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Journal of Algebra 274 (2004) 271–308

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Zeta functions of integral representations of cyclic p-groups

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Received 7 February 2003

Communicated by Gordon James

Abstract

For a prime number p and C_{p^k}, the cyclic group of order p^k, we consider the group ring $\mathbb{Z}_p[C_{p^k}]$ over the p-adic integers. Following L. Solomon, one can define the zeta function of the free $\mathbb{Z}_p[C_{p^k}]$-module $\mathbb{Z}_p[C_{p^k}]^n$, which counts submodules of finite index in $\mathbb{Z}_p[C_{p^k}]^n$. In this article we develop a recursion formula (relating submodules of $\mathbb{Z}_p[C_{p^k}]^n$ to certain submodules of $\mathbb{Z}_p[C_{p^{k-1}}]^n$), which yields some new explicit formulas for the zeta function of $\mathbb{Z}_p[C_{p^k}]^n$ in the cases $k = 1, 2$ and $n \geqslant 1$, and $k = 3$, $n = 1$. An important combinatorial tool for these computations is the Möbius function of a partially ordered set.

Keywords: Zeta function; Modules over group rings; Integral representation; Möbius function

1. Introduction

Let p be a prime number and C_{p^k} be the (multiplicative) cyclic group of order p^k, where k is a non-negative integer. Let L be a finitely generated torsion free $\mathbb{Z}[C_{p^k}]$-module. In [8] L. Solomon introduced a zeta function attached to L, defined as

$$\zeta_L(s) = \sum_{U \subseteq L} [L:U]^{-s};$$

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0021-8693/$ – see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/S0021-8693(03)00422-8

here the sum extends over all submodules U of finite index in L, and the series converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large (although we will regard this series rather as a formal sum; therefore, we will not care about questions of convergence in the sequel). Solomon showed that there is an Euler product

$$\zeta_L(s) = \prod_l \zeta_{L_l}(s),$$

where l ranges over all prime numbers, $L_l = L \otimes \mathbb{Z}_l$ ($\mathbb{Z}_l$ being the ring of l-adic integers) and

$$\zeta_{L_l} = \sum_{U_l \subseteq L_l} [L_l : U_l]^{-s},$$

this time summing over all $\mathbb{Z}_l[C_{p^k}]$-submodules U_l of finite index in L_l. Thus we can work locally in order to compute $\zeta_L(s)$.

If $l \neq p$ the order $\mathbb{Z}_l[C_{p^k}]$ is the maximal order in $\mathbb{Q}_l[C_{p^k}]$, and $\zeta_{L_l}(s)$ is a product of Dedekind zeta functions of number fields, cf. [8, (1.2)]. The case $l = p$ is more delicate. Let $\widetilde{R} \subseteq \mathbb{Q}_p[C_{p^k}]$ be the maximal order (containing $\mathbb{Z}_p[C_{p^k}]$). Then

$$\zeta_{L_p}(s) = \delta_{L_p}(s)\zeta_{L\otimes\widetilde{R}}(s),$$

where again $\zeta_{L\otimes\widetilde{R}}(s)$ is a product of Dedekind zeta functions of number fields, and $\delta_{L_p}(s) \in \mathbb{Z}[p^{-s}]$ is a polynomial in p^{-s} with integral coefficients, according to *Solomon's First Conjecture* proved in [1, Theorem 1]. The difficulty in calculating the zeta function of L consists in determining this polynomial.

The goal of this paper is to find explicit formulas for the zeta function of $L = \mathbb{Z}[C_{p^k}]^n$, where n is a positive integer. Put $R_k := \mathbb{Z}_p[C_{p^k}]$, and let $\widetilde{R}_k$ be the maximal order of the group algebra $\mathbb{Q}_p[C_{p^k}]$. Since $\mathbb{Q}_p[C_{p^k}] \cong \mathbb{Q}_p \oplus \mathbb{Q}_p(\omega_1) \oplus \cdots \oplus \mathbb{Q}_p(\omega_k)$ (ω_i being a primitive p^ith root of unity) we get $\widetilde{R}_k \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[\omega_1] \oplus \cdots \oplus \mathbb{Z}_p[\omega_k]$. In this case [8, (1.2)] reads

$$\zeta_{\mathbb{Z}[C_{p^k}]^n}(s) = \delta_{R_k^n}(s) \prod_{j=0}^{n-1} \left(\zeta(s-j) \prod_{i=1}^{k} \zeta_{\mathbb{Q}(\omega_i)}(s-j) \right),$$

where $\zeta(s)$ is the Riemann zeta function, and $\zeta_{\mathbb{Q}(\omega_i)}(s)$ is the Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\omega_i)$. Furthermore the zeta function of $\widetilde{R}_k^n$ is the "p-part" of the above product of zeta functions,

$$\zeta_{\widetilde{R}_k^n}(s) = \frac{1}{((1-p^{-s})(1-p^{1-s})\cdots(1-p^{n-1-s}))^{k+1}},$$

and it suffices to compute

$$\zeta_{R_k^n}(s) = \delta_{R_k^n}(s)\zeta_{\widetilde{R}_k^n}(s).$$

The only cases treated so far are $n = 1$ and $k = 1$ by Solomon (cf. [8], and an easier proof by I. Reiner in [6]), and $n = 1$ and $k = 2$ by Reiner in [6]. In this paper we will generalize Reiner's method. The main idea is that R_k can be written as a fibre product of R_{k-1} and some discrete valuation ring. This allows one to reduce the zeta function of R_k^n (a sum over submodules of R_k^n) to a sum over submodules of R_{k-1}^n. More accurately we get the following formula:

$$\zeta_{R_k^n}(s) = \zeta_{\mathbb{Z}_p^n}(s) \sum_{\overline{U} \subseteq R_{k-1}^n} \left(p^{r(\overline{U})}\right)^{n-s} \left[R_{k-1}^n : \overline{U}\right]^{-s},$$

where $\zeta_{\mathbb{Z}_p^n}(s)$ is easy to calculate, and $r(\overline{U})$ is a certain non-negative integer, depending on the submodule $\overline{U}$ of finite index in R_{k-1}^n. Unfortunately the sum on the right-hand side is not quite the zeta function of R_{k-1}^n, but the terms can be rearranged in order to obtain a $\mathbb{Z}[p^{-s}]$-linear combination of the zeta functions of the submodules $\overline{U}$ for $pR_{k-1}^n \subseteq \overline{U} \subseteq R_{k-1}^n$. This transformation is accomplished by means of a combinatorial tool: the Möbius function of the lattice of those submodules $\overline{U}$. In addition we mention that it is evidently not enough to consider zeta functions of free R_k-modules. We should rather find a recursive formula, similar to the one above, for the zeta function of any submodule $U \subseteq R_k^n$ of finite index.

This will be done in the sequel, and thereby we are able to find explicit formulas for $\zeta_{R_k^n}(s)$ in the following cases

- $k = 1$ and $n \geqslant 1$,
- $k = 2$ and $n \geqslant 1$ (where our result in the case $n = 1$ is "more explicit" than the one in [6]),
- $k = 3$ and $n = 1$.

The paper is organized as follows: in Section 2 we briefly describe two algebraic concepts that will be useful for the derivation of the recursion formula in Section 3. In Section 4 we define the Möbius function of a partially ordered set, and apply it to our case. In Section 5 we discuss a functional equation for the zeta function of $\mathbb{Z}_p[C_{p^k}]^n$. Eventually, Sections 6–8 are devoted to the development of explicit formulas for the zeta functions of $\mathbb{Z}_p[C_{p^k}]^n$ in the cases mentioned above.

Throughout this paper, we fix the following notation: p is a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, C_{p^k} with $k \geqslant 0$ is the cyclic group of order p^k, and σ is a fixed generator of this group. Furthermore $R_k = \mathbb{Z}_p[C_{p^k}]$, and

$$\phi_k = \sigma^{p^{k-1}(p-1)} + \sigma^{p^{k-1}(p-2)} + \cdots + \sigma^{p^{k-1}} + 1 \in R_k,$$

and for $i \geqslant 1$ we let ω_i be a primitive p^ith root of unity.

2. Zeta functions of matrix rings and Reiner's method

If R is any commutative ring, we denote by $\mathrm{M}_n(R)$ the ring of $n \times n$ matrices over R. We define the zeta function of $\mathrm{M}_n(R)$ as the formal sum

$$\zeta_{\mathrm{M}_n(R)}(s) = \sum_{I \subseteq \mathrm{M}_n(R)} \left[\mathrm{M}_n(R) : I\right]^{-s}.$$

Here the summation is over all left ideals I of finite index in $\mathrm{M}_n(R)$ (or equivalently, all right ideals I of finite index in $\mathrm{M}_n(R)$). In a similar way we can generalize the definition of a zeta function from the introduction: let V be an R-module, and put

$$\zeta_V(s) = \sum_{U \subseteq V} [V : U]^{-s},$$

where U runs through all R-submodules of finite index in V.

Theorem 2.1. *The zeta function of* $\mathrm{M}_n(R)$ *and the zeta function of the* R*-module* R^n *satisfy*

$$\zeta_{\mathrm{M}_n(R)}(s) = \zeta_{R^n}(ns).$$

Proof. By Morita's theorem (cf. [4, Section 3.12]) there is a lattice isomorphism between the lattice of left ideals I of $\mathrm{M}_n(R)$ and the lattice of R-submodules U of R^n. This induces an isomorphism between the lattices of ideals/submodules of *finite index*. Moreover, if I and U correspond to each other under this isomorphism, one can easily show that

$$\left[\mathrm{M}_n(R) : I\right] = \left[R^n : U\right]^n,$$

whence the desired equality. $\square$

Form the proof of the theorem we immediately infer:

Corollary 2.2. *If* $I \subseteq \mathrm{M}_n(R)$ *and* $U \subseteq R^n$ *are of finite index and correspond to each other under Morita's isomorphism, then*

$$\zeta_I(s) = \zeta_U(ns),$$

where $\zeta_I(s) = \sum_{J \subseteq I} [I : J]^{-s}$, J *running through all left* $\mathrm{M}_n(R)$*-ideals of finite index, contained in* I.

The above theorem allows us to compute the zeta function of the ring $\mathrm{M}_n(R_k)$ instead of the zeta function of the module R_k^n. To this ring, Reiner's method is applicable, as we will see in the next section. We will now briefly describe this method introduced by Reiner in [6].

Suppose we are given a fibre product of rings

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"] \\ A_2 \arrow[r, "g_2"'] & A' \end{tikzcd}$$

where all maps are ring surjections. Thus

$$A = \{(x_1, x_2) \in A_1 \oplus A_2 \mid g_1(x_1) = g_2(x_2)\}.$$

We further assume that every left ideal of A_1 is principal. The aim is to describe all left ideals of A in terms of the left ideals of A_1 and A_2.

Let $I \subseteq A$ be a left ideal. Then $f_1(I) = A_1\alpha$ for some $\alpha \in A_1$. Hence there exists $\beta \in f_2(I)$ such that $(\alpha, \beta) \in A$ and $g_1(\alpha) = g_2(\beta)$. Put

$$I_2 := \{x \in A_2 \mid (0, x) \in I\},$$

which is a left ideal of A_2 satisfying

$$I = A(\alpha, \beta) + (0, I_2).$$

Therefore, every left ideal $I \subseteq A$ has the form $I = A(\alpha, \beta) + (0, I_2)$ with

- a uniquely determined left ideal $I_2 \subseteq A_2$ with $g_2(I_2) = 0$,
- a generator $\alpha \in A_1$ of $f_1(I)$,
- an element $\beta \in A_2$ with $g_1(\alpha) = g_2(\beta)$; here β is uniquely determined mod I_2.

3. A recursion formula for the zeta function of R_k^n

In this section we will derive a formula for $\zeta_{\mathrm{M}_n(R_k)}(s)$ (which amounts to the same as $\zeta_{R_k^n}(s)$, by Theorem 2.1). To this end, as we already mentioned in the introduction, we have to find a formula for $\zeta_V(s)$, where $V \subseteq \mathrm{M}_n(R_k)$ is an arbitrary ideal (from now on all ideals are of finite index, unless stated otherwise). Put

$$R = R_k, \qquad \overline{R} = R_{k-1}, \qquad S = \mathbb{Z}_p[\omega_k], \qquad \overline{F} := \mathbb{F}_p[C_{p^{k-1}}]$$

and $\phi := \phi_k$. There is a canonical fibre product diagram linking these rings

$$\begin{tikzcd} R \arrow[r, "\bmod \phi_k"] \arrow[d, "\sigma \mapsto \tau"'] & S \arrow[d, "\bmod\ (1-\omega_k)^{p^{k-1}}"] \\ \overline{R} \arrow[r, "\bmod p"'] & \overline{F} \end{tikzcd}$$

where all maps are ring surjections and τ is a generator of $C_{p^{k-1}}$ in $\overline{R}$. This leads to a fibre product diagram

$$\begin{tikzcd} \mathrm{M}_n(R) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathrm{M}_n(S) \arrow[d, "g_1"] \\ \mathrm{M}_n(\overline{R}) \arrow[r, "g_2"'] & \mathrm{M}_n(\overline{F}) \end{tikzcd} \tag{1}$$

Since S is a principal ideal domain (even a discrete valuation ring), we know that every left ideal of $\mathrm{M}_n(S)$ is principal (cf. [5, Theorem 17.24]). Therefore, Reiner's method, introduced in the preceding section, is applicable to this situation.

Let $V \subseteq \mathrm{M}_n(R)$ be a left ideal. Then

$$V = \mathrm{M}_n(R)(\alpha_V, \beta_V) + (0, V_2), \tag{2}$$

where $\alpha_V \in f_1(V)$, $\beta_V \in f_2(V)$ with $g_1(\alpha_V) = g_2(\beta_V)$ and

$$V_2 = \bigl\{x \in \mathrm{M}_n(\overline{R}) \mid (0, x) \in V\bigr\}.$$

Lemma 3.1. *The left ideal* $V_2 \subseteq \mathrm{M}_n(\overline{R})$ *is given by*

$$V_2 = f_2\bigl(V \cap \phi\mathrm{M}_n(R)\bigr).$$

Proof. This follows from $\phi\mathrm{M}_n(R) = \ker(f_1) = \{(0, x) \in \mathrm{M}_n(R) \mid x \in \mathrm{M}_n(\overline{R})\}$. $\square$

Lemma 3.2.

(a) *There exists a left* $\mathrm{M}_n(\overline{R})$*-ideal* $V^\circ \supseteq f_2(V)$ *such that* $V_2 = pV^\circ$.
(b) $\beta_V \in V^\circ$, *where* β_V *is as in* (2).

Proof. The canonical map $R \to \overline{R}$ maps ϕ to p. Hence

$$V_2 = f_2\bigl(V \cap \phi\mathrm{M}_n(R)\bigr) \subseteq f_2\bigl(\phi\mathrm{M}_n(R)\bigr) = p\mathrm{M}_n(\overline{R}),$$

i.e., there is a $V^\circ \subseteq \mathrm{M}_n(\overline{R})$ such that $V_2 = pV^\circ$. Now (a) follows because of

$$pV^\circ = V_2 \supseteq f_2(\phi \cdot V) = pf_2(V).$$

Part (b) follows from

$$(0, p\beta_V) = (0, p)(\alpha_V, \beta_V) = \phi(\alpha_V, \beta_V) \in V,$$

i.e., $p\beta_V \in V_2 = pV^\circ$. $\square$

We summarize: the left ideal V can be written as

$$V = \mathrm{M}_n(R)(\alpha_V, \beta_V) + \big(0, pV^\circ\big) \tag{3}$$

with $\alpha_V \in f_1(V)$, $\beta_V \in V^\circ$, $g_1(\alpha_V) = g_2(\beta_V)$ and $pV^\circ = f_2(V \cap \phi\mathrm{M}_n(R))$.

Our goal is the computation of

$$\zeta_V(s) = \sum_{N \subseteq V} [V : N]^{-s}.$$

By Reiner's method and the above lemma, every left ideal of $\mathrm{M}_n(R)$ has the form

$$\mathrm{M}_n(R)(\alpha, \beta) + (0, p\overline{N}), \tag{4}$$

where

- $\overline{N} \subseteq \mathrm{M}_n(\overline{R})$ is a left ideal,
- $\alpha \in \mathrm{M}_n(S)$ with $\det(\alpha) \neq 0$ (which means $[\mathrm{M}_n(S) : \mathrm{M}_n(S)\alpha] < \infty$),
- $\beta \in \overline{N}$ such that $g_1(\alpha) = g_2(\beta)$.

Let $\mathcal{R}$ denote a system of representatives of generators $\alpha \in \mathrm{M}_n(S)$ with $\det(\alpha) \neq 0$ of all left ideals of $\mathrm{M}_n(S)$. Every left ideal of $\mathrm{M}_n(R)$ determines a unique pair $(\alpha, \overline{N})$ with $\alpha \in \mathcal{R}$ and a left ideals $\overline{N} \subseteq \mathrm{M}_n(\overline{R})$. On the other hand, if $\alpha \in \mathcal{R}$ and $\overline{N} \subseteq \mathrm{M}_n(\overline{R})$ are fixed, the number of left ideals of $\mathrm{M}_n(R)$ belonging to $(\alpha, \overline{N})$ equals the number of elements $\beta \in \overline{N}$ distinct mod $p\overline{N}$, satisfying $g_1(\alpha) = g_2(\beta)$.

For our purpose we have to determine the left ideals of the form (4) of $\mathrm{M}_n(R)$ *contained in* V, and we denote the number of these by $\nu(\alpha, \overline{N})$. A necessary condition for such a left ideal to be contained in V is $p\overline{N} \subseteq pV^\circ$, that is $\overline{N} \subseteq V^\circ$.

Lemma 3.3. *Let $\alpha \in \mathcal{R}$ and $\overline{N} \subseteq V^\circ$.*

(a) $\nu(\alpha, \overline{N}) \neq 0 \Longleftrightarrow \alpha \in f_1(V \cap f_2^{-1}(\overline{N}))$.
(b) *If $\nu(\alpha, \overline{N}) \neq 0$, then $\nu(\alpha, \overline{N}) = [pV^\circ \cap \overline{N} : p\overline{N}]$.*

Proof. (a) $\nu(\alpha, \overline{N}) \neq 0$ iff there is a left ideal of the form (4) contained in V, i.e., there is a $\beta \in \overline{N}$ such that $(\alpha, \beta) \in V$. This is the case iff $\alpha = f_1(v)$ for some $v \in V \cap f_2^{-1}(\overline{N})$.

(b) Suppose $\nu(\alpha, \overline{N}) \neq 0$. Let $v_0 \in V$ with $\alpha = f_1(v_0)$ and $\beta_0 := f_2(v_0) \in \overline{N}$, according to (a). The possible β, defining a left ideal as in (4) contained in V, are exactly all

$$\beta \in \overline{N} \quad \text{such that} \quad \exists v \in V\colon \alpha = f_1(v) \text{ and } \beta = f_2(v).$$

Since $v - v_0 \in \ker(f_1)$, this means $\beta \in f_2(v_0 + (\ker(f_1) \cap V)) \cap \overline{N}$. Now

$$\begin{aligned} f_2\big(v_0 + \big(\ker(f_1) \cap V\big)\big) \cap \overline{N} &= \big(\beta_0 + f_2\big(\ker(f_1) \cap V\big)\big) \cap \overline{N} \\ &= \beta_0 + \big(f_2\big(\ker(f_1) \cap V\big) \cap \overline{N}\big) \quad \text{since } \beta_0 \in \overline{N} \end{aligned}$$

$$= \beta_0 + \bigl(pV^\circ \cap \overline{N}\bigr).$$

Since we have to count the number $\nu(\alpha, \overline{N})$ of those β that are distinct mod $p\overline{N}$, the claim follows. $\square$

Lemma 3.4. *Let* $N := \mathrm{M}_n(R)(\alpha, \beta) + (0, p\overline{N})$ *be a left ideal as in* (4)*. Then*

$$\bigl[\mathrm{M}_n(R) : N\bigr] = \bigl[\mathrm{M}_n(S) : \mathrm{M}_n(S)\alpha\bigr]\bigl[\mathrm{M}_n(\overline{R}) : \overline{N}\bigr],$$

and in particular this index does not depend on β*.*

Proof. Consider the surjective map

$$\varphi : \mathrm{M}_n(R) \to \mathrm{M}_n(S)/\mathrm{M}_n(S)\alpha = \mathrm{M}_n(S)/f_1(N)$$

induced by f_1. Then $N \subseteq \ker(\varphi)$, and because of $\ker(\varphi) = f_1^{-1}(f_1(N)) = N + \ker(f_1)$ we get

$$\bigl[\ker(\varphi) : N\bigr] = \bigl[\ker(f_1) : N \cap \ker(f_1)\bigr] = \bigl[\bigl(0, p\mathrm{M}_n(\overline{R})\bigr) : (0, p\overline{N})\bigr] = \bigl[\mathrm{M}_n(\overline{R}) : \overline{N}\bigr]. \qquad \square$$

We now obtain a first formula for the zeta function of V:

$$\zeta_V(s) = \sum_{\overline{N} \subseteq V^\circ} \sum_{\alpha \in \mathcal{R}} \nu(\alpha, \overline{N}) \left(\frac{[\mathrm{M}_n(S) : \mathrm{M}_n(S)\alpha][\mathrm{M}_n(\overline{R}) : \overline{N}]}{[\mathrm{M}_n(R) : V]} \right)^{-s}, \tag{5}$$

and using Lemma 3.3

$$\begin{aligned} \zeta_V(s) = \bigl[\mathrm{M}_n(R) : V\bigr]^s \sum_{\overline{N} \subseteq V^\circ} \sum_{\substack{\alpha \in \mathcal{R} \\ \alpha \in f_1(V \cap f_2^{-1}(\overline{N}))}} &\bigl[pV^\circ \cap \overline{N} : p\overline{N}\bigr] \\ \times \bigl(\bigl[\mathrm{M}_n(S) : \mathrm{M}_n(S)\alpha\bigr]&\bigl[\mathrm{M}_n(\overline{R}) : \overline{N}\bigr]\bigr)^{-s}. \end{aligned} \tag{6}$$

Lemma 3.5.

$$\sum_{\substack{\alpha \in \mathcal{R} \\ \alpha \in f_1(V \cap f_2^{-1}(\overline{N}))}} \bigl[\mathrm{M}_n(S) : \mathrm{M}_n(S)\alpha\bigr]^{-s} = \bigl[\mathrm{M}_n(S) : f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr)\bigr]^{-s} \zeta_{\mathrm{M}_n(S)}(s).$$

Proof. Every left ideal of $\mathrm{M}_n(S)$ is principal, hence we have an isomorphism

$$\mathrm{M}_n(S) \cong f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr)$$

of left $\mathrm{M}_n(S)$-modules. Thus their zeta functions coincide, and the lemma is proved. $\square$

Note that V°/pV° is an $\mathbb{F}_p$-vector space in a natural way. Therefore, we may put

$$l_{V^\circ}(\overline{N}) := \dim_{\mathbb{F}_p}\bigl(\overline{N} + pV^\circ/pV^\circ\bigr)$$

if $\overline{N} \subseteq V^\circ$, and the value of $\nu(\alpha, \overline{N})$ determined in Lemma 3.3(b) can be transformed into

$$\bigl[pV^\circ \cap \overline{N} : p\overline{N}\bigr] = \frac{[\overline{N} : p\overline{N}]}{[\overline{N} : pV^\circ \cap \overline{N}]} = \frac{p^{n^2 p^{k-1}}}{[\overline{N} + pV^\circ : pV^\circ]} = p^{n^2 p^{k-1} - l_{V^\circ}(\overline{N})}.$$

Together with

$$\bigl[\mathrm{M}_n(R) : V\bigr] = \bigl[\mathrm{M}_n(S) : f_1(V)\bigr]\bigl[\mathrm{M}(\overline{R}) : V^\circ\bigr]$$

(cf. Lemma 3.4) we infer from (6) the new formula

$$\zeta_V(s) = \zeta_{\mathrm{M}_n(S)}(s) \sum_{\overline{N} \subseteq V^\circ} p^{n^2 p^{k-1} - l_{V^\circ}(\overline{N})} \bigl(\bigl[f_1(V) : f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr)\bigr]\bigl[V^\circ : \overline{N}\bigr]\bigr)^{-s} \tag{7}$$

for the zeta function of V.

Lemma 3.6.

$$\bigl[f_1(V) : f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr)\bigr]\bigl[V^\circ : \overline{N}\bigr] = p^{n^2 p^{k-1} - l_{V^\circ}(\overline{N})}\bigl[\overline{N} + f_2(V) : \overline{N}\bigr].$$

Proof. The following exact sequence is immediate:

$$0 \to \bigl(V \cap f_2^{-1}(\overline{N})\bigr) + \bigl(V \cap \ker(f_1)\bigr) \hookrightarrow V \xrightarrow{\pi_1} f_1(V)/f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr) \to 0,$$

where π_1 is induced by f_1. The definition of V° implies $f_2(V \cap \ker(f_1)) = pV^\circ$, and using $f_2(V \cap f_2^{-1}(\overline{N})) = f_2(V) \cap \overline{N}$ we get another exact sequence

$$0 \to \bigl(V \cap f_2^{-1}(\overline{N})\bigr) + \bigl(V \cap \ker(f_1)\bigr) \hookrightarrow V \xrightarrow{\pi_2} f_2(V)/\bigl(\bigl(f_2(V) \cap \overline{N}\bigr) + pV^\circ\bigr) \to 0,$$

where π_2 is induced by f_2.

Hence

$$\begin{aligned}\bigl[f_1(V) : f_1\bigl(V \cap f_2^{-1}(\overline{N})\bigr)\bigr] &= \bigl[f_2(V) : \bigl(f_2(V) \cap \overline{N}\bigr) + pV^\circ\bigr] \\ &= \frac{[f_2(V) : f_2(V) \cap \overline{N}]}{[(f_2(V) \cap \overline{N}) + pV^\circ : f_2(V) \cap \overline{N}]} = \frac{[\overline{N} + f_2(V) : \overline{N}]}{[pV^\circ : pV^\circ \cap \overline{N}]}\end{aligned}$$

and

$$\begin{aligned}\left[pV^\circ : pV^\circ \cap \overline{N}\right] &= \left[\overline{N} + pV^\circ : \overline{N}\right] = \frac{[V^\circ : \overline{N}]}{[V^\circ : \overline{N} + pV^\circ]} \\ &= \frac{[V^\circ : \overline{N}]}{[V^\circ : pV^\circ]/[\overline{N} + pV^\circ : pV^\circ]} = \frac{[V^\circ : \overline{N}]}{p^{n^2 p^{k-1} - l_{V^\circ}(\overline{N})}},\end{aligned}$$

and the assertion follows. □

With the help of this lemma formula (7) can be further simplified, and this yields the following theorem.

Theorem 3.7. *The following recursion formula holds for the zeta function of a left ideal $V \subseteq \mathrm{M}_n(R)$:*

$$\zeta_V(s) = \zeta_{\mathrm{M}_n(S)}(s) \sum_{\overline{N} \subseteq V^\circ} \left(p^{n^2 p^{k-1} - l_{V^\circ}(\overline{N})}\right)^{1-s} \left[\overline{N} + f_2(V) : \overline{N}\right]^{-s}; \tag{8}$$

here V° is given by $pV^\circ = f_2(V \cap \phi \mathrm{M}_n(R))$ and $l_{V^\circ}(\overline{N}) = \dim_{\mathbb{F}_p}(\overline{N} + pV^\circ / pV^\circ)$.

We will now retranslate this formula back to submodules of R^n, using Morita's theorem (cf. Corollary 2.2). We will mainly use the same notation as before, which should not be confusing. All submodules of R^n are again understood to be of finite index, unless stated otherwise. We have a fibre product diagram analogous to (1)

$$\begin{array}{ccc} R^n & \xrightarrow{f_1} & S^n \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ \overline{R}^n & \xrightarrow[g_2]{} & \overline{F}^n \end{array} \tag{9}$$

with surjective R-module homomorphisms.

Theorem 3.8. *The following recursion formula holds for the zeta function of a submodule $V \subseteq R^n$:*

$$\zeta_V(s) = \zeta_{S^n}(s) \sum_{\overline{N} \subseteq V^\circ} p^{(np^{k-1} - e_{V^\circ}(\overline{N}))(n-s)} \left[\overline{N} + f_2(V) : \overline{N}\right]^{-s}; \tag{10}$$

here V° is given by $pV^\circ = f_2(V \cap \phi R^n)$ and $e_{V^\circ}(\overline{N}) = \dim_{\mathbb{F}_p}(\overline{N} + pV^\circ / pV^\circ)$.

The term "recursion formula" is justified by the fact that the sum on the right-hand side extends over submodules of $\overline{R}^n$, hence we descended from k to $k-1$. We will reorder this sum in the next section and thereby obtain more practical results, at least in some cases. Note that the factor $\zeta_{S^n}(s)$ can be easily evaluated, for S is a discrete valuation ring (e.g., [1, Section 1]):

Theorem 3.9. *Let C be a discrete valuation ring with residue field $\mathbb{F}_p$, the field with p elements. Then*

$$\zeta_{C^n}(s) = \prod_{j=0}^{n-1} \left(1 - p^{j-s}\right)^{-1}.$$

Notation. Let $e \in \{0, \ldots, np^{k-1}\}$. We put

$$t_V(e; s) := \sum_{\substack{\overline{N} \subseteq V^\circ \\ e_{V^\circ}(\overline{N}) = e}} \left[\overline{N} + f_2(V) : \overline{N}\right]^{-s}. \tag{11}$$

Then (10) can be rewritten as

$$\zeta_V(s) = \zeta_{S^n}(s) \sum_{e=0}^{np^{k-1}} p^{(np^{k-1}-e)(n-s)} t_V(e; s). \tag{12}$$

We remark that it is particularly interesting to apply the recursion formula for $s = n$. This is discussed in [10] together with some applications.

4. Möbius functions

Our intention is to express the right-hand side of (10) in terms of the zeta functions $\zeta_{\overline{U}}(s)$ for certain submodules $\overline{U} \subseteq \overline{R}^n$. This can be done by a combinatorial tool, the Möbius function of a partially ordered set, which we will introduce first. The standard reference for details and examples is [7].

Thus let $(P, \leqslant)$ be a partially ordered set, which we require to be *locally finite*, i.e., for all $x, y \in P$ there are only finitely many $z \in P$ with $x \leqslant z \leqslant y$. The *Möbius function* $\mu : P \times P \to \mathbb{Z}$ is inductively defined by

$$\mu(x, y) = \begin{cases} 0 & x \nleqslant y, \\ 1 & x = y, \\ -\sum_{x \leqslant z < y} \mu(x, z) & x < y. \end{cases}$$

We recover the classical Möbius function μ' from number theory by letting P be the set of positive integers, partially ordered by the divisibility relation $|$. If $m \mid n$ are integers, then $\mu(m, n) = \mu'(n/m)$.

The reason for studying μ is the following useful inversion formula, which follows immediately from the definition.

Theorem 4.1 (Möbius inversion formula). *Let $(P, \leqslant)$ be a locally finite partially ordered set and let $a \in P$. Let $f : P \to \mathbb{C}$ be a function such that $f(x) = 0$ if $a \nleqslant x$. Define $g : P \to \mathbb{C}$ by*

$$g(x) := \sum_{a \leqslant y \leqslant x} f(y).$$

Then

$$f(x) = \sum_{a \leqslant y \leqslant x} \mu(y, x) g(y).$$

In the rest of this section we will deal with the following situation. Let R be a commutative ring. If V is an R-module, we can consider the lattice of all submodules of finite index in V, which is partially ordered by inclusion and locally finite. Hence we can study the Möbius function μ of that lattice.

Theorem 4.2. *Let U, V be R-modules with $U \subseteq V$ and $[V : U] < \infty$. If the radical of V/U satisfies* $\operatorname{rad}(V/U) \neq 0$, *then* $\mu(U, V) = 0$.

Proof. Let $\bar{\mu}$ be the Möbius function of the finite lattice of submodules of V/U. Then obviously

$$\mu(U, V) = \bar{\mu}(0, V/U).$$

If $\mathcal{M}$ is a non-empty subset of all maximal submodules of V/U, then $\bigcap \mathcal{M} \neq 0$, because $\operatorname{rad}(V/U) \neq 0$. Thus the set of all maximal submodules of V/U is a *cross cut* of the lattice (cf. [7, Section 6]), and the claim follows from [7, Theorem 3]. $\square$

Theorem 4.3. *The Möbius function μ of the lattice of subspaces of an r-dimensional $\mathbb{F}_p$-vector space satisfies*

$$\mu\big(0, \mathbb{F}_p^r\big) = (-1)^r p^{\binom{r}{2}}.$$

This is proved in [7, Section 5, Example 2]. We are now able to calculate the Möbius function of the lattice of submodules of a finitely generated module over a discrete valuation ring.

Corollary 4.4. *Let C be a discrete valuation ring with prime element π and residue class field $\mathbb{F}_p$. Let M be a finite C-module,*

$$M \cong C/\big(\pi^{a_1}\big) \oplus \cdots \oplus C/\big(\pi^{a_r}\big) \quad (a_i \geqslant 1).$$

Then the Möbius function μ of the lattice of submodules of M satisfies

$$\mu(0, M) = \begin{cases} (-1)^r p^{\binom{r}{2}} & a_1 = \cdots = a_r = 1, \\ 0 & \textit{otherwise}. \end{cases}$$

We continue letting R be a commutative ring. Moreover, the submodules of R^n considered here are always of finite index.

Notation. Let $U \subseteq X \subseteq R^n$ be submodules. We put

$$z_U(X;s) := \sum_{\substack{N \subseteq X \\ N+U=X}} [X:N]^{-s}. \tag{13}$$

We clearly have

$$z_U(U;s) = \zeta_U(s).$$

We will now show that $z_U(X;s)$ for $U \subseteq X$ can always be expressed in terms of $\zeta_W(s)$ for $U \subseteq W \subseteq X$. More precisely:

Theorem 4.5. *Let $U \subseteq X \subseteq R^n$ submodules, and let μ be the Möbius junction of all submodules of finite index in R^n. Then*

$$z_U(X;s) = \sum_{U \subseteq W \subseteq X} \mu(W,X)[X:W]^{-s}\zeta_W(s).$$

Proof. We define

$$f(W) := [X:W]^{-s} z_U(W;s)$$

for all $U \subseteq W \subseteq X$, and

$$\begin{aligned} g(W) &:= \sum_{U \subseteq W' \subseteq W} f(W') = \sum_{U \subseteq W' \subseteq W} [X:W']^{-s} z_U(W';s) \\ &= [X:W]^{-s} \sum_{U \subseteq W' \subseteq W} [W:W']^{-s} \sum_{\substack{N \subseteq W' \\ N+U=W'}} [W':N]^{-s} = [X:W]^{-s}\zeta_W(s). \end{aligned}$$

Now the inversion formula (Theorem 4.1) implies

$$z_U(X;s) = f(X) = \sum_{U \subseteq W \subseteq X} \mu(W,X) g(W),$$

and the theorem is proved. □

We now return to the notations of Section 3. In particular we let $R = R_k = \mathbb{Z}_p[C_{p^k}]$, $\overline{R} = R_{k-1} = \mathbb{Z}_p[C_{p^{k-1}}]$ and $e_V(N) = \dim_{\mathbb{F}_p}(N + pV/pV)$.

Lemma 4.6. *Let $V \subseteq R^n$ and $pV \subseteq Y \subseteq V$. Then*

$$\sum_{\substack{N \subseteq V \\ e_V(N)=e}} [N+Y:N]^{-s} = \sum_{\substack{pV \subseteq X \subseteq V \\ [X:pV]=p^e}} [X+Y:X]^{-s} z_{pV}(X;s)$$

for all $e \in \{0,\dots,np^k\}$.

Proof. Since $N \subseteq V$ we have $0 \leqslant e_V(N) = \dim_{\mathbb{F}_p}(N+pV/pV) \leqslant np^k$, and

$$e_V(N) = e \quad \Leftrightarrow \quad \exists pV \subseteq X \subseteq V: \quad N + pV = X \quad \text{and} \quad [X:pV] = p^e.$$

Hence

$$\sum_{\substack{N \subseteq V \\ e_V(N)=e}} [N+Y:N]^{-s} = \sum_{\substack{pV \subseteq X \subseteq V \\ [X:pV]=p^e}} \sum_{\substack{N \subseteq V \\ N+pV=X}} [N+Y:N]^{-s}.$$

Now $N + pV = X$ implies $N + Y = X + Y$, whence the latter sum equals

$$\sum_{\substack{pV \subseteq X \subseteq V \\ [X:pV]=p^e}} [X+Y:X]^{-s} \sum_{\substack{N \subseteq X \\ N+pV=X}} [X:N]^{-s} = \sum_{\substack{pV \subseteq X \subseteq V \\ [X:pV]=p^e}} [X+Y:X]^{-s} z_{pV}(X;s). \qquad \square$$

We can use the results of this section in order to compute $t_V(e;s)$ for $e \in \{0,\dots,np^{k-1}\}$ (defined in (11)), which are the building blocks of the zeta function of $V \subseteq R^n$. By Lemma 3.2 we have $pV^\circ \subseteq f_2(V) \subseteq V^\circ$, and Lemma 4.6 yields

$$t_V(e;s) = \sum_{\substack{pV^\circ \subseteq \overline{X} \subseteq V^\circ \\ [\overline{X}:pV^\circ]=p^e}} [X + f_2(V) : \overline{X}]^{-s} z_{pV^\circ}(\overline{X};s). \tag{14}$$

The expressions $z_{pV^\circ}(\overline{X};s)$ for $pV^\circ \subseteq \overline{X} \subseteq V^\circ$ equal (according to Theorem 4.5)

$$z_{pV^\circ}(\overline{X};s) = \sum_{pV^\circ \subseteq \overline{W} \subseteq \overline{X}} \mu(\overline{W},\overline{X})[\overline{X}:\overline{W}]^{-s} \zeta_{\overline{W}}(s), \tag{15}$$

where μ is the Möbius function of the lattice of submodules of finite index in $\overline{R}^n$. Therefore, we have proved the following important corollary.

Corollary 4.7. *Let $V \subseteq R^n$ be a submodule. By* (12), (14), (15) *the quotient $\zeta_V(s)/\zeta_{S^n}(s)$ is a $\mathbb{Z}[p^{-s}]$-linear combination of the zeta functions*

$$\zeta_{\overline{W}}(s) \quad \textit{for} \quad pV^\circ \subseteq \overline{W} \subseteq V^\circ.$$

5. A functional equation

Let $R = R_k$ and $V \subseteq R^n$ be an R-submodule. Furthermore, we let $\widetilde{R}$ be the maximal order of $R \otimes \mathbb{Q}_p = \mathbb{Q}_p[C_{p^k}]$, hence

$$\widetilde{R} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[\omega_1] \oplus \cdots \oplus \mathbb{Z}_p[\omega_k].$$

In the introduction we defined

$$\delta_V(s) := \frac{\zeta_V(s)}{\zeta_{\widetilde{R}^n}(s)} = \frac{\zeta_V(s)}{(\zeta_{\mathbb{Z}_p^n}(s))^{k+1}},$$

where $\zeta_{\mathbb{Z}_p^n}(s)$ is given by Theorem 3.9, and $\delta_V(s) \in \mathbb{Z}[p^{-s}]$, which is a special case of Solomon's first conjecture proved in [1]. Note that this assertion also follows from our recursion formula (12) together with Corollary 4.7 (even though the conjecture is proved in [1] in a much more general situation).

The main result of this section is the following functional equation for $\delta_{R^n}(s)$.

Theorem 5.1. *The function $\delta_{R^n}(s)$ satisfies the following functional equation*

$$\delta_{R^n}(s) = p^{(n^2-2ns)(1+p+\cdots+p^{k-1})}\delta_{R^n}(n-s).$$

Proof. Put $\Lambda := \mathrm{M}_n(R)$. Then $\tilde{\Lambda} := \mathrm{M}_n(\widetilde{R}) \supseteq \Lambda$ is a maximal order in $\mathrm{M}_n(R \otimes \mathbb{Q}_p)$. The proof of the functional equation for the zeta function of a group ring $\mathbb{Z}_p[G]$ presented in [1, Theorem 2] can be generalized to the matrix ring $\mathrm{M}_n(\mathbb{Z}_p[G])$, and hence to our Λ (details are worked out in [9]). Thus we obtain in the same manner

$$\frac{\zeta_\Lambda(s)}{\zeta_\Lambda(1-s)} = [\tilde{\Lambda} : \Lambda]^{1-2s}\frac{\zeta_{\tilde{\Lambda}}(s)}{\zeta_{\tilde{\Lambda}}(1-s)}.$$

Now Theorem 2.1 yields

$$\frac{\zeta_{R^n}(ns)}{\zeta_{\widetilde{R}^n}(ns)} = [\widetilde{R} : R]^{n^2-2n^2s}\frac{\zeta_{R^n}(n-ns)}{\zeta_{\widetilde{R}^n}(n-ns)}.$$

If we substitute s for ns, the claim follows from the next lemma. □

Lemma 5.2. *Let $\widetilde{R}_k \supseteq R_k$ be the maximal order in $R_k \otimes \mathbb{Q}_p$. Then*

$$[\widetilde{R}_k : R_k] = p^{1+p+\cdots+p^{k-1}}.$$

Proof. The assertion is trivial for $k = 0$, since $\widetilde{R}_0 = R_0 = \mathbb{Z}_p$. If $k > 0$ we have the fibre product diagram

$$\begin{array}{ccc} R_k & \longrightarrow & \mathbb{Z}_p[\omega_k] \\ \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & \mathbb{F}_p[C_{p^{k-1}}] \end{array}$$

which gives rise to an exact sequence

$$0 \longrightarrow R_k \longrightarrow \mathbb{Z}_p[\omega_k] \oplus R_{k-1} \xrightarrow{h} \mathbb{F}_p[C_{p^{k-1}}] \longrightarrow 0.$$

Then

$$\begin{aligned}[]
[\widetilde{R}_k : R_k] &= \big[\mathbb{Z}_p[\omega_k] \oplus \widetilde{R}_{k-1} : \mathbb{Z}_p[\omega_k] \oplus R_{k-1}\big] \cdot \big[\mathbb{Z}_p[\omega_k] \oplus R_{k-1} : R_k\big] \\
&= [\widetilde{R}_{k-1} : R_{k-1}] \cdot |\mathrm{im}(h)| = [\widetilde{R}_{k-1} : R_{k-1}] \cdot p^{p^{k-1}},
\end{aligned}$$

and the claim follows by induction. □

Put $x := p^{-s}$ and define $\hat{\delta}_V(x) \in \mathbb{Z}[x]$ for a submodule $V \subseteq R^n$ by

$$\hat{\delta}_V\big(p^{-s}\big) = \delta_V(s). \tag{16}$$

Similarly we define $\hat{t}_V(e; x) \in \mathbb{Z}[x]$ for $0 \leqslant e \leqslant np^{k-1}$ by

$$\hat{t}_V\big(e; p^{-s}\big) = \frac{t_V(e; s)}{(\zeta_{\mathbb{Z}_p^n}(s))^k}, \tag{17}$$

where $t_V(e; s)$ is as in (11), and so we clearly may write (12) in the form

$$\hat{\delta}_V(x) = \sum_{e=0}^{np^{k-1}} p^{n(np^{k-1}-e)} x^{np^{k-1}-e} \hat{t}_V(e; x). \tag{18}$$

Corollary 5.3. *$\hat{\delta}_{R^n}(x)$ is a polynomial in $\mathbb{Z}[x]$ of degree $2n(1 + p + \cdots + p^{k-1})$, having constant term 1 and satisfying the functional equation*

$$\hat{\delta}_{R^n}(x) = \big(p^{n^2} x^{2n}\big)^{1+p+\cdots+p^{k-1}} \hat{\delta}_{R^n}\left(\frac{1}{p^n x}\right).$$

Proof. The functional equation follows immediately from Theorem 5.1, and this implies the degree statement as soon as we have shown that $\hat{\delta}_{R^n}(0) = 1 \neq 0$. Now

$$\hat{\delta}_{R^n}(0) = \lim_{s\to\infty} \frac{\zeta_{R^n}(s)}{\zeta_{\mathbb{Z}_p^n}(s)^{k+1}},$$

and the denominator tends to 1 by Theorem 3.9. But the same holds for the nominator by formula (12), since all terms disappear for $s \to \infty$, except the one having e-value np^{k-1}, i.e., $\overline{N} = \overline{R}^n$, contributing 1 to the sum. □

6. The case $n = 1$

In this section we fix $n = 1$ and $k \geqslant 1$. Let $R = R_k = \mathbb{Z}_p[\sigma]$ with $\sigma^{p^k} = 1$, and $\overline{R} = R_{k-1} = \mathbb{Z}_p[\tau]$ with $\tau^{p^{k-1}} = 1$. Then we have a surjective homomorphism

$$\begin{aligned} \pi : R \to R/pR &\cong \mathbb{F}_p[C_{p^k}] \cong \mathbb{F}_p[y]/\big(y^{p^k}\big), \\ \sigma \mapsto \sigma \bmod p &\mapsto \bar{y} + 1. \end{aligned} \tag{19}$$

Therefore, we have a unique filtration

$$R = V_0^{(k)} \underset{\neq}{\supseteq} V_1^{(k)} \underset{\neq}{\supseteq} \cdots \underset{\neq}{\supseteq} V_{p^k}^{(k)} = pR.$$

The ideals of the ring $\mathbb{F}_p[y]/(y^{p^k})$ are the ones generated by $\bar{y}^l$ for $0 \leqslant l \leqslant p^k$, whence

$$V_l^{(k)} = \big(p, (\sigma - 1)^l\big) \quad \big(0 \leqslant l \leqslant p^k\big).$$

We begin by computing $\zeta_V(s)$ for $V = V_l^{(k)}$ $(0 \leqslant l \leqslant p^k)$. This will be done by applying the recursion formula from Section 3, together with (14), (15), so we need some information concerning the $\overline{R}$-ideals $f_2(V)$ and V°. Now $pR \subseteq V \subseteq R$ implies $p\overline{R} \subseteq f_2(V) \subseteq V^\circ \subseteq \overline{R}$, hence these ideals occur in the unique filtration

$$\overline{R} = V_0^{(k-1)} \underset{\neq}{\supseteq} V_1^{(k-1)} \underset{\neq}{\supseteq} \cdots \underset{\neq}{\supseteq} V_{p^{k-1}}^{(k-1)} = p\overline{R}. \tag{20}$$

Lemma 6.1. *Let $pR \subseteq V \subseteq R$, i.e., $V = V_l^{(k)}$ for some $0 \leqslant l \leqslant p^k$. Then*

(a)
$$f_2(V) = \begin{cases} V_l^{(k-1)} & l < p^{k-1}, \\ p\overline{R} & l \geqslant p^{k-1}. \end{cases}$$

(b)
$$V^\circ = \begin{cases} \overline{R} & l \leqslant p^{k-1}(p-1), \\ V_{l-p^{k-1}(p-1)}^{(k-1)} & l > p^{k-1}(p-1). \end{cases}$$

Proof. (a) Recall that $f_2(\sigma) = \tau$, by definition of f_2. Thus the claim is trivial for $l < p^{k-1}$, and the rest follows from the identity

$$f_2\big(V^{(k)}_{p^{k-1}}\big) = V^{(k-1)}_{p^{k-1}} = p\overline{R} = f_2(pR).$$

(b) It suffices to determine the index $[\overline{R} : V^\circ]$, for $p\overline{R} \subseteq V^\circ \subseteq \overline{R}$, and $\overline{R}/p\overline{R}$ has a unique composition series. We have

$$p\overline{R}/pV^\circ = f_2(\phi R)/f_2(V \cap \phi R) \cong \phi R/(V \cap \phi R) \cong (\phi R + V)/V \cong \pi(\phi R + V)/\pi(V),$$

where the first isomorphism is induced by f_2 (note that $\ker(f_2) \cap \phi R = \ker(f_2) \cap \ker(f_1) = 0$ by the fibre product diagram (9)). Moreover

$$\pi(\phi) = \pi\big(\sigma^{p^{k-1}(p-1)} + \cdots + \sigma^{p^{k-1}} + 1\big) = (\bar{y}+1)^{p^{k-1}(p-1)} + \cdots + (\bar{y}+1)^{p^{k-1}} + 1,$$

and since $u^{p-1} + \cdots + u + 1 = (u-1)^{p-1}$ holds in the polynomial ring $\mathbb{F}_p[u]$, we find

$$\pi(\phi) = \big((\bar{y}+1)^{p^{k-1}} - 1\big)^{p-1} = \bar{y}^{p^{k-1}(p-1)}.$$

This leads to

$$\begin{aligned}\pi(\phi R + V) &= \pi(\phi R) + \pi(V) = \big(\bar{y}^{p^{k-1}(p-1)}\big) + \big(\bar{y}^l\big)\\ &= \begin{cases} \big(\bar{y}^l\big) & l \leqslant p^{k-1}(p-1),\\ \big(\bar{y}^{p^{k-1}(p-1)}\big) & l > p^{k-1}(p-1),\end{cases}\end{aligned}$$

and the assertion follows from $[\overline{R} : V^\circ] = [\pi(\phi R + V) : \pi(V)]$. □

Using (14), (15) we obtain the following formulas for $t_V(e; s)$ $(0 \leqslant e \leqslant p^k)$.

Theorem 6.2. *Let $V = V^{(k)}_l$ for some $0 \leqslant l \leqslant p^k$.*

(a) *If* $0 \leqslant l < p^{k-1}$:

$$t_V(e; s) = \begin{cases} \big(p^{p^{k-1}-l}\big)^{-s}\zeta_{\overline{R}}(s) & e = 0,\\ \big(p^{p^{k-1}-l-e}\big)^{-s}\Big(\zeta_{V^{(k-1)}_{p^{k-1}-e}}(s) - p^{-s}\zeta_{V^{(k-1)}_{p^{k-1}-e+1}}(s)\Big) & 0 < e \leqslant p^{k-1} - l,\\ \zeta_{V^{(k-1)}_{p^{k-1}-e}}(s) - p^{-s}\zeta_{V^{(k-1)}_{p^{k-1}-e+1}}(s) & p^{k-1} - l < e \leqslant p^{k-1}.\end{cases}$$

(b) *If* $p^{k-1} \leqslant l \leqslant p^{k-1}(p-1)$:

$$t_V(e; s) = \begin{cases} \zeta_{\overline{R}}(s) & e = 0,\\ \zeta_{V^{(k-1)}_{p^{k-1}-e}}(s) - p^{-s}\zeta_{V^{(k-1)}_{p^{k-1}-e+1}}(s) & 0 < e \leqslant p^{k-1}.\end{cases}$$

(c) *If* $l = p^k$, *then* $t_V(e;s)$ *is identical to* $t_{V_0^{(k)}}(e;s)$ *for all* $0 \leqslant e \leqslant p^{k-1}$.

Proof. Since $V_{p^k}^{(k)} = pR \cong R = V_0^{(k)}$, their zeta functions and consequently their t-functions must coincide for all e. Hence part (c) is proved.

For (a) and (b) we note that $V^\circ = \overline{R}$ for all values of l under consideration, according to the preceding lemma. Now the case $e = 0$ follows directly from (14) and the calculation of $f_2(V)$ in Lemma 6.1. If $e > 0$ we will have to compute in addition the functions

$$z_{p\overline{R}}(\overline{X};s) \quad \text{for } p\overline{R} \subseteq \overline{X} \subseteq \overline{R} \text{ such that } [\overline{X} : p\overline{R}] = p^e,$$

according to (15). But $\overline{X} = V_{p^{k-1}-e}^{(k-1)}$ in view of the unique filtration (20), and since $\overline{X}$ has a unique maximal submodule $\overline{M}$ containing $p\overline{R}$ we infer

$$\mu(\overline{X},\overline{X}) = 1, \qquad \mu(\overline{M},\overline{X}) = -1 \quad \text{and} \quad \mu(\overline{W},\overline{X}) = 0 \qquad \forall p\overline{R} \subseteq \overline{W} \subsetneqq \overline{M}$$

(cf. Theorem 4.2). This proves the theorem. □

Unfortunately the computation of $t_V(e;s)$ is much more complicated for $p^{k-1}(p-1) < l < p^k$ (the case not covered in the above theorem). The reason is that for those l the ideal V° is no longer isomorphic to $\overline{R}$, and the lattice of submodules of V°/pV° seems to be too complex for (14), (15) to yield explicit results in general.

Nonetheless, we are now able to compute the zeta function of the ring R_k in some cases. For $k = 1$ and $k = 2$ Theorem 6.2 will provide us the answer, while for $k = 3$ we will have to make an additional effort.

The case $k = 1$

Let $R = R_1 = \mathbb{Z}_p[C_p]$. Fix an ideal $pR \subseteq V \subseteq R$, i.e., $V = V_l^{(1)}$ for some $0 \leqslant l \leqslant p$. Since $\mathbb{Z}_p = V_0^{(0)} \cong V_1^{(0)} = p\mathbb{Z}_p$ and $\zeta_{\mathbb{Z}_p}(s) = (1-p^{-s})^{-1}$, Theorem 6.2 and formula (12) yield

- If $l = 0$ or $l = p$:

$$t_V(e;s) = \begin{cases} p^{-s}\zeta_{\mathbb{Z}_p}(s) & e = 0, \\ (1-p^{-s})\zeta_{\mathbb{Z}_p}(s) & e = 1, \end{cases}$$

  that is $\zeta_V(s) = (\zeta_{\mathbb{Z}_p}(s))^2(p^{1-s}p^{-s} + (1-p^{-s})) = (\zeta_{\mathbb{Z}_p}(s))^2(1 - p^{-s} + p^{1-2s})$.
- If $0 < l < p$:

$$t_V(e;s) = \begin{cases} \zeta_{\mathbb{Z}_p}(s) & e = 0, \\ (1-p^{-s})\zeta_{\mathbb{Z}_p}(s) & e = 1, \end{cases}$$

  that is $\zeta_V(s) = (\zeta_{\mathbb{Z}_p}(s))^2(p^{1-s} + (1-p^{-s})) = (\zeta_{\mathbb{Z}_p}(s))^2(1 + (p-1)p^{-s})$.

Up to isomorphism there are two ideals of finite index in R, viz R and $J := \mathrm{rad}(R)$. The above formulas imply that we must have $V_1^{(1)} \cong \cdots \cong V_{p-1}^{(1)} = J$, and if $V \subseteq R$ is an arbitrary ideal of finite index, then

$$\hat{\delta}_V(x) = \begin{cases} px^2 - x + 1 & V \cong R, \\ (p-1)x + 1 & V \cong J, \end{cases}$$

holds, where the notation is as in Section 5. We finally remark that the zeta function of J can be derived in a completely elementary way from the zeta function of R, for the ring R is a local ring, J is its maximal ideal, and we thus have the formula

$$\zeta_J(s) = p^s\bigl(\zeta_R(s) - 1\bigr).$$

The case $k = 2$

Let $R = R_2 = \mathbb{Z}_p[C_{p^2}]$. Our goal is the computation of $\hat{\delta}_R(x)$ (which gives a formula for $\zeta_R(s)$). Since $R = V_0^{(2)}$, Theorem 6.2 is applicable with $l = 0$.

We therefore find

$$t_R(e; s) = \begin{cases} \bigl(p^p\bigr)^{-s}\zeta_{R_1}(s) & e = 0, \\ \bigl(p^{p-e}\bigr)^{-s}\bigl(\zeta_{V_{p-e}^{(1)}}(s) - p^{-s}\zeta_{V_{p-e+1}^{(1)}}(s)\bigr) & 0 < e \leqslant p. \end{cases} \tag{21}$$

The zeta functions occurring here have been calculated in the preceding subsection. Putting $x := p^{-s}$ and

$$\hat{\delta}_0(x) := px^2 - x + 1, \qquad \hat{\delta}_1(x) := (p-1)x + 1, \tag{22}$$

we can translate (21) into

$$\hat{t}_R(e; x) = \begin{cases} x^p\hat{\delta}_0(x) & e = 0, \\ x^{p-1}\bigl(\hat{\delta}_1(x) - x\hat{\delta}_0(x)\bigr) & e = 1, \\ x^{p-e}(1-x)\hat{\delta}_1(x) & 2 \leqslant e \leqslant p-1, \\ \hat{\delta}_0(x) - x\hat{\delta}_1(x) & e = p, \end{cases}$$

where $\hat{t}_R(e; x)$ is as in (17), and by (18) we get

$$\hat{\delta}_R(x) = \sum_{e=0}^{p} p^{p-e}x^{p-e}\hat{t}_R(e; x).$$

After some trivial transformations we obtain the following polynomial:

$$\begin{aligned} \hat{\delta}_R(x) = {} & p^{p+1}x^{2p+2} - 2p^px^{2p+1} + \bigl(p^p + p^{p-1}\bigr)x^{2p} \\ & + \sum_{i=2}^{p}\bigl(\bigl(p^i - 2p^{i-1}\bigr)x^{2i-1}\bigr) + \sum_{i=2}^{p-1}\bigl(p^{i-1}x^{2i}\bigr) + (p+1)x^2 - 2x + 1. \end{aligned}$$

Note that one may easily verify the functional equation

$$\hat{\delta}_R(x) = p^{p+1}x^{2p+2}\hat{\delta}_R\left(\frac{1}{px}\right)$$

predicted by Corollary 5.3.

The case $k = 3$

We now consider $R_3 = \mathbb{Z}_p[C_{p^3}]$. This time we have to compute

$$\hat{\delta}_{R_3}(x) = \sum_{f=0}^{p^2} p^{p^2-f}x^{p^2-f}\hat{t}_{R_3}(f;x),$$

cf. (18). By Theorem 6.2 we find

$$\hat{t}_{R_3}(f;x) = \begin{cases} x^{p^2}\hat{\delta}_{R_2}(x) & f = 0, \\ x^{p^2-f}\Big(\hat{\delta}_{V^{(2)}_{p^2-f}}(x) - x\cdot\hat{\delta}_{V^{(2)}_{p^2-f+1}}(x)\Big) & 1 \leqslant f \leqslant p^2, \end{cases}$$

using once more the filtration

$$R_2 = V_0^{(2)} \underset{\neq}{\supseteq} \cdots \underset{\neq}{\supseteq} V_{p^2}^{(2)} = pR_2.$$

It remains to determine the polynomials $\hat{\delta}_{V_l^{(2)}}$ for $0 \leqslant l \leqslant p^2$.

For $l = 0$ (as well as for $l = p^2$ because of $R_2 \cong pR_2$) this has been accomplished in the preceding subsection. We recall that

$$\hat{\delta}_{V_l^{(2)}}(x) = \sum_{e=0}^{p} p^{p-e}x^{p-e}\hat{t}_l(e;x),$$

using the abbreviation

$$\hat{t}_l(e;x) := \hat{t}_{V_l^{(2)}}(e;x)$$

and similarly

$$t_l(e;s) := t_{V_l^{(2)}}(e;s)$$

for the rest of this section. Therefore, we "only" have to determine these polynomials $\hat{t}_l(e;x)$ for $0 < l < p^2$.

The cases $1 \leqslant l \leqslant p(p-1)$ are covered by Theorem 6.2:

- If $1 \leqslant l \leqslant p-1$:

$$\hat{t}_l(e;x) = \begin{cases} x^{p-l}\hat{\delta}_0(x) & e=0, \\ x^{p-l-1}\big(\hat{\delta}_1(x) - x\hat{\delta}_0(x)\big) & e=1, \\ x^{p-l-e}(1-x)\hat{\delta}_1(x) & 2 \leqslant e \leqslant p-l, \\ (1-x)\hat{\delta}_1(x) & p-l < e \leqslant p-1, \\ \hat{\delta}_0(x) - x\hat{\delta}_1(x) & e=p. \end{cases}$$

- If $p \leqslant l \leqslant p^2 - p$:

$$\hat{t}_l(e;x) = \begin{cases} \hat{\delta}_0(x) & e=0, \\ \hat{\delta}_1(x) - x\hat{\delta}_0(x) & e=1, \\ (1-x)\hat{\delta}_1(x) & 2 \leqslant e \leqslant p-1, \\ \hat{\delta}_0(x) - x\hat{\delta}_1(x) & e=p. \end{cases}$$

Here $\hat{\delta}_0$, $\hat{\delta}_1$ are the polynomials introduced earlier in (22).

The remaining case $p^2 - p < l < p^2$ is the hardest part. We define

$$l' := l - \big(p^2 - p\big) \in \{1, \ldots, p-1\}$$

and furthermore set $V := V_l^{(2)}$. We have $f_2(V) = pR_1$ and $V^\circ = V_{l'}^{(1)}$ by Lemma 6.1. For the rest of this section, define

$$R := R_1 = \mathbb{Z}_p[\sigma] \quad \text{with } \sigma^p = 1,$$

and $\phi := \sigma^{p-1} + \cdots + \sigma + 1 \in R$. Then $V_{l'}^{(1)} \cong J := \mathrm{rad}(R)$ as we noted in the first subsection. Since we want to apply formula (14) again, in order to calculate $t_l(e;s)$, we first need some information on the lattice of R-submodules of $V^\circ/pV^\circ \cong J/pJ$. Put

$$R/pR \cong \mathbb{F}_p[y]/\big(y^p\big) =: F.$$

Then J/pJ is an F-module satisfying

$$J/pJ \cong \mathbb{F}_p \oplus \mathbb{F}_p[y]/\big(y^{p-1}\big),$$

by the following lemma.

Lemma 6.3.

(a) *The elements p and $\phi - (\sigma - 1)^{p-1}$ are associates in R, as well as $(\sigma - 1)^p$ and $p(\sigma - 1)$.*

(b) *We have*

$$J = \phi R \oplus (\sigma - 1)R,$$

and the surjective R-module homomorphism

$$\begin{aligned} \psi : J &\to \mathbb{F}_p \oplus \mathbb{F}_p[y]/\big(y^{p-1}\big), \\ \phi &\mapsto (1,0), \\ \sigma - 1 &\mapsto (0,1) \end{aligned}$$

has $\ker(\psi) = pJ$*. Moreover,* J *has exactly* $p+1$ *maximal submodules, two being isomorphic to* J *and the* $p-1$ *remaining ones being isomorphic to* R*.*

Proof. (a) Since

$$\phi = \sigma^{p-1} + \cdots + \sigma + 1 \equiv (\sigma - 1)^{p-1} \mod p,$$

there is an element $\alpha \in R$ such that $\alpha p = \phi - (\sigma - 1)^{p-1}$. Note that α must be invertible, as can be seen by applying the augmentation map $\varepsilon : R \to \mathbb{Z}_p$, $\sum_{i=0}^{p-1} \lambda_i \sigma^i \mapsto \sum_{i=0}^{p-1} \lambda_i$ to this equation. Now the first claim follows, and the second one is a direct consequence (multiply both elements by $\sigma - 1$).

(b) Since $J = pR + (\sigma - 1)R$, part (a) yields $J = \phi R \oplus (\sigma - 1)R$. For $a, b \in R$ may write

$$\psi\big(a\phi + b(\sigma - 1)\big) = \big(\pi_1(a), \pi_2(b)\big),$$

with $\pi_1 := \rho_1 \circ \pi$ and $\pi_2 := \rho_2 \circ \pi$. Here π is defined as in (19), and $\rho_1 : F \to \mathbb{F}_p$, $\rho_2 : F \to \mathbb{F}_p[y]/(y^{p-1})$ are the canonical projections. From $\ker(\pi_1) = J$ and $\ker(\pi_2) = pR + (\sigma - 1)^{p-1}R$ we infer $\ker(\psi) = pJ$, using (a) once again. For the statement concerning the maximal submodules of J we quote the proof of [8, Lemma 14]. $\square$

We next have to determine the lattice of F-submodules of $\mathbb{F}_p \oplus \mathbb{F}_p[y]/(y^{p-1})$. To this end we begin by counting the submodules U having $|U| = p^e$ elements ($0 \leqslant e \leqslant p$). For $e = 0$ and $e = p$ there is clearly only one such submodule. For $0 < e < p$, U must be contained in $W := \mathbb{F}_p \oplus (\bar{y}^{p-1-e})$, or more precisely be a maximal submodule of W. There are $p+1$ such submodules, for $W/\mathrm{rad}(W) \cong \mathbb{F}_p \oplus \mathbb{F}_p$.

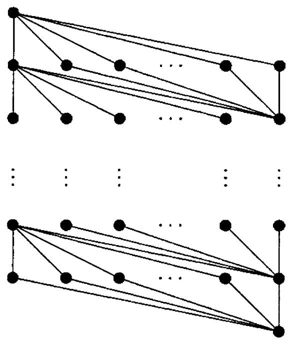
We are now able to state the lattice of submodules of $\mathbb{F}_p \oplus \mathbb{F}_p[y]/(y^{p-1})$:

$$\begin{array}{c|cccccc}
e = p & \mathbb{F}_p \oplus (1) & & & & & \\
e = p-1 & \mathbb{F}_p \oplus (\bar{y}) & \langle(1,1)\rangle_F & \langle(2,1)\rangle_F & \ldots & \langle(p-1,1)\rangle_F & 0 \oplus (1) \\
e = p-2 & \mathbb{F}_p \oplus \big(\bar{y}^2\big) & \langle(1,\bar{y})\rangle_F & \langle(2,\bar{y})\rangle_F & \ldots & \langle(p-1,\bar{y})\rangle_F & 0 \oplus (\bar{y}) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e = 2 & \mathbb{F}_p \oplus \big(\bar{y}^{p-2}\big) & \big\langle\big(1,\bar{y}^{p-3}\big)\big\rangle_F & \big\langle\big(2,\bar{y}^{p-3}\big)\big\rangle_F & \ldots & \big\langle\big(p-1,\bar{y}^{p-3}\big)\big\rangle_F & 0 \oplus \big(\bar{y}^{p-3}\big) \\
e = 1 & \mathbb{F}_p \oplus 0 & \big\langle\big(1,\bar{y}^{p-2}\big)\big\rangle_F & \big\langle\big(2,\bar{y}^{p-2}\big)\big\rangle_F & \ldots & \big\langle\big(p-1,\bar{y}^{p-2}\big)\big\rangle_F & 0 \oplus \big(\bar{y}^{p-2}\big) \\
e = 0 & & & & & & 0
\end{array} \tag{23}$$

In this diagram, the submodules in each row have p^e elements, where e is indicated in the leftmost column. The lattice structure is settled by the following inclusions:

- Each $\mathbb{F}_p \oplus (\bar{y}^i)$ $(i = 0, \ldots, p-1)$ contains all modules occurring in the subsequent row.
- Each $0 \oplus (\bar{y}^i)$ $(i = 0, \ldots, p-1)$ is contained in all modules of the preceding row.

The lattice structure can be visualized as follows:



We will now transfer this diagram to the lattice of R-submodules of J containing pJ. Thus we obviously have to replace each $U \subseteq \mathbb{F}_p \oplus \mathbb{F}_p[y]/(y^{p-1})$ by $\psi^{-1}(U)$ (cf. Lemma 6.3). However, for the applications we have in mind, it suffices to know whether the corresponding module is isomorphic to R or to J (which are the only possibilities, as we noted above). For $i = 0, \ldots, p-1$ we have $\psi((\sigma-1)^{i+1}) = (0, \bar{y}^i)$, and consequently

$$\psi^{-1}\big(\mathbb{F}_p \oplus \big(\bar{y}^i\big)\big) = \phi R \oplus (\sigma-1)^{i+1} R \quad \text{and} \quad \psi^{-1}\big(0 \oplus \big(\bar{y}^i\big)\big) = p\phi R \oplus (\sigma-1)^{i+1} R$$

are non-cyclic R-modules, that is isomorphic to J. The last assertion of Lemma 6.3(b) and diagram (23) now imply the following diagram of isomorphism types:

$$\begin{array}{c|cccccc}
e = p & J & & & & & \\
e = p-1 & J & R & R & \ldots & R & J \\
e = p-2 & J & R & R & \ldots & R & J \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e = 2 & J & R & R & \ldots & R & J \\
e = 1 & J & R & R & \ldots & R & J \\
e = 0 & & & & & & J
\end{array} \tag{24}$$

After this discussion, we return to our original task, viz the calculation of

$$t_l(e; s) = \sum_{\substack{pV_{l'}^{(1)} \subseteq X \subseteq V_{l'}^{(1)} \\ [X : pV_{l'}^{(1)}] = p^e}} [X + pR : X]^{-s} z_{pV_{l'}^{(1)}}(X; s)$$

for $0 \leqslant e \leqslant p$. Since $V_{l'}^{(1)} = (p, (\sigma - 1)^{l'}) = \phi R \oplus (\sigma - 1)^{l'} R \cong J$ by Lemma 3.6(a) we can rewrite this sum as

$$t_l(e; s) = \sum_{\substack{pJ \subseteq X \subseteq J \\ [X:pJ]=p^e}} \bigl[X + \iota(pR) : X\bigr]^{-s} z_{pJ}(X; s), \tag{25}$$

where ι is the R-module isomorphism

$$\iota : V_{l'}^{(1)} = \phi R \oplus (\sigma - 1)^{l'} R \to J = \phi R \oplus (\sigma - 1) R,$$
$$\phi \mapsto \phi, \quad (\sigma - 1)^{l'} \mapsto \sigma - 1.$$

Thus it remains to determine $\iota(pR)$. From Lemma 6.3(a) we infer

$$\iota(pR) = \iota\bigl(\bigl(\phi - (\sigma - 1)^{p-1}\bigr)R\bigr) = \bigl(\phi - (\sigma - 1)^{p-l'}\bigr)R,$$

and applying ψ defined as in Lemma 6.3 yields

$$\psi\iota(pR) = \bigl\langle\bigl(1, -\bar{y}^{p-l'-1}\bigr)\bigr\rangle_F = \bigl\langle\bigl(p - 1, \bar{y}^{p-l'-1}\bigr)\bigr\rangle_F. \tag{26}$$

Now if $pJ \subseteq X \subseteq J$ such that $[X : pJ] = p^e$, we can read off the functions $z_{pJ}(X; s)$ from diagram (24), using formula (15):

- If $e = 0$, then $X = pJ$ and

$$z_{pJ}(X; s) = \zeta_{pJ}(s) = \zeta_J(s).$$

- If $e \geqslant 2$ and $\psi(X) = \mathbb{F}_p \oplus (\bar{y}^{p-e})$, then

$$z_{pJ}(X; s) = \zeta_J(s) - p^{-s}\bigl(2\zeta_J(s) + (p - 1)\zeta_R(s)\bigr) + p \cdot p^{-2s}\zeta_J(s).$$

  In order to compute the μ-factors in formula (15) we may use Theorem 4.4, for $X/pJ \cong \mathbb{F}_p \oplus (\bar{y}^{p-e}) \cong \mathbb{F}_p \oplus \mathbb{F}_p[u]/(u^{e-1}) \cong \mathbb{F}_p \oplus \mathbb{F}_p[\![u]\!]/(u^{e-1})$, i.e., we can apply this theorem with $C := \mathbb{F}_p[\![u]\!]$.
- Otherwise X has a unique maximal submodule lying over pJ, and Theorem 4.2 implies

$$z_{pJ}(X; s) = \begin{cases} \zeta_R(s) - p^{-s}\zeta_J(s) & X \cong R, \\ \bigl(1 - p^{-s}\bigr)\zeta_J(s) & X \cong J. \end{cases}$$

Finally we just have to determine the indices $[X + \iota(pR) : X]$ for $pJ \subseteq X \subseteq J$ occurring in (25). After "shifting" this problem via ψ we may equivalently compute the indices $[U + \psi\iota(pR) : U]$ for $U := \psi(X) \subseteq \mathbb{F}_p \oplus \mathbb{F}_p[y]/(y^{p-1})$. We already know $\psi\iota(pR)$ by

(26), and we record these indices in a diagram corresponding to (23), namely the index $[U + \psi\iota(pR) : U]$ at the very position of U in (23):

$$
\begin{array}{l|cccccc}
e = p & 1 & & & & & \\
e = p-1 & 1 & p & p & \dots & p & p \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e = l'+1 & 1 & p & p & \dots & p & p \\
e = l' & p & p & p & \dots & 1 & p \\
e = l'-1 & p^2 & p^2 & p^2 & \dots & p^2 & p \\
e = l'-2 & p^3 & p^3 & p^3 & \dots & p^3 & p^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e = 1 & p^{l'} & p^{l'} & p^{l'} & \dots & p^{l'} & p^{l'-1} \\
e = 0 & & & & & & p^{l'}
\end{array}
$$

Now we are ready to state the result of the sum $t_l(e; s)$ according to (25). Therefore, we complete the list of polynomials $\hat{t}_l(e; x)$ from the beginning of this subsection for $p^2 - p < l < p^2$. Again we will make use of the polynomials $\hat{\delta}_0(x)$, $\hat{\delta}_1(x)$ defined in (22), and with $l' = l - (p^2 - p)$ we get

- $\hat{t}_l(0; x) = x^{l'}\hat{\delta}_1(x)$.
- $\hat{t}_l(p; x) = (1 - 2x + px^2)\hat{\delta}_1(x) - (p-1)x\hat{\delta}_0(x)$.
- If $l' = 1$:

$$
\hat{t}_l(e; x) = \begin{cases} \big(x - px^2\big)\hat{\delta}_1(x) + \big(1 + (p-2)x\big)\hat{\delta}_0(x) & e = 1, \\ (1-x)\hat{\delta}_1(x) & 2 \leqslant e \leqslant p-1. \end{cases}
$$

- If $l' > 1$:

$$
\hat{t}_l(e; x) = \begin{cases} (p-1)x^{l'}\hat{\delta}_0(x) + \big(x^{l'-1} - px^{l'+1}\big)\hat{\delta}_1(x) & e = 1, \\ (p-1)\big(x^{l'+1-e} - x^{l'+2-e}\big)\hat{\delta}_0(x) & \\ \quad + \big(x^{l'-e} - (p+1)x^{l'+2-e} + px^{l'+3-e}\big)\hat{\delta}_1(x) & 2 \leqslant e \leqslant l'-1, \\ \big(1 + (p-2)x - (p-1)x^2\big)\hat{\delta}_0(x) & \\ \quad + \big(x - (p+1)x^2 + px^3\big)\hat{\delta}_1(x) & e = l', \\ (1-x)\hat{\delta}_1(x) & l'+1 \leqslant e \leqslant p-1. \end{cases}
$$

We conclude this section by some numerical examples. We list the polynomials $\hat{\delta}_{\mathbb{Z}_p[C_{p^3}]}(x)$ for $p \in \{2, 3, 5\}$:

$$
\begin{aligned} \hat{\delta}_{\mathbb{Z}_2[C_8]}(x) = {} & 128x^{14} - 192x^{13} + 160x^{12} - 48x^{11} + 48x^{10} - 8x^9 + 32x^8 \\ & - 12x^7 + 16x^6 - 2x^5 + 6x^4 - 3x^3 + 5x^2 - 3x + 1, \end{aligned}
$$

$$
\begin{aligned}
&\hat{\delta}_{\mathbb{Z}_3[C_{27}]}(x)\\
&\quad = 1594323x^{26} - 1594323x^{25} + 1062882x^{24} - 59049x^{23} + 354294x^{22} - 196830x^{21}\\
&\qquad + 334611x^{20} - 65610x^{19} + 111537x^{18} + 10935x^{17} + 4374x^{16} + 29889x^{15}\\
&\qquad - 5103x^{14} + 9963x^{13} - 1701x^{12} + 3321x^{11} + 162x^{10} + 135x^{9} + 459x^{8}\\
&\qquad - 90x^{7} + 153x^{6} - 30x^{5} + 18x^{4} - x^{3} + 6x^{2} - 3x + 1,\\
&\hat{\delta}_{\mathbb{Z}_5[C_{125}]}(x)\\
&\quad = 465661287307739257812 5x^{62} - 2793967723846435546875x^{61}\\
&\qquad + 1490116119384765625000x^{60} + 3352761268615722656 25x^{59}\\
&\qquad + 670552253723144531250x^{58} - 193715095520019531250x^{57}\\
&\qquad + 350177288055419921875x^{56} + 357627868652343750 00x^{55}\\
&\qquad + 178813934326171875 00x^{54} + 503659248352050781 25x^{53}\\
&\qquad + 18477439880371093750x^{52} - 3576278686523437 50x^{51}\\
&\qquad + 12338161468505859375x^{50} + 2908706665039062500x^{49}\\
&\qquad + 1871585845947265625x^{48} + 1714229583740234375x^{47}\\
&\qquad - 162124633789062500x^{46} + 700473785400390625x^{45}\\
&\qquad - 92029571533203125x^{44} + 140094757080078125x^{43}\\
&\qquad - 18405914306640625x^{42} + 28018951416015625x^{41}\\
&\qquad - 3681182861328125x^{40} + 5603790283203125x^{39}\\
&\qquad - 736236572265625x^{38} + 1120758056640625x^{37} - 147247314453125x^{36}\\
&\qquad + 224151611328125x^{35} - 29449462890625x^{34} + 44830322265625x^{33}\\
&\qquad - 5889892578125x^{32} + 8966064453125x^{31} - 1177978515625x^{30}\\
&\qquad + 1793212890625x^{29} - 235595703125x^{28} + 358642578125x^{27}\\
&\qquad - 47119140625x^{26} + 71728515625x^{25} - 9423828125x^{24} + 14345703125x^{23}\\
&\qquad - 1884765625x^{22} + 2869140625x^{21} - 376953125x^{20} + 573828125x^{19}\\
&\qquad - 75390625x^{18} + 114765625x^{17} - 5312500x^{16} + 11234375x^{15} + 2453425x^{14}\\
&\qquad + 762500x^{13} + 646875x^{12} - 3750x^{11} + 38750x^{10} + 21125x^{9} + 1500x^{8}\\
&\qquad + 600x^{7} + 1175x^{6} - 130x^{5} + 90x^{4} + 9x^{3} + 8x^{2} - 3x + 1.
\end{aligned}
$$

The functional equation

$$\hat{\delta}_{\mathbb{Z}_p[C_{p^3}]}(x) = \left(px^2\right)^{1+p+p^2} \cdot \hat{\delta}_{\mathbb{Z}_p[C_{p^3}]}\left(\frac{1}{px}\right)$$

predicted by Corollary 5.3 is indeed satisfied in all cases (this has been checked as a test of the correctness of the computations).

7. The case $k = 1$ and $n \geqslant 1$

We put $R := R_1 = \mathbb{Z}_p[C_p]$ in this section. Our intention is to compute the zeta function of R^n, or equivalently the polynomial $\hat{\delta}_{R^n}(x)$. In view of the next section we determine more generally $\hat{\delta}_V(x)$, where $V \subseteq R^n$ is a submodule of finite index (as always). We will see below that the zeta function of V only depends on

$$m := \dim_{\mathbb{F}_p}\bigl(f_2(V)/pV^\circ\bigr).$$

Here $f_2 : R^n \to \mathbb{Z}_p^n$ is the map defined in (9), which in fact is the augmentation map in our case. Let V° be defined as in Section 3. Since $f_2(V) \subseteq V^\circ \cong \mathbb{Z}_p^n$ by Lemma 3.2, and since $f_2(V)/pV^\circ$ injects into $V^\circ/pV^\circ \cong \mathbb{F}_p^n$, we have $0 \leqslant m \leqslant n$.

We introduce the following notation.

Notation. We put

$$q := p^{-1}$$

for the rest of this article. For every non-negative integer m we set

$$(q)_m := \prod_{j=1}^{m}\bigl(1 - q^j\bigr).$$

Furthermore, if l, m are non-negative integers, we denote by

$$\begin{bmatrix} m \\ l \end{bmatrix}_p$$

the number of l-dimensional subspaces of an m-dimensional $\mathbb{F}_p$-vector space. It is well-known that in the case $m \geqslant l$ this number equals

$$\frac{(p^m - 1)(p^m - p)\cdots(p^m - p^{l-1})}{(p^l - 1)(p^l - p)\cdots(p^l - p^{l-1})} = p^{l(m-l)}\frac{(q)_m}{(q)_l(q)_{m-l}}.$$

The following combinatorial lemma is due to Cauchy (cf. [3, III.8.5] for a proof).

Lemma 7.1. *The following identity of polynomials holds*:

$$\prod_{l=0}^{m-1}\bigl(1 - p^l x\bigr) = \sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_p (-1)^i p^{\binom{i}{2}} x^i.$$

Lemma 7.2. *Let $Y \subseteq \mathbb{F}_p^n$ be a subspace with* $\dim(Y) = m$.

(a) *For $l \in \{0, \dots, n\}$ let $\chi(l)$ be the number of subspaces $U \subseteq \mathbb{F}_p^n$ with* $\dim(U) = l$ *and* $U \cap Y = 0$. *Then*

$$\chi(l) = p^{ml} \begin{bmatrix} n-m \\ l \end{bmatrix}_p,$$

in particular, $\chi(l) = 0$ if $l > n - m$.

(b) *If $e \in \{0, \dots, n\}$, then*

$$\sum_{\substack{X \subseteq \mathbb{F}_p^n \\ \dim(X)=e}} [X+Y:X]^{-s} = \sum_{f=\max\{0,e+m-n\}}^{\min\{e,m\}} \begin{bmatrix} m \\ f \end{bmatrix}_p \begin{bmatrix} n-m \\ e-f \end{bmatrix}_p p^{(m-f)(e-f)} \left(p^{m-f}\right)^{-s}.$$

Proof. (a) Comparing dimensions clearly implies $\chi(l) = 0$ if $l + m > n$. Hence we can restrict ourselves to the case $l + m \leqslant n$. Fix a basis $y_1, \dots, y_m$ of Y. We begin by counting the number of possibilities to choose l linearly independent vectors $u_1, \dots, u_l \in \mathbb{F}_p^n$ satisfying

$$\langle y_1, \dots, y_m \rangle \cap \langle u_1, \dots, u_l \rangle = 0.$$

For u_1 there are $p^n - p^m$ possibilities, because $u_1 \notin \langle y_1, \dots, y_m \rangle$. Since u_2 has to be linearly independent of u_1, we must have $u_2 \notin \langle y_1, \dots, y_m, u_1 \rangle$, hence there are $p^n - p^{m+1}$ choices for u_2, and so on. Finally there are $p^n - p^{m+l-1}$ choices for u_l because of $u_l \notin \langle y_1, \dots, y_m, u_1, \dots, u_{l-1} \rangle$.

On the other hand, a subspace $U \subseteq \mathbb{F}_p^n$ of dimension l has exactly $|\mathrm{GL}_l(\mathbb{F}_p)| = (p^l - 1) \cdots (p^l - p^{l-1})$ distinct bases. Hence

$$\chi(l) = \frac{(p^n - p^m) \cdots (p^n - p^{m+l-1})}{(p^l - 1) \cdots (p^l - p^{l-1})} = p^{ml} \begin{bmatrix} n-m \\ l \end{bmatrix}_p.$$

(b) We have

$$\begin{aligned}
&\sum_{\substack{X \subseteq \mathbb{F}_p^n \\ \dim(X)=e}} [X+Y:X]^{-s} \\
&= \sum_{\substack{X \subseteq \mathbb{F}_p^n \\ \dim(X)=e}} [Y : X \cap Y]^{-s} = \sum_{f=0}^{\min\{e,m\}} \sum_{\substack{X \subseteq \mathbb{F}_p^n \\ \dim(X)=e \\ \dim(X\cap Y)=f}} \left(p^{m-f}\right)^{-s} \\
&= \sum_{f=0}^{\min\{e,m\}} \sum_{\substack{X_0 \subseteq Y \\ \dim(X_0)=f}} \left|\left\{X \subseteq \mathbb{F}_p^n \mid \dim(X) = e \text{ and } X \cap Y = X_0\right\}\right| \left(p^{m-f}\right)^{-s}.
\end{aligned}$$

Fix a subspace $X_0 \subseteq Y$ such that $\dim(X_0) = f$, and put $\overline{Y} := Y/X_0$. Then:

$$\begin{aligned}
&\bigl|\bigl\{X \subseteq \mathbb{F}_p^n \mid \dim(X) = e \text{ and } X \cap Y = X_0\bigr\}\bigr| \\
&\quad = \bigl|\bigl\{\overline{X} \subseteq \mathbb{F}_p^n/X_0 \cong \mathbb{F}_p^{n-f} \mid \dim(\overline{X}) = e - f \text{ and } \overline{X} \cap \overline{Y} = 0\bigr\}\bigr| \\
&\quad = p^{(m-f)(e-f)} \begin{bmatrix} n-m \\ e-f \end{bmatrix}_p,
\end{aligned}$$

where the last equation follows from part (a). Since there are $\left[\begin{smallmatrix} m \\ f \end{smallmatrix}\right]_p$ subspaces $X_0 \subseteq Y$ of dimension f, we get

$$\sum_{\substack{X \subseteq \mathbb{F}_p^n \\ \dim(X)=e}} [X+Y : X]^{-s} = \sum_{f=0}^{\min\{e,m\}} \begin{bmatrix} m \\ f \end{bmatrix}_p \begin{bmatrix} n-m \\ e-f \end{bmatrix}_p p^{(m-f)(e-f)} \bigl(p^{m-f}\bigr)^{-s}.$$

Note that $\left[\begin{smallmatrix} n-m \\ e-f \end{smallmatrix}\right]_p \neq 0$ only if $e - f \leqslant n - m$, so the summation may start with $f = \max\{0, e+m-n\}$, and the proof is complete. $\square$

Lemma 7.3. *Let $pV^\circ \subseteq \overline{X} \subseteq V^\circ$ with $[\overline{X} : pV^\circ] = p^e$, $0 \leqslant e \leqslant n$. Then*

$$z_{pV^\circ}(\overline{X}; s) = \prod_{j=e}^{n-1} \bigl(1 - p^{j-s}\bigr)^{-1}.$$

Proof. Since $V^\circ \cong \mathbb{Z}_p^n$, formula (15) implies

$$z_{pV^\circ}(\overline{X}; s) = \sum_{p\mathbb{Z}_p^n \subseteq \overline{W} \subseteq \overline{X}'} \mu(\overline{W}, \overline{X}')[\overline{X}' : \overline{W}]^{-s} \zeta_{\overline{W}}(s),$$

where $p\mathbb{Z}_p^n \subseteq \overline{X}' \subseteq \mathbb{Z}_p^n$ with $[\overline{X}' : p\mathbb{Z}_p^n] = p^e$. We also have $\overline{W} \cong \mathbb{Z}_p^n$, and thus by Theorem 3.9:

$$\zeta_{\overline{W}}(s) = \zeta_{\mathbb{Z}_p^n}(s) = \prod_{j=0}^{n-1} \bigl(1 - p^{j-s}\bigr)^{-1}.$$

Denote by $\tilde{\mu}$ the Möbius function of the lattice of subspaces of $\mathbb{F}_p^e$. Using $\overline{X}'/p\mathbb{Z}_p^n \cong \mathbb{F}_p^e$ we get

$$\sum_{p\mathbb{Z}_p^n \subseteq \overline{W} \subseteq \overline{X}'} \mu(\overline{W}, \overline{X}')[\overline{X}' : \overline{W}]^{-s}$$

$$= \sum_{U \subseteq \mathbb{F}_p^e} \tilde{\mu}\big(U, \mathbb{F}_p^e\big)\big[\mathbb{F}_p^e : U\big]^{-s} = \sum_{i=0}^{e} \begin{bmatrix} e \\ i \end{bmatrix}_p (-1)^{e-i} p^{\binom{e-i}{2}} p^{-(e-i)s}$$

$$= \sum_{i=0}^{e} \begin{bmatrix} e \\ i \end{bmatrix}_p (-1)^i p^{\binom{i}{2}} p^{-is} = \prod_{j=0}^{e-1} \big(1 - p^{j-s}\big),$$

by Lemma 7.1. Here we made use of the fact that $\tilde{\mu}(U, \mathbb{F}_p^e) = (-1)^{e-i} p^{\binom{e-i}{2}}$, if $U \subseteq \mathbb{F}_p^e$ is a subspace such that $\dim(U) = i$ (cf. Theorem 4.3). Now multiplying by $\zeta_{\mathbb{Z}_p^n}(s)$ proves the assertion. $\square$

The above results allow us to compute $t_V(e; s)$ as in (14) for $0 \leqslant e \leqslant n$:

$$t_V(e; s) = \Bigg(\sum_{\substack{pV^\circ \subseteq \overline{X} \subseteq V^\circ \\ [\overline{X} : pV^\circ] = p^e}} [\overline{X} + f_2(V) : \overline{X}]^{-s} \Bigg) \prod_{j=e}^{n-1} \big(1 - p^{j-s}\big)^{-1}$$

$$= \Bigg(\sum_{f=\max\{0, e+m-n\}}^{\min\{e,m\}} \begin{bmatrix} m \\ f \end{bmatrix}_p \begin{bmatrix} n-m \\ e-f \end{bmatrix}_p p^{(m-f)(e-f)} \big(p^{m-f}\big)^{-s} \Bigg)$$

$$\cdot \prod_{j=e}^{n-1} \big(1 - p^{j-s}\big)^{-1},$$

because $m = \dim(f_2(V)/pV^\circ)$. Now (18) implies

$$\hat{\delta}_V(x) = \sum_{e=0}^{n} p^{n(n-e)} x^{n-e} \hat{t}_V(e; x)$$

with

$$\hat{t}_V(e; x) = \prod_{j=0}^{e-1} \big(1 - p^j x\big) \sum_{f=\max\{0, e+m-n\}}^{\min\{e,m\}} \begin{bmatrix} m \\ f \end{bmatrix}_p \begin{bmatrix} n-m \\ e-f \end{bmatrix}_p p^{(m-f)(e-f)} x^{m-f}.$$

We recall that $R = R_1 = \mathbb{Z}_p[C_p]$, hence $f_2(R^n) = (R^n)^\circ = \mathbb{Z}_p^n$ with the notations of Section 3. Thus $m = n$ for $V = R^n$, and we have the following formula.

Theorem 7.4. *If $R = R_1 = \mathbb{Z}_p[C_p]$, then*

$$\hat{\delta}_{R^n}(x) = \sum_{e=0}^{n} \Bigg(\begin{bmatrix} n \\ e \end{bmatrix}_p p^{n(n-e)} x^{2(n-e)} \prod_{j=0}^{e-1} \big(1 - p^j x\big) \Bigg).$$

8. The case $k = 2$ and $n \geqslant 1$

In this section we will compute the zeta function of $R_2^n (R_2 = \mathbb{Z}_p[C_{p^2}])$, i.e., the polynomial,

$$\hat{\delta}_{R_2^n}(x) = \sum_{e=0}^{np} p^{n(np-e)} x^{np-e} \hat{t}_{R_2^n}(e; x),$$

where $\hat{t}_{R_2^n}(e; x)$ is defined in (17) by means of (cf. (14))

$$t_{R_2^n}(e; s) \sum_{\substack{pR^n \subseteq X \subseteq R^n \\ [X : pR^n] = p^e}} \left[R^n : X\right]^{-s} z_{pR^n}(X; s) = \left(p^{np-e}\right)^{-s} \sum_{\substack{pR^n \subseteq X \subseteq R^n \\ [X : pR^n] = p^e}} z_{pR^n}(X; s).$$

Here we put

$$R := R_1 = \mathbb{Z}_p[C_p],$$

and we will keep this notation for the rest of this section. It therefore remains to determine the sums

$$t'(e; s) := \sum_{\substack{pR^n \subseteq X \subseteq R^n \\ [X : pR^n] = p^e}} z_{pR^n}(X; s)$$

for $0 \leqslant e \leqslant np$. Utilizing (15) yields

$$\begin{aligned} t'(e; s) &= \sum_{\substack{pR^n \subseteq X \subseteq R^n \\ [X : pR^n] = p^e}} \sum_{pR^n \subseteq V \subseteq X} \mu(V, X)[X : V]^{-s} \zeta_V(s) \\ &= \sum_{pR^n \subseteq V \subseteq R^n} \left(\sum_{\substack{V \subseteq X \subseteq R^n \\ [X : pR^n] = p^e}} \mu(V, X)[X : V]^{-s} \right) \zeta_V(s). \end{aligned} \tag{27}$$

The functions $\zeta_V(s)$ have been calculated in the preceding section, and we recall that they only depend on

$$m(V) := \dim_{\mathbb{F}_p}\left(f_2(V)/pV^\circ\right),$$

where again $f_2 : R^n \to \mathbb{Z}_p^n$ and V° are defined as in Section 3.

Fix $pR^n \subseteq V \subseteq R^n$. We will first evaluate the parenthetical expression in (27) belonging to V. This computation can be performed in R^n/pR^n. More precisely, let

$$F := \mathbb{F}_p[y]/\left(y^p\right) \cong R/pR,$$

and let $U \subseteq F^n$ be the image of V under the canonical projection $\pi : R^n \to F^n$. Then

$$\sum_{\substack{V\subseteq X\subseteq R^n\\ [X:pR^n]=p^e}} \mu(V,X)[X:V]^{-s} = \sum_{\substack{U\subseteq Y\subseteq F^n\\ |Y|=p^e}} \mu(U,Y)[Y:U]^{-s}, \tag{28}$$

where the Möbius function of the lattice of submodules of F^n is denoted by μ as well. There is an F-module isomorphism

$$U \cong \mathbb{F}_p[y]/\big(y^{r_1}\big) \oplus \cdots \oplus \mathbb{F}_p[y]/\big(y^{r_n}\big),$$

and the integers $0 \leqslant r_1 \leqslant \cdots \leqslant r_n \leqslant p$ are uniquely determined. $(r_1, \ldots, r_n)$ will be referred to as the *isomorphism type* of U.

Lemma 8.1. *Let $pR^n \subseteq V \subseteq R^n$ and let $U \subseteq F^n$ be the image of V under the projection $\pi : R^n \to F^n$. Let $(r_1, \ldots, r_n)$ be the isomorphism type of U. Then*

$$m(V) = |\{1 \leqslant i \leqslant n \mid r_i = 0 \text{ or } r_i = p\}|.$$

Proof. By definition $m(V) = \dim(f_2(V)/pV^\circ) = \dim(f_2(V)/p\mathbb{Z}_p^n) + \dim(p\mathbb{Z}_p^n/pV^\circ)$, and we write

$$U = \big(\bar{y}^{p-r_1}\big) \oplus \cdots \oplus \big(\bar{y}^{p-r_n}\big) \subseteq F^n$$

in the sequel. If we put $K = (\bar{y}) \oplus \cdots \oplus (\bar{y}) \subseteq F^n$, then obviously

$$f_2(V)/p\mathbb{Z}_p^n \cong \big(\pi(V) + K\big)/K \cong (U + K)/K,$$

and thus $\dim(f_2(V)/p\mathbb{Z}_p^n) = |\{i \mid r_i = p\}|$.

In addition we have

$$p\mathbb{Z}_p^n/pV^\circ = f_2\big(\phi R^n\big)\big/ f_2\big(V \cap \phi R^n\big) \cong \big(\pi(V) + \pi\big(\phi R^n\big)\big)/\pi(V)$$

with $\phi = \sigma^{p-1} + \cdots + \sigma + 1$ (cf. proof of Lemma 6.1). From $\pi(V) = U$ and $\pi(\phi R^n) = (\bar{y}^{p-1}) \oplus \cdots \oplus (\bar{y}^{p-1})$ we infer $\dim(p\mathbb{Z}_p^n/pV^\circ) = |\{i \mid r_i = 0\}|$, and the lemma is proved. $\square$

The next step is the computation of the sum (28). For the rest of this section, we continue to use the notation introduced at the beginning of Section 7.

Lemma 8.2. *Let $U \subseteq F^n$ be a submodule of isomorphism type $(r_1, \ldots, r_n)$. Set $n' := |\{1 \leqslant i \leqslant n \mid r_i \neq p\}|$ and $f := r_1 + \cdots + r_n$. Then the following holds for $e \geqslant f$:*

$$\sum_{\substack{U\subseteq Y\subseteq F^n\\ |Y|=p^e}} \mu(U,Y)[Y:U]^{-s} = \begin{bmatrix} n' \\ e-f \end{bmatrix}_p (-1)^{e-f} p^{\binom{e-f}{2}} \big(p^{e-f}\big)^{-s},$$

in particular $= 0$ *if* $e > f + n'$.

Proof. The sum extends over all $Y \supseteq U$ such that $[Y : U] = |Y|/|U| = p^{e-f}$. Hence it suffices to calculate

$$\sum_{\substack{U \subseteq Y \subseteq F^n \\ |Y| = p^e}} \mu(U, Y) = \sum_{\substack{U \subseteq Y \subseteq F^n \\ |Y| = p^e}} \bar{\mu}(0, Y/U) = \sum_{\substack{H \subseteq F^n/U \\ |H| = p^{e-f}}} \bar{\mu}(0, H),$$

where $\bar{\mu}$ is the Möbius function of the lattice of submodules of F^n/U. Now

$$\begin{aligned} F^n/U &\cong \mathbb{F}_p[y]/\left(y^{p-r_1}\right) \oplus \cdots \oplus \mathbb{F}_p[y]/\left(y^{p-r_n}\right) \\ &\cong \mathbb{F}_p[y]/\left(y^{p-r_1}\right) \oplus \cdots \oplus \mathbb{F}_p[y]/\left(y^{p-r_{n'}}\right) \quad \text{by definition of } n' \\ &\cong \mathbb{F}_p[\![y]\!]/\left(y^{p-r_1}\right) \oplus \cdots \oplus \mathbb{F}_p[\![y]\!]/\left(y^{p-r_{n'}}\right). \end{aligned}$$

Applying Theorem 4.4 yields

$$\sum_{\substack{U \subseteq Y \subseteq F^n \\ |Y| = p^e}} \mu(U, Y) = \sum_{\substack{H \subseteq \mathbb{F}_p^{n'} \\ |H'| = p^{e-f}}} \tilde{\mu}(0, H') = \begin{bmatrix} n' \\ e-f \end{bmatrix}_p (-1)^{e-f} p^{\binom{e-f}{2}},$$

$\tilde{\mu}$ being the Möbius function of the lattice of subspaces of $\mathbb{F}_p^{n'}$, and the claim follows. □

We briefly review the steps made towards the computation of (27). Instead of summing over $pR^n \subseteq V \subseteq R^n$, we sum over all submodules $U \subseteq F^n$. Because of the preceding lemma, the parenthetical expression in (27) only depends on the isomorphism type of U, as well as the zeta function of V by Lemma 8.1. Accordingly we only require the number of submodules $\subseteq F^n$ of given isomorphism type. This problem is considered in the following theorem (for a proof, see [2, Theorems 2.10, 2.11, Proposition 3.2]).

Theorem 8.3. *Let C be a discrete valuation ring with prime element π and residue class field $\mathbb{F}_p$. Let $a_1 \leqslant \cdots \leqslant a_n$ be positive integers, to be partitioned as follows*:

$$\begin{array}{ccccc} a_1 & = \cdots = & a_{k_1} & =: & u_1, \\ a_{k_1+1} & = \cdots = & a_{k_1+k_2} & =: & u_2, \\ \vdots & & \vdots & & \vdots \\ a_{k_1+\cdots+k_{d-1}+1} & = \cdots = & a_{k_1+\cdots+k_d} & =: & u_d \end{array}$$

such that $u_1 < \cdots < u_d$ and $k_1 + \cdots + k_d = n$ (all $k_i \geqslant 1$). There are precisely

$$\nu\begin{pmatrix} u_1 \cdots u_d \\ k_1 \cdots k_d \end{pmatrix} := \frac{(q)_n}{(q)_{k_1} \cdots (q)_{k_d}} p^{\sum_{i<j} k_i k_j (u_j - u_i)}$$

submodules $U \subseteq C^n$ satisfying

$$C^n/U \cong C/\left(\pi^{a_1}\right) \oplus \cdots \oplus C/\left(\pi^{a_n}\right).$$

For given integers $0 \leqslant r_1 \leqslant \cdots \leqslant r_n \leqslant p$ we will apply this theorem with $C := \mathbb{F}_p[\![y]\!]$ and $a_i := p - r_{n-i}$ $(i = 0, \ldots, n)$. We have indeed $0 \leqslant a_1 \leqslant \cdots \leqslant a_n \leqslant p$, and after defining u_i and k_i as in the statement of the theorem, the submodules $U \subseteq F^n$ of isomorphism type $(r_1, \ldots, r_n)$ correspond to the submodules $U' \subseteq (\mathbb{F}_p[\![y]\!])^n$ such that

$$(\mathbb{F}_p[\![y]\!])^n/U' \cong \mathbb{F}_p[\![y]\!]/\left(y^{a_1}\right) \oplus \cdots \oplus \mathbb{F}_p[\![y]\!]/\left(y^{a_n}\right),$$

and there are $\nu\binom{u_1 \cdots u_d}{k_1 \cdots k_d}$ of those.

We are now able to state the result of the sums $t'(e; s)$ from (27).

Theorem 8.4. *Let $0 \leqslant e \leqslant np$ be an integer. Then*

$$t'(e; s) = \sum_{d=1}^{n} \sum_{\substack{0 \leqslant u_1 < \cdots < u_d \leqslant p \\ k_1 + \cdots + k_d = n \\ f \leqslant e \leqslant f + n'}} \nu\begin{pmatrix} u_1 \cdots u_d \\ k_1 \cdots k_d \end{pmatrix} \begin{bmatrix} n' \\ e - f \end{bmatrix}_p (-1)^{e-f} p^{\binom{e-f}{2}} \left(p^{e-f}\right)^{-s} \zeta_m(s),$$

where f, n' and m depend on u_i, k_i, viz:

$$f = np - \sum_{i=1}^{d} k_i u_i,$$

$$n' = \begin{cases} n - k_1 & u_1 = 0, \\ n & \textit{otherwise}, \end{cases}$$

$$m = \begin{cases} k_1 + k_d & u_1 = 0 \textit{ and } u_d = p, \\ k_1 & u_1 = 0 \textit{ and } u_d < p, \\ k_d & u_1 > 0 \textit{ and } u_d = p, \\ 0 & \textit{otherwise}. \end{cases}$$

Furthermore

$$\zeta_m(s) := \zeta_V(s)$$

for any submodule $V \subseteq R^n$ with $m(V) = m$ (cf. preceding section).

From this theorem we immediately infer the polynomials $\hat{t}_{R_2^n}(e; x)$ and $\hat{\delta}_{R_2^n}(x)$, and hence the computation of the zeta function of R^n is complete in this case.

We conclude this section by some numerical examples: the polynomials $\hat{\delta}_{\mathbb{Z}_p[C_{p^2}]^n}(x)$ for $p \in \{2, 3, 5\}$ and $n \in \{1, 2, 3\}$.

$$\begin{aligned}
\hat{\delta}_{\mathbb{Z}_2[C_4]}(x) &= 8x^6 - 8x^5 + 6x^4 + 3x^3 - 2x + 1, \\
\hat{\delta}_{\mathbb{Z}_2[C_4]^2}(x) &= 4096x^{12} - 6144x^{11} + 6400x^{10} - 2304x^9 + 2816x^8 - 2304x^7 + 195x^6 \\
&\quad - 576x^5 + 176x^4 - 36x^3 + 25x^2 - 6x + 1, \\
\hat{\delta}_{\mathbb{Z}_2[C_4]^3}(x) &= 134217728x^{13} - 234881024x^{17} + 278921216x^{16} - 143654912x^{15} \\
&\quad + 136708096x^{14} - 110100480x^{13} + 102023168x^{12} - 43696128x^{11} \\
&\quad + 17389568x^{10} - 4376576x^9 + 2173696x^8 - 682752x^7 + 199264x^6 \\
&\quad - 26880x^5 + 4172x^4 - 548x^3 + 133x^2 - 14x + 1, \\
\hat{\delta}_{\mathbb{Z}_3[C_9]}(x) &= 81x^8 - 54x^7 + 36x^6 + 9x^5 + 3x^4 + 3x^3 + 4x^2 - 2x + 1, \\
\hat{\delta}_{\mathbb{Z}_3[C_9]^2}(x) &= 43046721x^{16} - 38263752x^{15} + 30823578x^{14} + 708588x^{13} \\
&\quad + 1535274x^{12} + 3385476x^{11} + 2119203x^{10} - 1355940x^9 + 1167129x^8 \\
&\quad - 150660x^7 + 26163x^6 + 4644x^5 + 234x^4 + 12x^3 + 58x^2 - 8x + 1,
\end{aligned}$$

$$\begin{aligned}
&\hat{\delta}_{\mathbb{Z}_3[C_9]^3}(x) \\
&\quad = 150094635296999121x^{24} - 144535574730443598x^{23} + 123122896992600102x^{22} \\
&\qquad - 5467553396735679x^{21} + 6377541363277461x^{20} + 14396280763046013x^{19} \\
&\qquad + 6581267202259089x^{18} - 4740422864535540x^{17} + 4916296269721980x^{16} \\
&\qquad - 868287187367289x^{15} + 200488295095218x^{14} + 15476501507688x^{13} \\
&\qquad + 777744240183x^{12} + 573203759544x^{11} + 275018237442x^{10} \\
&\qquad - 44113559283x^9 + 9250878780x^8 - 330368220x^7 + 16987401x^6 \\
&\qquad + 1376271x^5 + 22581x^4 - 717x^3 + 598x^2 - 26x + 1,
\end{aligned}$$

$$\begin{aligned}
\hat{\delta}_{\mathbb{Z}_5[C_{25}]}(x) &= 15625x^{12} - 6250x^{11} + 3750x^{10} + 1875x^9 + 125x^8 \\
&\quad + 375x^7 + 25x^6 + 75x^5 + 5x^4 + 15x^3 + 6x^2 - 2x + 1, \\
\hat{\delta}_{\mathbb{Z}_5[C_{25}]^2}(x) &= 59604644775390625x^{24} - 28610229492187500x^{23} \\
&\quad + 18692016601562500x^{22} + 7209777832031250x^{21} \\
&\quad + 7629394531250x^{20} + 2210998535156250x^{19} \\
&\quad - 56915283203125x^{18} + 450073242187500x^{17} \\
&\quad - 12957763671875x^{16} + 90329589843750x^{15} \\
&\quad + 20233642578125x^{14} - 6640722656250x^{13} \\
&\quad + 5911884765625x^{12} - 265628906250x^{11} \\
&\quad + 32373828125x^{10} + 5781093750x^9 - 33171875x^8 + 46087500x^7 \\
&\quad - 233125x^6 + 362250x^5 + 50x^4 + 1890x^3 + 196x^2 - 12x + 1,
\end{aligned}$$

$$
\begin{aligned}
&\hat{\delta}_{\mathbb{Z}_5[C_{25}]^3}(x) \\
&\quad = 555111512312578270211815834045410015625x^{36} \\
&\qquad - 27533531010703882202506065368652343750x^{35} \\
&\qquad + 1828226459110737778246402740478515625 0x^{34} \\
&\qquad + 66582117597135948017239570617675781 25x^{33} \\
&\qquad - 844067926664138212800025939941 40625x^{32} \\
&\qquad + 222914991354628000408411026000976 5625x^{31} \\
&\qquad - 971346707956399768590927124023437 50x^{30} \\
&\qquad + 4569092288875253871083259582519531 25x^{29} \\
&\qquad - 207712633709888905286788940429687 50x^{28} \\
&\qquad + 916390800557564944028854370117187 50x^{27} \\
&\qquad + 178267331968527287244796752929687 50x^{26} \\
&\qquad - 58045001490972936153411865234375 00x^{25} \\
&\qquad + 59192114890553057193756103515625 00x^{24} \\
&\qquad - 32667064666282385587692260742187 5x^{23} \\
&\qquad + 48173992687091231346130371093750x^{22} \\
&\qquad + 6197919498234987258911132812500x^{21} \\
&\qquad - 475388055503368377685546875 00x^{20} \\
&\qquad + 659950478899478912353515625 00x^{19} - 5604645519137382507324218 75x^{18} \\
&\qquad + 52796038311958312988281250 0x^{17} - 30424835552215576171875 00x^{16} \\
&\qquad + 31733347830963134765625 00x^{15} + 1973206740463256835937 50x^{14} \\
&\qquad - 107043437498474121093 75x^{13} + 15516857765869140625 00x^{12} \\
&\qquad - 121729190966796875 00x^{11} + 2990829534179687 50x^{10} \\
&\qquad + 122995891210937 50^{9} - 22302974218750x^{8} + 3924820390625x^{7} \\
&\qquad - 6675043750x^{6}1225488125x^{5} - 371225x^{4} + 234265x^{3} + 5146x^{2} - 62x + 1.
\end{aligned}
$$

Note that we checked in each case the validity of the functional equation

$$
\hat{\delta}_{\mathbb{Z}_p[C_{p^2}]^n}(x) = \left(p^{n^2}x^{2n}\right)^{1+p} \cdot \hat{\delta}_{\mathbb{Z}_p[C_{p^2}]^n}\left(\frac{1}{p^n x}\right)
$$

predicted by Corollary 5.3.

Acknowledgment

This article is part of my doctoral thesis under the supervision of Prof. Cornelius Greither. I am grateful to him for many valuable suggestions.

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