

Prime filtrations of monomial ideals and polarizations

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Abstract

We show that an arbitrary monomial ideal I is pretty clean if and only if its polarization I^P is clean. This yields a new characterization of pretty clean monomial ideals in terms of the arithmetic degree, and it also implies that a multicomplex is shellable if and only if the simplicial complex corresponding to its polarization is (non-pure) shellable. We also discuss Stanley decompositions in relation to prime filtrations.

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Introduction

Let R be a Noetherian ring, and M a finitely generated R -module. A basic fact in commutative algebra [9, Theorem 6.4] says that there exists a finite filtration

$$\mathcal{F}: \quad 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

with cyclic quotients $M_i/M_{i-1} \cong R/P_i$ and $P_i \in \text{Supp}(M)$. We call any such filtration of M a prime filtration. The set of prime ideals P_1, \dots, P_r which define the cyclic quotients of \mathcal{F} will be denoted by $\text{Supp}(\mathcal{F})$. Another basic fact [9, Theorem 6.5] implies that $\text{Ass}(M) \subset \text{Supp}(\mathcal{F}) \subset \text{Supp}(M)$. Let $\text{Min}(M)$ denote the set of minimal prime ideals in $\text{Supp}(M)$. Dress [4] calls a prime filtration \mathcal{F} of M *clean* if $\text{Supp}(\mathcal{F}) = \text{Min}(M)$. The R -module M is called *clean* if it admits a clean filtration.

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Herzog and Popescu [6] introduced the concept of *pretty clean modules*. A prime filtration

$$\mathcal{F}: 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M with $M_i/M_{i-1} \cong R/P_i$ is called *pretty clean*, if for all $i < j$ for which $P_i \subseteq P_j$ it follows that $P_i = P_j$. In other words, a proper inclusion $P_i \subset P_j$ is only possible if $i > j$. The module M is called *pretty clean*, if it has a pretty clean filtration. We say an ideal $I \subset R$ is pretty clean if R/I is pretty clean.

A prime filtration which is pretty clean has the nice property that $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$, see [6, Corollary 3.6]. It is still an open problem to characterize the modules which have a prime filtration \mathcal{F} with $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$, see [5, p. 93]. In Section 4 we give an example of a module which is not pretty clean but nevertheless has a prime filtration whose support coincides with the set of associated prime ideals of M .

Dress showed [4] that a simplicial complex is shellable if and only if its Stanley–Reisner ideal is clean, and Herzog and Popescu generalized this result by showing that the multicomplex associated with a monomial ideal I is shellable if and only if I is pretty clean. The main result of this paper is Theorem 4.3 which shows for a monomial ideal $I \subset S$ the following conditions are equivalent:

- (a) I is pretty clean;
- (b) I^p , the polarization of I is clean;
- (c) there exists a prime filtration \mathcal{F} of I with $\ell(\mathcal{F}) = \text{adeg}(I)$;
- (d) Γ , the multicomplex associated to I is shellable;
- (e) if Δ be the simplicial complex associated to I^p , then Δ is shellable.

In the first section of this paper we show that all monomial ideals in $K[x_1, \dots, x_n]$ of height $\geq n - 1$ are pretty clean and use this fact to show that any monomial ideal in the polynomial ring in three variables is pretty clean; see Proposition 1.7 and Theorem 1.10. However for all $n \geq 4$ there exists a monomial ideal of height 2 which is not pretty clean, see Example 1.11.

In Section 2 we discuss the Stanley conjecture concerning Stanley decompositions. In [6, Theorem 6.5] it was shown that the Stanley conjecture holds for any pretty clean monomial ideal. Therefore using the results of Section 1 we recover the result of Apel [1, Theorem 5.1] that the Stanley conjecture holds for any monomial ideal in the polynomial ring in three variables. Similarly we conclude that the Stanley conjecture holds for any monomial ideal of codimension 1.

We notice (Proposition 2.2) that for a monomial ideal, instead of requiring that I is pretty clean, it suffice to require that there exists a prime filtration \mathcal{F} with $\text{Ass}(S/I) = \text{Supp}(\mathcal{F})$ in order to conclude that the Stanley conjecture holds for S/I .

Unfortunately it is not true that each Stanley decomposition corresponds to a prime filtration as shown by an example of MacLagan and Smith [8, Example 3.8]. However we characterize in Proposition 2.7 those Stanley decomposition of S/I that correspond to prime filtrations. Using this characterization we show in Corollary 2.8 that in the polynomial ring in two variables Stanley decompositions and prime filtrations are in bijective correspondence.

In Section 3 we prove that a monomial ideal I is pretty clean if and only if its polarization, I^p is clean. One important step in the proof (see Proposition 3.8) is to show that there is a bijection between the facets of the multicomplex defined by the monomial ideal I and the facets of the simplicial complex defined by the polarization of I , this shows that I and I^p have the same arithmetic degree.

The final section is devoted to prove the new characterization of pretty clean monomial ideals in terms of the arithmetic degree.

1. Pretty clean monomial ideals and multicomplexes

We denote by $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K . Let $I \subset S$ be a monomial ideal. In this paper a prime filtration of I is always assumed to be a monomial prime filtration. This means a prime filtration

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

with $I_j/I_{j-1} \cong S/P_j$, for $j = 1, \dots, r$ such that all I_j are monomial ideals.

Recall that the prime filtration \mathcal{F} is called *pretty clean*, if for all $i < j$ which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The monomial ideal I is called *pretty clean*, if it has a pretty clean filtration.

In this section we will show that monomial ideals in at most three variables are pretty clean. Let $I \subset S$ be a monomial ideal. The saturation \tilde{I} of I is defined to be

$$\tilde{I} = I : \mathfrak{m}^\infty = \bigcup_k (I : \mathfrak{m}^k),$$

where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S .

We first note the following

Lemma 1.1. *Let $I \subset S$ be a monomial ideal of S . The ideal I is pretty clean if and only if \tilde{I} is pretty clean.*

Proof. The K -vector space \tilde{I}/I has a finite dimension, and we can choose monomials $u_1, \dots, u_t \in \tilde{I}$ whose residue classes modulo I form a K -basis of \tilde{I}/I . Moreover the basis can be chosen such that for all $j = 1, \dots, t$ one has $I_j/I_{j-1} \cong S/\mathfrak{m}$ where $I_0 = I$ and $I_j = (I_{j-1}, u_j)$, and where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S . Indeed, we have $\tilde{I} = I : \mathfrak{m}^k$ for some k . For each $i \in [k]$, where $[k] = \{1, \dots, k\}$, the K -vector space $(I : \mathfrak{m}^i)/(I : \mathfrak{m}^{i-1})$ has finite dimension. If

$$\dim_K(I : \mathfrak{m}^i / I : \mathfrak{m}^{i-1}) = r_i,$$

then we can choose monomials $u_{i,1}, \dots, u_{i,r_i} \in I : \mathfrak{m}^i$ whose residue classes modulo $I : \mathfrak{m}^{i-1}$ form a basis for this K -vector space. Composing these bases we obtain the required basis for \tilde{I}/I .

So we have

$$\mathcal{F}_1: \quad I = I_0 \subset I_1 \subset \dots \subset I_t = \tilde{I}$$

with $I_i/I_{i-1} \cong S/\mathfrak{m}$, for all $i = 1, \dots, t$. Now if \tilde{I} is a pretty clean and \mathcal{G} is pretty the clean filtration of \tilde{I} , then the prime filtration \mathcal{F} which is obtained by composing \mathcal{F}_1 and \mathcal{G} yields a pretty clean filtration of I .

For the converse, let $I = I_0 \subset I_1 \subset \dots \subset I_r = S$ be pretty clean filtration of I . We will show that \tilde{I} is pretty clean by induction on $\dim_K \tilde{I}/I = t$. If $t = 0$ the assertion is trivially true. Assume now that $t > 0$. It is clear that I_1 is also pretty clean and that $I_1/I \cong S/\mathfrak{m}$, since $I \neq \tilde{I}$. It

follows that $\tilde{I}_1 = \tilde{I}$ and that $\dim_K \tilde{I}_1/I_1 = t - 1$. So by the induction hypothesis $\tilde{I} = \tilde{I}_1$ is pretty clean. \square

Corollary 1.2. *If $S = K[x_1, \dots, x_n]$ is the polynomial ring in n variables, then any monomial ideal in S of height n is pretty clean.*

Our next goal is to show that even the monomial ideals in $S = K[x_1, \dots, x_n]$ of height at least $n - 1$ are pretty clean. To this end we have to recall the concept of multicomplexes and shellings.

Stanley [11] calls a subset $\Gamma \subseteq \mathbb{N}^n$ a *multicomplex* if for all $a \in \Gamma$ and for all $b \leq a$, i.e., $a(i) \leq b(i)$ for all $i = 1, \dots, n$, one has $b \in \Gamma$. Herzog and Popescu [6] give the following modification of Stanley's definition of multicomplex which will be used in this paper. Before we give this definition we introduce some notation. We set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Let Γ be a subset of \mathbb{N}_∞^n . An element $m \in \Gamma$ is called *maximal* if there is no $a \in \Gamma$ with $a > m$. We denote by $M(\Gamma)$ the set of maximal elements of Γ . If $a \in \Gamma$, we call

$$\text{infpt}(a) = \{i: a(i) = \infty\},$$

the *infinite part* of a .

Definition 1.3. A subset $\Gamma \subset \mathbb{N}_\infty^n$ is called a *multicomplex* if

- (i) for all $a \in \Gamma$ and for all $b \leq a$ it follows that $b \in \Gamma$,
- (ii) for all $a \in \Gamma$ there exists an element $m \in M(\Gamma)$ such that $a \leq m$.

The elements of a multicomplex are called *faces*. An element $a \in \Gamma$ is called a *facet* of Γ if for all $m \in M(\Gamma)$ with $a \leq m$ one has $\text{infpt}(a) = \text{infpt}(m)$. The set of all facets of Γ will be denoted by $F(\Gamma)$. The facets in $M(\Gamma)$ are called *maximal facets*. It is clear that $M(\Gamma) \subset F(\Gamma)$. We recall that for each multicomplex Γ the set of facets of Γ is a finite set, see [6, Lemma 9.6].

Let Γ be a multicomplex, and let $I(\Gamma)$ be the K -vector space in $S = K[x_1, \dots, x_n]$ spanned by all monomials x^a such that $a \notin \Gamma$. Note that $I(\Gamma)$ is a monomial ideal, and called the monomial ideal associated to Γ . Conversely, let $I \subset S$ be any monomial ideal, then there exists a unique multicomplex $\Gamma(I)$ with $I(\Gamma(I)) = I$. Indeed, let $A = \{a \in \mathbb{N}^n: x^a \notin I\}$; then $\Gamma(I) = \Gamma(A)$ is called the multicomplex associated to I , where $\Gamma(A)$ is the unique smallest multicomplex containing A .

A subset $S \subset \mathbb{N}_\infty^n$ is called a *Stanley set* if there exists $a \in \mathbb{N}^n$ and $m \in \mathbb{N}_\infty^n$ with $m(i) \in \{0, \infty\}$ such that $S = a + S^*$, where $S^* = \Gamma(m)$.

In [6] the concept of *shelling* of multicomplexes was introduced as in the following.

Definition 1.4. A multicomplex Γ is *shellable* if the facets of Γ can be ordered a_1, \dots, a_r such that

- (i) $S_i = \Gamma(a_i) \setminus \Gamma(a_1, \dots, a_{i-1})$ is a Stanley set for all $i = 1, \dots, r$, and
- (ii) whenever $S_i^* \subseteq S_j^*$, then $S_i^* = S_j^*$ or $i > j$.

Any order of the facets satisfying (i) and (ii) is called a *shelling* of Γ .

In [6, Theorem 10.5] the following has been proved.

Theorem 1.5. *The multicomplex Γ is shellable if and only if $S/I(\Gamma)$ is a pretty clean S -module.*

Remark 1.6. Let $\Gamma \subset \mathbb{N}_\infty^n$ be a shellable multicomplex with shelling a_1, \dots, a_r , then $a_1(i) \in \{0, \infty\}$ and therefore a_1 is one of the minimal elements in $F(\Gamma)$ with respect to its partially order. Indeed, since a_1, \dots, a_r is a shelling, it follows that $S_1 = \Gamma(a_1)$ is a Stanley set and therefore there exists a vector $b \in \mathbb{N}^n$ and a vector $m \in \{0, \infty\}^n$ such that

$$\Gamma(a_1) = b + \Gamma(m).$$

It is clear that $\text{infpt}(a_1) = \text{infpt}(m)$. If $\text{infpt}(m) = [n]$, then there is nothing to show. Suppose now that $\text{infpt}(m) \neq [n]$, and choose $i \in [n] \setminus \text{infpt}(m)$. If $a_1(i) \neq 0$ there exists $c \in \Gamma(a_1)$ with $c(i) < a_1(i)$. Since c and $a_1 \in b + \Gamma(m) = \Gamma(a_1)$, and since $m(i) = 0$, it follows that $c(i) = b(i) = a_1(i)$, a contradiction.

Furthermore, if Γ has only one maximal facet, then $F(\Gamma)$ has only one minimal element, also any shelling of Γ must start with this minimal element and end by the maximal one. In fact, suppose a_1 and a_2 are minimal elements in $F(\Gamma)$. By the first part of this remark it follows that a_1 and a_2 are vectors in $\{0, \infty\}^n$. Hence since $\text{infpt}(a_1) = \text{infpt}(a_2)$, we see that $a_1 = a_2$. Now let a_1, \dots, a_r be any shelling of Γ . Then, by what we have shown, it follows that a_1 is the unique minimal element in $F(\Gamma)$. Let m be the maximal element of $F(\Gamma)$. Suppose $m = a_k$ for some $k < r$, then

$$S_{k+1} = \Gamma(a_{k+1}) \setminus \Gamma(a_1, \dots, a_k) = \Gamma(a_{k+1}) \setminus \Gamma(m) = \emptyset,$$

which is not a Stanley set, a contradiction. Moreover in this case for each i there exists a $d_i \in \mathbb{N}^n$ such that $S_i = d_i + \Gamma(a_1)$.

Now we are ready to show that in $S = K[x_1, \dots, x_n]$, any ideal of height $n - 1$ is pretty clean.

Proposition 1.7. *If $I \subset S = K[x_1, \dots, x_n]$ is any monomial ideal of height $\geq n - 1$, then I is pretty clean.*

Proof. We may assume that I is a monomial ideal of height $n - 1$, and by Lemma 1.1 that I is saturated, i.e., $I = \tilde{I}$. It follows that $I = \bigcap I_j$, where $I_j = (x_1^{c_{j,1}}, \dots, x_{j-1}^{c_{j,j-1}}, x_{j+1}^{c_{j,j+1}}, \dots, x_n^{c_{j,n}})$, and where $c_{j,k} > 0$ for $k \neq j$. We denote by Γ and Γ_j the multicomplexes associated to I and I_j , and by F and F_j the sets of facets of Γ and Γ_j , respectively. The sets F and F_j are finite, see [6, Lemma 9.6]. Suppose $|F| = t$ and $|F_j| = t_j$. Since I_j is P_j -primary where $P_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, it follows from [6, Proposition 5.1] that I_j is pretty clean, and hence Γ_j is shellable. Moreover $a \in \mathbb{N}_\infty^n$ is a facet of Γ_j if and only if $a(j) = \infty$ and $a(k) < c_{j,k}$ for $k \neq j$. Let $a_{j,1}, \dots, a_{j,t_j}$ be a shelling of Γ_j .

For showing I is pretty clean it is enough to show that Γ is shellable. By [6, Lemma 9.9(b)] we have $\Gamma = \bigcup_{j=1}^n \Gamma_j$. Also by [6, Lemma 9.10], each F_j has only one maximal facet, say m_j , where

$$m_j(k) = \begin{cases} \infty, & \text{if } k = j, \\ c_{j,k} - 1, & \text{otherwise.} \end{cases}$$

It follows that $F = \bigcup F_j$ and that the union is disjoint, since $a \in F$ belongs to F_j if and only if $a(j) = \infty$ and $a(k) < \infty$ for $k \neq j$. In particular one has $(\bigcup_{i=1}^{j-1} F_i) \cap F_j = \emptyset$ for $j = 2, \dots, n$.

We claim that

$$a_{1,1}, \dots, a_{1,t_1}, a_{2,1}, \dots, a_{2,t_2}, \dots, a_{n,1}, \dots, a_{n,t_n}$$

is a shelling for Γ . Indeed, for all j and all k with $1 < k \leq t_j$ we have

$$S_{j,k} = \Gamma(a_{j,k}) \setminus \Gamma(a_{1,1}, \dots, a_{j,k-1}) = \Gamma(a_{j,k}) \setminus \Gamma(a_{j,1}, \dots, a_{j,k-1}),$$

and if $k = 1$, then

$$S_{j,1} = \Gamma(a_{j,1}) \setminus \Gamma(a_{1,1}, \dots, a_{j-1,t_{j-1}}) = \Gamma(a_{j,1}).$$

Since $a_{j,1}, \dots, a_{j,t_j}$ is a shelling of Γ_j , it follows that $S_{j,k}$ is a Stanley set for all j and all k .

Condition (ii) in the definition of shellability is obviously satisfied. In fact, since Γ_j is shellable and has only one maximal facet, it follows by Remark 1.6 that for all $k = 1, \dots, t_j$, there exists some $d_{j,k} \in \mathbb{N}^n$ such that $S_{j,k} = d_{j,k} + S_j^*$, where $S_j^* = \Gamma(a_{j,1})$. Moreover if $j \neq t$ then $a_{j,1}$ and $a_{t,1}$ are not comparable, and hence in this case there is no inclusion among S_j^* and S_t^* . \square

As a consequence of Proposition 1.7 we have

Corollary 1.8. *Any monomial ideal $I \subset S = K[x, y]$ is pretty clean.*

Next we will show that any monomial ideal in $S = K[x_1, x_2, x_3]$ is also pretty clean. First we need

Lemma 1.9. *If $I \subset S = K[x_1, x_2, x_3]$ is a monomial ideal of height 1, then $I = uJ$, where u is a monomial in S , and J is a monomial ideal of height ≥ 2 . Moreover, I is pretty clean if and only if J is pretty clean.*

Proof. The first statement of the lemma is obvious. Assume now that J is pretty clean with pretty clean filtration

$$\mathcal{F}: \quad J = J_0 \subset J_1 \subset \dots \subset J_r = S$$

such that $J_i/J_{i-1} \cong S/P_i$, where $P_i \in \text{Ass } J$. Then height $P_i \geq 2$. It follows that

$$\mathcal{F}_1: \quad I = uJ \subset uJ_1 \subset \dots \subset uJ_r = (u)$$

is a prime filtration of $(u)/I$ with factors $uJ_i/uJ_{i-1} \cong S/P_i$.

There exists a prime filtration

$$\mathcal{F}_2: \quad (u) = J_r \subset J_{r+1} \subset \dots \subset J_{r+t} = S$$

of the principal monomial ideal $I_1 = (u)$, where the J_{r+k} are again principal monomial ideals with $J_{r+k}/J_{r+k-1} \cong S/Q_k$ and where $Q_k \in \text{Ass}(u)$ has height 1 for all k . In fact, if $u = u_0 = \prod_{i=1}^k x_{i_r}^{a_{i_r}}$ and $u_j = \prod_{r=j+1}^k x_{i_r}^{a_{i_r}}$ for $j = 1, \dots, k-1$, then the prime filtration \mathcal{F}_2 is the following:

$$\mathcal{F}_2: \quad J_r = (u) \subset (x_{i_1}^{a_1-1} u_1) \subset (x_{i_1}^{a_1-2} u_1) \subset \cdots \subset (u_1) \subset (x_{i_2}^{a_2-1} u_2) \subset \cdots \subset (u_2) \subset \cdots \\ \subset (x_{i_k}) \subset S.$$

Therefore this filtration of $I_1 = (u)$ is pretty clean. Now composing the above filtrations \mathcal{F}_1 and \mathcal{F}_2 we obtain a pretty clean filtration of I .

The converse follows from Proposition 1.7, because $\text{height}(J) \geq 2$. \square

Combining Lemma 1.9 with Proposition 1.7 we get

Theorem 1.10. *Any monomial ideal in a polynomial ring in at most three variables is pretty clean.*

The following example shows that this theorem cannot be extended to polynomial rings in more than three variables, and it also shows that monomial ideals of $\text{height} < n - 1$ may not be pretty clean.

Example 1.11. Let $n = 4$, and Γ be the multicomplex with facets $(\infty, \infty, 0, 0)$ and $(0, 0, \infty, \infty)$. Then Γ is not shellable, and so the monomial ideal

$$I(\Gamma) = (x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4) \subset K[x_1, x_2, x_3, x_4]$$

is not pretty clean.

More generally, let $n > 3$ and $a = (0, 0, \infty, \dots, \infty)$ and $b = (\infty, \infty, 0, \dots, 0)$ be two elements in \mathbb{N}_∞^n . Then $\Gamma = \Gamma(a, b)$ is not a shellable multicomplex, hence $I = (x_1, x_2) \cap (x_3, \dots, x_n)$ is a squarefree monomial ideal in $S = K[x_1, \dots, x_n]$ which is not clean.

2. Prime filtrations and Stanley decompositions

Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal, any decomposition of S/I as a direct sum of K -vector spaces of the form $uK[Z]$ where u is a monomial in $K[X]$, and $Z \subset X = \{x_1, \dots, x_n\}$ is called a *Stanley decomposition*. In this paper we will call $uK[Z]$ a Stanley space of dimension $|Z|$, where $|Z|$ denotes the cardinality of Z . Stanley decomposition have been studied in various combinatorial and algebraic contexts, see [1,7], and [8]. Stanley sets and Stanley decompositions appeared already in the PhD thesis of Dave Bayer [2] under the name wild cards and wild card partitions.

Let R be a finitely generated standard graded K -algebra where K is a field, and let M be a finitely generated graded R -module. The Hilbert series of M is defined to be $\text{Hilb}(M) = \sum_{i \in \mathbb{Z}} (\dim_K M_i) t^i$. It is known that if $\dim(M) = d$, then there exists a $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$\text{Hilb}(M) = Q_M(t)/(1-t)^d$$

and $Q_M(1) \neq 0$. The number $Q_M(1)$ is called the multiplicity of M , and is denoted by $e(M)$.

Let $I \subset S$ be a monomial ideal. The number of Stanley spaces of a given dimension in a Stanley decomposition may depend on this particular decomposition. For example, if $I = (xy) \subset K[x, y]$, then for all integers $k > 0$ and $l > 0$ we have the Stanley decomposition

$$S/I = x^l K[x] \oplus y^k K[y] \oplus \left(\bigoplus_{i=0}^{l-1} x^i K \right) \oplus \left(\bigoplus_{j=1}^{k-1} y^j K \right),$$

for S/I with as many Stanley spaces of dimension 0 as we want, however only 2 Stanley spaces of dimension 1 in any Stanley decomposition.

This is a general fact. Indeed, the number of Stanley spaces of maximal dimension is independent of the special Stanley decomposition. In fact, this number is equal to the multiplicity, $e(S/I)$, of S/I .

Let

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i]$$

be an arbitrary Stanley decomposition of S/I , and $d = \max\{|Z_i| : i = 1, \dots, r\}$. Then

$$\text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(u_i K[Z_i]) = \sum_{i=1}^r t^{\deg(u_i)} / (1-t)^{|Z_i|} = Q_{S/I}(t) / (1-t)^d$$

with $Q_{S/I}(t) = \sum_{i=1}^r (1-t)^{d-|Z_i|} t^{\deg(u_i)}$. It follows that $e(S/I) = Q_{S/I}(1)$ is equal to the number of Stanley space of dimension d in this Stanley decomposition of S/I .

We also note that for each monomial $u \in \tilde{I} \setminus I$ the 0-dimensional Stanley space uK belongs to any Stanley decomposition of S/I . In fact $um^k \subset I$ for some k . Now if u belongs to some Stanley space $vK[Z]$ with $|Z| \geq 1$, then $vK[Z] \cap I \neq \emptyset$, a contradiction.

Stanley [12] conjectured that there always exists a Stanley decomposition

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i],$$

such that $|Z_i| \geq \text{depth}(S/I)$ for all i .

Apel [1] studied some cases in which Stanley's conjecture holds. Theorem 6.5 in [6] proves that for all pretty clean monomial ideals Stanley's conjecture holds. Therefore combining Theorem 1.10 and Lemma 1.7 with [6, Theorem 6.5] we get

Proposition 2.1.

- (a) If $I \subset S = K[x_1, \dots, x_n]$ is a monomial ideal of height $\geq n - 1$, then Stanley's conjecture holds for S/I .
- (b) (Apel, [1, Theorem 5.1]) If I is any monomial ideal in the polynomial ring in at most three variables, then Stanley's conjecture holds for S/I .

In the proof of [6, Theorem 6.5] it is used that Stanley decompositions of S/I arise from prime filtrations. In fact, if \mathcal{F} is a prime filtration of S/I with factors $(S/P_i)(-a_i)$, $i = 1, \dots, r$. Then if we set $u_i = \prod_{j=1}^n x_j^{a_i(j)}$ and $Z_i = \{x_j : x_j \notin P_i\}$, then

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i].$$

Recall that if \mathcal{F} is a pretty clean filtration of S/I , then $\text{Ass}(S/I) = \text{Supp}(\mathcal{F})$. The converse of this statement is not always true, see Example 4.4. As a generalization of [6, Theorem 6.5] we show

Proposition 2.2. *Suppose $I \subset S$ is a monomial ideal, and \mathcal{F} is a prime filtration of S/I with $\text{Supp}(\mathcal{F}) = \text{Ass}(S/I)$. Then the Stanley decomposition of S/I which is obtained from this prime filtration satisfies the condition of Stanley's conjecture.*

Proof. The Stanley decomposition which is obtained from \mathcal{F} has the property that $|Z_i| = \dim S/P_i$. By [3, Proposition 1.2.13] we have $\text{depth}(S/I) \leq \dim(S/P_i)$ for all $P_i \in \text{Ass}(S/I)$, and hence the assertion follows. \square

In all cases discussed above we found a Stanley decomposition corresponding to a prime filtration and satisfying the Stanley conjecture. However we will show that there exist examples of monomial ideals such that *all* Stanley decompositions arising from a prime filtration may fail to satisfy the Stanley conjecture.

Remark 2.3. Let $I \subset S$ be a Cohen–Macaulay monomial ideal, and

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

be a prime filtration of S/I . We claim that if the Stanley decomposition of S/I corresponding to \mathcal{F} satisfies the Stanley conjecture, then $\text{Ass}(I) = \text{Supp}(\mathcal{F})$. In particular I is clean, since $\text{Min}(I) = \text{Ass}(I)$.

Indeed, since I is Cohen–Macaulay we have $\text{depth}(S/I) = \dim(S/I) = \dim(S/P)$ for all $P \in \text{Ass}(I)$. We recall that $I_i/I_{i-1} \cong S/P_i(-a_i)$ for suitable a_i and that $P_i \in \text{Ass}(I_{i-1})$ for $i = 1, \dots, r$. Let $T_i = u_i K[Z_i]$ be the Stanley space corresponding to $S/P_i(-a_i)$ as explained as above. Then $|Z_i| = \dim(S/P_i)$. Assume that $P_i \notin \text{Ass}(I)$ for some $i > 1$. Since $I \subset I_{i-1} \subset P_i$, there exists a $P_j \in \text{Ass}(I)$ such that $P_j \subsetneq P_i$. It follows that $|Z_i| = \dim(S/P_i) < \dim(S/P_j) = \text{depth}(S/I)$, a contradiction.

Example 2.4. Let K be a field and

$$I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def) \subset S = K[a, b, c, d, e, f].$$

The ideal I is the Stanley–Reisner ideal corresponding to the simplicial complex Δ which is the triangulation of the real projective plane \mathbb{P}^2 , see [3, Figure 5.8]. It is known that S/I is Cohen–Macaulay if and only if $\text{char}(K) \neq 2$. This implies S/I is not clean, since otherwise Δ would be shellable and S/I would be Cohen–Macaulay for any field K . Hence by Remark 2.3,

if $\text{char}(K) \neq 2$, no Stanley decomposition of S/I which corresponds to a prime filtration of S/I satisfies the Stanley conjecture.

Unfortunately not all Stanley decompositions of S/I correspond to prime filtrations, even if S/I is pretty clean. Such an example is given by MacLagan and Smith in [8]. Let $I = (x_1 x_2 x_3) \subset K[x_1, x_2, x_3]$. Then

$$S/I = 1 \oplus x_1 K[x_1, x_2] \oplus x_2 K[x_2, x_3] \oplus x_3 K[x_1, x_3]$$

is a Stanley decomposition of S/I which does not correspond to a prime filtration of S/I . On the other hand, by Theorem 1.10 we know that S/I is a pretty clean.

Now we want to characterize those Stanley decompositions of S/I which correspond to a prime filtration of S/I .

Lemma 2.5. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal, and $T = uK[Z]$ be a Stanley space in a Stanley decomposition of S/I . The K -vector space $I_1 = I \oplus T$ is a monomial ideal if and only if $I_1 = (I, u)$. In this case, $I : u = P$, where $P = (x_i : x_i \notin Z)$.*

Proof. We have $I \subset I_1$ and $u \in I_1$. Suppose now that I_1 is a monomial ideal. Since (I, u) is the smallest monomial ideal that contains I and u , it follows that $(I, u) \subset I_1$. On the other hand, $I_1 = I + uK[Z] \subset I + uK[x_1, \dots, x_n] = (I, u)$. Hence $I_1 = (I, u)$.

Since for each $x_i \notin Z$ we have $x_i u \in I_1 = I \oplus T$ and $x_i u \notin uK[Z] = T$, it follows that $x_i u \in I$ and hence $x_i \in I : u$. On the other hand, if $v \in K[Z]$ is a monomial, then $vu \notin I$, since $uK[Z]$ is a Stanley space of S/I . Therefore $I : u = P = (x_i : x_i \notin Z)$. \square

Corollary 2.6. *The monomial ideal $I \subset S$ is a prime ideal if and only if there exists a Stanley decomposition of S/I consisting of only one Stanley space.*

As a consequence of this lemma we have

Proposition 2.7. *Let $I \subset S$ be a monomial ideal, and $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$ be a Stanley decomposition of S/I . The given Stanley decomposition corresponds to a prime filtration of S/I if and only if the Stanley spaces $T_i = u_i K[Z_i]$ can be ordered T_1, \dots, T_r , such that*

$$I_k = I \oplus T_1 \oplus \dots \oplus T_k$$

is a monomial ideal for $k = 1, \dots, r$.

Proof. We prove “if” by induction on r . If $r = 0$ then the assertion is trivially true. Let $r \geq 1$. By assumption $I_1 = I \oplus T_1$ is a monomial ideal. Hence by Lemma 2.5 we have $I_1 = (I, u_1)$ and $I : u_1 = P_1 = (x_i : x_i \notin Z_1)$. We notice that in this case $I_1/I \cong S/P_1(-a_1)$ and $u_1 = \prod_{j=1}^n x_j^{a_1(j)}$, and that $S/I_1 = \bigoplus_{i=2}^r T_i$. Now by the induction hypothesis this Stanley decomposition of S/I_1 corresponds to a prime filtration, say \mathcal{F}_1

$$\mathcal{F}_1: \quad I_1 \subset I_2 \subset \dots \subset I_r = S.$$

Therefore the given Stanley decomposition of S/I corresponds to the prime filtration

$$\mathcal{F}: I \subset I_1 \subset I_2 \subset \cdots \subset I_r = S.$$

The converse follows immediately if we order the Stanley spaces of S/I which are obtained from a prime filtration according to the order of the ideals in this filtration. \square

We conclude this section by showing

Corollary 2.8. *If $I \subset S = K[x, y]$ is a monomial ideal, then each Stanley decomposition of S/I corresponds to a prime filtration of S/I .*

Proof. The K -vector space \tilde{I}/I has finite dimension, say m . So we can choose monomials $v_1, \dots, v_m \in \tilde{I}$ whose residue classes modulo I form a K -basis for \tilde{I}/I . As observed in the discussions before Proposition 2.1, in any Stanley decomposition of S/I these monomials have to appear as 0-dimensional Stanley spaces. In the proof of Lemma 1.1 we showed that it is possible to order the monomials v_1, \dots, v_m in such a way that

$$I_i = I \oplus v_1 K \oplus \cdots \oplus v_i K = (I, v_1, \dots, v_i)$$

is a monomial ideal for $i = 1, \dots, m$. If we remove in the given Stanley decomposition of S/I the Stanley spaces $v_i K$, $i = 1, \dots, m$, the remaining summands establish a Stanley decomposition of S/\tilde{I} . Thus we may assume that I is saturated. Hence $I = (x^\alpha y^\beta)$.

Let $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$ be a Stanley decomposition of S/I . We will prove by induction on $\alpha + \beta$ that the given Stanley decomposition can be ordered such that $I_k = I \oplus (\bigoplus_{i=1}^k u_i K[Z_i])$ is a monomial ideal for all k . If $\alpha + \beta = 0$ the assertion is trivially true. Let $\alpha + \beta > 0$. The Stanley decomposition of S/I contains at least one summand of the form $x^{\alpha-1} y^\gamma K[y]$, where $\gamma \geq \beta$, or $x^\theta y^{\beta-1} K[x]$, where $\theta \geq \alpha$.

We may assume that $x^{\alpha-1} y^\gamma K[y]$ is one of the summands. Let $t = \gamma - \beta$, and set $v_i = x^{\alpha-1} y^{\gamma-i+1}$ for $i = 1, \dots, t+1$. If we set $T_1 = v_1 K[y]$, then $I_1 = I \oplus T_1 = (I, v_1)$ is a monomial ideal. If we remove the Stanley space T_1 from the given Stanley decomposition of S/I , the remaining establish a Stanley decomposition of S/I_1 . Since v_2, \dots, v_{t+1} belong to $\tilde{I}_1 \setminus I_1$, these monomials have to appear in any Stanley decomposition of S/I_1 as 0-dimensional Stanley spaces. In particular these monomials appear as 0-dimensional Stanley space, $T_2 = v_2 K, \dots, T_{t+1} = v_{t+1} K$ in the given Stanley decomposition of S/I . Now it is clear that $I_i = I_{i-1} \oplus T_i = (I_{i-1}, v_i)$ is a monomial ideal for $i = 1, \dots, t+1$, where $I_0 = I$.

Removing the Stanley spaces T_1, \dots, T_{t+1} from the given Stanley decomposition of S/I , the remaining summands establish a Stanley decomposition of S/I_{t+1} . Since $I_{t+1} = (x^{\alpha-1} y^\beta)$ is a saturated ideal, the assertion follows by the induction hypothesis applied to S/I_{t+1} . \square

3. A characterization of pretty clean monomial ideals in terms of polarizations

In this section we consider polarizations of monomial ideals and of prime filtrations. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field K , and $u = \prod_{i=1}^n x_i^{a_i}$ be a monomial in S . Then

$$u^p = \prod_{i=1}^n \prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \dots, x_{1,a_1}, \dots, x_{n,1}, \dots, x_{n,a_n}]$$

is called the *polarization* of u .

Let I be a monomial ideal in S with monomial generators u_1, \dots, u_m . Then (u_1^p, \dots, u_m^p) is called a *polarization* of I . Note that if v_1, \dots, v_k is another set of monomial generators of I and if T is the polynomial with sufficiently many variables $x_{i,j}$ such that all the monomials u_i^p and v_j^p belong to T , then

$$(u_1^p, \dots, u_m^p)T = (v_1^p, \dots, v_k^p)T.$$

Therefore we denote any polarization of I by I^p , since in a common polynomial ring extension all polarizations are the same, and we write $I^p = J^p$ if a polarization of I and a polarization of J coincide in a common polynomial ring extension.

Now let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial. Furthermore let T be the polynomial ring in variables $x_{i,j}$ such that:

- (1) for all $i \in [n]$ there exists $k_i \geq 1$ such that $x_{i,1}, \dots, x_{i,k_i}$ are in T ,
- (2) $I^p \subset T$, and $u^p \in T$.

We consider the K -algebra homomorphism

$$\pi : T \rightarrow S, \quad x_{i,j} \mapsto x_i.$$

Then π is an epimorphism with $\pi(u^p) = u$ for all monomials $u \in S$, and u^p is the unique square-free monomial in T of the form $\prod_{i=1}^n \prod_{j=1}^{k_i} x_{i,j}$ with this property. In particular, $\pi(I^p) = I$. We call π the specialization map attached with the polarization.

Remark 3.1. Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial. Then

- (a) $I : u = (u_i / \gcd(u_i, u))_{i=1}^m$, and it is again a monomial ideal in S .
- (b) $I : u$ is a prime ideal if and only if for each $i \in [m]$, there exists a $j \in [m]$ such that $u_j / \gcd(u_j, u)$ is a monomial of degree one, and $u_j / \gcd(u_j, u)$ divides $u_i / \gcd(u_i, u)$.
- (c) Let $u = \prod_{i=1}^n x_i^{a_i}$ and $u_j = \prod_{i=1}^n x_i^{b_i}$. If $u_j / \gcd(u_j, u) = x_i$, then $b_i = a_i + 1$ and $b_t \leq a_t$ for all $t \neq i$. Therefore $u_j / \gcd(u_j, u) = x_i$ if and only if $u_j^p / \gcd(u_j^p, u^p) = x_{i,b_i}$.

Lemma 3.2. Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal and $u \in S$ a monomial. If $I^p : u^p$ is a prime ideal, then $I^p : u^p = (x_{i_1, j_1}, \dots, x_{i_k, j_k})$ with $i_r \neq i_s$ for $r \neq s$.

Proof. Since $I^p : u^p$ is a monomial prime ideal in polynomial ring T it must be generated by variables. If $x_{i,j}$ and $x_{i,k}$ are two generators of $I^p : u^p$, then there exist $r_j \in [m]$, and $r_k \in [m]$ such that $x_{i,j} = u_{r_j}^p / \gcd(u_{r_j}^p, u^p)$ and $x_{i,k} = u_{r_k}^p / \gcd(u_{r_k}^p, u^p)$. It follows from Remark 3.1(c) that $j - 1 = k - 1$ is equal to the exponent of x_i in u . Hence $x_{i,j} = x_{i,k}$. \square

Lemma 3.3. Let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal, and $u \in S$ a monomial in S . Then $I : u$ is a prime ideal if and only if $I^p : u^p$ is a prime ideal. In this case $I : u = \pi(I^p : u^p)$.

Proof. Let $I : u$ be a prime ideal. We may assume that $I : u = (x_1, \dots, x_k)$ for some $k \in [n]$. Therefore for each $i \in [k]$ there exists some u_{j_i} , with $j_i \in [m]$, such that $x_i = u_{j_i} / \gcd(u_{j_i}, u)$

and for each $t \in [m]$, there exists $i_t \in [k]$, such that x_{i_t} divides $(u_t / \gcd(u_t, u))$. Therefore by Remark 3.1(c) we have $u_{j_i}^p / \gcd(u_{j_i}^p, u^p) = x_{i,t_i}$, where t_i is the exponent of x_i in u_{j_i} and $t_i - 1$ is the exponent of x_i in u .

Also for each $s \in [m]$, the monomial $u_s^p / \gcd(u_s^p, u^p)$ is divisible by one of these x_{i,t_i} , where $i \in [k]$. Indeed, since $I : u$ is a prime ideal there exists some $i \in [k]$ such that x_i divides $(u_s / \gcd(u_s, u))$, where $x_i = u_{j_i} / \gcd(u_{j_i}, u)$. Let $t_i - 1$ be the exponent of x_i in u . Then it follows that the exponent of x_i in u_s is $> t_i - 1$. Hence x_{i,t_i} divides $u_s^p / \gcd(u_s^p, u^p)$, and $I^p : u^p = (x_{1,t_1}, \dots, x_{k,t_k})$.

For the converse, let $I^p : u^p$ be a prime ideal. By Lemma 3.2 we may assume that $I^p : u^p = (x_{1,t_1}, \dots, x_{k,t_k})$. This means that for each $i \in [k]$ there is a monomial u_{j_i} with $j_i \in [m]$ such that $x_{i,t_i} = u_{j_i}^p / \gcd(u_{j_i}^p, u^p)$ and for each $s \in [m]$, the squarefree monomial $u_s^p / \gcd(u_s^p, u^p)$ is divisible by one of these x_{i,t_i} . Therefore by Remark 3.1(c) we have $x_i = u_{j_i} / \gcd(u_{j_i}, u)$ for $i \in [k]$, and for each $s \in [m]$, one of these variables divides $u_s / \gcd(u_s, u)$. Hence $I : u = (x_1, \dots, x_k)$. \square

Let $I \subset S$ be a monomial ideal and

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

a filtration of S/I . We call r the *length of filtration* \mathcal{F} and denote it by $\ell(\mathcal{F})$.

Assume now that for all j we have $I_{j+1} = (I_j, u_j)$ where $u_j \in S$ is a monomial. We will define the *polarization* \mathcal{F}^p of \mathcal{F} inductively as follow: set $J_0 = I^p$; assuming that J_i is already defined, we set $J_{i+1} = (J_i, u_i^p)$. So $J_i = (I^p, u_1^p, \dots, u_i^p)$, and

$$\mathcal{F}^p: \quad I^p = J_0 \subset J_1 \subset \dots \subset J_r = T$$

is a filtration of T/I^p .

We have the following

Proposition 3.4. *Suppose $I \subset S$ is a monomial ideal, and*

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

a filtration of S/I as above. Then \mathcal{F} is a prime filtration of S/I if and only if \mathcal{F}^p is a prime filtration of T/I^p .

Proof. Let

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

be a prime filtration of S/I . We use induction on $r = \ell(\mathcal{F})$ the length of prime filtration. If $r = 1$, then I is a monomial prime ideal and $I^p = I$.

Let $r > 1$. Then $\mathcal{F}_1: I_1 \subset \dots \subset I_r = S$ is a prime filtration of S/I_1 , and $\ell(\mathcal{F}_1) = r - 1$. By our induction hypothesis, \mathcal{F}_1^p is a prime filtration of $I_1^p = (I^p, u_1^p)$. Since $I_1/I \cong I_1 : u_1$ is a prime ideal, it follows from Lemma 3.3 that $J_0/J_1 \cong I_1^p : u_1^p$ is a prime ideal too. Hence \mathcal{F}^p is a prime filtration of T/I^p .

The other direction of the statement is proved similarly. \square

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring, and $u, v \in S$ be monomials. We notice that

$$\text{lcm}(u, v)^p = \text{lcm}(u^p, v^p).$$

Therefore we have

Lemma 3.5. *Let I, J be two monomial ideals in S . Then $(I \cap J)^p = I^p \cap J^p$.*

Proof. Let $I = (u_1, \dots, u_m)$ and $J = (v_1, \dots, v_t)$. Then $I \cap J = (\text{lcm}(u_i, v_j))$, where $1 \leq i \leq m$ and $1 \leq j \leq t$. Therefore $(I \cap J)^p = (\text{lcm}(u_i, v_j)^p) = (\text{lcm}(u_i^p, v_j^p)) = I^p \cap J^p$. \square

We recall that a monomial ideal $I \subset S$ is an irreducible monomial ideal if and only if there exists a subset $A \subset [n]$ and for each $i \in A$ an integer $a_i > 0$ such that $I = (x_i^{a_i} : i \in A)$, see [14, Theorem 5.1.16]. It is known that for each monomial ideal I there exists a decomposition $I = \bigcap_{i=1}^r J_i$ such that J_i are irreducible monomial ideals, see [14, Theorem 5.1.17].

Corollary 3.6. *Suppose J_1, \dots, J_r are monomial ideals in the polynomial ring S , and $I = \bigcap_{i=1}^r J_i$. Then $I^p = \bigcap_{i=1}^r J_i^p$. In particular the minimal prime ideals of I^p are of the form $(x_{i_1, t_1}, \dots, x_{i_k, t_k})$, with $i_r \neq i_s$ for $r \neq s$.*

Next we show that if $I \subset S$ is a monomial ideal and I^p the polarization of I , then $|F(\Gamma(I))| = |F(\Gamma(I^p))|$. First we notice the following:

Lemma 3.7. *Let $I \subset S$ be an irreducible monomial ideal and I^p the polarization of I . Furthermore, let F and F^p be the sets of facets of $\Gamma(I)$ and $\Gamma(I^p)$, respectively. Then there exists a bijection between F and F^p .*

Proof. By [14, Theorem 5.1.16] there exists a subset $A \subset [n]$ and for each $i \in A$ an integer $a_i > 0$ such that $I = (x_i^{a_i} : i \in A)$. We may assume $A = [k]$ for some $k \leq n$. In this case $\Gamma(I) = \Gamma(m)$, where

$$m(i) = \begin{cases} a_i - 1, & \text{if } i \in [k], \\ \infty, & \text{otherwise,} \end{cases}$$

and $a \in F$ if and only if $a \leq m$ and $a(i) = \infty$ for $i > k$. We have

$$I^p = \left(\prod_{j=1}^{a_1} x_{1,j}, \prod_{j=1}^{a_2} x_{2,j}, \dots, \prod_{j=1}^{a_k} x_{k,j} \right),$$

and we know that the facets in F^p correspond to the minimal prime ideals of I^p . Indeed, if $a \in F^p$ is a facet of Γ^p , then $P_a = (x_i : a(i) = 0)$ is a minimal prime ideal of I^p . Each minimal prime ideal of I^p is of the form $(x_{1,t_1}, \dots, x_{k,t_k})$, with $t_i \leq a_i$.

Now we define

$$\theta : F \rightarrow F^p, \quad a \mapsto \bar{a}$$

as follows: if $k < i \leq n$, then $\bar{a}(ij) = \infty$ for all j , and if $i \in [k]$ we have $a(i) = t_i < a_i$, and we set

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = t_i + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Obviously $\bar{a} \in F^p$, since $P_{\bar{a}} = (x_{1,t_1+1}, \dots, x_{k,t_k+1})$ is a minimal prime ideal of I^p , and it is also clear that θ is an injective map.

Let $\bar{a} \in F^p$. Then \bar{a} corresponds to the minimal prime ideal $P_{\bar{a}} = (x_{1,t_1}, \dots, x_{k,t_k})$, where $t_i \leq a_i$. Therefore if $k < i \leq n$, we have $\bar{a}(ij) = \infty$ for all j , and if $i \in [k]$, then

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = t_i, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $a \in \mathbb{N}_{\infty}^n$ be the following:

$$a(i) = \begin{cases} t_i - 1, & \text{if } i \in [k], \\ \infty, & \text{otherwise,} \end{cases}$$

then a is a facet in F , since $a \leq m$ and $\text{infpt}(a) = n - k = \text{infpt}(m)$, and moreover $\theta(a) = \bar{a}$. \square

Now let $I = (u_1, \dots, u_m) \subset S$ be a monomial ideal and let $D \subset [n]$ be the set of elements $i \in [n]$ such that x_i divides u_j for at least one $j = 1, \dots, m$. Then we set

$$r_i = \max\{t: x_i^t \text{ divides } u_j \text{ at least for one } j \in [m]\}$$

if $i \in D$ and $r_i = 1$, otherwise. Moreover we set $r = \sum_{i=1}^n r_i$.

Note that I has a decomposition $I = \bigcap_{i=1}^t J_i$ where the ideals J_i are irreducible monomial ideals. In other words, each J_i is generated by pure powers of some of the variables. Then $I^p = \bigcap_{i=1}^t J_i^p$ is an ideal in the polynomial ring

$$T = K[x_{1,1}, \dots, x_{1,r_1}, x_{2,1}, \dots, \dots, x_{n,1}, \dots, x_{n,r_n}]$$

in r variables.

We denote by Γ , Γ^p , Γ_i and Γ_i^p the multicomplexes associated to I , I^p , J_i and J_i^p , respectively, and by F , F^p , F_i and F_i^p the sets of facets of Γ , Γ^p , Γ_i and Γ_i^p , respectively.

It is clear that $F \subset \bigcup_{i=1}^t F_i$ since $\Gamma = \bigcup_{i=1}^t \Gamma_i$, and also that $F^p \subset \bigcup_{i=1}^t F_i^p$. Each Γ_i has only one maximal facet, say m_i , and $m_i(k) \leq r_k - 1$ if $m_i(k) \neq \infty$.

Let $A \subset \mathbb{N}_{\infty}^n$ be the following set:

$$A = \{a \in \mathbb{N}_{\infty}^n: a(i) < r_i \text{ if } a(i) \neq \infty\}.$$

We define the map

$$\beta: A \rightarrow \{0, \infty\}^r, \quad a \mapsto \bar{a}$$

as follows: if $a(i) = \infty$, then $\bar{a}(ij) = \infty$ for all j , and if $a(i) = e$ where $e \leq r_i - 1$, then

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } j = e + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition 3.8. *With the above assumptions and notation the restriction of the map β to F is a bijection from F to F^p .*

Proof. First of all we want to show that $\bar{a} \in F^p$. Indeed, $a \in F \subset \bigcup_{i=1}^t F_i$. Therefore there exists an integer $j \in [n]$ such that $a \in F_j$, and since the restriction of β to F_j is the map θ defined in Lemma 3.7, it follows that $\bar{a} \in F_j^p$. Therefore there exists a subset $\{j_1, \dots, j_s\} \subset [n]$ and positive integers t_k with $t_k \leq r_{j_k}$ for $k = 1, \dots, s$ such that $P_{\bar{a}} = (x_{j_1, t_1}, \dots, x_{j_s, t_s})$. It is clear that $P_{\bar{a}}$ is a prime ideal which contains I^p and $\beta(a) = \bar{a}$, where

$$a(i) = \begin{cases} t_k - 1, & \text{if } i = j_k \text{ for some } k, \\ \infty, & \text{otherwise.} \end{cases}$$

Now $\bar{a} \in F^p$ if and only if $P_{\bar{a}} \in \text{Min}(I^p)$. Assume $P_{\bar{a}} \notin \text{Min}(I^p)$. Then there is a prime ideal $Q \in \text{Min}(I^p)$ such that $Q \subset P_{\bar{a}}$. Suppose $Q = (x_{i_1, e_1}, \dots, x_{i_h, e_h})$. Then $\{i_1, \dots, i_h\} \subset \{j_1, \dots, j_s\}$ and $\{e_1, \dots, e_h\} \subset \{t_1, \dots, t_s\}$. On the other hand, since Q is a minimal prime ideal of $I^p = \bigcap_{i=1}^t J_i^p$, there exists an integer $e \in [t]$ such that Q is one of the minimal prime ideals of

$$J_e^p = (x_{i_1}^{b_1}, \dots, x_{i_h}^{b_h})^p.$$

It follows that $1 \leq e_i \leq b_i$ for $i = 1, \dots, h$. Therefore there exists $b \in F_e$ with

$$b(i) = \begin{cases} e_k - 1, & \text{if } i \in \{i_1, \dots, i_h\}, \\ \infty, & \text{otherwise.} \end{cases}$$

This implies that $a < b \leq m_e$, and $\text{infpt}(a) < \text{infpt}(b) = \text{infpt}(m_e)$, a contradiction.

Next we show that β is injective: let $a, b \in F$ and $a \neq b$. Then there exists an integer i such that $a(i) \neq b(i)$. We have to show $\bar{a} \neq \bar{b}$. We consider different cases:

- (i) If $a(i) = 0$, and $b(i) \neq 0$, then $\bar{b}(i1) = \infty$ and $\bar{a}(i1) = 0$.
- (ii) If $a(i) = \infty$, and $b(i) = t - 1$ where $t \neq \infty$, then $\bar{a}(it) = \infty$ and $\bar{b}(it) = 0$.
- (iii) Suppose $0 < t - 1 = a(i) \neq \infty$. If $b(i) = 0$, then we have case (i). If $b(i) = \infty$ then we have case (ii). Finally if $0 < s - 1 = b(i) \neq \infty$, then $t \neq s$ since $a(i) \neq b(i)$ and hence $\bar{a}(it) = 0$ and $\bar{b}(it) = \infty$.

In all cases it follows that $\bar{a} \neq \bar{b}$.

Finally we show that β is surjective: let $\bar{a} \in F^p \subset \bigcup_{i=1}^t F_i^p$ be any facet of Γ^p . Then there exists an integer $i \in [t]$ such that $\bar{a} \in F_i^p$. Therefore $P_{\bar{a}}$ is a minimal prime ideal of

$$J_i^p = (x_{i_1}^{a_1}, \dots, x_{i_k}^{a_k})^p,$$

and hence there exists $t_i \leq a_i$ such that $P_{\bar{a}} = (x_{i_1, t_1}, \dots, x_{i_k, t_k})$. Therefore

$$\bar{a}(ij) = \begin{cases} 0, & \text{if } i = i_r \text{ and } j = t_r \text{ for some } r \in [k], \\ \infty, & \text{otherwise.} \end{cases}$$

By our definition we have $\bar{a} = \beta(a)$, where $a \in A$ with

$$a(i) = \begin{cases} t_r - 1, & \text{if } i = i_r \in \{i_1, \dots, i_k\}, \\ \infty, & \text{otherwise.} \end{cases}$$

It will be enough to show that $a \in F$. Since $\bar{a} \in F_i^P$ and the restriction of β to F_i is a bijection from F_i to F_i^P , it follows that $a \in F_i$. If $a \notin F$, then there exists some $j \neq i$, such that $a \leq m_j$, and $\text{infpt}(a) < \text{infpt}(m_j)$. Therefore there exists an element $b \in F_j$, such that $b(i) = a(i)$ for all i with $b(i) \neq \infty$. This implies that $a < b$, and $\text{infpt}(a) < \text{infpt}(b) = \text{infpt}(m_j)$. It follows from the definition of the map β that $\bar{a} < \bar{b}$, and that $P_{\bar{b}}$ is a prime ideal with $I^P \subset P_{\bar{b}} \subsetneq P_{\bar{a}}$, a contradiction. \square

Now let $I \subset S$ be a monomial ideal and $I^P \subset T$ be the polarization of I . Furthermore let

$$\pi: T \rightarrow S, \quad x_{i,j} \mapsto x_i.$$

be the epimorphism which attached to the polarization. Note that

$$\ker(\pi) = (x_{1,1} - x_{1,2}, \dots, x_{1,1} - x_{1,r_1}, \dots, x_{n,1} - x_{n,2}, \dots, x_{n,1} - x_{n,r_n})$$

where r_i is the number of variables of the form $x_{i,j}$ which are needed for polarization. Set

$$y := x_{1,1} - x_{1,2}, \dots, x_{1,1} - x_{1,r_1}, \dots, x_{n,1} - x_{n,2}, \dots, x_{n,1} - x_{n,r_n},$$

then y is a sequence of linear forms in T .

Proposition 3.9. *Let $I \subset S$ be a monomial ideal and I^P be the polarization of I . Assume that*

$$\mathcal{G}: \quad I^P = J_0 \subset J_1 \subset \dots \subset J_r = T$$

is a clean filtration of I^P , and that

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

is the specialization of \mathcal{G} , that is, $\pi(J_i) = I_i$ for all i . Then \mathcal{F} is a pretty clean filtration of I with $I_k/I_{k-1} \cong S/\pi(Q_k)$, where $J_k/J_{k-1} \cong T/Q_k$.

Proof. For each $k \in [r]$ the S -module I_k/I_{k-1} is a cyclic module since J_k/J_{k-1} is cyclic for all k . Let $I_k/I_{k-1} \cong S/L_k$, where L_k is a monomial ideal in S . It is clear that $\pi(Q_k) \subset L_k$. Indeed, $Q_k = J_{k-1} : u_k$, where $J_k = (J_{k-1}, u_k)$ and where $J_k/J_{k-1} \cong T/Q_k$. If $v \in Q_k$, then $vu_k \in J_{k-1}$. It follows that $\pi(vu_k) = \pi(v)\pi(u_k) \in \pi(J_{k-1}) = I_{k-1}$, and hence $\pi(v) \in I_{k-1} : \pi(u_k) = L_k$.

We want to show that $\pi(Q_k) = L_k$. S and T are standard graded with $\deg(x_i) = \deg(x_{i,j}) = 1$ for all i and j , and \mathcal{G} is a graded prime filtration of I^P . Therefore \mathcal{F} is a graded filtration

of I , and we have the following isomorphisms of graded modules $J_i/J_{i-1} \cong T/Q_i(-a_i)$ and $I_i/I_{i-1} \cong S/L_i(-a_i)$, where $a_i = \deg(u_i) = \deg(\pi(u_i))$.

The filtrations \mathcal{G} and \mathcal{F} yield the following Hilbert series of T/I^P and S/I :

$$\text{Hilb}(T/I^P) = \sum_{i=1}^r \text{Hilb}(T/Q_i)t^{a_i} \quad \text{and} \quad \text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(S/L_i)t^{a_i}.$$

Since y is a regular sequence of linear forms on T/I^P and on T/Q_i for each $i \in [r]$, we have

$$\begin{aligned} \text{Hilb}(S/I) &= (1-t)^l \text{Hilb}(T/I^P) = (1-t)^l \sum_{i=1}^r \text{Hilb}(T/Q_i)t^{a_i} \\ &= \sum_{i=1}^r (1-t)^l \text{Hilb}(T/Q_i)t^{a_i} = \sum_{i=1}^r \text{Hilb}(S/\pi(Q_i))t^{a_i}, \end{aligned}$$

where $l = |y|$.

On the other hand, since $\pi(Q_i) \subset L_i$, we have the coefficientwise inequality $\text{Hilb}(S/L_i) \leq \text{Hilb}(S/\pi(Q_i))$, in other words, $\dim_K(S/L_i)_j \leq \dim_K(S/\pi(Q_i))_j$ for all j , and equality holds if and only if $L_i = \pi(Q_i)$. Therefore we have

$$\text{Hilb}(S/I) = \sum_{i=1}^r \text{Hilb}(S/\pi(Q_i))t^{a_i} \geq \sum_{i=1}^r \text{Hilb}(S/L_i)t^{a_i} = \text{Hilb}(S/I).$$

It follows that $L_i = \pi(Q_i)$ is a prime ideal for $i = 1, \dots, r$.

We know that Γ^P the multicomplex associated to I^P is shellable, since I^P is clean. Therefore we may assume that \mathcal{G} is obtained from a shelling of Γ^P . By [6, Corollary 10.7] and its proof (or directly from the definition of shellings of a simplicial complex) it follows that $\mu(Q_i) \geq \mu(Q_{i-1})$ for all $i \in [r]$, where $\mu(Q_i)$ is the number of generators of Q_i . Since by Corollary 3.6 each Q_i is of the form $(x_{i_1, t_1}, \dots, x_{i_k, t_k})$ with $i_r \neq i_s$ for $r \neq s$, it follows that $\mu(Q_i) = \mu(\pi(Q_i)) = \mu(L_i)$. Therefore $\mu(L_i) \geq \mu(L_{i-1})$ for all i . This implies that \mathcal{F} is a pretty clean filtration of S/I . \square

As the main result of this section we have

Theorem 3.10. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal and I^P its polarization. Then the following are equivalent:*

- (a) I is pretty clean.
- (b) I^P is clean.

Proof. (a) \Rightarrow (b). Assume I is pretty clean. Then the multicomplex Γ associated with I is shellable. Let a_1, \dots, a_r be a shelling of Γ , and

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \dots \subset I_r = S$$

the pretty clean filtration of I which is obtain from this shelling, i.e., $I_i = \bigcap_{k=1}^{r-i} I(\Gamma(a_k))$. Let \mathcal{F}^p be the polarization of \mathcal{F} . By Proposition 3.4, \mathcal{F}^p is a prime filtration of I^p with $\ell(\mathcal{F}) = \ell(\mathcal{F}^p)$. Using Proposition 3.8 we have

$$|F(\Gamma^p)| = |F(\Gamma)|.$$

On the other hand, since I is pretty clean we know that $\ell(\mathcal{F}) = |F(\Gamma)|$. Hence we conclude that

$$|F(\Gamma^p)| = \ell(\mathcal{F}^p).$$

Therefore, since $\text{Min}(I^p) = \text{Ass}(I^p) \subset \text{Supp}(\mathcal{F}^p)$, it follows that $\text{Min}(I^p) = \text{Supp}(\mathcal{F}^p)$, which implies that I^p is clean.

(b) \Rightarrow (a). This follows from Proposition 3.9. \square

As an immediate consequence we obtain the following result of [6, Corollary 10.7]:

Corollary 3.11. *Let $I \subset S$ be a monomial ideal, and*

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

a prime filtration of S/I with $I_j/I_{j-1} \cong S/P_j$. Then the following are equivalent:

- (a) \mathcal{F} is a pretty clean filtration of S/I .
- (b) $\mu(P_i) \geq \mu(P_{i+1})$ for all $i = 0, \dots, r-1$.

4. A new characterization of pretty clean monomial ideals

Let R be a Noetherian ring, and M a finitely generated R -module. For $P \in \text{Spec}(R)$ the number $\text{mult}_M(P) = \ell(H_P^0(M_P))$ is called the *length multiplicity* of P with respect to M . Obviously, one has $\text{mult}_M(P) > 0$ if and only if $P \in \text{Ass}(M)$. Assume now that (R, \mathfrak{m}) is a local ring. Recall that the *arithmetic degree* of M is defined to be

$$\text{adeg}(M) = \sum_{P \in \text{Ass}(M)} \text{mult}_M(P) \cdot e(R/P),$$

where $e(R/P)$ is the *multiplicity* of the associated graded ring of R/P .

First we notice the following

Lemma 4.1. *Suppose R is a Noetherian ring, and M a finitely generated R -module. Let*

$$\mathcal{F}: \quad 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

be a prime filtration of M with $M_i/M_{i-1} \cong R/P_i$. Then

$$\text{mult}_M(P) \leq |\{i \in [r-1]: M_{i+1}/M_i \cong R/P\}|$$

for all $P \in \text{Spec}(R)$.

Proof. If $P \notin \text{Ass}(M)$, the assertion is trivial. So now let $P \in \text{Ass}(M)$. Localizing at P we may assume that P is the maximal ideal of M .

Now we will prove the assertion by induction on $\ell(\mathcal{F})$. If $\ell(\mathcal{F}) = 1$, then the assertion is obviously true. Let $\ell(\mathcal{F}) > 1$. From the following short exact sequence

$$0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$$

we get the following long exact sequence

$$0 \rightarrow H_P^0(M_1) \rightarrow H_P^0(M) \rightarrow H_P^0(M/M_1) \rightarrow \cdots.$$

Therefore $\text{mult}_M(P) = \ell(H_P^0(M)) \leq \ell(H_P^0(M_1)) + \ell(H_P^0(M/M_1))$. By induction hypothesis

$$\text{mult}_{M/M_1}(P) = \ell(H_P^0(M/M_1)) \leq |\{i \in [r-1] \setminus \{1\}: M_{i+1}/M_i \cong R/P\}|.$$

Now consider the following two cases:

(i) If $M_1 \cong R/P$, then $\ell(H_P^0(M_1)) = 1$. Therefore

$$\text{mult}_M(P) \leq 1 + \text{mult}_{M/M_1}(P) \leq |\{i \in [r-1]: M_{i+1}/M_i \cong R/P\}|.$$

(ii) If $M_1 \not\cong R/P$, then $\ell(H_P^0(M_1)) = 0$. Hence

$$\text{mult}_M(P) \leq |\{i \in [r-1]: M_{i+1}/M_i \cong R/P\}|. \quad \square$$

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field K . Let $I \subset S$ be a monomial ideal and Γ be the multicomplex associated to I . We denote the arithmetic degree of S/I by $\text{adeg}(I)$. Since $e(S/P) = 1$ for all $P \in \text{Ass}(I)$, it follows that $\text{adeg}(I) = \sum_{P \in \text{Ass}(I)} \text{mult}_I(P)$, where $\text{mult}_I(P) = \text{mult}_{S/I}(P)$. By [13, Lemma 3.3] $\text{adeg}(I) = |\text{Std}(I)|$, where $\text{Std}(I)$ is the set of standard pairs with respect to I . Also by [6, Lemma 9.14] $|\text{Std}(I)| = |F(\Gamma)|$. Since $|F(\Gamma)| = |F(\Gamma^p)|$, see Proposition 3.8, it follows that $\text{adeg}(I) = \text{adeg}(I^p)$, where I^p is the polarization of I and Γ^p the multicomplex associated to I^p .

In this part we want to show that $\text{adeg}(I)$ is a lower bound for the length of any prime filtration of S/I and the equality holds if and only if S/I is a pretty clean module.

Lemma 4.2. *Let $I \subset S$ be a monomial ideal and \mathcal{F} a prime filtration of I . One has*

- (1) $\text{adeg}(I) \leq \ell(\mathcal{F})$;
- (2) $\ell(\mathcal{F}) = \text{adeg}(I) \Leftrightarrow \mathcal{F}$ is a pretty clean filtration of I .

Proof. Part (1) is clear by Lemma 4.1.

One direction of (2) is [6, Corollary 6.4]. For the other direction assume $\ell(\mathcal{F}) = \text{adeg}(I) = |F(\Gamma)| = |F(\Gamma^p)|$. By Proposition 3.4 \mathcal{F}^p is a prime filtration of I^p with $\ell(\mathcal{F}^p) = |F(\Gamma^p)| = \text{adeg}(I^p)$. Therefore \mathcal{F}^p is a clean filtration of I^p , so by Theorem 3.10 \mathcal{F} is a pretty clean filtration of I . \square

Combining Lemma 4.2 with Theorem 3.10 we get

Theorem 4.3. Let $I \subset S$ be a monomial ideal. Assume Γ is the multicomplex associated to I and I^p the polarization of I . The following are equivalent:

- (a) Γ is shellable;
- (b) I is pretty clean;
- (c) there exists a prime filtration \mathcal{F} of I with $\ell(\mathcal{F}) = \text{adeg}(I)$;
- (d) I^p is clean;
- (e) if Δ be the simplicial complex associated to I^p , then Δ is shellable.

If R is a Noetherian ring and M a finitely generated R -module with pretty clean filtration \mathcal{F} , then $\text{Ass}(M) = \text{Supp}(\mathcal{F})$, see [6, Corollary 3.6]. The converse is not true in general as shown in [6, Example 4.4]. The example given there is a cyclic module defined by a non-monomial ideal. The following example shows that even in the monomial case the converse does not hold in general.

Example 4.4. Let $S = K[a, b, c, d]$ be the polynomial ring over the field K , $I \subset S$ the ideal

$$I = (a, b) \cdot (c, d) \cdot (a, c, d) = (abc, abd, acd, ad^2, a^2d, ac^2, a^2c, bcd, bc^2, bd^2)$$

and $M = S/I$. We claim that the module $M = S/I$ is not pretty clean, but that M has a prime filtration \mathcal{F} with $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$.

Note that $(a, b) \cap (c, d) \cap (a, c, d^2) \cap (a, c^2, d) \cap (a^2, b, c, d^2) \cap (a^2, b, c^2, d)$ modulo I is an irredundant primary decomposition of (0) in M .

We see that $\text{Ass}(M) = \{(a, b), (c, d), (a, c, d), (a, b, c, d)\}$. It is clear that

$$\begin{aligned} \mathcal{F}: \quad I &= I_0 \subset I_1 = (I, ac) \subset I_2 = (I_1, ad) \subset I_3 = (I_2, bd) \\ &\subset I_4 = (I_3, bc) \subset I_5 = (I_4, a) \subset I_6 = (a, b) \subset S \end{aligned}$$

is a prime filtration of M with $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$. Indeed $I_1/I \cong I_2/I_1 \cong S/(a, b, c, d)$, $I_3/I_2 \cong I_4/I_3 \cong I_6/I_5 \cong S/(a, c, d)$ and $I_5/I_4 \cong S/(c, d)$.

From the above irredundant primary decomposition of I it follows that $\text{adeg}(I) = 6$. But the length of any prime filtration of I is at least 7. Therefore I cannot be pretty clean. In other words, from [6, Corollary 1.2] it follows that $D_1(M) = ((a, b) \cap (c, d))/I$ and that $D_2(M) = M$, where $D_i(M)$ is the largest submodule of M with $\dim(D_i(M)) \leq i$, for $i = 0, \dots, \dim(M)$, see [10, Section 2] for definition and basic fact about dimension filtrations. It follows that $D_2(M)/D_1(M) \cong S/(a, b) \cap (c, d)$ is not clean. Knowing now $D_2(M)/D_1(M)$ is not clean, we conclude from [6, Corollary 4.2] that $M = S/I$ is not pretty clean.

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