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Group actions on algebras and the graded Lie structure of Hochschild cohomology ☆

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ABSTRACT

Hochschild cohomology governs deformations of algebras, and its graded Lie structure plays a vital role. We study this structure for the Hochschild cohomology of the skew group algebra formed by a finite group acting on an algebra by automorphisms. We examine the Gerstenhaber bracket with a view toward deformations and developing bracket formulas. We then focus on the linear group actions and polynomial algebras that arise in orbifold theory and representation theory; deformations in this context include graded Hecke algebras and symplectic reflection algebras. We give some general results describing when brackets are zero for polynomial skew group algebras, which allow us in particular to find noncommutative Poisson structures. For abelian groups, we express the bracket using inner products of group characters. Lastly, we interpret results for graded Hecke algebras.

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1. Introduction

Let G be a finite group acting on a $\mathbb{C}$-algebra S by automorphisms. Deformations of the natural semi-direct product $S\#G$, the *skew group algebra*, include many compelling and influential algebras. For example, when G acts linearly on a finite dimensional, complex vector space V, it induces an action on the symmetric algebra $S(V)$ (a polynomial ring). Deformations of $S(V)\#G$ play a profound

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role in representation theory and connect diverse areas of mathematics. Graded Hecke algebras were originally defined independently by Drinfeld [6] and (in the special case of Coxeter groups) by Lusztig (see [15–17]), who described them as graded versions of the affine Hecke algebra. (They are often called *Drinfeld Hecke algebras* for this reason.) These deformations of $S(V)\#G$ include symplectic reflection algebras (investigated by Etingof and Ginzburg [7] in the study of orbifolds) and rational Cherednik algebras (degenerations of the algebras introduced by Cherednik [5] to solve Macdonald's inner product conjectures). Gordon [11] used these algebras to prove a version of the $n!$ conjecture for Weyl groups due to Haiman.

The deformation theory of an algebra is governed by its Hochschild cohomology as a graded Lie algebra under the Gerstenhaber bracket. Each deformation of the algebra arises from a (noncommutative) *Poisson structure*, that is, an element of Hochschild cohomology in degree 2 whose Gerstenhaber square bracket is zero. Thus, a first step in understanding an algebra's deformation theory is a depiction of the Gerstenhaber bracket. In some situations, every noncommutative Poisson structure *integrates*, i.e., lifts to a deformation (e.g., see Kontsevich [13]). Halbout and Tang [12] investigate some of these structures for algebras $C^\infty(M)\#G$ where M is a real manifold, concentrating on the case that M has a G-invariant symplectic structure.

In this paper, we explore the rich algebraic structure of the Hochschild cohomology of $S\#G$ with an eye toward deformation theory. We describe the Gerstenhaber bracket on the Hochschild cohomology of $S\#G$, for a general algebra S. We then specialize to the case $S = S(V)$ to continue an analysis begun in two previous articles. Hochschild cohomology is a Gerstenhaber algebra under two operations, a cup product and a graded Lie (Gerstenhaber) bracket. In [20], we examined the cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ as a graded algebra under cup product. In this paper, we study its Gerstenhaber bracket and find noncommutative Poisson structures. These structures catalog potential deformations of $S(V)\#G$, most of which have not yet been explored.

For any algebra A over a field k, both the cup product and Gerstenhaber bracket on the Hochschild cohomology $\mathrm{HH}^\bullet(A) := \mathrm{Ext}^\bullet_{A\otimes A^{op}}(A, A)$ are defined initially on the bar resolution, a natural $A \otimes A^{op}$-free resolution of A. The cup product has another description as Yoneda composition of extensions of modules, which can be transported to any other projective resolution. However, the Gerstenhaber bracket has resisted such a general description. In this paper, we use isomorphisms of cohomology which encode traffic between resolutions to analyze $\mathrm{HH}^\bullet(S\#G)$ and unearth its Gerstenhaber bracket.

For the case $S = S(V)$, we use Demazure operators on one hand and quantum partial differentiation on the other hand to render the Gerstenhaber bracket and gain theorems which predict its vanishing. For example, the cohomology $\mathrm{HH}^\bullet(S\#G)$ breaks into a direct sum over G. By invoking Demazure operators to implement automorphisms of cohomology, we procure the following main result in Section 9:

Theorem 9.2. *The bracket of any two elements α, β in $\mathrm{HH}^2(S(V)\#G)$ supported on group elements acting nontrivially on V is zero*: $[\alpha, \beta] = 0$.

As a consequence of the theorem, the Gerstenhaber square bracket of every element $\mathrm{HH}^2(S(V)\#G)$ supported off the kernel of the action of G on V defines a noncommutative Poisson structure. In particular, if G acts faithfully, any Hochschild 2-cocycle with zero contribution from the identity group element defines a noncommutative Poisson structure.

The cohomology $\mathrm{HH}^\bullet(S\#G)$ is graded not only by cohomological degree, but also by polynomial degree. In [18], we showed that every *constant* Hochschild 2-cocycle (i.e., of polynomial degree 0) defines a graded Hecke algebra. Since graded Hecke algebras are deformations of $S(V)\#G$ (see Section 11), this immediately implies that every constant Hochschild 2-cocycle has square bracket zero. We articulate our automorphisms of cohomology using quantum partial differentiation to extend this result in Section 8 to cocycles of *arbitrary* cohomological degree:

Theorem 8.1. *The bracket of any two* constant *cocycles α, β in $\mathrm{HH}^\bullet(S(V)\#G)$ is zero*: $[\alpha, \beta] = 0$.

Supporting these two main "zero bracket" theorems, we give formulas for calculating brackets at the cochain level in Theorems 6.12 and 7.2. We apply these formulas to the abelian case in Exam-

ple 7.6 and express the bracket of 2-cocycles in terms of inner products of characters. We use this example in Theorem 10.2 to show that the hypotheses of Theorem 9.2 cannot be weakened and that its converse is false for abelian groups.

We briefly outline our program. In Section 2, we establish notation and definitions. In Sections 3 and 4, we construct an explicit, instrumental isomorphism between $\mathrm{HH}^\bullet(S\#G)$ and $\mathrm{HH}^\bullet(S, S\#G)^G$ and lift the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S\#G)$ to an arbitrary resolution used to compute $\mathrm{HH}^\bullet(S, S\#G)^G$. In Sections 5 and 6, we turn to the case $S = S(V)$ and examine isomorphisms of cohomology developed in [19]. We lift the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$ from the bar resolution of $S(V)\#G$ to the Koszul resolution of $S(V)$. In Section 7, we give our closed formulas for the bracket in terms of quantum differentiation and recover the classical Schouten–Nijenhuis bracket in case $G = \{1\}$. These formulas allow us to characterize geometrically some cocycles with square bracket zero in Section 8.

Mindful of applications to deformation theory, we turn our attention to cohomological degree two in Section 9. We prove the above Theorem 9.2 (which applies to cocycles of *arbitrary* polynomial degree) and focus on abelian groups in Section 10. We then apply our approach to the deformation theory of $S(V)\#G$ and graded Hecke algebras. When a Hochschild 2-cocycle is a graded map of negative degree, corresponding deformations are well understood and include graded Hecke algebras (which encompass symplectic reflection algebras and rational Cherednik algebras). In Section 11, we explicitly present graded Hecke algebras as deformations defined using Hochschild cohomology, thus fitting our previous work on graded Hecke algebras into a more general program to understand all deformations of $S(V)\#G$. We show how to render a constant Hochschild 2-cocycle into the defining relations of a graded Hecke algebra concretely and vice versa. In fact, we use our results to explain how to convert *any* Hochschild 2-cocycle into the explicit multiplication map of an infinitesimal deformation of $S(V)\#G$, allowing for exploration of a wide class of algebras which include graded Hecke algebras as examples.

2. Preliminary material

Let G be a finite group. We work over the complex numbers $\mathbb{C}$; all tensor products will be taken over $\mathbb{C}$ unless otherwise indicated. Let S be any $\mathbb{C}$-algebra on which G acts by automorphisms. Denote by ${}^g s$ the result of applying an element g of G to an element s of S. The *skew group algebra* $S\#G$ is the vector space $S \otimes \mathbb{C}G$ with multiplication given by

$$(r \otimes g)(s \otimes h) = r\big({}^g s\big) \otimes gh$$

for all r, s in S and g, h in G. We abbreviate $s \otimes g$ by $s\overline{g}$ ($s \in S$, $g \in G$) and $s \otimes 1$, $1 \otimes g$ simply by s, $\overline{g}$, respectively. An element g in G acts on S by an inner automorphism in $S\#G$: $\overline{g}s(\overline{g})^{-1} = ({}^g s)\overline{g}(\overline{g})^{-1} = {}^g s$ for all s in S. We work with the induced group action on all maps throughout this article: For any map θ and element h in $\mathrm{GL}(V)$, we define ${}^h\theta$ by $({}^h\theta)(v) := {}^h(\theta({}^{h^{-1}}v))$ for all v.

Hochschild cohomology and deformations. The *Hochschild cohomology* of a $\mathbb{C}$-algebra A with coefficients in an A-bimodule M is the graded vector space $\mathrm{HH}^\bullet(A, M) = \mathrm{Ext}^\bullet_{A^e}(A, M)$, where $A^e = A \otimes A^{op}$ acts on A by left and right multiplication. We abbreviate $\mathrm{HH}^\bullet(A) = \mathrm{HH}^\bullet(A, A)$.

One projective resolution of A as an A^e-module is the *bar resolution*

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{2.1}$$

where $\delta_p(a_0 \otimes \cdots \otimes a_{p+1}) = \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{p+1}$, and $\delta_0 = m$ is multiplication. For each p, $\mathrm{Hom}_{A^e}(A^{\otimes(p+2)}, *) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, *)$, and we identify these two spaces of p-cochains throughout this article.

The *Gerstenhaber bracket* on Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is defined at the chain level on the bar complex. Let $f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A)$ and $f' \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes q}, A)$. The Gerstenhaber bracket $[f, f']$ in $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes(p+q-1)}, A)$ is defined as

$$\left[f, f'\right] = f \,\overline{\circ}\, f' - (-1)^{(p-1)(q-1)} f' \,\overline{\circ}\, f, \tag{2.2}$$

where

$$\begin{aligned} &f \,\overline{\circ}\, f'(a_1 \otimes \cdots \otimes a_{p+q-1}) \\ &= \sum_{k=1}^{p} (-1)^{(q-1)(k-1)} f\left(a_1 \otimes \cdots \otimes a_{k-1} \otimes f'(a_k \otimes \cdots \otimes a_{k+q-1}) \otimes a_{k+q} \otimes \cdots \otimes a_{p+q-1}\right) \end{aligned}$$

for all $a_1, \ldots, a_{p+q+1}$ in A. This induces a bracket on $\mathrm{HH}^\bullet(A)$ under which it is a graded Lie algebra. The bracket is compatible with the cup product on $\mathrm{HH}^\bullet(A)$, in the sense that if $\alpha \in \mathrm{HH}^p(A)$, $\beta \in \mathrm{HH}^q(A)$, and $\gamma \in \mathrm{HH}^r(A)$, then

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{p(r-1)} \alpha \smile [\beta, \gamma].$$

Thus $\mathrm{HH}^\bullet(A)$ becomes a Gerstenhaber algebra.

Let t be an indeterminate. A *formal deformation* of A is an associative $\mathbb{C}[[t]]$-algebra structure on formal power series $A[[t]]$ with multiplication determined by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots$$

for all a, b in A, where $ab = m(a \otimes b)$ and the $\mu_i : A \otimes A \to A$ are linear maps. In case the above sum is finite for each a, b in A, we may consider the subalgebra $A[t]$, a deformation of A over the polynomial ring $\mathbb{C}[t]$. Graded Hecke algebras (see Section 11) are examples of deformations over $\mathbb{C}[t]$.

Associativity of the product $*$ implies in particular that μ_1 is a Hochschild 2-cocycle (i.e., $\delta_3^*(\mu_1) = 0$) and that the Gerstenhaber bracket $[\mu_1, \mu_1]$ is a coboundary (in fact, $[\mu_1, \mu_1] = 2\delta_3^*(\mu_2)$, see [9, (4_2)]). If f is a Hochschild 2-cocycle, then f may or may not be the first multiplication map μ_1 for some formal deformation. The *primary obstruction* to lifting f to a deformation is the Gerstenhaber bracket $[f, f]$ considered as an element of $\mathrm{HH}^3(A)$. A (noncommutative) *Poisson structure* on A is an element α in $\mathrm{HH}^2(A)$ such that $[\alpha, \alpha] = 0$ as an element of $\mathrm{HH}^3(A)$. The set of noncommutative Poisson structures includes the cohomology classes of the first multiplication maps μ_1 of all deformations of A, and thus every deformation of A defines a noncommutative Poisson structure. (For more details on Hochschild cohomology, see Weibel [22], and on deformations, see Gerstenhaber [9]; Poisson structures for noncommutative algebras are introduced in Block and Getzler [2] and Xu [24].)

3. Hochschild cohomology of skew group algebras

We briefly describe various isomorphisms used to compute Hochschild cohomology. Let S be any algebra upon which a finite group G acts by automorphisms, and let $A := S\#G$ denote the resulting skew group algebra. Let $\mathcal{C}$ be a set of representatives of the conjugacy classes of G. For any g in G, let $Z(g)$ be the centralizer of g. A result of Ştefan [21, Corollary 3.4] implies that there is a G-action giving the first in a series of isomorphisms of graded vector spaces:

$$\begin{aligned} \mathrm{HH}^\bullet(S\#G) &\cong \mathrm{HH}^\bullet(S, S\#G)^G \\ &\cong \left(\bigoplus_{g \in G} \mathrm{HH}^\bullet(S, S\overline{g})\right)^G \\ &\cong \bigoplus_{g \in \mathcal{C}} \mathrm{HH}^\bullet(S, S\overline{g})^{Z(g)}. \end{aligned} \tag{3.1}$$

The first isomorphism is in fact an isomorphism of graded algebras (under the cup product); it follows from applying a spectral sequence. We take a direct approach in this paper to gain results on the Gerstenhaber bracket. The second isomorphism results from decomposing the S^e-module $S\#G$ into the direct sum of components $S\overline{g}$. The action of G permutes these components via the conjugation action of G on itself. The third isomorphism canonically projects onto a set of representative summands.

In the next theorem, we begin transporting Gerstenhaber brackets on $\mathrm{HH}^\bullet(S\#G)$ to the other spaces in (3.1) by etching an *explicit isomorphism* between $\mathrm{HH}^\bullet(S\#G)$ and $\mathrm{HH}^\bullet(S, S\#G)^G$. First, some definitions. We say that a projective resolution $P_\bullet$ of S^e-modules is *G-compatible* if G acts on each term in the resolution and the action commutes with the differentials. Let $\mathcal{R}$ be the *Reynolds operator* on $\mathrm{Hom}_{\mathbb{C}}(S^{\otimes\bullet}, A)$,

$$\mathcal{R}(\gamma) := \frac{1}{|G|}\sum_{g\in G} {}^g\gamma.$$

We shall use the same notation $\mathcal{R}$ to denote the analogous operator on any vector space carrying an action of G. Let Θ^* be the map "move group elements far right, applying them along the way": Define $\Theta^* : \mathrm{Hom}_{\mathbb{C}}(S^{\otimes p}, A)^G \to \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A)$ by

$$\Theta^*(\kappa)(f_1\overline{g}_1 \otimes \cdots \otimes f_p\overline{g}_p) = \kappa\big(f_1 \otimes {}^{g_1}f_2 \otimes \cdots \otimes {}^{(g_1\cdots g_{p-1})}f_p\big)\overline{g_1\cdots g_p} \tag{3.2}$$

for all $f_1, \ldots, f_p$ in S and $g_1, \ldots, g_p$ in G. A similar map appears in [12]. In the next theorem, we give an elementary explanation for the appearance of Θ^* in an explicit isomorphism of cohomology; the proof shows that Θ^* is induced from a map Θ at the chain level.

Remark 3.3. The notion of G-invariant cohomology here is well formulated. If S and B are algebras on which G acts by automorphisms and B is an S-bimodule, we may define G-invariant cohomology $\mathrm{HH}^\bullet(S, B)^G$ via any G-compatible resolution. This definition does not depend on choice of resolution. Indeed, since $|G|$ is invertible in our field, any chain map between resolutions can be averaged over the group to produce a G-invariant chain map. By the Comparison Theorem [22, Theorem 2.2.6], this yields not only an isomorphism on cohomology $\mathrm{HH}^\bullet(S, B)$ arising from two different resolutions, but a G-invariant isomorphism.

We adapt ideas of Theorem 5.1 of [4] to our general situation. It was assumed implicitly in that theorem that the resolution defining cohomology was G-compatible and that the cochains fed into the given map were G-invariant. We generalize that theorem while simultaneously making it explicit. We apply a Reynolds operator to turn the cochains fed into the map Θ^* invariant, as they might not be so a priori. This allows us to apply chain maps to arbitrary cocycles, in fact, to cocycles that may not even represent invariant cohomology classes.

Theorem 3.4. *Let S be an arbitrary $\mathbb{C}$-algebra upon which a finite group G acts by automorphisms, and let $A = S\#G$. The map*

$$\Theta^* \circ \mathcal{R} : \mathrm{Hom}_{\mathbb{C}}\big(S^{\otimes p}, A\big) \to \mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes p}, A\big),$$

given by

$$\big(\big(\Theta^* \circ \mathcal{R}\big)(\beta)\big)(f_1\overline{g}_1 \otimes \cdots \otimes f_p\overline{g}_p) = \frac{1}{|G|}\sum_{g\in G} {}^g\beta\big(f_1 \otimes {}^{g_1}f_2 \otimes \cdots \otimes {}^{g_1\cdots g_{p-1}}f_p\big)\overline{g_1\cdots g_p}$$

for all $f_1, \ldots, f_p \in S$, $g_1, \ldots, g_p \in G$, and $\beta \in \mathrm{Hom}_{\mathbb{C}}(S^{\otimes p}, A)$, induces an isomorphism

$$\mathrm{HH}^p(S, A)^G \xrightarrow{\sim} \mathrm{HH}^p(A).$$

We note that the following proof does not require the base field to be $\mathbb{C}$, only that $|G|$ be invertible in the field.

Proof. We express every G-compatible resolution as a resolution over a ring that succinctly absorbs the group and its action: Let

$$\Delta := \bigoplus_{g\in G} S\overline{g} \otimes S\overline{g^{-1}},$$

a natural subalgebra of A^e containing S^e. Any S^e-module carrying an action of G is naturally also a Δ-module: Each group element g acts as the ring element $\overline{g} \otimes \overline{g^{-1}}$ in Δ. In fact, an S^e-resolution of a module is G-compatible if and only if it is simultaneously a Δ-resolution.

The bar resolution for S is G-compatible, i.e., extends to a Δ-resolution. We define a different resolution of S as a Δ-module that interpolates between the bar resolution of S and the bar resolution of A. For each $p \geqslant 0$, let

$$\Delta_p := \left\{ \sum f_0\overline{g}_0 \otimes \cdots \otimes f_{p+1}\overline{g}_{p+1} \;\middle|\; f_i \in S,\ g_i \in G \text{ and } g_0 \cdots g_{p+1} = 1 \right\},$$

a Δ-submodule of $A^{\otimes(p+2)}$. Note that $\Delta_0 = \Delta$, each Δ_p is a projective Δ-module, and Δ_p is also a projective S^e-module by restriction. The complex

$$\cdots \xrightarrow{\delta_3} \Delta_2 \xrightarrow{\delta_2} \Delta_1 \xrightarrow{\delta_1} \Delta_0 \xrightarrow{m} S(V) \to 0$$

(where m is multiplication and δ_i is the restriction of the differential on the bar resolution of A) is a Δ-projective resolution of S. By restriction it is also an S^e-projective resolution of S. The inclusion $S^{\otimes(p+2)} \hookrightarrow \Delta_p$ induces a restriction map

$$\mathrm{Hom}_{S^e}(\Delta_p, A) \to \mathrm{Hom}_{S^e}\left(S^{\otimes(p+2)}, A\right)$$

that in turn induces an automorphism on the cohomology $\mathrm{HH}^p(S, A)$. The subspace of G-invariant elements of either of the above Hom spaces is precisely the subspace of Δ-homomorphisms. Therefore, a G-invariant element of $\mathrm{Hom}_{S^e}(S^{\otimes(p+2)}, A)$ identifies with an element of $\mathrm{Hom}_\Delta(\Delta_p, A)$, and its application to an element of Δ_p is found via the Δ-chain map $\Delta_\bullet \to S^{\otimes(\bullet+2)}$ given by

$$f_0\overline{g}_0 \otimes \cdots \otimes f_{p+1}\overline{g}_{p+1} \mapsto f_0 \otimes {}^{g_0}f_1 \otimes {}^{g_0g_1}f_2 \otimes \cdots \otimes {}^{(g_0g_1\cdots g_p)}f_{p+1} \tag{3.5}$$

for all $f_0, \ldots, f_{p+1} \in S$ and $g_0, \ldots, g_{p+1} \in G$. Finally, one may check that the following map is an isomorphism of A^e-modules $A^{\otimes(p+2)} \xrightarrow{\sim} A^e \otimes_\Delta \Delta_p$:

$$f_0\overline{g}_0 \otimes \cdots \otimes f_{p+1}\overline{g}_{p+1} \mapsto (1 \otimes \overline{g_0 \cdots g_{p+1}}) \otimes \left(f_0\overline{g}_0 \otimes \cdots \otimes f_{p+1}\overline{g}_{p+1}(\overline{g_0 \cdots g_{p+1}})^{-1}\right). \tag{3.6}$$

This gives rise to an isomorphism $\mathrm{Hom}_\Delta(\Delta_p, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes(p+2)}, A)$. In fact, one may obtain the bar resolution of A from the Δ-resolution $\Delta_\bullet$ directly by applying the functor $A^e \otimes_\Delta \otimes -$; the map (3.6) realizes the corresponding Eckmann–Shapiro isomorphism on cohomology, $\mathrm{Ext}^\bullet_\Delta(S, A) \cong \mathrm{Ext}^\bullet_{A^e}(A, A)$, at the chain level. Let $\Theta : A^{\otimes(p+2)} \to A^e \otimes_\Delta S^{\otimes(p+2)}$ denote the composition of (3.6) and (3.5), that is,

$$\Theta(f_0\overline{g}_0 \otimes \cdots \otimes f_{p+1}\overline{g}_{p+1}) = (1 \otimes \overline{g_0 \cdots g_{p+1}}) \otimes f_0 \otimes {}^{g_0}f_1 \otimes {}^{g_0g_1}f_2 \otimes \cdots \otimes {}^{(g_0g_1\cdots g_p)}f_{p+1}$$

for all $f_0, \dots, f_{p+1} \in S$ and $g_0, \dots, g_{p+1} \in G$. The induced map Θ^* on cochains is indeed that defined by Eq. (3.2). The above arguments show that this induced map Θ^* gives an explicit isomorphism, at the chain level, from $\mathrm{HH}^p(S, A)^G$ to $\mathrm{HH}^p(A, A)$. □

4. Lifting brackets to other resolutions

The Gerstenhaber bracket on the Hochschild cohomology of an algebra is defined using the bar resolution of that algebra. But cohomology is often computed using some other projective resolution. One seeks to express the induced Gerstenhaber bracket on any other projective resolution giving cohomology. Again, let S be any algebra upon which a finite group G acts by automorphisms, and let $A := S\#G$ denote the resulting skew group algebra. In this section, we lift the Gerstenhaber bracket on $\mathrm{HH}^\bullet(A)$ to other resolutions used to compute cohomology. This task is complicated by the fact that we compute the cohomology $\mathrm{HH}^\bullet(A)$ as the space $\mathrm{HH}^\bullet(S, A)^G$ (using the isomorphisms of (3.1)). Hence, we consider alternate resolutions of S, not A.

Suppose the Hochschild cohomology of S has been determined using some S^e-projective resolution $P_\bullet$ of S. We assume that $P_\bullet$ is G-compatible, so that $P_\bullet$ defines the G-invariant cohomology $\mathrm{HH}^\bullet(S, A)^G$. Let Ψ and Φ be chain maps between the bar resolution and $P_\bullet$, i.e.,

$$\Psi_p : S^{\otimes(p+2)} \to P_p,$$
$$\Phi_p : P_p \to S^{\otimes(p+2)},$$

for all $p \geqslant 1$, and the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S^{\otimes(p+2)} & \xrightarrow{\delta_p} & S^{\otimes(p+1)} & \longrightarrow & \cdots \\ & & \Psi_p \downarrow \uparrow \Phi_p & & \Psi_{p-1} \downarrow \uparrow \Phi_{p-1} & & \\ \cdots & \longrightarrow & P_p & \xrightarrow{d_p} & P_{p-1} & \longrightarrow & \cdots . \end{array} \tag{4.1}$$

Let Ψ^* and Φ^* denote the maps induced by application of the functor $\mathrm{Hom}_{S^e}(-, A)$.

Lemma 4.2. *Any cochain maps Ψ^* and Φ^* induce G-invariant, inverse isomorphisms on the cohomology $\mathrm{HH}^\bullet(S, A)$.*

Proof. Since both $P_\bullet$ and the bar resolution are G-compatible, ${}^g\Psi^*$ and ${}^g\Phi^*$ are chain maps for any g in G. Such chain maps are unique up to chain homotopy equivalence, and so induce the same maps on cohomology. □

We represent an element of $\mathrm{HH}^\bullet(S, A)$ at the chain level by a function $f : P_p \to A$. By Theorem 3.4, the corresponding function from $A^{\otimes(p+2)}$ to A is given by $\Theta^* \circ \mathcal{R} \circ \Psi_p^*(f)$. Lemma 4.2 thus implies:

Theorem 4.3. *Let $A = S\#G$. Let $P_\bullet$ be any G-compatible S^e-resolution of S and let Ψ be a chain map from the bar resolution of S to $P_\bullet$. The map*

$$\Theta^* \circ \mathcal{R} \circ \Psi^* : \mathrm{Hom}_{S^e}(P_p, A) \to \mathrm{Hom}_{\mathbb{C}}\left(A^{\otimes p}, A\right)$$

induces an isomorphism $\mathrm{HH}^p(S, A)^G \xrightarrow{\sim} \mathrm{HH}^p(A)$.

We now describe the inverse isomorphism. The inclusion map $S \hookrightarrow A$ induces a restriction map

$$\mathrm{res} : \mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes \bullet}, A\big) \to \mathrm{Hom}_{\mathbb{C}}\big(S^{\otimes \bullet}, A\big).$$

Theorem 4.4. *Let $A = S\#G$. Let Ψ, Φ be any chain maps converting between the bar resolution of S and $P_\bullet$, as above. The maps*

$$\Phi^* \circ \mathrm{res} \quad \textit{and} \quad \Theta^* \circ \mathcal{R} \circ \Psi^*$$

on cochains, $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A) \underset{\Theta^* \circ \mathcal{R} \circ \Psi^*}{\overset{\Phi^* \circ \mathrm{res}}{\rightleftarrows}} \mathrm{Hom}_{S^e}(P_p, A)$, *induce inverse isomorphisms on cohomology,*

$$\mathrm{HH}^\bullet(A) \overset{\sim}{\longleftrightarrow} \mathrm{HH}^\bullet(S, A)^G.$$

Proof. By Theorem 4.3, $\Theta^* \circ \mathcal{R} \circ \Psi^*$ induces an isomorphism from $\mathrm{HH}^p(S, A)^G$ to $\mathrm{HH}^p(A)$. We show that $\Phi^* \circ \mathrm{res}$ defines an inverse map on cohomology.

By functoriality of Hochschild cohomology (see [14, §1.5.1]), res induces a linear map:

$$\mathrm{res} : \mathrm{HH}^p(A) \to \mathrm{HH}^p(S, A).$$

We saw in the proof of Theorem 3.4 that the Eckmann–Shapiro isomorphism realized at the chain level in (3.6) induces a map from $\mathrm{Hom}_{A^e}(A^{\otimes(p+2)}, A)$ to $\mathrm{Hom}_{S^e}(\Delta_p, A)^G$. The restriction of (3.6) to $S^{\otimes(p+2)}$ is essentially the identity map, and it induces a map to $\mathrm{Hom}_{S^e}(S^{(p+2)}, A)^G$ agreeing with res. Hence the image of res is G-invariant. Note that $\mathrm{res} \circ \Theta^* = 1$ on cochains, and thus on cohomology. By the proof of Theorem 3.4, Θ^* is invertible on cohomology, and so we conclude that $\mathrm{res} = (\Theta^*)^{-1}$ and that res also yields an isomorphism on cohomology:

$$\mathrm{res} : \mathrm{HH}^\bullet(A) \overset{\sim}{\longrightarrow} \mathrm{HH}^\bullet(S, A)^G.$$

Lemma 4.2 then implies that $\Phi^* \circ \mathrm{res}$ induces a well-defined map on cohomology with G-invariant image. The fact that it is inverse to $\Theta^* \circ \mathcal{R} \circ \Psi^*$ follows from the observations that $\mathcal{R}$ commutes with Φ^* (as Φ^* is invariant), Φ^* and Ψ^* are inverses on cohomology, and $\mathcal{R}$ is the identity on G-invariants. □

Remark 4.5. We may replace Ψ^* in Theorems 4.3 and 4.4 above by any other cochain map (with the same domain and range), provided that map induces an automorphism inverse to Φ^* on the cohomology $\mathrm{HH}^\bullet(S, A)$, even if it is not induced by a map on the original projective resolution. (As Φ^* is G-invariant on cohomology, so too is its inverse.)

Remark 4.6. Some comments are in order before we give a formula for the Gerstenhaber bracket. By Theorem 4.4, a cocycle in $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A)$ is determined by its values on $S^{\otimes p}$, and so we may compute the bracket of two cocycles by determining how that bracket acts on elements in $S^{\otimes p}$. The ring S embeds canonically in A, and we identify this ring with its image when convenient. We also note that, for the purpose of using Theorem 4.4, it suffices to start with a (not necessarily G-invariant) cocycle α in $\mathrm{Hom}_{S^e}(P_\bullet, A)$ and apply $\Theta^* \circ \mathcal{R} \circ \Psi^*$, since $\mathcal{R} \circ \Psi^*(\alpha)$ is cohomologous to $\mathcal{R} \circ \Psi^* \circ \mathcal{R}(\alpha)$: Indeed, observe that $\mathcal{R}(\Psi^*(\mathcal{R}(\alpha))) = \mathcal{R}(\mathcal{R}(\Psi^*)(\alpha))$ and that $\mathcal{R}(\mathcal{R}(\Psi^*)(\alpha)) \sim \mathcal{R}(\Psi^*(\alpha))$ since Ψ^* and $\mathcal{R}(\Psi^*)$ are both chain maps.

We now lift the Gerstenhaber bracket on $\mathrm{HH}^\bullet(A)$ to the cohomology $\mathrm{HH}^\bullet(S,A)^G$ expressed in terms of any resolution of S. Again, we assume the resolution is G-compatible, or else it may not define $\mathrm{HH}^\bullet(S,A)^G$. Theorem 4.4 implies the following formula for the graded Lie bracket at the cochain level, on cocycles in $\mathrm{Hom}_{S^e}(P_\bullet, A)^G$. We use Definition 2.2 of the bracket on $\mathrm{HH}^\bullet(A)$ at the cochain level.

Theorem 4.7. *Let $P_\bullet$ be any G-compatible S^e-resolution of S. The isomorphism $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(S,A)^G$ induces the following bracket on $\mathrm{HH}^\bullet(S,A)^G$ (expressed via $P_\bullet$): For α, β in $\mathrm{Hom}_{S^e}(P_\bullet, A)$ representing G-invariant cohomology classes, the cohomology class of $[\alpha,\beta]$ is represented at the cochain level by*

$$\frac{1}{|G|^2}\sum_{a,b\in G} \Phi^*\big[\Theta^{*\,a}\big(\Psi^*\alpha\big), \Theta^{*\,b}\big(\Psi^*\beta\big)\big],$$

where the bracket $[\ ,\ ]$ on the right side is the Gerstenhaber bracket on $\mathrm{HH}^\bullet(A)$.

Proof. We apply the inverse isomorphisms of Theorem 4.4 to lift the bracket from the bar complex of A to $P_\bullet$: The above formula gives the resulting bracket $\Phi^*\mathrm{res}[\Theta^*\mathcal{R}\Psi^*(\alpha), \Theta^*\mathcal{R}\Psi^*(\beta)]$. Note that the restriction map res is not needed in the formula since the output of Φ is automatically in $S^{\otimes\bullet}$. Also note that it is not necessary to apply the Reynolds operator to α and β before applying Ψ^* since we are interested only in the bracket at the level of cohomology (see Remark 4.6). □

5. Koszul resolution

Let G be a finite group and V a (not necessarily faithful) $\mathbb{C}G$-module of finite dimension n. The Hochschild cohomology of the skew group algebra $\mathrm{HH}^\bullet(S(V)\#G)$ is computed using the Koszul resolution of the polynomial ring $S(V)$ while the cup product and Gerstenhaber bracket are defined on the bar resolution of $S(V)\#G$. In this section, we use machinery developed in Section 4 to translate between spaces and between resolutions.

First, some preliminaries. We denote the image of v in V under the action of g in G by ${}^g v$. Let V^* denote the contragredient (or dual) representation. For any basis $v_1,\dots,v_n$ of V, let $v_1^*,\dots,v_n^*$ be the dual basis of V^*. Denote the set of G-invariants in V by $V^G := \{v \in V\colon {}^g v = v \text{ for all } g \in G\}$ and the g-invariant subspace of V by $V^g := \{v \in V\colon {}^g v = v\}$ for any g in G. Since G is finite, we may assume G acts by isometries on V (i.e., G preserves a Hermitian form $\langle\ ,\ \rangle$). If h lies in the centralizer $Z(g)$, then h preserves both V^g and its orthogonal complement $(V^g)^\perp$ (defined with respect to the Hermitian form). We shall frequently use the observation that $(V^g)^\perp = \mathrm{im}(1-g)$.

The *Koszul resolution* $K_\bullet(S(V))$ is given by $K_0(S(V)) = S(V)^e$ and

$$K_p\big(S(V)\big) = S(V)^e \otimes \bigwedge^p (V) \tag{5.1}$$

for $p \geqslant 1$, with differentials defined by

$$d_p(1\otimes 1\otimes v_{j_1}\wedge\cdots\wedge v_{j_p}) = \sum_{i=1}^{p}(-1)^{i+1}(v_{j_i}\otimes 1 - 1\otimes v_{j_i})\otimes(v_{j_1}\wedge\cdots\wedge\hat{v}_{j_i}\wedge\cdots\wedge v_{j_p}) \tag{5.2}$$

for any choice of basis $v_1,\dots,v_n$ of V (e.g., see Weibel [22, §4.5]). We identify $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p V, S(V)\overline{g})$ with $S(V)\overline{g}\otimes\bigwedge^p V^*$ for each g in G.

We fix the set of *cochains* arising from the Koszul resolution (from which the cohomology classes emerge) as vector forms on V tagged by group elements: Let

$$C^\bullet = \bigoplus_{g\in G} C_g^\bullet, \quad \text{where } C_g^p := S(V)\overline{g}\otimes\bigwedge^p V^* \text{ for } g \in G. \tag{5.3}$$

We refer to $C^\bullet_g$ as the set of cochains *supported on* g. Similarly, if X is a subset of G, we set $C^\bullet_X := \oplus_{g\in X} C^\bullet_g$, the set of cochains *supported on* X. Note that group elements permute the summands of $C^\bullet$ via the conjugation action of G on itself.

We apply the notation of Section 4 to the case $S = S(V)$. Let $P_\bullet$ be the Koszul resolution (5.1) of $S(V)$. Since the bar and Koszul complexes of $S(V)$ are both $S(V)^e$-resolutions, there exist chain maps Φ and Ψ between the two complexes,

$$\Psi_p : S(V)^{\otimes(p+2)} \to S(V)^e \otimes \bigwedge^p V,$$
$$\Phi_p : S(V)^e \otimes \bigwedge^p V \to S(V)^{\otimes(p+2)},$$

for all $p \geqslant 1$, such that Diagram 4.1 commutes. Let Φ be the canonical inclusion of the Koszul resolution (5.1) into the bar resolution (2.1):

$$\Phi_p(1\otimes 1\otimes v_{j_1}\wedge\cdots\wedge v_{j_p}) = \sum_{\pi\in \mathrm{Sym}_p} \mathrm{sgn}(\pi)\otimes v_{j_{\pi(1)}}\otimes\cdots\otimes v_{j_{\pi(p)}}\otimes 1 \tag{5.4}$$

for all $v_{j_1},\dots,v_{j_p}$ in V. (See [19] for an explicit chain map Ψ in this case.) We obtain the following commutative diagram of induced cochain maps:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, A) & \xrightarrow{\delta^*} & \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes(p+1)}, A) \\ \Psi_p^* \uparrow \downarrow \Phi_p^* & & \Psi_{p+1}^* \uparrow \downarrow \Phi_{p+1}^* \\ C^p & \xrightarrow{\quad d^*\quad} & C^{p+1}. \end{array} \tag{5.5}$$

Note that both the Koszul and bar resolutions are G-compatible. Using Lemma 4.2, we identify Φ^* and Ψ^* with their restrictions to

$$\mathrm{HH}^\bullet\big(S(V), S(V)\#G\big)^G.$$

Given any basis $v_1,\dots,v_n$ of V, let $\partial/\partial v_i$ denote the usual partial differential operator with respect to v_i. In addition, given a complex number $\epsilon \neq 1$, we define the *ϵ-quantum partial differential operator* with respect to $v := v_i$ as the scaled Demazure (BGG) operator $\partial_{v,\epsilon} : S(V) \to S(V)$ given by

$$\partial_{v,\epsilon}(f) = (1-\epsilon)^{-1}\frac{f - {}^s f}{v} = \frac{f - {}^s f}{v - {}^s v}, \tag{5.6}$$

where $s \in \mathrm{GL}(V)$ is the reflection whose matrix with respect to the basis $v_1,\dots,v_n$ is $\mathrm{diag}(1,\dots,1,\epsilon,1,\dots,1)$ with ϵ in the i-th slot. Set $\partial_{v,\epsilon} = \partial/\partial v$ when $\epsilon = 1$. The operator $\partial_{v,\epsilon}$ coincides with the usual definition of quantum partial differentiation:

$$\partial_{v_1,\epsilon}\big(v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}\big) = [k_1]_\epsilon\, v_1^{k_1-1}v_2^{k_2}\cdots v_n^{k_n},$$

where $[k]_\epsilon$ is the quantum integer $[k]_\epsilon := 1+\epsilon+\epsilon^2+\cdots+\epsilon^{k-1}$.

Next, we recall an explicit map Υ, involving quantum differential operators, that replaces Ψ^*:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, A) & \xrightarrow{\delta^*} & \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes (p+1)}, A) \\ \Upsilon_p \uparrow \downarrow \Phi_p^* & & \Upsilon_{p+1} \uparrow \downarrow \Phi_{p+1}^* \\ C^p & \xrightarrow{d^*} & C^{p+1}. \end{array} \tag{5.7}$$

For each g in G, fix a basis $B_g = \{v_1, \dots, v_n\}$ of V consisting of eigenvectors of g with corresponding eigenvalues $\epsilon_1, \dots, \epsilon_n$. Decompose g into reflections according to this basis: Let $g = s_1 \cdots s_n$ where each s_i in $\mathrm{GL}(V)$ is the reflection (or the identity) defined by $s_i(v_j) = v_j$ for $j \neq i$ and $s_i(v_i) = \epsilon_i v_i$. Let $\partial_i := \partial_{v_i, \epsilon_i}$, the quantum partial derivative with respect to B_g.

Definition 5.8. We define a linear map Υ from the dual Koszul complex to the dual bar complex for $S(V)$ with coefficients in $A := S(V)\#G$,

$$\Upsilon_p : C^p \to \mathrm{Hom}_{\mathbb{C}}\big(S(V)^{\otimes p}, A\big).$$

Fix g in G with basis B_g of V as above. Let $\alpha = f_g \overline{g} \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^*$ lie in C^p with f_g in $S(V)$ and $1 \leqslant j_1 < \cdots < j_p \leqslant n$. Define $\Upsilon(\alpha) : S(V)^{\otimes p} \to S(V)\overline{g}$ by

$$\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) = \Bigg(\prod_{k=1,\dots,p} {}^{s_1 s_2 \cdots s_{j_k - 1}}(\partial_{j_k} f_k) \Bigg) f_g \overline{g}.$$

Then Υ is a cochain map (see [19]) and thus induces an endomorphism Υ of cohomology $\mathrm{HH}^\bullet(S(V), A)$. Denote the restriction of Υ to the g-component of $C^\bullet$ and of $\mathrm{HH}^\bullet(S(V), A) \cong \bigoplus_{g \in G} \mathrm{HH}^\bullet(S(V), S(V)\overline{g})$ by $\Upsilon_g = \Upsilon_{g, B_g}$ so that

$$\Upsilon = \bigoplus_{g \in G} \Upsilon_g.$$

In formulas, we wish to highlight group elements explicitly instead of leaving them hidden in the definition of Υ, and thus we find it convenient to define a version of Υ untagged by group elements:

Definition 5.9. Let $\mathrm{proj}_{G \to 1} : S(V)\#G \to S(V)$ be the projection map that drops group element tags: $f\overline{g} \mapsto f$ for all g in G and f in $S(V)$. Define

$$\Upsilon := \mathrm{proj}_{G \to 1} \circ \Upsilon : C^\bullet \to \bigoplus_{g \in G} \mathrm{Hom}_{\mathbb{C}}\big(S(V)^{\otimes \bullet}, S(V)\big).$$

We shall use the following observation often.

Remark 5.10. For the fixed basis $B_g = \{v_1, \dots, v_n\}$ and $\alpha = f_g \overline{g} \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^*$ in C_g^p (with $j_1 < \cdots < j_p$), note that

$$\Upsilon(\alpha)(v_{i_1} \otimes \cdots \otimes v_{i_p}) = 0 \quad \text{unless } i_1 = j_1, \dots, i_p = j_p.$$

Generally, $\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) = 0$ whenever $\frac{\partial}{\partial v_{j_k}}(f_k) = 0$ for some k.

The following proposition from [19] provides a cornerstone for our computations.

Proposition 5.11. *The map Υ induces an isomorphism on the Hochschild cohomology of the skew group algebra*

$$\mathrm{HH}^\bullet\big(S(V)\#G\big) \cong \left(\bigoplus_{g\in G} \mathrm{HH}^\bullet\big(S(V), S(V)\overline{g}\big)\right)^G.$$

Specifically, Υ and Φ^* are G-invariant, inverse isomorphisms on cohomology converting between expressions in terms of the Koszul resolution and the bar resolution.

In fact, the map Υ is easily seen to be a right inverse to Φ^* on cochains, not just on cohomology; see [20, Proposition 5.4].

Remark 5.12. The *cochain* map $\Upsilon = \bigoplus_{g\in G} \Upsilon_{g,B_g}$ depends on the choices of bases B_g of eigenvectors of g in G, but Υ induces an automorphism on cohomology which does *not* depend on choice of basis. (This follows from Proposition 5.11 above, as Φ is independent of basis.)

6. Brackets for polynomial skew group algebras

Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. We apply the previous results to determine the Gerstenhaber bracket of $\mathrm{HH}^\bullet(S(V)\#G)$ explicitly. We lift the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$, which is defined via the bar complex of $S(V)\#G$, to the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)^G$, which is computed via the Koszul complex of $S(V)$.

The next theorem allows one to replace the Hochschild cohomology of $S(V)\#G$ with a convenient space of forms. Set

$$H^\bullet_g := S\big(V^g\big)\overline{g} \otimes \bigwedge^{\bullet-\operatorname{codim} V^g} \big(V^g\big)^* \otimes \bigwedge^{\operatorname{codim} V^g} \big(\big(V^g\big)^\perp\big)^* \quad \text{and}$$
$$H^\bullet := \bigoplus_{g\in G} H^\bullet_g, \tag{6.1}$$

where a negative exterior power is defined to be 0. We regard these spaces as subsets of $C^\bullet$, the space of cochains arising from the Koszul complex (see (5.3)), after making canonical identifications. Note that for each g in G, the centralizer $Z(g)$ acts diagonally on the tensor factors in $H^\bullet_g$. The third factor $\bigwedge^{\operatorname{codim} V^g}((V^g)^\perp)^*$ of $H^\bullet_g$ is a vector space of dimension one with a possibly nontrivial $Z(g)$-action. The theorem below is due to Ginzburg and Kaledin [10, (6.4)] for faithful group actions in a more general geometric setting. See also Farinati [8, Section 3.2] in the algebraic setting. Our formulation is from [18, (3.3) and (3.4)].

Theorem 6.2. *Set $A := S(V)\#G$. Cohomology classes arise as tagged vector forms*:

- *$H^\bullet$ is a set of cohomology class representatives for $\mathrm{HH}^\bullet(S(V), A)$ arising from the Koszul resolution*: *The inclusion map $H^\bullet \hookrightarrow C^\bullet$ induces an isomorphism $H^\bullet \cong \mathrm{HH}^\bullet(S(V), A)$.*
- $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(S(V), A)^G \cong (H^\bullet)^G$.

Remark 6.3. The only contribution to degree two cohomology $\mathrm{HH}^2(S(V)\#G)$ comes from group elements acting either trivially or as bireflections:

$$H^2_g \neq 0 \quad \text{implies} \quad V^g = V \quad \text{or} \quad \operatorname{codim} V^g = 2.$$

Indeed, by Definition 6.1, $(H^\bullet)^G \cong \bigoplus_{g\in\mathcal{C}} (H_g^\bullet)^{Z(g)}$ where $\mathcal{C}$ is a set of representatives of the conjugacy classes of G. Since each group element g in G centralizes itself, the determinant of g on V is necessarily 1 whenever $(H_g^\bullet)^{Z(g)}$ is nonzero (since g acts on $\bigwedge^{\operatorname{codim} V^g}((V^g)^\perp)^*$ by the inverse of its determinant). In fact, if g acts nontrivially on V, it does not contribute to cohomology in degrees 0 and 1.

Our results from previous sections allow us to realize the isomorphism of Theorem 6.2 explicitly at the chain level in the next theorem, which will be used extensively in our bracket calculations:

Theorem 6.4. *Let $A = S(V)\#G$. The map*

$$\Gamma := \Theta^* \circ \mathcal{R} \circ \Upsilon : C^p \to \operatorname{Hom}_{\mathbb{C}}\big(A^{\otimes p}, A\big),$$
$$\alpha \mapsto \frac{1}{|G|} \sum_{g\in G} \Theta^*\big({}^g(\Upsilon\alpha)\big),$$

induces an isomorphism

$$\big(H^\bullet\big)^G \xrightarrow{\sim} \mathrm{HH}^\bullet(A).$$

Proof. The statement follows from Theorem 4.3, Remark 4.5, Proposition 5.11, and Theorem 6.2. □

Remark 6.5. We explained explicitly in [18] how a *constant* Hochschild 2-cocycle defines a graded Hecke algebra. More generally, Theorem 6.4 offers a direct conversion from vector forms in $(H^\bullet)^G$ to functions on tensor powers of $S(V)\#G$ representing elements in $\mathrm{HH}^\bullet(S(V)\#G)$: For all α in H^p,

$$\Gamma(\alpha)(f_1\overline{g}_1 \otimes \cdots \otimes f_p\overline{g}_p) = \frac{1}{|G|} \sum_{g\in G} {}^g(\Upsilon\alpha)\big(f_1 \otimes {}^{g_1}f_2 \otimes \cdots \otimes {}^{g_1\cdots g_{p-1}}f_p\big)\overline{g_1\cdots g_p}$$

for all $f_1, \ldots, f_p$ in $S(V)$ and $g_1, \ldots, g_p$ in G. In particular, if $p = 2$, then $\Gamma(\alpha)(f_1\overline{g}_1 \otimes f_2\overline{g}_2) = \frac{1}{|G|}\sum_{g\in G} {}^g(\Upsilon\alpha)(f_1 \otimes {}^{g_1}f_2)\overline{g_1 g_2}$.

We now record some inverse isomorphisms that will facilitate finding bracket formulas. For each g in G, let $\{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of g and let

$$\operatorname{Proj}_{H_g} : C_g^\bullet \to H_g^\bullet$$

be the map which takes any $\mathbb{C}$-basis element $(v_1^{m_1}\cdots v_n^{m_n})\overline{g} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p}$ of $C_g^\bullet$ to itself if it lies in $H_g^\bullet$ and to zero otherwise. Set $\operatorname{Proj}_H := \bigoplus_{g\in G} \operatorname{Proj}_{H_g}$. Then

$$\operatorname{Proj}_H : C^\bullet \to H^\bullet$$

projects each cocycle in $C^\bullet$ to its cohomology class representative in $H^\bullet$. (See, for example, the computation of [8, Section 3.2].) In fact, the map Proj_H splits with respect to the canonical embedding $H^\bullet \hookrightarrow C^\bullet$. Now consider projection just on the polynomial part of a form: Let

$$\operatorname{Proj}_{V^g} : S(V)\overline{g} \otimes \bigwedge\nolimits^\bullet V^* \to S\big(V^g\big)\overline{g} \otimes \bigwedge\nolimits^\bullet V^*$$

be the canonical projection arising from the isomorphism $S(V^g) \cong S(V)/I((V^g)^\perp)$ for each g in G. Then $\operatorname{Proj}_{H_g} = \operatorname{Proj}_{H_g} \circ \operatorname{Proj}_{V^g}$ and thus any cocycle in C_g which vanishes under $\operatorname{Proj}_{V^g}$ is zero in

cohomology. (In fact, we can write Proj_H as a composition of $\bigoplus_g \mathrm{Proj}_{V^g}$ with a similar projection map on the exterior algebra.)

The inclusion map $S(V) \hookrightarrow A$ again induces a restriction map (as in Section 4):

$$\mathrm{res} : \mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes \bullet}, A\big) \to \mathrm{Hom}_{\mathbb{C}}\big(S(V)^{\otimes \bullet}, A\big).$$

Theorem 4.4 implies that the following compositions (for $S = S(V)$) induce inverse isomorphisms on cohomology, as asserted in the next theorem:

$$\mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes p}, A\big) \underset{\Theta^*}{\overset{\mathrm{res}}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{C}}\big(S^{\otimes p}, A\big)^G \underset{\mathcal{R}}{\overset{}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{C}}\big(S^{\otimes p}, A\big) \underset{\Upsilon}{\overset{\Phi^*}{\rightleftarrows}} C^p \overset{\mathrm{Proj}_H}{\rightleftarrows} H^p.$$

Theorem 6.6. *Let $A = S(V)\#G$. The maps*

$$\Gamma' := \mathrm{Proj}_H \circ \Phi^* \circ \mathrm{res} \quad \textit{and} \quad \Gamma := \Theta^* \circ \mathcal{R} \circ \Upsilon$$

on cochains, $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A) \underset{\Gamma}{\overset{\Gamma'}{\rightleftarrows}} H^p$, induce inverse isomorphisms

$$\mathrm{HH}^p(A) \overset{\sim}{\longleftrightarrow} \big(H^p\big)^G.$$

The next theorem, a consequence of Theorem 4.7, describes the graded Lie bracket on $(H^\bullet)^G$ induced by the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$.

Theorem 6.7. *The Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$ induces the following graded Lie bracket on $(H^\bullet)^G$ under the isomorphism $(H^\bullet)^G \cong \mathrm{HH}^\bullet(S(V)\#G)$ of Theorem* 6.6*: For α, β in $(H^\bullet)^G$,*

$$[\alpha, \beta] = \frac{1}{|G|^2}\,\mathrm{Proj}_H \sum_{a,b \in G} \Phi^*\big[\Theta^{*\,a}(\Upsilon\alpha), \Theta^{*\,b}(\Upsilon\beta)\big]$$

where the bracket $[\ ,\ ]$ on the right is the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$ given at the cochain level.

In the remainder of this section, we use the above theorem to give formulas for the Gerstenhaber bracket on $(H^\bullet)^G \cong \mathrm{HH}^\bullet(S(V)\#G)$. We introduce notation for a prebracket at the cochain level to aid computations and allow for an explicit, closed formula. Recall that we have fixed a basis B_g of V consisting of eigenvectors of g for each g in G. These choices are for computational convenience; our results do not depend on the choices. For a multi-index $I = (i_1, \ldots, i_m)$, we write dv_I for $v_{i_1}^* \wedge \cdots \wedge v_{i_m}^*$. In formulas below, note that we sum over *all* multi-indices I of a given length, not just those with indices of increasing order. We use the untagged version $\mathfrak{T}$ of the map Υ with image in $S(V)$ (see Definition 5.9) to highlight the group elements appearing in various formulas.

Definition 6.8. Define a prebracket (bilinear map) on the set of cochains $C^\bullet$ defined in (5.3),

$$[[\ ,\ ]] : C_g^p \times C_h^q \to C_{gh}^{p+q-1} + C_{hg}^{p+q-1},$$

for g and h in G, depending on a basis of eigenvectors B_1 for g and a basis of eigenvectors B_2 for h as follows. For α in C_g^p and β in C_h^q, define $\alpha \,\overline{\circ}\, \beta$ to be

$$\sum_{\substack{I=(i_1,\dots,i_m)\\ 1\leqslant k\leqslant p}} (-1)^{(q-1)(k-1)} \mathcal{F}_{g,B_1}(\alpha)\big(v_{i_1}\otimes\cdots\otimes v_{i_{k-1}}\otimes f_h^{(i_k)}\otimes {}^h v_{i_{k+q}}\otimes\cdots\otimes {}^h v_{i_m}\big)\overline{gh}\otimes dv_I,$$

where $m=p+q-1$, $f_h^{(i_k)} := \mathcal{F}_{h,B_2}(\beta)(v_{i_k}\otimes\cdots\otimes v_{i_{k+q-1}})$, and $v_1,\dots,v_n$ is any basis of V. Define

$$[[\alpha,\beta]]_{(B_1,B_2)} = \alpha\,\bar{\circ}\,\beta - (-1)^{(p-1)(q-1)}\beta\,\bar{\circ}\,\alpha.$$

Remark 6.9. When computing brackets, one may be tempted to seek results by working with just $\alpha\,\bar{\circ}\,\beta$ (or just $\beta\,\bar{\circ}\,\alpha$) and extending by symmetry. However the operation $\bar{\circ}$ is not defined on cohomology. Furthermore, one must exercise care in treating the operation $\bar{\circ}$ alone (e.g., see the proof of Theorem 9.2 below), as one risks covertly changing the bases used to apply $\mathcal{F}$ in the middle of a bracket calculation. Similarly, one must exercise care when examining the bracket of two cocycles summand by summand, although the bracket is linear: The bases used to apply $\mathcal{F}$ should not depend on the pair of summands considered. The maps Υ and $\mathcal{F}$ are independent of the choices of bases used when taking brackets of cohomology classes, but a choice should be made once and for all throughout the whole calculation of a Gerstenhaber bracket.

We are particularly interested in brackets and prebrackets of elements of cohomological degree 2. For α in C_g^2 and β in C_h^2, the above definition gives

$$\begin{aligned}
&[[\alpha,\beta]]_{(B_1,B_2)}\\
&\quad=\sum_{1\leqslant i,j,k\leqslant n}\Big[\mathcal{F}_1\alpha\big(\mathcal{F}_2\beta(v_i\otimes v_j)\otimes {}^h v_k\big)\overline{gh} - \mathcal{F}_1\alpha\big(v_i\otimes\mathcal{F}_2\beta(v_j\otimes v_k)\big)\overline{gh}\\
&\qquad+\mathcal{F}_2\beta\big(\mathcal{F}_1\alpha(v_i\otimes v_j)\otimes {}^g v_k\big)\overline{hg} - \mathcal{F}_2\beta\big(v_i\otimes\mathcal{F}_1\alpha(v_j\otimes v_k)\big)\overline{hg}\Big]\otimes\big(v_i^*\wedge v_j^*\wedge v_k^*\big) \qquad (6.10)
\end{aligned}$$

where $v_1,\dots,v_n$ is any basis of V and $\Upsilon_1 := \Upsilon_{g,B_1}$ and $\Upsilon_2 := \Upsilon_{h,B_2}$.

Remark 6.11. The theorem below articulates the Gerstenhaber bracket in terms of a fixed basis B_g of V for each g in G. Although each individual prebracket (6.8) depends on these choices, the formula for the bracket given below is *independent of these choices* (by Remark 5.12).

Theorem 6.12. *Definition* 6.8 *gives a formula for the Gerstenhaber bracket on* $\mathrm{HH}^\bullet(S(V)\#G)$ *as realized on* $(H^\bullet)^G$*: For* g,h *in* G *and* α *in* $H_g^\bullet$ *and* β *in* $H_h^\bullet$,

$$[\alpha,\beta] = \frac{1}{|G|^2}\operatorname{Proj}_H\sum_{a,b\in G}\big[\big[{}^a\alpha,{}^b\beta\big]\big]_{({}^aB_g,{}^bB_h)}.$$

Proof. By Theorem 6.7, the Gerstenhaber bracket $[\ ,\ ]$ on $\mathrm{HH}^\bullet(S(V)\#G)$ induces the following bracket on $H^\bullet$:

$$[\alpha,\beta] = \frac{1}{|G|^2}\operatorname{Proj}_H\sum_{a,b\in G}\Phi^*\big[\Theta^{*\,a}(\Upsilon\alpha),\Theta^{*\,b}(\Upsilon\beta)\big].$$

We show that this formula is exactly that claimed in the special case when α and β have cohomological degree 2; the general case follows analogously.

Let $v_1,\dots,v_n$ be any basis of V. We determine $\Phi^*[\Theta^{*\,a}(\Upsilon\alpha),\Theta^{*\,b}(\Upsilon\beta)]$ explicitly in the case $a=b=1_G$ as an element of

$$\big(S(V)\overline{gh} + S(V)\overline{hg}\big) \otimes \bigwedge\nolimits^3 V^* \cong \mathrm{Hom}_{\mathbb{C}}\Big(\bigwedge\nolimits^3 V, S(V)\overline{gh} + S(V)\overline{hg}\Big)$$

by evaluating on input of the form $v_i \wedge v_j \wedge v_k$. The computation for general a, b in G is similar, using (for example) ${}^a(\Upsilon\alpha) = \Upsilon_{aga^{-1},{}^aB_g}({}^a\alpha)$ (see [19, Proposition 3.8]). For all i, j, k (see Eq. (5.4)),

$$\begin{aligned}
&\Phi^*\big[\Theta^*(\Upsilon\alpha), \Theta^*(\Upsilon\beta)\big](v_i \wedge v_j \wedge v_k)\\
&\quad = \big[\Theta^*(\Upsilon\alpha), \Theta^*(\Upsilon\beta)\big]\Phi(v_i \wedge v_j \wedge v_k)\\
&\quad = \sum_{\pi\in \mathrm{Sym}_3} \mathrm{sgn}(\pi)\big[\Theta^*(\Upsilon\alpha), \Theta^*(\Upsilon\beta)\big](v_{\pi(i)} \otimes v_{\pi(j)} \otimes v_{\pi(k)}).
\end{aligned}$$

We expand the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$ to obtain

$$\begin{aligned}
&\sum_{\pi\in \mathrm{Sym}_3} \mathrm{sgn}(\pi)\big[\Theta^*(\Upsilon\alpha)\big(\Theta^*(\Upsilon\beta)(v_{\pi(i)} \otimes v_{\pi(j)}) \otimes v_{\pi(k)}\big)\\
&\quad - \Theta^*(\Upsilon\alpha)\big(v_{\pi(i)} \otimes \Theta^*(\Upsilon\beta)(v_{\pi(j)} \otimes v_{\pi(k)})\big)\\
&\quad + \Theta^*(\Upsilon\beta)\big(\Theta^*(\Upsilon\alpha)(v_{\pi(i)} \otimes v_{\pi(j)}) \otimes v_{\pi(k)}\big)\\
&\quad - \Theta^*(\Upsilon\beta)\big(v_{\pi(i)} \otimes \Theta^*(\Upsilon\alpha)(v_{\pi(j)} \otimes v_{\pi(k)})\big)\big].
\end{aligned} \tag{6.13}$$

But for any w_1, w_2, w_3 in V,

$$\begin{aligned}
\Theta^*(\Upsilon\alpha)\big(\Theta^*(\Upsilon\beta)(w_1 \otimes w_2) \otimes w_3\big) &= \Theta^*(\Upsilon\alpha)\big(\Upsilon\beta(w_1 \otimes w_2) \otimes w_3\big)\\
&= \Upsilon\alpha\big(\Upsilon\beta(w_1 \otimes w_2) \otimes {}^hw_3\big)\overline{gh}.
\end{aligned}$$

We obtain similar expressions for the other terms arising in the bracket, and hence the above sum is simply

$$[[\alpha, \beta]]_{(B_g, B_h)}(v_i \wedge v_j \wedge v_k).$$

Similarly, we see that for all a, b in G,

$$\Phi^*\big[\Theta^{*\,a}(\Upsilon\alpha), \Theta^{*\,b}(\Upsilon\beta)\big] = \big[\big[{}^a\alpha, {}^b\beta\big]\big]_{({}^aB_g, {}^bB_h)},$$

and the result follows. □

Remark 6.14. The theorem above actually gives a formula for a bracket on $H^\bullet$, not just on $(H^\bullet)^G \cong \mathrm{HH}^\bullet(S(V)\#G)$. One can show that this extension to all of $H^\bullet$ agrees with the composition of the Reynolds operator on $H^\bullet$ with the Gerstenhaber bracket on $(H^\bullet)^G$: By Theorem 6.12 and Remark 4.6, $[\alpha, \beta]$ is G-invariant and cohomologous to $[\mathcal{R}(\alpha), \mathcal{R}(\beta)]$ for all α, β in $H^\bullet$. Hence, the extension to $H^\bullet$ is artifical in some sense. Indeed, a natural Gerstenhaber bracket on all of $H^\bullet$ does not make sense, as this space does not present itself as the Hochschild cohomology of an algebra with coefficients in that same algebra. This idea may be used to explain the formula of Theorem 6.12: When applying Theorem 6.12 to G-invariant elements α and β, we might write $\alpha = \sum_{c\in[G/Z(g)]} {}^c\alpha'$ and $\beta = \sum_{d\in[G/Z(h)]} {}^d\beta'$, where $\alpha' \in (H_g^\bullet)^{Z(g)}$ and $\beta' \in (H_h^\bullet)^{Z(h)}$ are representative summands. (Here, $[G/A]$ is a set of representatives of the cosets G/A for any subgroup A of G.) Then

$$[\alpha,\beta] = \frac{1}{|G|^2}\operatorname{Proj}_H \sum_{\substack{a,b\in G\\ c\in[G/Z(g)],\ d\in[G/Z(h)]}} \left[\left[{}^{ac}\alpha', {}^{bd}\beta'\right]\right]_{({}^aB_{cgc^{-1}}, {}^bB_{dhd^{-1}})}.$$

However this more complicated expression is cohomologous to that of Theorem 6.12, since $\mathcal{R}\circ\Upsilon(\alpha)$ is cohomologous to $\mathcal{R}\circ\Upsilon\circ\mathcal{R}(\alpha)$ (see Remark 4.6, noting that $\Psi^* = \Upsilon$).

7. Explicit bracket formulas

Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. In this section, we give a closed formula for the prebracket on $\mathrm{HH}^\bullet(S(V)\#G)$. We shall use this formula in later sections to obtain new results on zero brackets and to locate noncommutative Poisson structures. We also recover the classical Schouten–Nijenhuis bracket in this section and we give an example.

The prebracket of Definition 6.8 simplifies enormously when we work with a basis of simultaneous eigenvectors for g and h in G. Indeed, Remark 5.10 predicts that most terms of Definition 6.8 are zero. We capitalize on this idea in the next theorem and corollaries. When the actions of g and h are not simultaneously diagonalizable, we enact a change of basis at various points of the calculation of the prebracket to never-the-less take advantage of Remark 5.10. The following proof shows how to keep track of the effect on the map $\mathcal{F}$ (mindful of cautionary Remark 6.9).

First, some notation. For g, h in G, let $M = M^{g,h}$ be the change of basis matrix between B_g and B_h: For $B_g = \{w_1, \dots, w_n\}$ and $B_h = \{v_1, \dots, v_n\}$ (our fixed bases of eigenvectors of g and h, respectively), set $M = (a_{ij})$ where, for $i = 1, \dots, n$,

$$v_i = a_{1i}w_1 + \cdots + a_{ni}w_n. \tag{7.1}$$

The formula below involves determinants of certain submatrices of M. If I and J are (ordered) subsets of $\{1, \dots, n\}$, denote by $M_{I;J}$ the submatrix of M obtained by deleting all rows except those indexed by I and deleting all columns except those indexed by J. Recall that $dv_I := v_{i_1}^* \wedge \cdots \wedge v_{i_m}^*$ for a multi-index $I = (i_1, \dots, i_m)$, not necessarily in increasing order. Below we use the operation $\bar{\circ}$ giving the prebracket of Definition 6.8. We also use the quantum partial differentiation operators $\partial_{v,\epsilon}$ (see (5.6)) after decomposing g into reflections, $g = s_1 \cdots s_n$, with respect to B_g, i.e., each s_i in $\mathrm{GL}(V)$ is defined by $s_i(w_i) = {}^g w_i = \epsilon_i w_i$ and $s_i(w_j) = w_j$ for $j \neq i$.

Theorem 7.2. *Let g, h lie in G with change of basis matrix $M = M^{g,h}$ as above. Decompose g into reflections, $g = s_1 \cdots s_n$, with respect to B_g as above. Let*

$$\alpha = f_g\overline{g} \otimes dw_J \in H_g^p \quad \text{where } J = (j_1 < \cdots < j_p) \quad \text{and}$$
$$\beta = f'_h\overline{h} \otimes dv_L \in H_h^q \quad \text{where } L = (l_1 < \cdots < l_q).$$

Then $\alpha \,\bar{\circ}\, \beta$ is given as an element of C_{gh}^{p+q-1} by the following formula: For $I = (i_1 < \cdots < i_m)$ and $m = p + q - 1$,

$$(\alpha \,\bar{\circ}\, \beta)(v_{i_1} \wedge \cdots \wedge v_{i_m}) = \sum_{1\leqslant k\leqslant p} (-1)^{\nu(k)} \det(M_{J_k;I-L})^{s_1\cdots s_{j_k-1}}\left(\partial_{j_k} f'_h\right) f_g\overline{gh},$$

where $\det(M_{*;I-L}) = 0$ for $L \not\subset I$, $J_k := (j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_p)$, $\partial_i = \partial_{\epsilon_i, w_i}$, and $\nu(k) = 1 - q - k + \lambda(1) + \cdots + \lambda(q) - \binom{q}{2}$ for λ defined by $l_s = i_{\lambda(s)}$.

Proof. By Definition 6.8, $(\alpha \,\overline{\circ}\, \beta)(v_{i_1} \wedge \cdots \wedge v_{i_m})$ is equal to

$$\sum_{\substack{1 \leqslant k \leqslant p \\ \pi \in \mathrm{Sym}_m}} (-1)^{(q-1)(k-1)} (\operatorname{sgn} \pi)(\Upsilon_g \alpha)\big(v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(k-1)}} \otimes \big(f'_h\big)^{(i_{\pi(k)})} \otimes {}^h v_{i_{\pi(k+q)}} \otimes \cdots \otimes {}^h v_{i_{\pi(m)}}\big)\overline{h},$$

where $(f'_h)^{(i_{\pi(k)})} = (\Upsilon_h \beta)(v_{i_{\pi(k)}} \otimes \cdots \otimes v_{i_{\pi(k+q-1)}})$. Fix k and note that by Remark 5.10, $(f'_h)^{(i_{\pi(k)})}$ is nonzero only if $L = (i_{\pi(k)}, \ldots, i_{\pi(k+q-1)})$. Hence, we may restrict the sum to those permutations π for which $i_{\pi(k)} = l_1, \ldots, i_{\pi(k+q-1)} = l_q$. We identify this set of permutations with the symmetric group Sym_{p-1} (of permutations on the set $\{1, \ldots, k-1, k+q, \ldots, m\}$) in the standard way. Under this identification, the factor $\operatorname{sgn}(\pi)$ changes by

$$(-1)^{\lambda(1)-k}(-1)^{\lambda(2)-(k+1)} \cdots (-1)^{\lambda(q)-(k+q-1)}.$$

For such permutations π, the vectors $v_{i_{\pi(k+q)}}, \ldots, v_{i_{\pi(m)}}$ lie in V^h (as the exterior part of $\beta \in H_h^q$ includes a volume form on $(V^h)^\perp$, and hence $v_i \in V^h$ for $i \notin L$). Therefore ${}^h v_{i_{\pi(k+q)}} = v_{i_{\pi(k+q)}}, \ldots,$ ${}^h v_{i_{\pi(m)}} = v_{i_{\pi(m)}}$. Note also that $(f'_h)^{(i_{\pi(k)})} = f'_h$ for all such π. We now invoke the change of basis

$$v_{i_{\pi(s)}} = a_{1,i_{\pi(s)}} w_1 + \cdots + a_{n,i_{\pi(s)}} w_n$$

for $s = 1, \ldots, k-1, k+q, \ldots, m$ to obtain

$$\begin{aligned} &\Upsilon_g(\alpha)\big(v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(k-1)}} \otimes f'_h \otimes v_{i_{\pi(k+q)}} \otimes \cdots \otimes v_{i_{\pi(m)}}\big) \\ &\quad = a_{j_1,i_{\pi(1)}} \cdots a_{j_{k-1},i_{\pi(k-1)}} a_{j_{k+1},i_{\pi(k+q)}} \cdots a_{j_p,i_{\pi(m)}}{}^{s_1 \cdots s_{j_k}-1}\big(\partial_{j_k} f'_h\big) f_g \overline{g}. \end{aligned}$$

Thus the k-th summand of $(\alpha \,\overline{\circ}\, \beta)(v_{i_1} \wedge \cdots \wedge v_{i_m})$ is a sum over $\pi \in \mathrm{Sym}_{p-1}$ of

$$(-1)^{(q-1)(k-1)+\lambda(1)+\cdots+\lambda(q)-k-(k+1)-\cdots-(k+q-1)}(\operatorname{sgn} \pi)$$

times

$$a_{j_1,i_{\pi(1)}} \cdots a_{j_{k-1},i_{\pi(i_{k-1})}} a_{j_{k+1},i_{\pi(k+q)}} \cdots a_{j_p,i_{\pi(m)}}{}^{s_1 \cdots s_{j_k}-1}\big(\partial_{j_k} f'_h\big) f_g \overline{gh}.$$

By definition of the determinant and some sign simplifications, this is precisely the formula claimed. □

In the remainder of this section, we refine the formula of Theorem 7.2 in the special case that the actions of g and h commute. In this case, we may choose $B_g = B_h$ to be a simultaneous basis of eigenvectors for both g and h, and the change of basis matrix M in Theorem 7.2 is simply the identity matrix. We introduce some notation to capture the single summand that remains, for each k, after the prebracket formula in Definition 6.8 collapses. For any two multi-indices $J = (j_1, \ldots, j_p)$ and $L = (l_1, \ldots, l_q)$, and $k \leqslant p$, set

$$\begin{aligned} I_k &:= (j_1, \ldots, j_{k-1}, l_1, \ldots, l_q, j_{k+1}, \ldots, j_p), \\ I'_k &:= (l_1, \ldots, l_{k-1}, j_1, \ldots, j_p, l_{k+1}, \ldots, l_q). \end{aligned}$$

Corollary 7.3. *Suppose the actions of g, h in G on V commute and $B_g = B_h = \{v_1, \dots, v_n\}$ is a simultaneous basis of eigenvectors for g and h. Let $\alpha = f_g \overline{g} \otimes dv_J$ lie in H_g^p and $\beta = f_h' \overline{h} \otimes dv_L$ lie in H_h^q, for multi-indices $J = (j_1 < \cdots < j_p)$ and $L = (l_1 < \cdots < l_q)$. Then*

$$[[\alpha, \beta]]_{(B_g, B_h)} = \sum_{1 \leqslant k \leqslant p} (-1)^{(q-1)(k-1) s_1 \cdots s_{j_k - 1}} \left(\partial_{j_k} f_h'\right) f_g \overline{gh} \otimes dv_{I_k} - (-1)^{(p-1)(q-1)} \sum_{1 \leqslant k \leqslant q} (-1)^{(p-1)(k-1) s_1' \cdots s_{l_k - 1}'} \left(\partial_{l_k}' f_g\right) f_h' \overline{hg} \otimes dv_{I_k'}$$

where s_i, s_i' are diagonal reflections (or identity maps) with ${}^{s_i} v_i = {}^{g} v_i = \epsilon_i v_i$ and ${}^{s_i'} v_i = {}^{h} v_i = \epsilon_i' v_i$, and where $\partial_i = \partial_{v_i, \epsilon_i}$ and $\partial_i' = \partial_{v_i, \epsilon_i'}$.

If the group elements g and h act as the identity, we recover the classical Schouten–Nijenhuis bracket, as detailed in the next two corollaries. Indeed, we view Corollary 7.3 as giving a quantum version of the Schouten–Nijenhuis bracket.

Corollary 7.4. *Let g and h in G both act trivially on V (e.g., $g = h = 1_G$). Let $\alpha = f_g \overline{g} \otimes dv_J$ lie in H_g^p and $\beta = f_h' \overline{h} \otimes dv_L$ lie in H_h^q, for multi-indices $J = (j_1 < \cdots < j_p)$ and $L = (l_1 < \cdots < l_q)$. If $B_g = \{v_1, \dots, v_n\} = B_h$, then*

$$[[\alpha, \beta]]_{(B_g, B_h)} = \sum_{1 \leqslant k \leqslant p} (-1)^{(q-1)(k-1)} \frac{\partial}{\partial v_{j_k}} \left(f_h'\right) f_g \overline{gh} \otimes dv_{I_k} - (-1)^{(p-1)(q-1)} \sum_{1 \leqslant k \leqslant q} (-1)^{(p-1)(k-1)} \frac{\partial}{\partial v_{l_k}} (f_g) f_h' \overline{hg} \otimes dv_{I_k'}.$$

If the group G is trivial, then the bracket agrees with the prebracket:

Corollary 7.5. *If $G = \{1\}$, we recover the classical Schouten–Nijenhuis bracket: Let $\alpha = f_1 \otimes dv_J$ in degree p and $\beta = f_2 \otimes dv_L$ in degree q. Then*

$$[\alpha, \beta] = \sum_{1 \leqslant k \leqslant p} (-1)^{(q-1)(k-1)} \frac{\partial}{\partial v_{j_k}} (f_2) f_1 \otimes dv_{I_k} - (-1)^{(p-1)(q-1)} \sum_{1 \leqslant k \leqslant q} (-1)^{(p-1)(k-1)} \frac{\partial}{\partial v_{l_k}} (f_1) f_2 \otimes dv_{I_k'}.$$

For abelian groups, brackets enjoy a combinatorial description:

Example 7.6. Let G be an abelian group acting on V with $\dim V \geqslant 3$. Let $B = \{v_1, \dots, v_n\}$, a simultaneous basis of eigenvectors for G. For $i \in \{1, \dots, n\}$, let χ_i be the character defined by ${}^{g} v_i = \chi_i(g) v_i$ for each g in G. Fix $g, h \in G$ and

$$\alpha = \left(v_1^{c_1} v_2^{c_2} v_3^{c_3}\right) \overline{g} \otimes v_1^* \wedge v_2^*,$$
$$\beta = \left(v_1^{d_1} v_2^{d_2} v_3^{d_3}\right) \overline{h} \otimes v_2^* \wedge v_3^*,$$

elements of H_g^2, H_h^2, respectively. (Note that if g acts nontrivially on V, then $c_1 = c_2 = 0$, and if h acts nontrivially on V, then $d_2 = d_3 = 0$.) A calculation using Theorems 6.12 and 7.2 gives the bracket:

$$[\alpha, \beta] = \kappa \operatorname{Proj}_H v_1^{c_1+d_1} v_2^{c_2+d_2-1} v_3^{c_3+d_3} \overline{gh} \otimes v_1^* \wedge v_2^* \wedge v_3^*,$$

where

$$\kappa = \langle \chi_1^{c_1-1} \chi_2^{c_2-1}, \chi_3^{-c_3} \rangle \langle \chi_1^{d_1}, \chi_2^{1-d_2} \chi_3^{1-d_3} \rangle \left([c_2]_\epsilon \chi_1(h)^{c_1} - [d_2]_{\epsilon'} \chi_1(g)^{d_1} \right).$$

Here, $\epsilon = \chi_2(h)$, $\epsilon' = \chi_2(g)$, $[m]_\lambda$ is the quantum integer $1 + \lambda + \cdots + \lambda^{m-1}$ (or zero when $m = 0$), and $\langle , \rangle$ denotes the inner product of characters on G. The map Proj_H in particular projects the polynomial coefficient onto $S(V^{gh})$: $v_i \mapsto v_i$ if $\chi_i(gh) = 1$, $v_i \mapsto 0$ otherwise. Thus $[\alpha, \beta]$ is usually 0, but can be nonzero as a consequence of the orthogonality relations on characters of finite groups (see Proposition 10.1 below).

8. Zero brackets

Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. Every deformation arises from a Hochschild 2-cocycle α whose square bracket $[\alpha, \alpha]$ is a coboundary. We use the formulas of Section 7 to now determine some conditions under which brackets are zero. In the process, we take advantage of our depiction of cohomology automorphisms in terms of *quantum partial derivatives*. We begin with an easy corollary of our formulation of the Gerstenhaber bracket as a sum of prebrackets.

We say that a cochain α in $C^\bullet$ is *constant* if the polynomial coefficient of α is constant, i.e., if α lies in the subspace $\bigoplus_{g \in G} \mathbb{C}\overline{g} \otimes \bigwedge V^*$. We showed in [18] that any constant Hochschild 2-cocycle lifts to a graded Hecke algebra. As graded Hecke algebras are deformations of $S(V)\#G$ (see Section 11), every constant Hochschild 2-cocycle defines a noncommutative Poisson structure. The following consequence of Theorem 6.12 extends this result to *arbitrary* cohomological degree.

Theorem 8.1. *Suppose α and β in $H^\bullet$ are constant. Then $[\alpha, \beta] = 0$.*

Proof. Theorem 6.12 gives the Gerstenhaber bracket as a sum of prebrackets of cochains ${}^a\alpha$ and ${}^b\beta$ for a, b in G. Definition 6.8 expresses the prebracket in terms of quantum partial differentiation of polynomial coefficients of cochains (see Definitions 5.8 and 5.9). As the cochains ${}^a\alpha$ and ${}^b\beta$ are constant, any partial derivative of their coefficients is zero. Hence, each prebracket is zero. □

As an immediate consequence of the theorem, we obtain:

Corollary 8.2. *Any* constant *Hochschild cocycle in* $\mathrm{HH}^\bullet(S(V)\#G)$ *defines a noncommutative Poisson structure.*

We need a quick linear algebra lemma in order to give more results on zero brackets.

Lemma 8.3. *Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two bases of V with change of basis matrix M determined by Eq.* (7.1)*. If J and L are two multi-indices with*

$$\operatorname{Span}_{\mathbb{C}}\{w_j \mid j \in J\} \cap \operatorname{Span}_{\mathbb{C}}\{v_l \mid l \in L\} \neq \{0\},$$

then $\det(M_{J;L'}) = 0$ *for any L' with $L \cap L' = \emptyset$.*

Proof. Write $\sum_{j \in J} c_j w_j = \sum_{l \in L} d_l v_l$ for some scalars c_j, d_l, not all 0. We substitute $v_i = a_{1i} w_i + \cdots + a_{ni} w_n$ to see that $\sum_{l \in L} d_l a_{il} = 0$ for $i \notin J$. If we delete the rows of M indexed by J, these equations give a linear dependence relation among the columns indexed by L. Thus $\det(M_{J;L'}) = 0$. □

As a consequence, we have the following proposition.

Proposition 8.4. *Suppose g, h lie in G and $(V^g)^\perp \cap (V^h)^\perp$ is nontrivial and fixed setwise by G. Let α and β be elements of $(H^\bullet)^G$ supported on the conjugacy classes of g and h, respectively. Then*

$$[\alpha, \beta] = 0.$$

Proof. Let α' be the summand of α supported on g itself and let β' be the summand of β supported on h. Then $|Z(g)|\alpha = \mathcal{R}\alpha'$. (Indeed, since α is G-invariant, α' is $Z(g)$-invariant and $\alpha = \sum_c {}^c\alpha'$, a sum over coset representatives c of $G/Z(g)$.) Similarly, $|Z(h)|\beta = \mathcal{R}\beta'$. Remark 4.6 then implies that

$$\big(\big|Z(g)\big|\big|Z(h)\big|\big)[\alpha, \beta] = \big[\mathcal{R}\alpha', \mathcal{R}\beta'\big] = \big[\alpha', \beta'\big],$$

and hence we compute $[\alpha', \beta']$. Set $W := (V^g)^\perp \cap (V^h)^\perp$ and recall that the Hermitian form on V is G-invariant. By Remark 6.11, we may assume that $B_g = \{w_1, w_2, \dots, w_n\}$ and $B_h = \{v_1, v_2, \dots, v_n\}$ are orthogonal bases with w_1, v_1 in W. Without loss of generality, suppose

$$\alpha' = f_g \overline{g} \otimes w_{j_1}^* \wedge \cdots \wedge w_{j_p}^* \quad \text{and} \quad \beta' = f_h' \overline{h} \otimes v_{l_1}^* \wedge \cdots \wedge v_{l_q}^*$$

in $H^\bullet$ where $f_g \in S(V^g)$, $f_h' \in S(V^h)$ and $j_1 = l_1 = 1$. As β' lies in H_h^q, the linear span of $v_{l_1}, \dots, v_{l_q}$ contains $(V^h)^\perp$ and thus w_1. Hence, for $k \neq 1$,

$$\mathrm{Span}_{\mathbb{C}}\{w_{j_i} \mid i = 1, \dots, p;\ i \neq k\} \cap \mathrm{Span}_{\mathbb{C}}\{v_{l_i} \mid i = 1, \dots, q\} \neq \{0\}.$$

Theorem 7.2 and Lemma 8.3 then imply that the k-th summand of $\alpha' \,\overline{\circ}\, \beta'$ in the prebracket $[[\alpha', \beta']]_{(B_g, B_h)}$ is zero for $k \neq 1$ (as $\det M_{J_k; I-L} = 0$). But the first ($k = 1$) summand is also zero, since $w_1 = w_{j_1}$ lies in $(V^h)^\perp$ while f_h' lies in V^h, forcing the partial derivative of f_h' with respect to w_1 and B_g to be zero: $\partial_1(f_h') = 0$. (Indeed, since $w_1 \in (V^h)^\perp$ and B_g is orthogonal, $V^h \subset \mathrm{Span}_{\mathbb{C}}\{w_2, \dots, w_n\}$: Were some $w_1 + a_2 w_2 + \cdots a_n w_n$ to lie in V^h (with $a_i \in \mathbb{C}$), the inner product $\langle w_1, w_1\rangle = \langle w_1, w_1 + a_2 w_2 + \cdots + a_n w_n\rangle$ would be zero.) Hence $\alpha' \,\overline{\circ}\, \beta' = 0$. By a symmetric argument, interchanging the role of g and h, $\beta' \,\overline{\circ}\, \alpha'$ is also zero.

The same arguments apply when we compute the prebracket $[[{}^a\alpha', {}^b\beta']]$ with respect to the pair of bases $({}^aB_g, {}^bB_h)$ for a, b in G, as ${}^aw_1$ and ${}^bv_1$ both lie in $W = {}^aW = {}^bW$ by hypothesis and

$$W \subset {}^a\big((V^g)^\perp\big) \cap {}^b\big((V^h)^\perp\big) = \big(V^{aga^{-1}}\big)^\perp \cap \big(V^{bhb^{-1}}\big)^\perp. \tag{8.5}$$

Hence, Theorem 6.12 implies that $0 = [\alpha', \beta'] = [\alpha, \beta]$. □

We immediately obtain some interesting corollaries for square brackets:

Corollary 8.6. *Let g lie in G. If $V^g \neq V$ is fixed setwise by G, then any Hochschild cocycle α in $\mathrm{HH}^\bullet(S(V)\#G)$ supported on conjugates of g defines a noncommutative Poisson bracket*:

$$[\alpha, \alpha] = 0.$$

Note that the hypothesis of the corollary is automatically satisfied for any element g in the *center* of G acting nontrivially on V. Thus the square bracket of any Hochschild cocycle in $\mathrm{HH}^\bullet(S(V)\#G)$ supported on conjugates of such an element g is zero.

More generally, we have a corollary for brackets of possibly different cocycles:

Corollary 8.7. *Suppose g, h lie in the center of G with $(V^g)^\perp \cap (V^h)^\perp$ nontrivial. Then the bracket of any two Hochschild cocycles α, β in $\mathrm{HH}^\bullet(S(V)\#G)$ supported on conjugates of g, h, respectively, is zero*:

$$[\alpha, \beta] = 0.$$

In particular, if G is abelian, such a bracket is always 0 whenever $(V^g)^\perp \cap (V^h)^\perp$ is nontrivial.

9. Zero brackets in cohomological degree 2

Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. We are particularly interested in the Gerstenhaber bracket in cohomological degree 2, since every deformation of $S(V)\#G$ arises from a Hochschild 2-cocycle μ with square bracket $[\mu, \mu]$ zero in cohomology, i.e., from a noncommutative Poisson structure (see Section 2). Graded Hecke algebras are deformations of $S(V)\#G$ that arise from noncommutative Poisson structures of a particular form (see Section 11). In Theorem 9.2 below, we prove that Hochschild 2-cocycles supported on group elements acting nontrivially on V always have zero bracket. As an immediate consequence, if G acts faithfully, then any single Hochschild 2-cocycle supported on nonidentity group elements defines a noncommutative Poisson structure.

The proof of Theorem 9.2 rests on the following combinatorial lemma about Demazure operators and reflections. In fact, by enunciating our automorphisms of cohomology in terms of Demazure operators, we show in this lemma that the operation $\overline{\circ}$ (see Definitions 6.8 and 2.2) is essentially zero on 2-cocycles after projecting to cohomology classes under the hypotheses of the theorem.

Lemma 9.1. *Suppose g in G acts as a bireflection, i.e.,* $\operatorname{codim} V^g = 2$. *For all w in V, α in H^2_g, and $m \geqslant 0$, the difference $w - {}^g w$ divides*

$$\mathfrak{T}\alpha\big(w^m \otimes w\big) - \mathfrak{T}\alpha\big(w \otimes w^m\big).$$

Proof. Let $B_g = \{v_1, \dots, v_n\}$. Without loss of generality, assume v_1, v_2 span $(V^g)^\perp$. Let ϵ_1, ϵ_2 be the corresponding (nontrivial) eigenvalues of g. Write $\alpha = f_g \overline{g} \otimes v_1^* \wedge v_2^*$ for some f_g in $S(V)$. Let s_1 and s_2 in $\mathrm{GL}(V)$ be diagonal reflections decomposing g, i.e., ${}^{s_1}v_1 = {}^g v_1 = \epsilon_1 v_1$, ${}^{s_1}v_2 = v_2$, ${}^{s_2}v_1 = v_1$, ${}^{s_2}v_2 = {}^g v_2 = \epsilon_2 v_2$, and $g = s_1 s_2 = s_2 s_1$. By Definitions 5.8 and 5.9, $\mathfrak{T}\alpha(w^m \otimes w) - \mathfrak{T}\alpha(w \otimes w^m)$ is the following multiple of f_g:

$$\begin{aligned}
&\big(\partial_1 w^m\big)\,{}^{s_1}(\partial_2 w) - (\partial_1 w)\,{}^{s_1}\big(\partial_2 w^m\big) \\
&\quad = \left(\frac{w^m - {}^{s_1}w^m}{v_1 - {}^{s_1}v_1}\right)^{s_1}\left(\frac{w - {}^{s_2}w}{v_2 - {}^{s_2}v_2}\right) - \left(\frac{w - {}^{s_1}w}{v_1 - {}^{s_1}v_1}\right)^{s_1}\left(\frac{w^m - {}^{s_2}w^m}{v_2 - {}^{s_2}v_2}\right).
\end{aligned}$$

Since $v_1 - {}^{s_1}v_1 = (1-\epsilon_1)v_1$ and $v_2 - {}^{s_2}v_2 = (1-\epsilon_2)v_2$, we may factor out the scalar $(1-\epsilon_1)^{-1}(1-\epsilon_2)^{-1}$ from each summand, leaving us with $\frac{1}{v_1 v_2}$ times

$$\begin{aligned}
&\big(w^m - {}^{s_1}w^m\big)\big({}^{s_1}w - {}^g w\big) - \big(w - {}^{s_1}w\big)\big({}^{s_1}w^m - {}^g w^m\big) \\
&\quad = w^m\big({}^{s_1}w\big) - w^m\big({}^g w\big) + \big({}^{s_1}w^m\big)\big({}^g w\big) - w\big({}^{s_1}w^m\big) + w\big({}^g w^m\big) - \big({}^{s_1}w\big)\big({}^g w^m\big) \\
&\quad = \big({}^{s_1}w\big)\big(w^m - {}^g w^m\big) + \big({}^{s_1}w^m\big)\big({}^g w - w\big) - w^m\big({}^g w\big) + w^{m+1} - w^{m+1} + w\big({}^g w^m\big) \\
&\quad = \big(w - {}^g w\big)\big(w^m - {}^{s_1}w^m\big) - \big(w - {}^{s_1}w\big)\big(w^m - {}^g w^m\big) \\
&\quad = \big(w - {}^g w\big)\big(w - {}^{s_1}w\big)\big[w^{m-1} + w^{m-2}\big({}^{s_1}w\big) + \cdots + {}^{s_1}w^{m-1}\big]
\end{aligned}$$

$$
\begin{aligned}
&- \big(w - {}^{s_1}w\big)\big(w - {}^{g}w\big)\big[w^{m-1} + w^{m-2}\big({}^{g}w\big) + \cdots + {}^{g}w^{m-1}\big] \\
&= \big(w - {}^{g}w\big)\big(w - {}^{s_1}w\big) \\
&\quad \times \big(w^{m-2}\big({}^{s_1}w - {}^{g}w\big) + w^{m-3}\big({}^{s_1}w^2 - {}^{g}w^2\big) + \cdots + \big({}^{s_1}w^{m-1} - {}^{g}w^{m-1}\big)\big).
\end{aligned}
$$

Now $w - {}^{s_1}w$ lies in $\operatorname{im}(1 - s_1) = (V^{s_1})^{\perp} = \operatorname{Span}_{\mathbb{C}}\{v_1\}$. Similarly, for each i, ${}^{s_1}w^i - {}^{g}w^i = ({}^{s_1}w^i) - {}^{s_2}({}^{s_1}w^i)$ lies in $\operatorname{im}(1 - s_2) = \operatorname{Span}_{\mathbb{C}}\{v_2\}$. Hence, $v_1 v_2$ divides the above expression in $S(V)$ and the quotient by $v_1 v_2$ is divisible by $w - {}^{g}w$. □

We are now ready to show that the Gerstenhaber bracket is zero on cocycles supported on group elements acting nontrivially. In the next section, we show that the converse of this theorem is false and that the hypothesis in the theorem cannot be easily weakened. Let K denote the kernel of the action of G on V.

Theorem 9.2. *The bracket of any two elements α, β in $\mathrm{HH}^2(S(V)\#G)$ supported off K is zero*:

$$[\alpha, \beta] = 0.$$

Proof. For each k in G, we fix a basis B_k of V consisting of eigenvectors of k so that the first $\operatorname{codim}(V^k)$ vectors in that list span $(V^k)^{\perp}$. Then for all a in G, the first $\operatorname{codim}(V^{aka^{-1}})$ vectors in the basis ${}^{a}B_k$ span $(V^{aka^{-1}})^{\perp}$ as well.

Since the bracket is linear, we may assume without loss of generality that α and β are each supported on the conjugacy class of a single element in G. In fact, it is enough to consider single summands: Assume that α lies in H^2_g and β lies in H^2_h for some g and h in G. By Theorem 6.12, our Definition 6.8 gives a closed formula for the Gerstenhaber bracket on $\mathrm{HH}^{\bullet}(S(V)\#G)$ as realized on $H^{\bullet}$:

$$[\alpha, \beta] = \frac{1}{|G|^2} \operatorname{Proj}_H \sum_{a,b \in G} \big[\big[{}^{a}\alpha, {}^{b}\beta\big]\big]_{({}^{a}B_g, {}^{b}B_h)}.$$

(By Remark 6.11, this bracket does not depend on our choices of bases B_k for each k in G.) Write $B_1 := \{w_1, \ldots, w_n\}$ for ${}^{a}B_g$ and $B_2 := \{v_1, \ldots, v_n\}$ for ${}^{b}B_h$. We assume first that $a = b = 1_G$. Consider the prebracket $[[\alpha, \beta]] := [[\alpha, \beta]]_{(B_1, B_2)}$, a cochain in $C^3_{gh} + C^3_{hg}$, and suppose that H^3_{gh} or H^3_{hg} is nonzero. We show that this prebracket is either identically zero or projects to zero under the map Proj_H.

For any $i < j < k$ and $\mathfrak{T}_1 = \mathfrak{T}_{B_1, g}$ and $\mathfrak{T}_2 = \mathfrak{T}_{B_2, h}$, Eq. (6.10) gives

$$
\begin{aligned}
[[\alpha, \beta]](v_i \wedge v_j \wedge v_k) = \sum_{\pi \in \mathrm{Sym}_3} \operatorname{sgn}(\pi) \big(& \mathfrak{T}_1 \alpha\big(\mathfrak{T}_2 \beta(v_{\pi(i)} \otimes v_{\pi(j)}) \otimes {}^{h}v_{\pi(k)}\big)\overline{gh} \\
& - \mathfrak{T}_1 \alpha\big(v_{\pi(i)} \otimes \mathfrak{T}_2 \beta(v_{\pi(j)} \otimes v_{\pi(k)})\big)\overline{gh} \\
& + \mathfrak{T}_2 \beta\big(\mathfrak{T}_1 \alpha(v_{\pi(i)} \otimes v_{\pi(j)}) \otimes {}^{g}v_{\pi(k)}\big)\overline{hg} \\
& - \mathfrak{T}_2 \beta\big(v_{\pi(i)} \otimes \mathfrak{T}_1 \alpha(v_{\pi(j)} \otimes v_{\pi(k)})\big)\overline{hg}\big). \qquad (9.3)
\end{aligned}
$$

We may assume g and h act as bireflections, i.e., $\operatorname{codim} V^g = \operatorname{codim} V^h = 2$, else H^2_g or H^2_h is zero by Remark 6.3. We consider three cases, depending on whether the spaces $(V^g)^{\perp}$ and $(V^h)^{\perp}$ intersect in dimension 0, 1, or 2.

Case 1: Disjoint orthogonal complements. Assume that $(V^g)^\perp \cap (V^h)^\perp = 0$. Then

$$\operatorname{codim} V^g + \operatorname{codim} V^h = \operatorname{codim} V^{gh} = \operatorname{codim} V^{hg}$$

(e.g., see [19, Lemma 2.1]). Thus $\operatorname{codim} V^{gh} = 4 = \operatorname{codim} V^{hg}$, and by examination of Definition 6.1, $H^3_{gh} = 0 = H^3_{hg}$ (as $H^\bullet_{gh}$ "begins" in degree 4). But we have excluded this case from consideration.

Case 2. Equal orthogonal complements. Assume that $(V^g)^\perp$ and $(V^h)^\perp$ are equal, i.e., $\dim((V^g)^\perp \cap (V^h)^\perp) = 2$, and v_1, v_2 span $(V^g)^\perp = (V^h)^\perp$. Consider

$$\alpha = f_g \overline{g} \otimes \left(v_1^* \wedge v_2^*\right) \quad \text{and} \quad \beta = f'_h \overline{h} \otimes \left(v_1^* \wedge v_2^*\right),$$

where $f_g, f'_h \in S(V^g) = S(V^h)$. If nonzero, $\mathfrak{r}_1\alpha(\mathfrak{r}_2\beta(v_i \otimes v_j) \otimes^h v_k) = \mathfrak{r}_1\alpha(f'_h \otimes^h v_k)$ up to a constant. But $\mathfrak{r}_1\alpha(f'_h \otimes *) = 0$ by Remark 5.10, as $f'_h \in S(V^g)$ has zero partial derivative with respect to any basis element in $(V^g)^\perp$. Hence

$$0 = \mathfrak{r}_1\alpha\left(\mathfrak{r}_2\beta(v_i \otimes v_j) \otimes^h v_k\right).$$

Similarly,

$$0 = \mathfrak{r}_1\alpha\left(v_i \otimes \mathfrak{r}_2\beta(v_j \otimes v_k)\right) = \mathfrak{r}_2\beta\left(\mathfrak{r}_1\alpha(v_i \otimes v_j) \otimes^g v_k\right) = \mathfrak{r}_2\beta\left(v_i \otimes \mathfrak{r}_1\alpha(v_j \otimes v_k)\right)$$

for all distinct i, j, and k and thus Eq. (9.3) yields zero. Hence, $[[\alpha, \beta]] = 0$.

Case 3. Overlapping orthogonal complements. Assume the two spaces $(V^g)^\perp$ and $(V^h)^\perp$ overlap only partially, i.e., $\dim((V^g)^\perp \cap (V^h)^\perp) = 1$. The remainder of the proof is devoted to this last case.

We now refine our bases B_1 and B_2 to ease determination of the prebracket. Recall that $B_1 = \{w_1, \ldots, w_n\}$ and $B_2 = \{v_1, \ldots, v_n\}$ are bases of V with w_1, w_2 in $(V^g)^\perp$, v_1, v_2 in $(V^h)^\perp$, $w_3, \ldots, w_n$ in V^g, and $v_3, \ldots, v_n$ in V^h. We make additional assumptions. Let $W = (V^g)^\perp + (V^h)^\perp$. The space W has dimension 3 since $(V^g)^\perp$ and $(V^h)^\perp$ intersect in dimension 1. Notice that V^g intersects W nontrivially, as otherwise $\dim(W + V^g) = \dim(W) + \dim(V^g) = 3 + (n-2) = n + 1 > \dim V$. Also note that $W^\perp = V^g \cap V^h$. Thus, we may assume further that w_1, w_2, w_3 span W, $w_4, \ldots, w_n$ span $W^\perp$, and $v_4 = w_4, \ldots, v_n = w_n$.

These refining assumptions have no effect on the prebracket $[[\alpha, \beta]]$. Although Υ is independent of choice of bases as a map on cohomology, it depends on the choices B_1 and B_2 as a *cochain map*. (See cautionary Remark 6.9.) Yet in refining our choices of B_1 and B_2, we have *not* altered the values of $\Upsilon(\alpha)$ or $\Upsilon(\beta)$ as cochains. Indeed, quantum partial differentiation with respect to one subset of variables in a basis ignores any change of basis affecting only the other variables. The exterior part of α is an element of $\bigwedge((V^g)^\perp)^*$ (as α lies in H^2_g), so the map $\Upsilon(\alpha)$ differentiates with respect to vectors in $(V^g)^\perp$. Since we altered the basis of B_1 on V^g alone, the map $\Upsilon(\alpha)$ is unchanged. Similarly, $\Upsilon(\beta)$ is also unchanged.

We examine the coefficient of $\overline{gh}$ in Eq. (9.3). Consider

$$\alpha = f_g \overline{g} \otimes w_1^* \wedge w_2^* \quad \text{and} \quad \beta = f'_h \overline{h} \otimes v_1^* \wedge v_2^*,$$

where $f_g \in S(V^g)$ and $f'_h \in S(V^h)$. We rewrite the sum (giving the coefficient of $\overline{gh}$) with indices in a different order:

$$\sum_{\pi\in \mathrm{Sym}_3} \operatorname{sgn}(\pi)\big[\Upsilon_1\alpha\big(\Upsilon_2\beta(v_{\pi(i)}\otimes v_{\pi(j)})\otimes {}^h v_{\pi(k)}\big) - \Upsilon_1\alpha\big(v_{\pi(i)}\otimes \Upsilon_2\beta(v_{\pi(j)}\otimes v_{\pi(k)})\big)\big]$$

$$= \sum_{\pi\in \mathrm{Sym}_3} \operatorname{sgn}(\pi)\big[\Upsilon_1\alpha\big(\Upsilon_2\beta(v_{\pi(i)}\otimes v_{\pi(j)})\otimes {}^h v_{\pi(k)}\big) - \Upsilon_1\alpha\big(v_{\pi(k)}\otimes \Upsilon_2\beta(v_{\pi(i)}\otimes v_{\pi(j)})\big)\big]. \quad (9.4)$$

By Remark 5.10, each of the above summands is zero for every permutation save one. Indeed, the summand corresponding to π is zero unless $\pi(i)=1$, $\pi(j)=2$, and $\pi(k)=3$ (since $v_4 = w_4, \dots, v_n = w_n$), and we are left with

$$\begin{aligned}&\Upsilon_1\alpha\big(\Upsilon_2\beta(v_1\otimes v_2)\otimes {}^h v_3\big) - \Upsilon_1\alpha\big(v_3\otimes \Upsilon_2\beta(v_1\otimes v_2)\big)\\ &\quad = \Upsilon_1\alpha\big(f_h'\otimes v_3\big) - \Upsilon_1\alpha\big(v_3\otimes f_h'\big).\end{aligned} \quad (9.5)$$

We show that this difference maps to zero under the projection $S(V)\to S(V^{gh})$. We reduce to the case when f_h' is a power of v_3. Indeed, the above difference is just

$$\partial_1\big(f_h'\big)\,{}^{s_1}\partial_2(v_3) - \partial_1(v_3)\,{}^{s_1}\partial_2\big(f_h'\big)$$

where s_1 is some reflection and ∂_i is some partial quantum differentiation with respect to w_i in the basis $\{w_1, w_2, w_3, w_4 = v_4, \dots, w_n = v_n\}$. As ∂_1, ∂_2 are both $\mathbb{C}[v_4, \dots, v_n]$-linear, and f_h' lies in $S(V^h) = \mathbb{C}[v_3, \dots, v_n]$, we may break f_h' into its monomial summands (in the basis $\{v_1, \dots, v_n\}$) and pull out all factors from $\mathbb{C}[v_4, \dots, v_n]$ when evaluating the above difference. Thus, it suffices to consider the special case when $f_h' = v_3^m$ for some $m \geqslant 0$.

By Lemma 9.1, $u := v_3 - {}^g v_3$ divides

$$\Upsilon_1\alpha\big(v_3^m\otimes v_3\big) - \Upsilon_1\alpha\big(v_3\otimes v_3^m\big). \quad (9.6)$$

But notice that u lies in $(V^{gh})^\perp$, since

$${}^{gh}(u) = {}^{gh}\big(v_3 - {}^g v_3\big) = {}^g v_3 - {}^{ghg} v_3 = {}^{1-gh}\big({}^g v_3\big),$$

i.e., ${}^{gh}u$ lies in $\operatorname{im}(1-gh) = (V^{gh})^\perp$. Thus the difference (9.6) lies in the ideal $I((V^{gh})^\perp)$ of $S(V)$ and projects to zero under the map $S(V)\to S(V^{gh})$.

By a symmetric argument, the coefficient of $\overline{hg}$ in Eq. (9.3) projects to zero under the map $S(V)\to S(V^{hg})$. Thus, the prebracket $[[\alpha,\beta]]$ projects to zero under the map Proj_H. The same arguments apply to $[[{}^a\alpha, {}^b\beta]]_{({}^aB_g, {}^bB_h)}$ for arbitrary a, b in G. Hence,

$$[\alpha,\beta] := \frac{1}{|G|^2}\mathrm{Proj}_H \sum_{a,b\in G} \big[\big[{}^a\alpha, {}^b\beta\big]\big]_{({}^aB_g, {}^bB_h)} = 0. \qquad \square$$

The theorem above implies that if α lies in $\mathrm{HH}^2(S(V)\#G)$ with $[\alpha,\alpha]\neq 0$, then the support of α includes at least one group element acting as the identity on V:

Corollary 9.7. *The Gerstenhaber square bracket of every α in $\mathrm{HH}^2(S(V)\#G)$ supported off of K is zero, i.e., α defines a noncommutative Poisson structure on $S(V)\#G$:*

$$[\alpha,\alpha] = 0.$$

Next, we illustrate our results by giving an explicit example of a Gerstenhaber bracket in degree 2.

Example 9.8. Let $G = D_8$, the dihedral group of order 8, generated by g and h with relations $g^4 = 1 = h^2$, $hgh^{-1} = g^3$, realized as a subgroup of $\mathrm{GL}_3(\mathbb{C})$ in the following way:

$$g = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$. Let $B_h = \{v_1, v_2, v_3\}$ be the corresponding basis of $\mathbb{C}^3$, and set $B_g = \{w_1, w_2, w_3\}$ where $w_1 = v_1 + v_2$, $w_2 = -v_1 + v_2$, and $w_3 = v_3$, so that ${}^g w_1 = i w_1$ and ${}^g w_2 = -i w_2$. Define

$$\alpha = v_3 \overline{g} \otimes w_1^* \wedge w_2^* \quad \text{in } H_g^2,$$
$$\beta = v_1^3 \overline{h} \otimes v_2^* \wedge v_3^* \quad \text{in } H_h^2.$$

Note that $v_3 \in S(V^g)$, $v_1^3 \in S(V^h)$, and α, β are $Z(g)$-, $Z(h)$-invariant, respectively. We compute $[\alpha, \beta]$. By Definition 6.8, $(\alpha \,\overline{\circ}\, \beta)(v_1 \wedge v_2 \wedge v_3)$ is equal to

$$\sum_{\pi \in \mathrm{Sym}_3} (\operatorname{sgn} \pi) \mathcal{T}\alpha\big(\mathcal{T}\beta(v_{\pi(1)} \otimes v_{\pi(2)}) \otimes {}^h v_{\pi(3)}\big)\overline{gh} - \mathcal{T}\alpha\big(v_{\pi(1)} \otimes \mathcal{T}\beta(v_{\pi(2)} \otimes v_{\pi(3)})\big)\overline{gh}.$$

The $\pi = 1$ and $\pi = (123)$ terms are

$$\left(-\frac{3}{16}w_1^2 - \frac{3}{16}(1+i)w_1 w_2 - \frac{i}{16}w_2^2\right) v_3 \overline{gh}, \quad \text{and}$$
$$\left(-\frac{i}{16}w_1^2 + \frac{3}{16}(1+i)w_1 w_2 - \frac{3}{16}w_2^2\right) v_3 \overline{gh},$$

respectively, and the remaining terms are 0. Combining these, we find that the coefficient of $\overline{gh}$ in $(\alpha \,\overline{\circ}\, \beta)(v_1 \wedge v_2 \wedge v_3)$ is

$$\left(\frac{-3-i}{16}w_1^2 + \frac{-3-i}{16}w_2^2\right) v_3.$$

Thus $\alpha \,\overline{\circ}\, \beta$ is nonzero, as a function at the chain level, given our choices of bases. By Lemma 9.1, $(\alpha \,\overline{\circ}\, \beta)(v_1 \wedge v_2 \wedge v_3)$ is divisible by $u := (1-g)v_1 = v_1 - iv_2$, and indeed $w_1^2 + w_2^2$ is a scalar multiple of $(v_1 + iv_2)(v_1 - iv_2)$, the product of an element in V^{gh} with an element in $(V^{gh})^{\perp}$. Hence, $[\alpha, \beta]$ projects to zero under the map Proj_H, i.e., it is a coboundary. Similar calculations give ${}^a\alpha \,\overline{\circ}\, {}^b\beta$ and ${}^b\beta \,\overline{\circ}\, {}^a\alpha$ for all a, b in G. Thus $[\alpha, \beta] = 0$ in cohomology, as predicted by Theorem 9.2.

10. Abelian groups and inner products of characters

We now consider the Gerstenhaber bracket for abelian groups. Orthogonality relations on characters allow us to place Theorem 9.2 in context. We observe that the hypothesis of the theorem cannot be weakened and we show that its converse is false.

Let G be a finite abelian group and V a (not necessarily faithful) $\mathbb{C}G$-module of finite dimension n. We first explain how inner products of characters of G determine the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$. It would be interesting to know whether a similar description holds for arbitrary groups. We concentrate on cohomological degree 2 due to connections with deformation theory. Assume $\dim V \geqslant 3$, as otherwise the Hochschild cohomology of $S(V)\#G$ in degree 3 is always zero.

Fix a simultaneous basis of eigenvectors for G, say $B = \{v_1, \ldots, v_n\}$. For $i \in \{1, \ldots, n\}$, let χ_i be the character defined by ${}^g v_i = \chi_i(g) v_i$ for each g in G. The Gerstenhaber bracket of 2-cocycles in

$(H^\bullet)^G \cong \mathrm{HH}^\bullet(S(V)\#G)$ may be computed summand by summand, at the cochain level, as the bracket is linear and extends to $H^\bullet$ (see Remark 6.14). Thus, it suffices to describe the bracket on simple cochains of the form

$$\alpha = f_g \overline{g} \otimes v_i^* \wedge v_j^* \quad \text{and} \quad \beta = f_h' \overline{h} \otimes v_k^* \wedge v_m^*$$

where f_g, f_h' are monomials and g, h lie in G. Three cases arise.

Case 1. If the exterior parts of α and β agree up to sign (i.e., $\{i, j\} = \{k, m\}$), then the bracket $[\alpha, \beta]$ (given by Eq. (6.10)) is easily seen to be zero by Remark 5.10.

Case 2. If the exterior parts of α and β partially overlap, relabel indices so that

$$\alpha = f_g \overline{g} \otimes v_1^* \wedge v_2^* \quad \text{and} \quad \beta = f_h' \overline{h} \otimes v_2^* \wedge v_3^*.$$

The bracket operations $[*, \beta]$ and $[\alpha, *]$ are then both $\mathbb{C}[v_4, \dots, v_n]$-linear: $[v_i\alpha, \beta] = [\alpha, v_i\beta] = v_i[\alpha, \beta]$ for $i \geqslant 4$. The bracket computation thus reduces to that of Example 7.6, in which $f_g = v_1^{c_1} v_2^{c_2} v_3^{c_3}$ and $f_h' = v_1^{d_1} v_2^{d_2} v_3^{d_3}$. The formula in that example expresses the bracket $[\alpha, \beta]$ in terms of inner products of the group characters χ_i. Note that the bracket $[\alpha, \beta]$ is zero or $\chi_1^{c_1+d_1-1} \chi_2^{c_2+d_2-2} \chi_3^{c_3+d_3-1} = 1$, and thus the bracket $[\alpha, \beta]$ is G-invariant (see the formula), even if α and β are not (verifying Remark 6.14).

Case 3. If the exterior parts of α and β do not overlap (i.e., $\{i, j\} \cap \{k, m\} = \emptyset$), a similar but lengthier formula results. We may assume

$$\alpha = \big(v_1^{c_1} v_2^{c_2} v_3^{c_3} v_4^{c_4}\big)\overline{g} \otimes v_1^* \wedge v_2^* \quad \text{and} \quad \beta = \big(v_1^{d_1} v_2^{d_2} v_3^{d_3} v_4^{d_4}\big)\overline{h} \otimes v_3^* \wedge v_4^*.$$

The bracket $[\alpha, \beta]$ is given by

$$\mathrm{Proj}_H \sum_{1 \leqslant l \leqslant 4} \kappa_l \big(v_1^{c_1+d_1} v_2^{c_2+d_2} v_3^{c_3+d_3} v_4^{c_4+d_4} v_l^{-1}\big)\overline{gh} \otimes v_1^* \wedge \cdots \wedge \hat{v}_l^* \wedge \cdots \wedge v_4^*,$$

where

$$\begin{aligned}
\kappa_1 &:= +\big\langle \chi_1^{c_1-1} \chi_2^{c_2+1} \chi_3^{c_3} \chi_4^{c_4}, 1\big\rangle \big\langle \chi_1^{d_1} \chi_2^{d_2-2} \chi_3^{d_3-1} \chi_4^{d_4-1}, 1\big\rangle \cdot [d_1]_{\chi_1(g)}, \\
\kappa_2 &:= -\big\langle \chi_1^{c_1+1} \chi_2^{c_2-1} \chi_3^{c_3} \chi_4^{c_4}, 1\big\rangle \big\langle \chi_1^{d_1-2} \chi_2^{d_2} \chi_3^{d_3-1} \chi_4^{d_4-1}, 1\big\rangle \cdot [d_2]_{\chi_2(g)} \cdot \chi_1^{d_1}(g), \\
\kappa_3 &:= +\big\langle \chi_1^{c_1-1} \chi_2^{c_2-1} \chi_3^{c_3} \chi_4^{c_4-2}, 1\big\rangle \big\langle \chi_1^{d_1} \chi_2^{d_2} \chi_3^{d_3-1} \chi_4^{d_4+1}, 1\big\rangle \cdot [c_3]_{\chi_3(h)} \cdot \big(\chi_1^{c_1} \chi_2^{c_2}\big)(h), \\
\kappa_4 &:= -\big\langle \chi_1^{c_1-1} \chi_2^{c_2-1} \chi_3^{c_3-2} \chi_4^{c_4}, 1\big\rangle \big\langle \chi_1^{d_1} \chi_2^{d_2} \chi_3^{d_3+1} \chi_4^{d_4-1}, 1\big\rangle \cdot [c_4]_{\chi_4(h)} \cdot \big(\chi_1^{c_1} \chi_2^{c_2} \chi_3^{c_3}\big)(h),
\end{aligned}$$

$\langle , \rangle$ denotes inner product of group characters, and $[m]_\lambda$ is the quantum integer $1 + \lambda + \lambda^2 + \cdots + \lambda^{m-1}$ (or zero when $m = 0$). Note that $[\alpha, \beta]$ is either zero or $\chi_1^{c_1+d_1-1} \chi_2^{c_2+d_2-1} \chi_3^{c_3+d_3-1} \chi_4^{c_4+d_4-1} = 1$. Thus the bracket $[\alpha, \beta]$ is G-invariant, even if α and β are not, as results in previous sections predict.

We now use our analysis of the Gerstenhaber bracket for abelian groups to revisit Theorem 9.2 and Corollary 9.7. The next two results show that the hypotheses on Theorem 9.2 and Corollary 9.7 cannot be weakened. Theorem 10.2 below reveals that the converse of Theorem 9.2 is false for any abelian group. We do not know whether the statements below hold for nonabelian groups. Recall that $K = \{g \in G\colon\ V^g = V\}$, the kernel of the action of G on V.

Proposition 10.1. *Let G be an abelian group. Suppose g, h lie in K and* $\dim V$ *is at least* 3. *Then there are elements α, β in* $\mathrm{HH}^2(S(V)\#G)$ *supported on g, h, respectively, with nonzero Gerstenhaber bracket*: $[\alpha, \beta] \neq 0$.

Proof. We shall use the notation and formula of Example 7.6. Let

$$\alpha = \left(v_1 v_2^{|G|+1}\right)\overline{g} \otimes v_1^* \wedge v_2^* \quad \text{and} \quad \beta = \left(v_2 v_3^{|G|+1}\right)\overline{h} \otimes v_2^* \wedge v_3^*,$$

i.e., set $c_1 = 1$, $c_2 = |G| + 1$, $c_3 = 0$, $d_1 = 0$, $d_2 = 1$, and $d_3 = |G| + 1$. Note that α and β are invariant cocycles. Then

$$[\alpha, \beta] = \kappa\left(v_1 v_2^{|G|+1} v_3^{|G|+1}\right)\overline{gh} \otimes v_1^* \wedge v_2^* \wedge v_3^*$$

where

$$\kappa = \langle 1 \cdot 1, 1\rangle\langle 1, 1 \cdot 1\rangle\left(|G| + 1 - 1\right) = |G|$$

and 1 denotes the trivial character of G. Hence $[\alpha, \beta]$ is nonzero. □

The last proposition implies:

Theorem 10.2. *Let G be an abelian group. Then*

- *There is a* 2*-cocycle supported on K whose square bracket is nonzero.*
- *There is a* 2*-cocycle supported on K whose square bracket is zero.*

Proof. We prove a slightly stronger statement. Let k be any element of K. We apply Proposition 10.1 in the case that $g = h = k$. We obtain cocycles α, β in $(H_k^\bullet)^G \cong \mathrm{HH}^2(S(V)\#G)$ with $[\alpha, \beta] \neq 0$. Then as

$$[\alpha + \beta, \alpha + \beta] = [\alpha, \alpha] + 2[\alpha, \beta] + [\beta, \beta],$$

there must be a cocycle supported on k with square bracket nonzero.

Now set $\alpha = (v_1 v_2)\overline{k} \otimes v_1^* \wedge v_2^*$. Since G acts diagonally, α is G-invariant, i.e., α lies in $(H^2)^G \cong \mathrm{HH}^2(S(V)\#G)$. Yet $[\alpha, \alpha] = 0$ (see Case 1 above). □

We end this section by pointing out a direct and easy proof of Theorem 9.2 for abelian groups as follows. Suppose α and β in $(H^2)^G$ are supported off K but $[\alpha, \beta] \neq 0$. Then the bracket of some summand of α and some summand of β is nonzero. We consider the three cases at the beginning of this section. The bracket in Case 1 is always zero. The bracket in Case 3 is also zero: Remark 6.3 implies that v_1, v_2 span $(V^g)^\perp$ while v_3, v_4 span $(V^h)^\perp$; hence $\mathbb{C}[v_1, v_2, v_3, v_4]$ projects to zero under the map $S(V) \to S(V^{gh}) = S(V^{hg})$ and the bracket lies in the kernel of Proj_H. We thus reduce to Case 2 and Example 7.6: We assume v_1, v_2 span $(V^g)^\perp$ and v_2, v_3 span $(V^h)^\perp$. Then $[\alpha, \beta] = 0$ as the polynomial coefficient of α (in $S(V^g)$) and the polynomial coefficient of β (in $S(V^h)$) both vanish after taking the partial derivative with respect to v_2. (See Remark 5.10 or the formula of Example 7.6 with $c_1 = c_2 = d_2 = d_3 = 0$.)

11. Graded (Drinfeld) Hecke algebras

We end by briefly highlighting connections with graded Hecke algebras (often called Drinfeld Hecke algebras), which include both symplectic reflection algebras (and rational Cherednik algebras in particular) and Lusztig's graded version of the affine Hecke algebra as special cases. We show how

the maps in previous sections give explicit conversions among graded Hecke algebras, deformation theory, and Hochschild cocycles (expressed as vector forms).

Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. Let $T(V)$ denote the tensor algebra on V and let $\kappa : V \times V \to \mathbb{C}G$ be a bilinear, skew-symmetric function. A *graded Hecke algebra* is a quotient

$$\mathcal{H} = T(V)\#G/\big(v \otimes w - w \otimes v - \kappa(v, w) \;\big|\; v, w \in V\big) \tag{11.1}$$

that satisfies the Poincaré–Birkhoff–Witt property: Any linear splitting of the canonical projection $T(V) \to S(V)$ induces a vector space isomorphism $\mathcal{H} \cong S(V)\#G$. We extend scalars to the polynomial ring $\mathbb{C}[t]$ and consider every graded Hecke algebra as a quotient

$$T(V)\#G[t]/\big(v \otimes w - w \otimes v - \kappa(v, w)t \;\big|\; v, w \in V\big). \tag{11.2}$$

See [17,18] for basic definitions.

We first restate [23, Theorem 3.2] in our context. (Alternatively, results from [1] and [3] could be used to obtain this and related statements on deformations of $S(V)\#G$.) Recall that the i-th multiplication map for a given deformation of $S(V)\#G$ is denoted μ_i (see Section 2). Consider $S(V)\#G$ to be a graded algebra with $\deg v = 1$, $\deg g = 0$ for all v in V, g in G. We agree that the zero map has degree i for any integer i.

Theorem 11.3. *Every graded Hecke algebra is isomorphic to a deformation of $S(V)\#G$. In fact, up to isomorphism, the graded Hecke algebras are precisely the deformations of $S(V)\#G$ over $\mathbb{C}[t]$ for which the i-th multiplication map lowers degree by $2i$, i.e., μ_i is a (homogeneous) graded map with $\deg \mu_i = -2i$ $(i \geqslant 1)$.*

We now discuss the explicit conversions among graded Hecke algebras, deformations, and Hochschild cocycles by interpreting the above theorem and its proof using our results from previous sections.

Converting deformations to graded Hecke algebras. Given a fixed deformation of $S(V)\#G$ over $\mathbb{C}[t]$ for which $\deg \mu_i = -2i$ for $i \geqslant 1$ (see Section 2), we obtain a graded Hecke algebra by defining $\kappa : V \times V \to \mathbb{C}G$ by

$$\kappa(v, w) = \mu_1(v \otimes w) - \mu_1(w \otimes v).$$

Note that κ is skew-symmetric by definition, even if μ_1 is not skew-symmetric. It is shown in [23] that the graded Hecke algebra (11.1) corresponding to this choice is isomorphic to the deformation with which we started.

Converting Hochschild cocycles to graded Hecke algebras. We identify $\mathrm{HH}^\bullet(S(V)\#G)$ with a subset of $C^\bullet$, vector forms tagged by group elements, using Theorem 6.2. The algebra $S(V)$ is graded by polynomial degree: $S(V) = \bigoplus_{k \geqslant 0} S(V)_{(k)}$. This induces a grading on $S(V)\#G$ (after assigning degree 0 to each g in G) which is inherited by $C^\bullet$:

$$C^\bullet = \bigoplus_{k \geqslant 0,\ g \in G} S(V)_{(k)}\overline{g} \otimes \bigwedge\nolimits^{\bullet} V^*.$$

Recall that a Hochschild p-cocycle is said to be *constant* if it lies in the 0-th graded piece of C^p, i.e., defines a vector form with constant polynomial part. We rephrase Theorem 8.7 of [18], which uses Theorem 11.3 to determine that every graded Hecke algebra arises from a *constant* 2-form.

Theorem 11.4. *The parameter space of graded Hecke algebras is isomorphic to the space of* constant *Hochschild* 2*-cocycles,*

$$\left(\bigoplus_{g\in G} \overline{g} \otimes \bigwedge^{2-\operatorname{codim} V^g} \left(V^g\right)^* \otimes \bigwedge^{\operatorname{codim} V^g} \left(\left(V^g\right)^{\perp}\right)^*\right)^G.$$

We next give an explicit conversion from constant 2-cocycles to graded Hecke algebras.

Proposition 11.5. *The correspondence above is induced from the map*:

$$\{\textit{constant Hochschild 2-cocycles}\} \to \{\textit{graded Hecke algebras}\},$$
$$\alpha \mapsto \big(T(V)\#G\big)/I_\alpha,$$

where I_α *is the ideal generated by* $\{v\otimes w - w\otimes v - \alpha(v\wedge w) \mid v, w \in V\}$ *and* $\alpha \in C^2 \cong \operatorname{Hom}_{\mathbb{C}}(\bigwedge^2 V, \mathbb{C}G)$. (*I.e.,* α *defines a graded Hecke algebra with* $\kappa(v,w) = \alpha(v\wedge w)$.)

Proof. We use our conversion map $\Gamma : (H^\bullet)^G \to \mathrm{HH}^\bullet(S(V)\#G)$ of Theorem 6.4. The values of the multiplication map μ_1 in a deformation are given by the corresponding Hochschild 2-cocycle in $\mathrm{HH}^\bullet(S(V)\#G)$. Let α lie in $(H^2)^G$ with isomorphic image $\Gamma(\alpha)$ in $\mathrm{HH}^\bullet(S(V)\#G)$. Note that for any v, w in V,

$$(\Upsilon\alpha)(v\otimes w - w\otimes v) = \alpha(v\wedge w).$$

Then,

$$\begin{aligned}
\Gamma(\alpha)(v\otimes w - w\otimes v) &= \frac{1}{|G|}\sum_{g\in G} \Theta^{*\,g}(\Upsilon\alpha)(v\otimes w - w\otimes v)\\
&= \frac{1}{|G|}\sum_{g\in G} {}^g\big((\Upsilon\alpha)^{\,g^{-1}}(v\otimes w - w\otimes v)\big)\\
&= \frac{1}{|G|}\sum_{g\in G} {}^g\big(\alpha^{\,g^{-1}}(v\wedge w)\big)\\
&= \frac{1}{|G|}\sum_{g\in G} \big({}^g\alpha\big)(v\wedge w)\\
&= \alpha(v\wedge w).
\end{aligned}$$

By Theorem 11.3 and its proof, $\mu_i(v\otimes w) = 0$ for all $i \geqslant 2$. Thus we have $\Gamma(\alpha)(v\otimes w - w\otimes v)t = (\mu_1(v\otimes w) - \mu_1(w\otimes v))t = v*w - w*v$ in the deformation of $S(V)\#G$ over $\mathbb{C}[t]$. This corresponds to the relation in the graded Hecke algebra $\alpha(v\wedge w) = v\otimes w - w\otimes v$. □

Converting Hochschild cocycles to infinitesimal deformations. The interpretation in the last result and its proof hold for Hochschild cocycles of arbitrary polynomial degree, not just constant cocycles: The proof gives the conversion from *any* Hochschild 2-cocycle to an infinitesimal deformation. In fact, a closed form expresses the multiplication map μ_1 in terms of quantum differentiation. Let α in $(H^\bullet)^G$ be a Hochschild 2-cocycle; then α defines a multiplication map $A\otimes A \to A$ for $A = S(V)\#G$ given by

$$\mu_1(f_1\bar{h}_1 \otimes f_2\bar{h}_2) = \frac{1}{|G|}\sum_{g\in G} {}^g(\Upsilon\alpha)\big(f_1 \otimes {}^{h_1}f_2\big)\overline{h_1 h_2} \tag{11.6}$$

(see Remark 6.5). If μ_1 integrates, then the above formula gives the coefficient of t in the product of $f_1\bar{h}_1$, $f_2\bar{h}_2$ in the corresponding deformation of $S(V)\#G$ over $\mathbb{C}[t]$. In particular, when α is constant, the formula defines the first multiplication map μ_1 of a deformation of $S(V)\#G$ arising from a graded Hecke algebra. (We see directly in that case that μ_1 must lower degree by 2.)

Converting graded Hecke algebras to deformations and Hochschild cocycles. Consider a graded Hecke algebra defined by κ. Define α in $C^2 \cong \mathrm{Hom}_{\mathbb{C}}(\bigwedge^2 V, \mathbb{C}G)$ by

$$\alpha(v \wedge w) = \kappa(v, w) \in \mathbb{C}G.$$

Theorem 6.6 implies that α defines a constant Hochschild cocycle in $\mathrm{HH}^2(S(V)\#G)$, since κ defines the first multiplication map μ_1 of a deformation (i.e., a cocycle in $\mathrm{HH}^2(S(V)\#G)$) by Theorem 11.3. (Alternatively, we may use Theorem 11.4 above.) The second author [23] showed how to define functions μ_i $(i \geqslant 1)$ giving the corresponding deformation of $S(V)\#G$ over $\mathbb{C}[t]$. But the construction of the μ_i is iterative, involving repeated applications of the relations in the graded Hecke algebra. Our closed formula (11.6) improves this description by giving the multiplication map μ_1 in terms of quantum differentiation.

Faithful versus nonfaithful actions. We end by pointing out that it is not sufficient merely to consider G modulo the kernel of its representation in this theory: The Hochschild cohomology of $S(V)\#G$ for G acting nonfaithfully on V requires extra care. As an example, we explicitly point out the contribution from the kernel of the representation of G on V to the space of graded Hecke algebras. (The effect of the kernel on the ring structure of cohomology under cup product is described in [20].) Theorem 11.4 and Remark 6.3 imply (also see [17, Theorem 1.9] and [18, Corollary 8.17]):

Corollary 11.7. *The parameter space of graded Hecke algebras is isomorphic to*

$$\bigoplus_{\substack{g\in\mathcal{C}\\ \operatorname{codim} V^g=2\\ \det h|_{(V^g)^\perp}=1,\ \forall h\in Z(g)}} \big(\mathbb{C}\overline{g}\otimes \mathrm{vol}_g^\perp\big) \oplus \bigoplus_{\substack{g\in\mathcal{C}\\ V^g=V}} \Big(\overline{g}\otimes\bigwedge^2 V^*\Big)^{Z(g)},$$

where $\mathrm{vol}_g^\perp$ *is any fixed choice of volume form on* $((V^g)^\perp)^*$*, and* $\mathcal{C}$ *is a set of representatives of conjugacy classes of* G*.*

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