle for each $i = 1, \dots, m$, $\mathcal{R}(x_m, \dots, x_0)$ will be called a *rectangular m-chain*. A rectangular m-chain $\mathcal{R}(x_m, \dots, x_0)$ with the property that no rectangle $\mathcal{R}(x_i, x_{i-1})$ is a proper subquiver of a rectangle $\mathcal{R}(x', x_{i-1})$ will be called a *left maximal* rectangular m-chain. To each rectangular m-chain $\mathcal{R}(x_m, \dots, x_0)$, there is an associated exact sequence $\eta(x_m, \dots, x_0)$ which is defined to be the Yoneda product $\eta(x_m, x_{m-1}) \cdots \eta(x_2, x_1)\eta(x_1, x_0)$. Thus $\eta(x_m, \dots, x_0)$ represents an element of $\operatorname{Ext}^m_\Lambda(x_0, x_m)$. We have the following connection between left maximal rectangular m-chains and projective resolutions.

PROPOSITION 1.6. *Let y be in* ind Λ. *An indecomposable Λ module x is a summand of $\Omega^m y$ if and only if there exists a left maximal rectangular chain $\mathcal{R}(x_m, \dots, x_0)$ with $x_m = x$ and $x_0 = y$.*

Proof. Let x be an indecomposable summand of $\Omega^m y$. We proceed by induction on m. If $m = 1$, there exists a pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 y \arrow[r] \arrow[d, "\pi_x"] & \mathcal{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & x \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is non-split. By Proposition 1.4(c), $\mathcal{R}(x, y)$ is a rectangle. Suppose $\mathcal{R}(x', y)$ is a rectangle containing $\mathcal{R}(x, y)$. We would then have the commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & \Omega^1 y \arrow[r] \arrow[d, "\alpha"] & \mathcal{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x', y): & 0 \arrow[r] & x' \arrow[r] \arrow[d, "\beta"] & y_0' \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x, y): & 0 \arrow[r] & x \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

where the lower two rows are given by Proposition 1.5. By composing the vertical maps, we obtain a pushout along the map $\Omega^1 y \xrightarrow{\beta\alpha} x$, so by uniqueness of pushouts, $\beta\alpha$ is a splittable epimorphism. Since x and x' are indecomposable, it follows that β is an isomorphism, therefore $\mathcal{R}(x, y)$ is left maximal.

Now assume that x is an indecomposable summand of $\Omega^m y$. There exists an indecomposable summand x_{m-1} of $\Omega^{m-1} y$ such that $\operatorname{Ext}^1_\Lambda(x_{m-1}, x) \neq 0$, so $\mathcal{R}(x, x_{m-1})$ is a rectangle by Proposition 1.4(c). By the induction hypothesis, there exists a left maximal rectangular $(m-1)$-chain

$\mathscr{R}(x_{m-1}, \ldots, x_1, y)$. Thus, we have the commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \Omega^{m-1}y \arrow[r] \arrow[d, "\pi_{x_{m-1}}"] & \mathscr{P}_{m-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x_{m-1}, \ldots, y)\colon & 0 \arrow[r] & x_{m-1} \arrow[r] & y_{m-2} \arrow[r] & \cdots \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd} \tag{$*$}$$

Consider the commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \Omega^{m}y \arrow[r] \arrow[d, "\alpha"] & \mathscr{P}_{m-1} \arrow[r] \arrow[d] & \Omega^{m-1}y \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & x \arrow[r] \arrow[d, equal] & y'_{m-1} \arrow[r] \arrow[d] & \Omega^{m-1}y \arrow[r] \arrow[d, "\pi_{x_{m-1}}"] & 0 \\
\eta(x, x_{m-1})\colon & 0 \arrow[r] & x \arrow[r] & y_{m-1} \arrow[r] & x_{m-1} \arrow[r] & 0
\end{tikzcd} \tag{$**$}$$

By uniqueness of pushouts, α is a splittable epimorphism. By composing the vertical maps in $(**)$ and splicing with $(*)$ we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \Omega^{m}y \arrow[r] \arrow[d, "\alpha"] & \mathscr{P}_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x, x_{m-1}, \ldots, y)\colon & 0 \arrow[r] & x \arrow[r] & y_{m-1} \arrow[r] & \cdots \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

To show that $\mathscr{R}(x, x_{m-1}, \ldots, y)$ is a left maximal rectangular m-chain, it suffices to show that $\mathscr{R}(x, x_{m-1})$ is left maximal. If we had a rectangle $\mathscr{R}(x', x_{m-1})$ containing $\mathscr{R}(x, x_{m-1})$, we would have the commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \Omega^{m}y \arrow[r] \arrow[d, "\sigma"] & \mathscr{P}_{m-1} \arrow[r] \arrow[d] & \Omega^{m-1}y \arrow[r] \arrow[d] & 0 \\
\eta(x', x_{m-1})\colon & 0 \arrow[r] & x' \arrow[r] \arrow[d, "\rho"] & y_{m-1} \arrow[r] \arrow[d] & x_{m-1} \arrow[r] \arrow[d, equal] & 0 \\
\eta(x, x_{m-1})\colon & 0 \arrow[r] & x \arrow[r] & y'_{m-1} \arrow[r] & x_{m-1} \arrow[r] & 0
\end{tikzcd}$$

By composing the vertical arrows and splicing the resulting diagram with $(*)$, we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[r] & \Omega^{m}y \arrow[r] \arrow[d, "\rho\sigma"] & \mathscr{P}_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x, x_{m-1}, \ldots, y)\colon & 0 \arrow[r] & x \arrow[r] & y_{m-1} \arrow[r] & \cdots \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

By uniqueness, $\rho\sigma$ is a splittable epimorphism, so again x is isomorphic to x' and $\mathscr{R}(x, x_{m-1})$ is left maximal.

Now suppose $\mathscr{R}(x, y)$ is a left maximal rectangle. We have the commutative diagram with exact rows

$$\begin{tikzcd} & 0 \arrow[r] & \Omega^1 y \arrow[r] \arrow[d, "\alpha"] & \mathscr{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\ \eta(x, y)\colon & 0 \arrow[r] & x \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0 \end{tikzcd}$$

Thus, there exists an indecomposable summand x' of $\Omega^1 y$ such that $\operatorname{Hom}_\Lambda(x', x)$ is non-zero. Furthermore, $\operatorname{Ext}^1_\Lambda(y, x')$ is also non-zero. Thus, $\mathscr{R}(x', y)$ is a rectangle containing x, so by maximality, $x = x'$, and x is therefore a summand of $\Omega^1 y$.

Now suppose $\mathscr{R}(x_m, \ldots, y)$ is a left maximal rectangular m-chain. We have the commutative diagram with exact rows

$$\begin{tikzcd} & 0 \arrow[r] & \Omega^m y \arrow[r] \arrow[d, "\alpha"] & \mathscr{P}_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{P}_0 \arrow[r] \arrow[d] & y \arrow[r] \arrow[d, equal] & 0 \\ \eta(x_m, \ldots, y)\colon & 0 \arrow[r] & x_m \arrow[r] & y_{m-1} \arrow[r] & \cdots \arrow[r] & y_0 \arrow[r] & y \arrow[r] & 0 \end{tikzcd}$$

Thus there exists some indecomposable summand x' of $\Omega^m y$ such that $\operatorname{Hom}_\Lambda(x', x_m)$ is non-zero. By the induction hypothesis, x_{m-1} is a summand of $\Omega^{m-1} y$. Thus, $\operatorname{Ext}^1_\Lambda(x_{m-1}, x')$ is non-zero, and $\mathscr{R}(x', x_{m-1})$ is a rectangle containing x_m. By maximality, $x_m = x'$; hence x_m is a summand of $\Omega^m y$. ∎

We have the following corollary.

COROLLARY 1.7. $\Omega^m y = \coprod_{\mathscr{R}(x_m, \ldots, y)} x_m$, *where the sum runs over a complete collection of left maximal rectangular m-chains.*

Proof. For $m = 1$, the statement follows from Proposition 1.6, since for each summand x of $\Omega^1 y$, there is a unique maximal rectangle $\mathscr{R}(x, y)$. For $m > 1$, $\Omega^m y = \Omega^1(\Omega^{m-1} y) = \Omega^1(\coprod_{\mathscr{R}(x_{m-1}, \ldots, y)} x_{m-1})$, where the sum is taken over a complete collection of left maximal rectangular $(m-1)$-chains. ∎

Now we may describe a basis for $\operatorname{Ext}^m_\Lambda(y, x)$, where x and y are in $\operatorname{ind} \Lambda$.

THEOREM 1.8. *Let* $\mathscr{R}^m_{x,y}$ *be the collection of all exact sequences* $\eta(x_m, \ldots, x_0)$ *such that* $\mathscr{R}(x_m, \ldots, x_0)$ *is a rectangular m-chain with the properties that* $x_m = x$, $x_0 = y$, *and* $\mathscr{R}(x_{m-1}, \ldots, x_0)$ *is left maximal. Then* $\mathscr{R}^m_{x,y}$ *is a k-basis for* $\operatorname{Ext}^m_\Lambda(y, x)$.

Proof. By dimension shifting and the fact that Ext is additive in all dimensions we have

$$\operatorname{Ext}^m_\Lambda(y, x) \cong \operatorname{Ext}^1_\Lambda(\Omega^{m-1} y, x) \cong \coprod_{\mathscr{R}(x_{m-1}, \ldots, x_0)} \operatorname{Ext}^1_\Lambda(x_{m-1}, x),$$

where the sum is taken over a complete collection of left maximal rectangular $(m-1)$-chains. It follows that $\{\eta(x_m, x_{m-1}) \mid \eta(x_m, x_{m-1}, \ldots, x_0) \in \mathcal{R}^m_{x,y}\}$ forms a basis for the left hand side. We show that the image of this basis in $\operatorname{Ext}^m_\Lambda(y, x)$ under the above isomorphism is $\mathcal{R}^m_{x,y}$. If $\eta(x_m, \ldots, x_0)$ is in $\mathcal{R}^m_{x,y}$, the inclusion of the exact sequence $\eta(x_m, x_{m-1})$ in $\operatorname{Ext}^1_\Lambda(\Omega^{m-1}y, x)$ is given by the pullback diagram

$$\begin{tikzcd}
& 0 \arrow[r] & x \arrow[r] \arrow[d, equal] & u \arrow[r] \arrow[d] & \Omega^{m-1}y \arrow[r] \arrow[d, "p_{x_{m-1}}"] & 0 \\
\eta(x_m, x_{m-1}): & 0 \arrow[r] & x \arrow[r] & v \arrow[r] & x_{m-1} \arrow[r] & 0
\end{tikzcd}$$

By composing this diagram with the commutative diagram for $\eta(x_{m-1}, \ldots, x_0)$ given in Proposition 1.6(b), we obtain the commutative diagram with exact rows

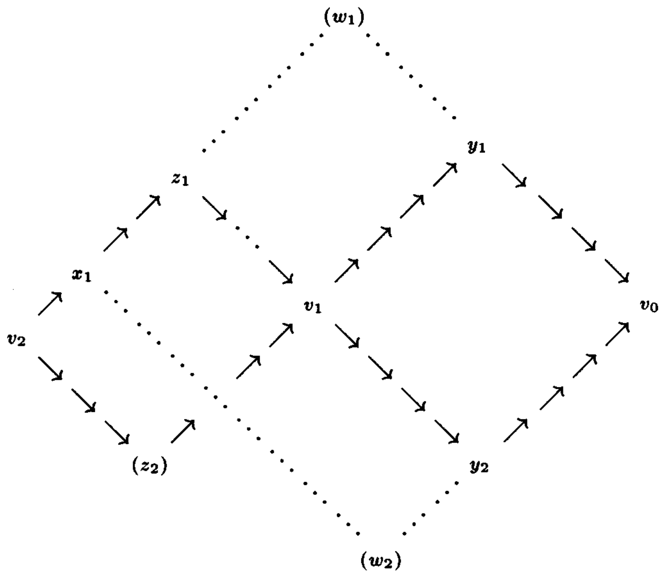
$$\begin{tikzcd}
& 0 \arrow[r] & x \arrow[r] \arrow[d, equal] & u \arrow[r] \arrow[d, "f_{m-1}"] & \mathcal{P}_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots\ \mathcal{P}_0 \arrow[r] \arrow[d, "f_0"] & y \arrow[r] \arrow[d, equal] & 0 \\
\eta(x_m, \ldots, x_0): & 0 \arrow[r] & x \arrow[r] & v \arrow[r] & y_{m-2} \arrow[r] & \cdots\ y_0 \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

The image of $\eta(x_m, x_{m-1})$ in $\operatorname{Ext}^m_\Lambda(y, x)$ is the top sequence, which is then congruent to $\eta(x_m, \ldots, x_0)$. ∎

We conclude the section by showing that any long exact sequence given by an arbitrary rectangular m-chain is either zero or congruent to a sequence in some $\mathcal{R}^m_{x,y}$.

LEMMA 1.9. *Let $\mathcal{R}(v_m, \ldots, v_0)$ be a rectangular m-chain. If there is an exact sequence $\eta(x_m, \ldots, x_0)$ in $\mathcal{R}^m_{x,y}$ such that $x = x_m = v_m$, $y = x_0 = v_0$, and v_i is in $\mathcal{R}(x_i, \tau x_{i-1})$ for all $i = 1, \ldots, m-1$, then $\eta(v_m, \ldots, v_0)$ is congruent to $\eta(x_m, \ldots, x_0)$. Otherwise, $\eta(v_m, \ldots, v_0)$ is zero.*

Proof. If v_i is in $\mathcal{R}(x_i, \tau x_{i-1})$ for every i, then we have by Proposition 1.5 that $\eta(v_{i+1}, v_i) f_i = f_{i+1} \eta(x_{i+1}, x_i)$, for each i, where right (respectively, left) multiplication of an exact sequence by a morphism indicates its pullback (respectively, pushout) along that morphism. By the definition of the congruence relation for exact sequences in [19] (which will be denoted below by $\equiv$), and by induction, the sequences are congruent. Now assume that $\eta(v_m, \ldots, v_0)$ is non-zero. We construct an element of $\mathcal{R}^m_{x,y}$ which is congruent to $\eta(v_m, \ldots, v_0)$. If $m > 0$ and $\mathcal{R}(v_1, v_0)$ is not maximal, we would like to extend it to a left maximal rectangle $\mathcal{R}(x_1, x_0)$ where x_1 is in $\mathcal{R}(v_2, v_1)$ and $x_0 = v_0$. We observe that such an x_1 cannot lie on a sectional path starting at v_2, for, if that were the case, we would have the following rectangles:



By Proposition 1.5, there is a morphism $v_1 \xrightarrow{h} y_1$ such that $\eta(v_2, v_1) = \eta(v_2, y_1)h$, and $\eta(y_1, v_0) = h\eta(v_1, v_0)$. Therefore,

$$\begin{aligned}\eta(v_2, v_1)\eta(v_1, v_0) &\equiv [\eta(v_2, y_1)h]\eta(v_1, v_0)\\ &\equiv \eta(v_2, y_1)[h\eta(v_1, v_0)]\\ &\equiv \eta(v_2, y_1)\eta(y_1, v_0)\\ &\equiv 0\end{aligned}$$

since $\mathscr{R}(y_1, v_0)$ is a sectional path, so this contradicts the assumption that $\eta(v_m, \dots, v_0) \not\equiv 0$.

Any two rectangles whose left endpoints are in $\mathscr{R}(v_2, v_1)$ are contained in another rectangle with left endpoint in $\mathscr{R}(v_2, v_1)$. Therefore, the union of all rectangles of the form $\mathscr{R}(x, x_0)$ with x in $\mathscr{R}(v_2, v_1)$ is a rectangle $\mathscr{R}(x_1, x_0)$ where x_1 is in $\mathscr{R}(\tau^- v_2, v_1)$, and since $\mathscr{R}(v_1, v_0)$ is a rectangle, v_1 is contained in $\mathscr{R}(x_1, x_0)$.

Now assume that we have a left maximal rectangular chain $\mathscr{R}(x_{j-1}, \dots, x_1, x_0)$ such that v_i is contained in $\mathscr{R}(x_i, \tau x_{i-1})$ for $i = 1, \dots, j-1$. Then x_{j-1} is contained in $\mathscr{R}(v_j, v_{j-1})$, and $\mathscr{R}(v_j, x_{j-1})$ is a rectangle. If $j = m$, we are done. Otherwise, as above, there is x_j in $\mathscr{R}(\tau^- v_{j+1}, v_j)$ such that $\mathscr{R}(x_j, x_{j-1})$ is maximal. By construction, $\mathscr{R}(x_{m-1}, \dots, x_0)$ is left maximal, and v_i is in $\mathscr{R}(x_i, \tau x_{i-1})$ for all i. By the above, $\eta(v_m, \dots, v_0)$ is congruent to $\eta(x_m, \dots, x_0)$. ∎

2. THE EXT-ALGEBRA

In this section, Λ will be a basic, connected, representation-finite, biserial finite-dimensional algebra over an algebraically closed field k. We use the results of Section 1, along with covering techniques to obtain a basis for $\mathrm{Ext}^m_\Lambda(N, S)$, where S and N are indecomposable Λ-modules, S is simple, and $m > 0$. A multiplicative basis for the Ext-algebra $E(\Lambda)$ is then described, along with the bound quiver of $E(\Lambda)$.

Let Γ_Λ be the Auslander–Reiten quiver of Λ, and let $\tilde{\Gamma}_\Lambda \xrightarrow{\pi} \Gamma_\Lambda$ be the universal cover. By Theorem 2.9 in [11], $\tilde{\Gamma}_\Lambda \cong \Gamma_{\tilde{\Lambda}}$, where $\tilde{\Lambda}$, the universal cover of Λ, is locally representation-finite, locally finite dimensional, and simply connected and satisfies $\beta(\tilde{\Lambda}) \leq 2$. Since $\tilde{\Lambda}$ is simply connected, it is standard. By [21, Theorem 2], Λ is also standard. Also by [11, Sect. 3.1], the covering map π induces a functor F on mesh categories, so we have the composite

$$\mathrm{ind}\ \tilde{\Lambda} \xrightarrow{\sim} k\tilde{\Gamma}_\Lambda/\tilde{m}_\Lambda \xrightarrow{F} k\Gamma_\Lambda/m_\Lambda \xrightarrow{\sim} \mathrm{ind}\ \Lambda.$$

The image of of an indecomposable $\tilde{\Lambda}$ module x under the above composition is πx, and the image of an irreducible morphism associated with an arrow $x \to y$ in $\tilde{\Gamma}_\Lambda$ is an irreducible morphism $\pi x \to \pi y$. Since $\tilde{\Lambda}$ and Λ are standard, we will assume that there are irreducible morphisms associated with the arrows in their Auslander–Reiten quivers such that compositions of these irreducible morphisms satisfy the mesh relations. It then follows [11, Sect. 3.1], that F is a *covering functor*, that is, for each fixed x in $\mathrm{ind}\ \tilde{\Lambda}$, where $\pi x = M$, there is an isomorphism

$$\coprod_{y \in \pi^{-1} N} \mathrm{Hom}_{\tilde{\Lambda}}(y, x) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(N, M).$$

Using this isomorphism, we obtain the following isomorphism of Ext spaces.

LEMMA 2.1. *Let S be a simple Λ-module, and let x be in* $\mathrm{ind}\ \tilde{\Lambda}$ *such that $S = \pi x$. Then for every N in* $\mathrm{ind}\ \Lambda$ *and every $m > 0$, F induces an isomorphism*

$$\coprod_{y \in \pi^{-1} N} \mathrm{Ext}^m_{\tilde{\Lambda}}(y, x) \xrightarrow{\sim} \mathrm{Ext}^m_\Lambda(N, S).$$

Proof. Since S is a simple Λ-module, it follows that $\mathrm{Ext}^m_\Lambda(N, S) \cong \mathrm{Hom}_\Lambda(\Omega^m N, S)$. A similar isomorphism holds for x since it is a simple

$\tilde{\Lambda}$-module, thus

$$\begin{aligned}\mathrm{Ext}^m_\Lambda(N,S) &\cong \mathrm{Hom}_\Lambda(\Omega^m N,S)\\ &\cong \coprod_{y\in\pi^{-1}N} \mathrm{Hom}_{\tilde{\Lambda}}(\Omega^m y,x)\\ &\cong \coprod_{y\in\pi^{-1}N} \mathrm{Ext}^m_{\tilde{\Lambda}}(y,x).\end{aligned}$$

$\blacksquare$

If $\eta(x_m,\ldots,x_0)$: $0\to x\to y_{m-1}\to\cdots\to y_1\to y_0\to y\to 0$ is an exact sequence in $\mathrm{Ext}^m_{\tilde{\Lambda}}(y,x)$ associated with a rectangular m-chain $\mathcal{R}(x_m,\ldots,x_0)$, then its image under the above isomorphism is an exact sequence in $\mathrm{Ext}^m_\Lambda(N,M)$ of the form $0\to S\to Y_{m-1}\to\cdots\to Y_1\to Y_0\to N\to 0$, where $\pi x=S$, $\pi y=N$, and $\pi y_i=Y_i$. We will denote this sequence also by $\eta(x_m,\ldots,x_0)$, which is consistent with our previous notation since in the simply connected case, π is the identity. By the isomorphism in Lemma 2.1, a basis for $\mathrm{Ext}^m_\Lambda(N,S)$ is given by the image of the union of $\mathcal{R}^m_{x,y}$, where x is a fixed indecomposable $\tilde{\Lambda}$-module such that $\pi x=S$ and y is in $\pi^{-1}N$. We now describe this basis.

PROPOSITION 2.2. *Let S and N be indecomposable Λ-modules, where S is simple, and let x be in* ind $\tilde{\Lambda}$ *such that $\pi x=S$. Let $\mathcal{C}^m_{S,N}$ be the collection of all exact sequences $\eta(x_m,\ldots,x_0)$ such that $\mathcal{R}(x_m,\ldots,x_0)$ is a rectangular m-chain in $\tilde{\Gamma}_\Lambda$ with $x_m=x$, $\pi x_0=N$, and such that $\mathcal{R}(x_{m-1},\ldots,x_0)$ is a left maximal rectangular chain. Then $\mathcal{C}^m_{S,N}$ is a k-basis for* $\mathrm{Ext}^m_\Lambda(N,S)$.

Remark. If Λ is self-injective, then $\mathrm{Ext}^m_\Lambda(N,M)\cong \underline{\mathrm{Hom}}_\Lambda(\Omega^m N,M)$, for all pairs of indecomposable Λ-modules M and N (without the assumption that one of the modules is simple). Using this isomorphism, we can show that $\mathcal{C}^m_{M,N}$ is a k-basis for $\mathrm{Ext}^m_\Lambda(N,M)$ for all pairs of indecomposable Λ-modules M,N. For this, we need to observe that the isomorphism for Hom induced by the covering functor in turn induces an isomorphism for $\underline{\mathrm{Hom}}$. This is clear, since the covering functor maps projectives to projectives. Now using the fact that the universal cover $\tilde{\Lambda}$ is also self-injective, we get

$$\begin{aligned}\mathrm{Ext}^m_\Lambda(N,M) &\cong \underline{\mathrm{Hom}}_\Lambda(\Omega^m N,M)\\ &\cong \coprod_{y\in\pi^{-1}N} \underline{\mathrm{Hom}}_{\tilde{\Lambda}}(\Omega^m y,x)\\ &\cong \coprod_{y\in\pi^{-1}N} \mathrm{Ext}^m_{\tilde{\Lambda}}(y,x)\end{aligned}$$

where $\pi x=M$, so the isomorphism in Lemma 2.1 holds for all pairs of indecomposable Λ-modules. It follows that $\mathcal{C}^m_{M,N}$ is a k-basis for $\mathrm{Ext}^m_\Lambda(N,M)$

when Λ is self-injective (in [10], Bleher and Chinburg give a method for computing Ext groups of Brauer tree algebras (which are self-injective) using Auslander–Reiten quivers).

Let $S_1, \ldots, S_n$ be a complete collection of non-isomorphic simple Λ-modules. Let $\mathscr{C}^0_{S_i,S_j}$ be a basis for $\mathrm{Hom}_\Lambda(S_j, S_i)$. A basis for the Ext-algebra $E(\Lambda) = \coprod_{m\geq 0} \mathrm{Ext}^m_\Lambda(\Lambda/\underline{\mathbf{r}}, \Lambda/\underline{\mathbf{r}}) \cong \coprod_{m\geq 0, 1\leq i,j\leq n} \mathrm{Ext}^m_\Lambda(S_i, S_j)$ can now be described.

THEOREM 2.3. *$\mathscr{E} = \bigcup_{m\geq 0, 1\leq i,j\leq n} \mathscr{C}^m_{S_i,S_j}$ is a multiplicative basis for $E(\Lambda)$.*

Proof. $\mathscr{E}$ is a basis by the above decomposition. If $\eta(v_q, \ldots, v_0)$ and $\eta(w_r, \ldots, w_0)$ are in $\mathscr{E}$ and their product is non-zero, then $v_0 = w_r$, and $\eta(v_q, \ldots, v_0)\eta(w_r, \ldots, w_0) = \eta(v_q, \ldots, v_1, w_r, \ldots, w_0)$. By Lemma 1.9, $\eta(v_q, \ldots, v_1, w_r, \ldots, w_0)$ is congruent to an element of $\mathscr{C}^{q+r}_{S,T} \subset \mathscr{E}$, where $S = \pi v_q$ and $T = \pi w_0$. Therefore, $\mathscr{E}$ is a multiplicative basis for $E(\Lambda)$. ∎

We would now like to describe the bound quiver of $E(\Lambda)$. We say that an element η of $\mathscr{E}$ is of *degree m* if it is contained in $\mathscr{C}^m_{S_i,S_j}$ for some i, j. Since $\mathscr{E}$ is a basis, there exists a unique minimal subset of $\mathscr{E}$ which generates $E(\Lambda)$. These minimal generators can be partitioned into the disjoint union $\mathscr{E}_0 \cup \mathscr{E}_1$, where the elements of $\mathscr{E}_0$ have degree 0, and the elements of $\mathscr{E}_1$ have positive degree. The elements of $\mathscr{E}_0$ can be identified with the vertex set Q_0 of the quiver Q of Λ. Now we have the following description of the quiver of $E(\Lambda)$.

PROPOSITION 2.4. *Let Δ be the quiver with vertices $\Delta_0 = Q_0$ and arrows $\Delta_1 = \{x_\eta \mid \eta \in \mathscr{E}_1\}$, such that if η is in $\mathscr{C}^m_{S_i,S_j}$, then x_η is the arrow $j \bullet \xrightarrow{x_\eta} \bullet\, i$. Let $k\Delta$ be the path algebra of Δ, and let $\mathscr{I}$ be the ideal of $k\Delta$ with generators*

(1) $x_{\eta_1} \cdots x_{\eta_p}$, *where* $\eta_p \eta_{p-1} \cdots \eta_1 \equiv 0$,

(2) $x_{\eta_1} \cdots x_{\eta_p} - x_{\zeta_1} \cdots x_{\zeta_p}$, *where* $\eta_p \eta_{p-1} \cdots \eta_1 \equiv \zeta_p \zeta_{p-1} \cdots \zeta_1$.

Then $E(\Lambda) \cong k\Delta/\mathscr{I}$.

Proof. Since the vertices and arrows of Δ correspond to minimal generators of $E(\Lambda)$, we have an epimorphism $k\Delta \xrightarrow{\Phi} E(\Lambda)$, where the image of a path $x_{\eta_1} \cdots x_{\eta_p}$ in Δ under Φ is the congruence class of the exact sequence $\eta_p \eta_{p-1} \cdots \eta_1$. By Lemma 1.9, such a sequence is either zero or congruent to some element of $\mathscr{E}$. The generators of $\mathscr{I}$ are clearly in $\ker \Phi$. If $\alpha_1\rho_1 + \cdots + \alpha_h\rho_h$ is in $\ker \Phi$, where $\alpha_1, \ldots, \alpha_h \in k$ are not all zero and $\rho_1, \ldots, \rho_h$ are paths in Δ, then using the fact that the image of each ρ_i is either zero or an element of the basis $\mathscr{E}$, we conclude that $\alpha_1\rho_1 + \cdots + \alpha_h\rho_h$ is in $\mathscr{I}$, which completes the proof. ∎

Remark. The Ext-algebra of a representation-finite biserial algebra need not be biserial, even if it is representation-finite. For example, let Λ be given by the bound quiver

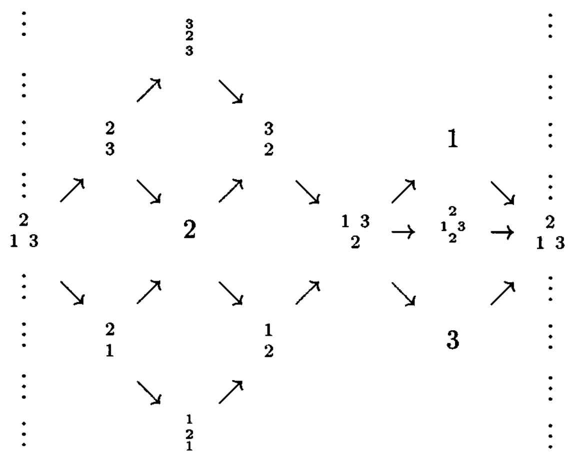
$$\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \begin{matrix} \xrightarrow{\beta} \overset{3}{\bullet} \\ \xrightarrow[\gamma]{} \underset{4}{\bullet} \end{matrix},$$

with relations $\beta\alpha = \gamma\alpha = 0$. By Theorem 2.2 in [17], $E(\Lambda)$ is given by the same quiver, but without the relations, so $E(\Lambda)$ is representation-finite hereditary, but not biserial.

EXAMPLE. Let Q be the quiver

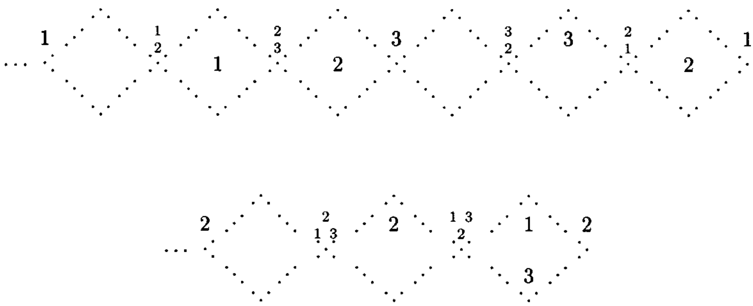
$$\underset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \underset{2}{\bullet} \underset{\delta}{\overset{\gamma}{\rightleftarrows}} \underset{3}{\bullet}.$$

Let I be the ideal of kQ generated by the relations $\alpha\gamma = \delta\beta = 0$, and $\beta\alpha = \gamma\delta$. Then $\Lambda = kQ/I$ is representation-finite special biserial. In fact, Λ is a Brauer quiver algebra. The Auslander–Reiten quiver Γ_Λ is



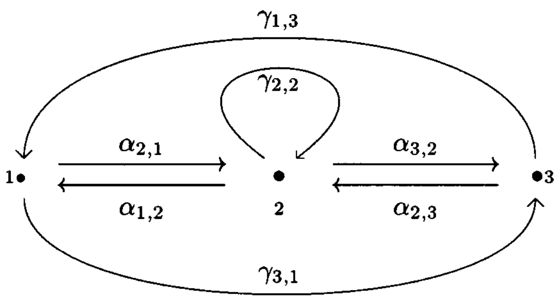
where identification is made along the dotted lines, and indecomposable modules are represented by their Loewy series. We now use the above results to determine a presentation of the Ext-algebra of Λ as a bound quiver algebra (a numerical algorithm for determining the minimal generators of the Ext-algebra of a Brauer quiver has been developed by Chasen [12]).

We have the left maximal rectangular chains



where simple modules contained in the rectangles are shown. Elements of $\mathscr{E}$ can be read from the chains by starting at a simple module inside a rectangle and continuing left to right with modules at the vertices of the rectangles. For example, in the first sequence, we have the sequence $\eta(S_1, \begin{smallmatrix}2\\3\end{smallmatrix}, S_3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, S_1)$, which is an element of $\operatorname{Ext}^5_\Lambda(S_1, S_1)$. Since each simple module is a syzygy of itself, the exact sequences obtained from the above chains, and their products will span $E(\Lambda)$.

To determine minimal generators, we observe first that $\eta(S_3, \begin{smallmatrix}2\\1\end{smallmatrix}, S_1) \equiv \eta(S_3, S_2)\eta(S_2, S_1)$ by Lemma 1.9. Similarly, all other 2-fold extensions are Yoneda products of short exact sequences. However, $\eta(S_3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, S_1)$ cannot be expressed as the product of extensions of smaller degree, so it is a minimal generator. The other minimal generators are $\eta(S_2, \begin{smallmatrix}2\\13\end{smallmatrix}, \begin{smallmatrix}13\\2\end{smallmatrix}, S_2)$, $\eta(S_1, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, S_3)$, and the generators corresponding to vertices and arrows of Q. Thus the quiver of $E(\Lambda)$ is



The arrows $\gamma_{i,j}$ correspond to the minimal generators in $\operatorname{Ext}^3_\Lambda(S_j, S_i)$, and the arrows $\alpha_{i,j}$ correspond to the minimal generators in $\operatorname{Ext}^1_\Lambda(S_j, S_i)$. Since $\operatorname{Ext}^2_\Lambda(S_1, S_1) = \operatorname{Ext}^2_\Lambda(S_3, S_3) = 0$, we get the relations $\alpha_{1,2}\alpha_{2,1} = \alpha_{3,2}\alpha_{2,3} = 0$. But since $\eta(S_2, S_1, S_2) \equiv \eta(S_2, S_3, S_2) \equiv \eta(S_2, \begin{smallmatrix}13\\2\end{smallmatrix}, S_2)$, we have the commutativity relation $\alpha_{2,1}\alpha_{1,2} = \alpha_{2,3}\alpha_{3,2}$. The other relations are

$\gamma_{1,3}\alpha_{3,2} = \alpha_{1,2}\gamma_{2,2}$, $\gamma_{3,1}\alpha_{1,2} = \alpha_{3,2}\gamma_{2,2}$, and $\gamma_{2,2}\alpha_{2,1} = \alpha_{2,3}\gamma_{3,1}$. For example, by Lemma 1.9, $\eta(S_2, S_1)\eta(S_1, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, S_3) \equiv \eta(S_2, \begin{smallmatrix}2\\1\,3\end{smallmatrix}, \begin{smallmatrix}1\,3\\2\end{smallmatrix}, S_2)\eta(S_2, S_3)$, so $\gamma_{1,3}\alpha_{2,1} = \alpha_{2,3}\gamma_{2,2}$.

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REFERENCES

1. I. Agoston, V. Dlab, and E. Lukács, Ext-algebras, *C. R. Math. Rep. Acad. Sci. Canada*, **XVI**, No. 5 1995, 215–220.
2. J. Alperin, "Local Representation Theory," Cambridge Studies in Advanced Mathematics Vol. 11, Cambridge Uni. Press, Cambridge, UK, 1986.
3. I. Assem and D. Happel, Generalized tilted algebras of type $\mathbb{A}_n$, *Comm. Algebra* **10** (1982), 2121–2139.
4. I. Assem and A. Skowroński, Iterated tilted algebras of type $\tilde{\mathbb{A}}_n$, *Math. Z.* **195** (1987), 269–290.
5. I. Assem and A. Skowroński, Minimal representation-infinite coil algebras, *Manuscripta Math.* **67** (1990), 305–311.
6. M. Auslander and I. Reiten, Representation theory of artin algebras. III. Almost split sequences, *Comm. Algebra* **3** (1975), 239–294.
7. M. Auslander and I. Reiten, Uniserial functors, *in* "Representation Theory II, Proceedings Ottawa 1979", Lecture Notes *in* Mathematics, Vol. 832, pp. 1–47, Springer-Verlag, Berlin/New York, 1980.
8. M. Auslander, I. Reiten, and S. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Mathematics Vol 36, Cambridge Uni. Press, Cambridge, UK, 1994.
9. D. J. Benson "Representations and Cohomology," Cambridge Studies in Advanced Mathematics, Vol 31, Cambridge Uni. Press, Cambridge, UK, 1991.
10. F. M. Bleher and T. Chinburg, Using quivers to compute Ext groups for Brauer tree algebras, submitted for publication, 1998.
11. K. Bongartz and P. Gabriel, Covering spaces in representation-theory, *Comment. Math. Helv.* **65** (1982), 331–378.
12. L. Chasen, "The Cohomology Ring of a Brauer Quiver Algebra," Ph.D. thesis, Virginia Polytechnic Institute, 1995.
13. K.R. Fuller, Biserial rings, *in* Lecture Notes in Mathematics, Vol. 734, pp. 64–87, Springer-Verlag, Berlin/New York, 1979.
14. P. Gabriel, Auslander–Reiten sequences and representation-finite algebras, in Lecture Notes in Mathematics, Vol. 831, pp. 1–71, Springer-Verlag, Berlin/New York, 1980.
15. P. Gabriel, The universal cover of a representation-finite algebra, *in* Lecture Notes in Mathematics, Vol. 903, pp. 68–105, Springer-Verlag, New York, 1981.
16. P. Gabriel and C. Riedtmann, Group representations without groups, *Comment. Math. Helv.* **54** (1979), 240–287.

17. E. L. Green and D. Zacharia, The cohomology ring of a monomial algebra, *Manuscripta Math.* **85** (1994), 11–23.
18. C. Löfall, On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra, *in* Lecture Notes in Mathematics, Vol. 1352, pp. 291–298, Springer-Verlag, Berlin/New York, 1988.
19. S. Mac Lane "Homology," Die Gründlehren der mathematischen Wissenschaften, Vol. 114, Springer-Verlag, Berlin/New York, 1967.
20. J. E. Roos, Relations between the Poincaré–Betti series of loop spaces and of local rings, *in* Lecture Notes in Mathematics, Vol. 740, pp. 285–322, Springer-Verlag, Berlin/New York, 1979.
21. A. Skowroński and J. Waschbüsch, Representation-finite biserial algebras, *J. Reine Angew. Math.* **345** (1983), 172–181.