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The number of lifts of a Brauer character with a normal vertex

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ABSTRACT

In this paper we examine the behavior of lifts of Brauer characters in p-solvable groups. In the main result, we show that if $\varphi \in \mathrm{IBr}(G)$ has a normal vertex Q and either p is odd or Q is abelian, then the number of lifts of φ is at most $|Q : Q'|$. As a corollary, we prove that if $\varphi \in \mathrm{IBr}(G)$ has an abelian vertex subgroup Q, then the number of lifts of φ in $\mathrm{Irr}(G)$ is at most $|Q|$.

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1. Introduction

Let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}(G)$. The celebrated Fong–Swan theorem asserts that there exists $\chi \in \mathrm{Irr}(G)$ such that the restriction χ^0 of χ to the p-regular elements of G is φ. We say that χ is a **lift** of φ. We write $\mathrm{L}(\varphi)$ for the set of all lifts of φ. In [1], the first author proposed that $|\mathrm{L}(\varphi)|$ should be less than or equal to $|Q/Q'|$ where Q is a vertex for φ. This global/local connection, if true, does not seem easy to prove. In [1], the first author proved that this conjecture was true for groups of odd order using some heavy machinery. In this note, we prove it in another case: when the vertex is normal in G. In view of the so called Green correspondence, this seems to be a natural key step in the right direction.

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Theorem A. *Let G be a p-solvable group and let $\varphi \in \mathrm{IBr}(G)$. Suppose φ has a normal vertex Q. If either p is odd or Q is abelian, then $|\mathrm{L}(\varphi)| \leqslant |Q : Q'|$.*

As a consequence, we can prove Cossey's conjecture whenever Q is abelian.

Corollary B. *Let G be a p-solvable group and let $\varphi \in \mathrm{IBr}(G)$. If the vertex Q for φ is abelian, then $|\mathrm{L}(\varphi)| \leqslant |Q|$.*

2. Proofs

Let G be a finite group. Recall that the defect zero characters of G are

$$\mathrm{dz}(G) = \bigl\{\chi \in \mathrm{Irr}(G) \bigm| \chi(1)_p = |G|_p\bigr\}.$$

If $\gamma^G = \chi \in \mathrm{Irr}(G)$, then it is easy to prove that γ has defect zero if and only if χ has defect zero.

Given a normal subgroup of G, we can also consider the relative defect zero characters. If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then

$$\mathrm{rdz}(G \mid \theta) = \bigl\{\chi \in \mathrm{Irr}(G \mid \theta) \bigm| \bigl(\chi(1)/\theta(1)\bigr)_p = |G : N|_p\bigr\}.$$

The following is Theorem (2.1) of [8], and gives a connection between defect zero characters and relative defect zero characters when N is a p-group.

Theorem 2.1. *If N is a normal p-subgroup of G and $\theta \in \mathrm{Irr}(N)$ is G-invariant, then there exists a natural bijection $\chi \mapsto \chi_\theta$ from $\mathrm{dz}(G/N) \to \mathrm{rdz}(G \mid \theta)$. If θ is linear, then $\chi_\theta(g) = \chi(g)$ for every p-regular $g \in G$.*

Proof. With the notation of [8], if θ is linear, then $\hat{\theta}(g) = 1$ for every p-regular element $g \in G$. Now apply the formula in Theorem (2.1.a) of [8]. □

We also need to consider π-special characters where π is a set of primes. Let G be a π-separable group. A character $\chi \in \mathrm{Irr}(G)$ is π-special if $\chi(1)$ is a π-number and for every subnormal group M of G, the irreducible constituents of χ_M have determinants that have π-order. Many of the basic results of π-special characters can be found in Section 40 of [2] and Chapter VI of [6]. One result that is proved is that if α is π-special and β is π'-special, then $\alpha\beta$ is necessarily irreducible. We are particularly interested in the case where $\pi = \{p\}$. We say that χ is **factored** if $\chi = \alpha\beta$ where α is p'-special and β is p-special.

The following is Lemma 2.1 of [9], which shows that irreducibility does not occur when we restrict a p-special character to the p-regular elements when p is odd. Note that $\mathrm{GL}(2,3)$ has a 2-special character of degree 2 whose restriction to the 2-regular elements yields an irreducible 2-Brauer character. Hence, this lemma is not true without the assumption that p is odd.

Lemma 2.2. *Let p be an odd prime and let G be a p-solvable group. Let $\chi \in \mathrm{Irr}(G)$ be p-special. If $\chi(1) > 1$, then χ^o is not in $\mathrm{IBr}(G)$.*

The next lemma is an easy observation.

Lemma 2.3. *If $\chi \in \mathrm{Irr}(G)$ and $U \subseteq G$, then the number of irreducible characters $\nu \in \mathrm{Irr}(U)$ inducing χ is less than or equal to $|G : U|$.*

Proof. Suppose that $\nu_i \in \mathrm{Irr}(U)$ are different characters inducing χ. In particular, $\nu_i(1) = \chi(1)/|G : U|$. Now, $\chi_U = \nu_1 + \cdots + \nu_s + \Delta$ where Δ is either a character of U or the zero function, and

$$\chi(1) \geqslant s\big(\chi(1)/|G:U|\big),$$

which proves the lemma. $\square$

We will need to know that a character $\chi \in \mathrm{Irr}(G)$ is induced from a character with certain properties.

Lemma 2.4. *Let G be a p-solvable group. If $\chi \in \mathrm{Irr}(G)$, then there exists a pair (W, γ) so that $W \subseteq G$ and $\gamma \in \mathrm{Irr}(W)$ satisfies $\gamma^G = \chi$, γ is factored, and W contains $\mathbf{O}_p(G)$.*

Proof. Consider a chief series extending from $\mathbf{O}_p(G)$ up to G. If the restriction of χ is homogeneous for every term of this series, then χ will be factored (see Theorem 21.7 of [6]), and we take $(W, \gamma) = (G, \chi)$. Thus, we may assume that χ_N is not homogeneous for some N in this series. Let θ be an irreducible constituent of χ_N. Take T to be the stabilizer of θ in G. Let $\tau \in \mathrm{Irr}(T \mid \theta)$ be the Clifford correspondent for χ (see Theorem 6.11 of [3]). We know $T < G$. Thus, by induction, there exists (W, γ) so that $\gamma^T = \tau$, γ is factored and $\mathbf{O}_p(T) \leqslant W$. It follows that $\gamma^G = (\gamma^T)^G = \tau^G = \chi$, and $\mathbf{O}_p(G) \leqslant \mathbf{O}_p(T) \leqslant W$. This proves the result. $\square$

For a p-solvable group, a (Green) vertex for $\varphi \in \mathrm{IBr}(G)$ can be characterized as a p-subgroup Q with the property that there is a subgroup U of G with a Brauer character of p' degree of U that induces φ and Q is a Sylow p-subgroup of U (see [5]).

Lemma 2.5. *Suppose that G is p-solvable. Let $N = \mathbf{O}_p(G)$. Suppose that $\varphi \in \mathrm{IBr}(G)$ has vertex N. Then there exists a unique character $\chi \in \mathrm{Irr}(G/N)$ that is a lift of φ.*

Proof. Since $\varphi \in \mathrm{IBr}(G)$ has vertex N, we have $\varphi(1)_p = |G/N|_p$. We may consider $\varphi \in \mathrm{IBr}(G/N)$, and let $\chi \in \mathrm{Irr}(G/N)$ be a lift of φ. Then χ has defect zero in G/N. If $\psi \in \mathrm{Irr}(G/N)$ is another lift of φ, then it follows that ψ has also defect zero in G/N. Since χ and ψ coincide on p-regular elements of G/N and are zero on the p-singular elements of G/N, we have that $\chi = \psi$, as claimed. $\square$

We now prove the following theorem which includes Theorem A.

Theorem 2.6. *Suppose that G is p-solvable. Let $N = \mathbf{O}_p(G)$. Assume that p is odd or that N is abelian. Suppose that $\varphi \in \mathrm{IBr}(G)$ has vertex N, and write χ for the unique lift of φ in $\mathrm{Irr}(G/N)$. Let $\mathcal{A}$ be a complete set of representatives of the G-action on $\mathrm{Irr}(N/N')$. For $\lambda \in \mathcal{A}$, let $\mathcal{B}_\lambda$ be the set of the irreducible defect zero characters of T_λ/N inducing χ, where T_λ is the stabilizer of λ in G, and let $\mathcal{C}_\lambda = \{(\nu_\lambda)^G \mid \nu \in \mathcal{B}_\lambda\}$. Then $\mathrm{L}(\varphi) = \bigcup_{\lambda \in \mathcal{A}} \mathcal{C}_\lambda$ is a disjoint union. In addition, $|\mathcal{C}_\lambda| = |\mathcal{B}_\lambda|$ for each $\lambda \in \mathcal{A}$. In particular, $|\mathrm{L}(\varphi)| = \sum_{\lambda \in \mathcal{A}} |\mathcal{B}_\lambda| \leqslant \sum_{\lambda \in \mathcal{A}} |G : T_\lambda| = |N/N'|$.*

Proof. If $\nu \in \mathcal{B}_\lambda$, then $\nu^G = \chi$ by hypothesis. Then

$$\big((\nu_\lambda)^G\big)^o = \big((\nu_\lambda)^o\big)^G = \big(\nu^o\big)^G = \big(\nu^G\big)^o = \chi^o = \varphi.$$

Hence, $\mathcal{C}_\lambda$ is contained in $\mathrm{L}(\varphi)$. Also, since ν_λ lies over λ, it follows that $(\nu_\lambda)^G \in \mathrm{Irr}(G \mid \lambda)$. In particular, $\mathcal{C}_\lambda \cap \mathcal{C}_\tau$ is empty if $\lambda \neq \tau$. (This is Clifford's theorem, Theorem 6.11 of [3].)

Now, if $\nu, \mu \in \mathcal{B}_\lambda$ and $(\nu_\lambda)^G = (\mu_\lambda)^G$, then $\nu_\lambda = \mu_\lambda$ by the Clifford correspondence. Hence, $\nu = \mu$ by Theorem 2.1, and we conclude that $|\mathcal{C}_\lambda| = |\mathcal{B}_\lambda|$.

Let $\psi \in \mathrm{L}(\varphi)$. We need to show that $\psi \in \mathcal{C}_\lambda$ for some $\lambda \in \mathcal{A}$. We have that $\psi(1) = \varphi(1)$ and therefore $\psi(1)_p = |G/N|_p$. Now, let $\theta \in \mathrm{Irr}(N)$ be under ψ. We claim that θ is linear. This is clear if N is abelian. So let us assume that p is odd.

To prove the claim, find (W, γ) for ψ as in Lemma 2.4. Write γ_p for the p-special factor of ψ. Since $((\gamma)^G)^o = \psi^o$ is irreducible, it follows that γ^o is irreducible. Therefore, $(\gamma_p)^o \in \mathrm{IBr}(W)$. Hence,

γ_p is linear by Lemma 2.2. Since W contains N, the claim follows. In particular, we have now that $\psi \in \mathrm{rdz}(G \mid \theta)$.

Therefore, we may assume that $\theta \in \mathcal{A}$. Let $\eta \in \mathrm{Irr}(T_\theta \mid \theta)$ be the Clifford correspondent of ψ over θ. Then $\eta \in \mathrm{rdz}(T_\theta \mid \theta)$. By Theorem 2.1, we have that $\eta = \nu_\theta$ for some $\nu \in \mathrm{dz}(T_\theta/N)$. Now

$$\varphi = \psi^o = \left(\eta^o\right)^G = \left((\nu_\theta)^o\right)^G = \left(\nu^o\right)^G = \left(\nu^G\right)^o.$$

We conclude that ν^G is a lift of φ in G/N and therefore $\nu^G = \chi$. Thus, $\nu \in \mathcal{B}_\theta$, and the first part follows.

By Lemma 2.3, we have that $|\mathcal{B}_\lambda| \leqslant |G : T_\lambda|$, and then

$$\left|\mathrm{L}(\varphi)\right| \leqslant \sum_{\lambda \in \mathcal{A}} |G : T_\lambda| = \left|N/N'\right|,$$

as desired. □

Proof of Corollary B. Let $M = \mathbf{O}_{p'}(G)$, and let $\tau \in \mathrm{Irr}(M)$ be such that the Clifford correspondent μ of φ over τ has vertex Q. (Clifford's theorem for Brauer characters is Theorem 8.9 of [7].) We may view τ as both a Brauer character and an ordinary character of M. Using Clifford's theorem of ordinary characters, we see that $|\mathrm{L}(\varphi)| = |\mathrm{L}(\mu)|$. Thus, we can assume that τ is G-invariant.

By Theorem 5.2 of [4], there is a character triple (G^*, M^*, τ^*) which is isomorphic to (G, M, τ) and where M^* is a central, p'-subgroup. Take H to be a Hall p-complement of G. Let χ^* correspond to χ and H^* correspond to H, and note that H^* is a Hall p-complement of G^*. By the Fong–Swan theorem, χ^o is not irreducible if and only if there exist characters α, β such that $\chi^o = \alpha^o + \beta^o$. This occurs if and only if $\chi_H = \alpha_H + \beta_H$. Using the character triple isomorphism, this is equivalent to $(\chi^*)_{H^*} = (\alpha^*)_{H^*} + (\beta^*)_{H^*}$ and to $(\chi^*)^o = (\alpha^*)^o + (\beta^*)^o$. We conclude that χ^o is irreducible if and only if $(\chi^*)^o$ is irreducible. Suppose ψ is a lift of φ, then we define $\varphi^* = (\psi^*)^o$. Notice that $\chi \in \mathrm{Irr}(G)$ is a lift of φ if and only if $\chi_H = \varphi_H$. It follows that χ is a lift of φ if and only if χ^* is a lift of φ^*, and so, $|\mathrm{L}(\varphi)| = |\mathrm{L}(\varphi^*)|$. Thus, we may assume that M is central in G.

Let $N = \mathbf{O}_p(G)$. By Lemma 2.4, we know that $N \subseteq Q$. Using the Hall–Higman 1.2.3 Lemma, we determine that $\mathbf{C}_G(N) \subseteq MN$. Since Q is abelian, it follows that $Q \subseteq \mathbf{C}_G(N)$. As N is the unique Sylow p-subgroup of NM, it follows that $Q = N$. Then we apply Theorem 2.6. □

We conclude by noting that Corollary B could be analogously proved if φ is an Isaacs' π-partial character of G whose vertex is abelian where G is a π-separable group and π is a set of primes. Theorem A can be analogously proved for a π-partial character φ whose vertex is normal and either $2 \in \pi$ or the vertex is also abelian.

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