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Symplectic alternating nil-algebras

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ABSTRACT

In this paper we continue developing the theory of symplectic alternating algebras that was started in Traustason (2008) [3]. We focus on nilpotency, solubility and nil-algebras. We show in particular that symplectic alternating nil-2 algebras are always nilpotent and classify all nil-algebras of dimension up to 8.

1. Introduction

Symplectic alternating algebras have arisen in the study of 2-Engel groups (see [1,2]) but seem also to be of interest in their own right, with many beautiful properties. Some general theory was developed in [3].

Definition. Let F be a field. A *symplectic alternating algebra* over F is a triple $L = (V, (\,,\,), \cdot)$ where V is a symplectic vector space over F with respect to a non-degenerate alternating form $(\,,\,)$ and $\cdot$ is a bilinear and alternating binary operation on V such that

$$(u \cdot v, w) = (v \cdot w, u)$$

for all $u, v, w \in V$.

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Notice that $(u \cdot x, v) = (x \cdot v, u) = -(v \cdot x, u) = (u, v \cdot x)$. The multiplication by x from the right is therefore a *self-adjoint* linear operation with respect to the alternating form. We know that the dimension of a symplectic alternating algebra must be even and we will refer to a basis $x_1, y_1, \ldots, x_r, y_r$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij}$ as a *standard basis*. We will also adopt the *left-normed* convention for multiple products. Thus $x_1x_2 \cdots x_n$ stands for $(\cdots(x_1x_2)\cdots)x_n$. If $x_1, x_2, \ldots, x_{2r}$ is a basis for the symplectic vector space, then the alternating product is determined from the values of all triples $(x_ix_j, x_k) = (x_jx_k, x_i) = (x_kx_i, x_j)$ for $1 \leqslant i < j < k \leqslant 2r$.

Given a standard basis $x_1, y_1, \ldots, x_r, y_r$ for a symplectic alternating algebra L, we can describe L, as follows. Consider the two isotropic subspaces $Fx_1 + \cdots + Fx_r$ and $Fy_1 + \cdots + Fy_r$. It suffices then to write only down the products of x_ix_j, y_iy_j, $1 \leqslant i < j \leqslant r$. The reason for this is that having determined these products we have determined (uv, w) for all triples u, v, w of basis vectors, since two of those are either some x_i, x_j or some y_i, y_j in which case the triple is determined from x_ix_j or y_iy_j. The only restraints on the products x_ix_j and y_iy_j come from $(x_ix_j, x_k) = (x_jx_k, x_i) = (x_kx_i, x_j)$ and $(y_iy_j, y_k) = (y_jy_k, y_i) = (y_ky_i, y_j)$.

It is clear that the only symplectic alternating algebra of dimension 2 is the abelian one. Furthermore, it is easily seen that up to isomorphism there are two symplectic alternating algebras of dimension 4: one is abelian whereas the other one has the following multiplication table (see [3])

$$L:\quad \begin{aligned} x_1x_2 &= 0,\\ y_1y_2 &= -y_1,\\ x_1y_1 &= x_2,\\ x_1y_2 &= -x_1,\\ x_2y_1 &= 0,\\ x_2y_2 &= 0. \end{aligned}$$

Of course, the presentation is determined by $x_1x_2 = 0$ and $y_1y_2 = -y_1$ as the other products are consequences of these two. The symplectic alternating algebras of dimension 6 have been classified in [3], when the field has three elements: there are 31 such algebras of which 15 are simple.

As we said before, some general theory was developed in [3]. In particular it was shown that a symplectic alternating algebra is either semisimple or has an abelian ideal. In this paper we continue developing a structure theory for symplectic alternating algebras and we are motivated by the following question that was posed in [3]:

Question. What can one say about the structure of symplectic alternating nil-algebras? In particular, does a symplectic alternating nil-algebra have to be nilpotent?

If k is a positive integer, we say that a symplectic alternating algebra L is *nil-k* if $xy^k = 0$ for all $x, y \in L$. More generally, a *symplectic alternating nil-algebra* is a symplectic alternating nil-k algebra for some positive integer k. Also, we define $a \in L$ to be a *right nil-k element* if $ax^k = 0$ for all $x \in L$ and to be a *right nil-element* if it is right nil-k for some k. Similarly, $a \in L$ is a *left nil-k element* when $xa^k = 0$ for all $x \in L$ and a *left nil-element* if it is left nil-k for some k.

Furthermore, we say that a symplectic alternating algebra is *nilpotent* if $x_1x_2 \cdots x_n = 0$ for all $x_1, x_2, \ldots, x_n \in L$ and for some integer $n \geqslant 1$. As usual, the *nilpotency class* of L is the smallest $c \geqslant 0$ such that $x_1x_2 \cdots x_{c+1} = 0$ for all $x_1, x_2, \ldots, x_{c+1} \in L$.

In the following, we first discuss connections between nilpotency and solubility of a symplectic alternating algebra. We will see in particular that every symplectic alternating algebra that is abelian-by-nilpotent is nilpotent. We then move to nil-k elements and to symplectic alternating nil-k algebras. We get a positive answer to the question above for $k = 2$ and, when the dimension is $\leqslant 8$, also for $k = 3$. We finish with the classification of all nil-algebras of dimension up to 8.

2. Nilpotency and solubility

For subspaces U, V of a symplectic alternating algebra L, we define UV in the usual way as the subspace consisting of all linear spans of elements of the form uv where $u \in U$ and $v \in V$. We define the *lower central series* $(L^i)_{i \geqslant 1}$ inductively by $L^1 = L$ and $L^{i+1} = L^i \cdot L$. Clearly

$$L^1 \geqslant L^2 \geqslant \cdots$$

which implies in particular that every L^i is an ideal. We can also define the *upper central series* $(Z^i(L))_{i \geqslant 0}$ naturally by $Z^0(L) = \{0\}$, $Z^1(L) = Z(L) = \{a \in L\colon\ ax = 0 \text{ for all } x \in L\}$ and $Z^{i+1}(L) = \{a \in L\colon ax \in Z^i(L) \text{ for all } x \in L\}$. In [3, Lemma 2.2], the author proves that the lower and the upper central series are related as follows:

$$Z^i(L) = \left(L^{i+1}\right)^{\perp}.$$

It follows that $Z^i(L)$ is an ideal since, in a symplectic alternating algebra, $I^{\perp}$ is an ideal whenever I is an ideal (see [3, Lemma 2.1]); but this also follows directly from $Z^{i+1}(L) \cdot L \leqslant Z^i(L)$. Notice also that the $\dim(Z^i(L)) + \dim(L^{i+1}) = \dim(L)$. We then have that L is nilpotent of class $c \geqslant 0$ if and only if c is the smallest integer such that $Z^c(L) = L$ or, equivalently, $L^{c+1} = \{0\}$. One more way to characterize the nilpotency in terms of the lower central series is given by the following result.

Proposition 2.1. *Let L be a symplectic alternating algebra. Then L is nilpotent if and only if there exists $i \geqslant 1$ such that L^i is isotropic.*

Proof. Let L be nilpotent and denote by c its nilpotency class. Then $L = Z^c(L) = (L^{c+1})^{\perp}$ and hence L^{c+1} is isotropic. Conversely, let L^i be isotropic for some $i \geqslant 1$. Then

$$(u_1 \cdots u_i, v_1 \cdots v_i) = 0$$

whenever $u_1, \ldots, u_i, v_1, \ldots, v_i$ belong to L. It follows

$$(u_1, v_1 \cdots v_i u_i \cdots u_2) = 0$$

and thus L is nilpotent of class at most $2i - 2$ since the symplectic form is non-degenerate. $\square$

As usual, the *derived series* $(L^{(i)})_{i \geqslant 0}$ is defined inductively by $L^{(0)} = L$, $L^{(1)} = L \cdot L = L^2$ and $L^{(i+1)} = L^{(i)} \cdot L^{(i)}$. Then

$$L^{(0)} \geqslant L^{(1)} \geqslant \cdots$$

and we say that a symplectic alternating algebra L is *soluble* if there exists an integer $n \geqslant 0$ such that $L^{(n)} = \{0\}$. The smallest n enjoying this property is then referred to as the *derived length* of L. Thus L has derived length 0 if and only if it has order one. Also, the symplectic alternating algebras with derived length at most 1 are just the abelian ones. A symplectic alternating algebra which is soluble of derived length at most 2 is said to be *metabelian*.

Lemma 2.2. *If L is a symplectic alternating algebra then $L^{(i)} \subseteq L^{i+1}$. In particular, if L is nilpotent of class i then L is soluble of derived length at most i.*

Proof. We argue by induction on i. The claim is obviously true when $i = 0$ being $L^{(0)} = L = L^1$. Assuming $i > 0$ and $L^{(i)} \subseteq L^{i+1}$, we get $L^{(i+1)} = L^{(i)} \cdot L^{(i)} \subseteq L^{i+1} \cdot L = L^{i+2}$, as required. $\square$

Next result is rather odd and shows that all metabelian symplectic alternating algebras are nilpotent. It also shows that the inclusion in last lemma is not optimal.

Proposition 2.3. *Let L be a symplectic alternating algebra. Then L is metabelian if and only if it is nilpotent of class at most* 3.

Proof. We have that L is metabelian if and only if $xy(zw) = 0$ for all $x, y, z, w \in L$, that is $(xy(zw), t) = 0$ for all $t \in L$. This means $0 = (xy, zwt) = (x, zwty)$ and L is nilpotent of class at most 3. □

Not all soluble symplectic alternating algebras are however nilpotent as the following example shows.

Example 2.4. Consider

$$L\colon \begin{array}{l} x_1x_2 = 0, \\ y_1y_2 = -y_1, \end{array}$$

the only nonabelian symplectic alternating algebra of dimension 4 over a field F. We have

$$Z(L) = Fx_2 \quad \text{and} \quad L^2 = Z(L)^{\perp} = Fx_1 + Fx_2 + Fy_1.$$

Here $L^{(3)} = L^{(2)} \cdot L^{(2)} = Fx_2 \cdot Fx_2 = \{0\}$ and L is soluble of derived length 3 but it is not nilpotent. In fact $y_1y_2^n = (-1)^n y_1$ for any integer $n \geqslant 1$.

However, we have the following strong generalization of Proposition 2.3.

Proposition 2.5. *Let L be a symplectic alternating algebra. If L is abelian-by-(nilpotent of class $\leqslant c$) then it is nilpotent of class at most $2c+1$.*

Proof. Let I be an abelian ideal of L such that L/I is nilpotent of class at most c. Then $L^{c+1} \subseteq I$ and

$$\bigl(x_1 \cdots x_{c+1} \cdot (y_1 \cdots y_{c+1}), z\bigr) = 0$$

for all $x_1, \ldots, x_{c+1}, y_1, \ldots, y_{c+1}, z \in L$. Thus

$$(x_1, y_1 \cdots y_{c+1} z x_{c+1} \cdots x_2) = 0$$

and L is nilpotent of class at most $2c+1$. □

This result fails if we assume that our algebra is nilpotent-by-abelian. The example above still provides a counterexample, for L^2 is nilpotent and L/L^2 is abelian.

3. Nil-elements

Let L be a symplectic alternating algebra and x be a left nil-element of L. We say that an element $a \in L$ has *nil-x degree m* if m is the smallest positive integer such that $ax^m = 0$. Pick $a \in L$ of maximal nil-x degree k and let

$$V(a) = \bigl\langle a, ax, ax^2, \ldots, ax^{k-1}\bigr\rangle.$$

We know that this is an isotropic subspace in L (see [3, Lemma 2.10]). Then there exists $b \in L$ such that

$$(a,b) = (ax,b) = \cdots = \left(ax^{k-2}, b\right) = 0 \quad \text{and} \quad \left(ax^{k-1}, b\right) = 1.$$

Since $(a, bx^{k-1}) = (ax^{k-1}, b) = 1$, we have that the nil-$x$ degree of b is k. Notice also that

$$\left(ax^r, bx^s\right) = \left(ax^{r+s}, b\right)$$

which is 1 if $r + s = k - 1$ but 0 otherwise. So that the subspace

$$V(a) + V(b) = V(a) \oplus V(b) = \left\langle a, bx^{k-1}\right\rangle \oplus \left\langle ax, bx^{k-2}\right\rangle \oplus \cdots \oplus \left\langle ax^{k-1}, b\right\rangle$$

is a perpendicular direct sum of hyperbolic subspaces.

Let $W = W(a,b) = V(a) + V(b)$. The multiplication by x from the right gives us a linear map on L. Then W is invariant under the right multiplication by x and the same is then true for the orthogonal complement $W^{\perp}$: in fact, for all $y \in W^{\perp}$ and $z \in W$ we have $(yx, z) = -(y, zx) = 0$ as $zx \in W$. Now, we can take $c \in W^{\perp}$ of maximal nil-x degree, say m. Then, as before, we get $d \in L$ of nil-x degree m and $W(c,d) = V(c) + V(d)$ is a perpendicular direct sum. Thus we inductively see that L splits up into a perpendicular direct sum

$$L = W(a_1, b_1) \oplus \cdots \oplus W(a_n, b_n). \tag{1}$$

We will refer to such a decomposition as a *primary decomposition* of L with respect to multiplication by x from the right. We will also use the notation

$$\begin{pmatrix} a & bx^{k-1} \\ ax & bx^{k-2} \\ \vdots & \vdots \\ ax^{k-1} & b \end{pmatrix}$$

for the subspace $W(a,b)$.

Proposition 3.1. *Let L be a symplectic alternating algebra. If $x \in L$ is a left nil-element, then $C_L(x)$ is even dimensional.*

Proof. Consider a decomposition as above with respect to right multiplication by x. We have seen that the cyclic subspaces come in pairs, say that

$$L = V(a_1) \oplus V(b_1) \oplus \cdots \oplus V(a_n) \oplus V(b_n).$$

The kernel of each of these is one-dimensional, hence $C_L(x)$ has dimension $2n$. $\square$

For the remainder of this section we focus on right nil-2 elements. In general, a left nil-2 element needs not to be a right nil-2 element. In Example 2.4, y_1 is a left nil-2 element that is not a right nil-element. However, the converse is always true.

Lemma 3.2. *Let L be a symplectic alternating algebra. If a is a right nil-2 element of L, then*:

(i) $ayz = -azy$ *for all* $y, z \in L$;
(ii) *a is left nil*-2;

(iii) *$C_L(a)$ is an ideal*;
(iv) *La and $Fa + La$ are abelian ideals and the latter is the smallest ideal containing a.*

Proof. (i) We have

$$0 = a(y+z)(y+z) = (ay+az)(y+z) = ayz + azy$$

and $ayz = -azy$.

(ii) For all $x \in L$, we have $0 = -a(a+x)^2 = xa(a+x) = xa^2$.

(iii) Let $x, y \in L$ and $b \in C_L(a)$. Then $0 = a(x+b)^2 = ax(x+b) = axb$ which implies $0 = (axb, y) = (a(by), x)$. Thus $a(by) = 0$ and $by \in C_L(a)$.

(iv) That La is an ideal follows immediately from $uax = -uxa$ and of course it follows then that $Fa + La$ is an ideal, the smallest ideal containing a. As a is left nil-2 and since $ax(ya) = -a(ya)x = 0$, it is clear that both the ideals are abelian. □

Theorem 3.3. *Let X be a set of right nil-2 elements in a symplectic alternating algebra L and denote by $I(X)$ the smallest ideal of L containing X. Then*

$$I(X) = \sum_{a \in X} Fa + La.$$

Furthermore, if $|X| = c$ then $I(X)$ is nilpotent of class at most c.

Proof. Let $a \in X$. By Lemma 3.2(iv) we know that $I(a) = Fa + La$ is the smallest ideal containing a and that $I(a)$ is abelian. It follows that $I(X) = \sum_{a \in X} I(a)$. Since each of these ideals is abelian it is clear that $I(X)^{c+1} = \{0\}$, here $c = |X|$. □

It follows in particular that the ideal generated by all the right nil-2 elements is always a nilpotent ideal.

4. Nil-2 algebras

The results concerning right nil-2 elements lead to the following characterization of symplectic alternating nil-2 algebras.

Theorem 4.1. *Let L be a symplectic alternating algebra. Then the following are equivalent*:

(i) *L is nil-2*;
(ii) *$C_L(x)$ is an ideal for any $x \in L$*;
(iii) *$I(x)$ is abelian for any $x \in L$*;
(iv) *the identity $xyz = -xzy$ holds in L*;
(v) *the identity $x(yz) = xzy$ holds in L.*

Proof. First we show that (i) ⇔ (ii) ⇔ (iii). From Lemma 3.2, we know that (i) implies (ii) and (iii). To see that (iii) implies (i), take any $a, x \in L$. As $I(x)$ is abelian and $ax, x \in I(x)$, it follows that $ax^2 = 0$. Finally to show that (ii) implies (i), notice that $x \in C_L(x)$ and as $C_L(x)$ is an ideal we also have $ax \in C_L(x)$. The latter gives $ax^2 = 0$.

We finish the proof by showing that (i) ⇒ (iv) ⇒ (v) ⇒ (i). The fact that (i) implies (iv) follows from Lemma 3.2. If (iv) holds, then $x(yz) = -yzx = yxz = -xyz = xzy$ that gives us (v). Finally (i) follows from (v) by taking $y = z$. □

It follows from Theorem 3.3 that all symplectic alternating nil-2 algebras are nilpotent. We next analyze this in more details.

Theorem 4.2. *Let L be a symplectic alternating algebra over a field F of characteristic $\neq 2$. If L is nil-2, then L is nilpotent of class at most 3.*

Proof. Let $x, y, z, t \in L$. By Theorem 4.1, $xy(tz) = xyzt$ and $xy(tz) = -x(tz)y = -xzty = xzyt = -xyzt$. It follows that $2xyzt = 0$ and, since $char\, F \neq 2$, we conclude that $xyzt = 0$. $\square$

Moreover, the bound provided is optimal as there exists a nil-2 algebra which is nilpotent of class 3.

Example 4.3. Let F be any field and L be the linear span of

$$\begin{array}{ll} x_1 = a, & y_1 = tcb, \\ x_2 = b, & y_2 = tac, \\ x_3 = c, & y_3 = tba, \\ x_4 = ab, & y_4 = tc, \\ x_5 = ca, & y_5 = tb, \\ x_6 = bc, & y_6 = ta, \\ x_7 = abc, & y_7 = t. \end{array}$$

As a symplectic vector space let $L = (Fx_1 + Fy_1) \oplus \cdots \oplus (Fx_7 + Fy_7)$ be a perpendicular direct sum of hyperbolic subspaces (where $(x_i, y_i) = 1$ for $i = 1, \ldots, 7$). We turn this into a symplectic alternating nil-2 algebra by adding an alternating product satisfying condition (iv) of Theorem 4.1. As the identity (iv) is multilinear it suffices that $xyz = -xzy$ whenever x, y, z are generators. The condition implies that the only nontrivial triples $(uv, w) = (vw, u) = (wu, v)$ are

$$(x_1x_2, y_4) = 1,$$

$$(x_3x_1, y_5) = 1,$$

$$(x_2x_3, y_6) = 1,$$

$$(x_4x_3, y_7) = 1,$$

$$(x_5x_2, y_7) = 1,$$

$$(x_6x_1, y_7) = 1.$$

Conversely one can easily check that this alternating product turns L into a symplectic alternating nil-2 algebra that is nilpotent of class 3.

Theorem 4.4. *Let F be a field of characteristic 2 and let L be a symplectic alternating algebra of dimension $n = 2m$. If L is nil-2, then L is nilpotent of class at most $\lfloor \log_2(m+1) \rfloor$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of L. If $char\, F = 2$, then L is commutative and, by Theorem 4.1, it is also associative. It follows that

$$u_1 \cdots u_n = 0 \quad \text{for all } u_1, \ldots, u_n \in L \quad \text{if and only if} \quad x_1 \cdots x_n = 0.$$

But $(x_1 \cdots x_n, x_i) = 0$ for any $i \in \{1, \ldots, n\}$. Hence $x_1 \cdots x_n = 0$ and L is nilpotent of class at most $n - 1$. So, if we denote by c the nilpotency class of L, then $c < n$. Since the class is c there is a non-zero product $x_{i_1} \cdots x_{i_c}$ and without loss of generality we can suppose that $x_1 \cdots x_c \neq 0$. Now, let

$$x_I = x_{i_1} \cdots x_{i_r}$$

for any $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, c\}$ and let

$$X = \{x_I \colon \emptyset \subset I \subseteq \{1, \ldots, c\}\}.$$

We prove that X is a linearly independent subset of L. Assume

$$\alpha_1 x_{I_1} + \cdots + \alpha_m x_{I_m} = 0$$

where $m \leqslant 2^c - 1$ and $|I_1| \leqslant \cdots \leqslant |I_m|$. Let α_j be the least non-zero coefficient and $J = \{1, \ldots, c\} \backslash I_j$. Then, multiplying by $\prod_{k \in J} x_k$, we get

$$\alpha_j x_1 \cdots x_c = 0$$

and thus $x_1 \cdots x_c = 0$ which is a contradiction. Thus X is linearly independent and $|X| = 2^c - 1$. Hence $2^c - 1 \leqslant 2m$ and $2^c < 2m + 2$. Then $c < \log_2(2(m+1)) = 1 + \log_2(m+1)$ and so $c \leqslant \log_2(m+1)$, as we claimed. $\square$

Indeed, the bound we have just got is the best possible, as shown in the following construction:

Example 4.5. Let F be the field with 2 elements and let $r > 3$. There exists a symplectic alternating nil-2 algebra L over F of dimension $2(2^{r-1} - 1)$ which is nilpotent of class $r - 1$. In fact, define L to be the linear span of all monomials in $x_1, \ldots, x_r$ with no repeated entries and of weight less than r. Then L has dimension $2^r - 2$ over F. Let

$$(x_{i_1} \ldots x_{i_n}, x_{j_1} \ldots x_{j_m}) = 0$$

except if $n + m = r$ and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\} = \{1, \ldots, r\}$, and 1 otherwise. This gives a symplectic vector space. Let

$$x_{i_1} \ldots x_{i_n} \cdot x_{j_1} \ldots x_{j_m} = x_{i_1} \ldots x_{i_n} x_{j_1} \ldots x_{j_m}$$

if $i_1, \ldots, i_n, j_1, \ldots, j_m$ are distinct and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\} \subset \{1, \ldots, r\}$, and 0 otherwise. Then L is a symplectic alternating algebra that is nilpotent of class $r - 1$. Since L is commutative and associative, it is also nil-2.

5. Nil-3 algebras

In this section we describe some general properties of a symplectic alternating nil-3 algebra L.

Lemma 5.1. *For any $x, y_i, z \in L$ the following identities hold*:

(i) $\sum_{\sigma \in S_3} x y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} = 0$;
(ii) $\sum_{\sigma \in S_2} x y_{\sigma(1)} y_{\sigma(2)} z + x y_{\sigma(1)} (z y_{\sigma(2)}) + x(z y_{\sigma(1)} y_{\sigma(2)}) = 0$.

Proof. The proof of (i) is straightforward. To see why (ii) holds notice that, for any $u \in L$, from (i) we have

$$\begin{aligned}0 &= \left(\sum_{\sigma\in S_2} xy_{\sigma(1)}y_{\sigma(2)}u + xy_{\sigma(1)}uy_{\sigma(2)} + xuy_{\sigma(1)}y_{\sigma(2)}, z\right)\\ &= \sum_{\sigma\in S_2} (xy_{\sigma(1)}y_{\sigma(2)}, zu) + (xy_{\sigma(1)}, zy_{\sigma(2)}u) + (x, zy_{\sigma(2)}y_{\sigma(1)}u)\\ &= -\left(\sum_{\sigma\in S_2} xy_{\sigma(1)}y_{\sigma(2)}z + xy_{\sigma(1)}(zy_{\sigma(2)}) + x(zy_{\sigma(2)}y_{\sigma(1)}), u\right). \qquad \square\end{aligned}$$

In the following we will use the notation

$$x\{y_1, y_2, y_3\}$$

for the first sum in Lemma 5.1 and similarly

$$x\{y_1, y_2\} = xy_1y_2 + xy_2y_1.$$

Lemma 5.2. *For any $x, y, z \in L$ the following hold*:

(i) $yx^2y = -yxyx \in Lx$;
(ii) *if* $zx^2y = 0$ *then* $yx^2z \in Lx$;
(iii) $yx^2(zx^2) \in Lx \cap C_L(x)$;
(iv) *if* $yx^2(zx^2) = 0$ *then* $yx^2(zx) \in Lx \cap C_L(x)$.

Proof. (i) First we have

$$0 = y(x+y)^3 = yx(x+y)^2 = \big(yx^2 + yxy\big)(x+y) = yx^2y + yxyx.$$

(ii) Assume $zx^2y = 0$. Then we get

$$\begin{aligned}0 &= x\{x, y, z\}\\ &= xy\{x, z\} + xz\{x, y\}\\ &= xyxz + xyzx + xzyx\end{aligned}$$

that gives $yx^2z \in Lx$.

(iii) We see that

$$0 = -x\big\{x, yx, zx^2\big\} = yx^2\big\{x, zx^2\big\} = yx^2\big(zx^2\big)x.$$

Then also

$$\begin{aligned}0 &= x\big\{x, y, zx^2\big\}\\ &= xy\big\{x, zx^2\big\}\\ &= xyx\big(zx^2\big) + xy\big(zx^2\big)x\end{aligned}$$

that implies $yx^2(zx^2) \in Lx \cap C_L(x)$.

(iv) Let $yx^2(zx^2)=0$. Since

$$0=x\{x, yx^2, z\}=xz(yx^2)x,$$

it follows

$$yx^2(zx)x=0.$$

Notice also

$$\begin{aligned}0&=x\{x, y, zx\}\\&=xy\{x, zx\}+x(zx)\{x, y\}\\&=xyx(zx)+xy(zx)x+x(zx)yx.\end{aligned}$$

Thus $yx^2(zx)\in Lx\cap C_L(x)$. $\square$

6. Classification of nil-algebras of dimension $\leqslant 8$

Before embarking on the classification of the symplectic alternating nil-algebras of dimension $\leqslant 8$, we prove the following result.

Proposition 6.1. *If L is a symplectic alternating nil-k algebra, then* $\dim(L)\geqslant 2(k+1)$.

Proof. Suppose by contradiction $\dim(L)=2k$ and take $x\in L$ which is not left nil-$(k-1)$. By (1), there is only one possible primary decomposition for the multiplication by x from the right. This is

$$\begin{pmatrix} a & bx^{k-1} \\ ax & bx^{k-2} \\ \vdots & \vdots \\ ax^{k-1} & b \end{pmatrix}.$$

It is easy to see that $x=cx^{k-1}$ for some $c\in L$. Then $0=x(-cx^{k-2})^k=x$, which is impossible. $\square$

As a consequence, all the nonabelian nil-algebras of dimension $\leqslant 8$ are the nil-2 algebras of dimension either 6 or 8 and the nil-3 of dimension 8.

6.1. Nil-2 algebras of dimension 6

Let L be a symplectic alternating nil-2 algebra of dimension 6 over a field F. Assume that L is not abelian and let $x\in L\setminus Z(L)$. Because of (1), we have that the only primary decomposition of L with respect to multiplication by x from the right is

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix}\oplus\begin{pmatrix} c & d \end{pmatrix}$$

where $cx=dx=0$.

By Theorem 4.1, $axc=-xac=xca=0$ and similarly ax commutes with d, a, ax, bx. As $C_L(ax)$ is even dimensional, it follows that ax commutes also with b and thus $ax\in Z(L)$. Similarly $bx\in Z(L)$ and $Lx\subseteq Z(L)$. Of course this is also true if $x\in Z(L)$. We have thus shown that $Ly\subseteq Z(L)$ for all $y\in L$ and thus L is nilpotent of class 2.

Now we have

$$x = \alpha ax + \beta bx + u$$

for some $\alpha, \beta \in F$ and $u \in Fc + Fd$. As $x \notin Lx$ we must have that u is nontrivial. Also $au = ax$ and $bu = bx$. We can thus, without loss of generality, replace x by u and suppose that x is orthogonal to a, ax, b, bx. Next we turn to ab. Notice that ab is orthogonal to a, b, ax, bx and $(x, ab) = (-bx, a) = (a, bx) = 1$. Hence we have the primary decomposition

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} x & ab \end{pmatrix}$$

with respect to multiplication by x from the right. The structure is now completely determined. So there is just one nonabelian nil-2 algebra of dimension 6.

6.2. Nil-2 algebras of dimension 8

Let L be a symplectic alternating nil-2 algebra of dimension 8 over a field F. Assume that L is not abelian and let $x \in L \setminus Z(L)$. We cannot have $x \in Lx$ as this would imply that $x = xz$ for some $z \in L$ and then $x = xz^2 = 0$. By (1), this implies that there is only one possible primary decomposition of L with respect to multiplication by x from the right. This is

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} c & d \end{pmatrix} \oplus \begin{pmatrix} e & f \end{pmatrix}$$

where $cx = dx = ex = fx = 0$.

By Theorem 4.1, $axc = -xac = xca = 0$ and similarly we see that ax commutes with d, e, f, bx as well as, of course, with a and ax. Since $C_L(ax)$ is even dimensional, it follows that ax commutes also with b and $ax \in Z(L)$. The same argument shows that $bx \in Z(L)$. So $Lx \subseteq Z(L)$ and obviously this is also true if $x \in Z(L)$. We have thus shown that $Ly \subseteq Z(L)$ for all $y \in L$ and L is nilpotent of class 2. Now we have that

$$x = \alpha ax + \beta bx + u$$

for some $\alpha, \beta \in F$ and for $u \in Fc + Fd + Fe + Ff$. As x cannot be in Lx we must have that u is nontrivial. Now $au = ax$ and $bu = bx$ so we can, without loss of generality, replace x by u and so we can suppose that x is orthogonal to a, b, ax, bx. Next consider the element ab. We have that ab is orthogonal to a, b and as $ab \in Z(L)$, we also have that ab is orthogonal to ax and bx. Furthermore $(x, ab) = (-bx, a) = (a, bx) = 1$. So we have a primary decomposition

$$\begin{pmatrix} a & bx \\ ax & b \end{pmatrix} \oplus \begin{pmatrix} x & ab \end{pmatrix} \oplus \begin{pmatrix} c & d \end{pmatrix} \tag{2}$$

with $cx = dx = 0$. But now $Fa + Fax + Fbx + Fb + Fx + Fab$ is invariant under multiplication by a and b. It follows that its orthogonal complement, $Fc + Fd$, is also invariant under multiplication by a and b. The only possibility then is that $ca = da = cb = db = 0$. Notice, finally, that cd is orthogonal to a, ax, b, bx, x, ab as well as to c, d and thus $cd = 0$. The structure of L is thus determined. All triples (uv, w) involving ax, bx, ab, c, d are trivial and $(ax, b) = (xb, a) = (ba, x) = 1$. So there is only one nonabelian nil-2 algebra of dimension 8.

6.3. Nil-3 algebras of dimension 8

Let L be a symplectic alternating nil-3 algebra of dimension 8 over a field F. Suppose that $x \in L$ is not left nil-2. By (1), there is only one possible primary decomposition for the multiplication by x from the right. This is

$$L = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} u & t \end{pmatrix}$$

where $ux = tx = 0$.

Lemma 6.2. *The following properties hold*:

(i) *Lx^2 is abelian*;
(ii) *$Lx^2(Lx) \subseteq Lx^2$*;
(iii) *$ax^2(ax) = -ax^2ax$ and $bx^2(bx) = -bx^2bx$*;
(iv) *if $bx^2(ax) = 0$ then $ax^2(ax) = rbx^2$ for some $r \in F$*;
(v) *if $ax^2(bx) = 0$ then $bx^2(bx) = sax^2$ for some $s \in F$.*

Proof. (i) As $Lx \cap C_L(x) = Lx^2$, it follows from Lemma 5.2(iii) that $ax^2(bx^2) \in Lx^2 = Fax^2 \oplus Fbx^2$. Suppose

$$ax^2\left(bx^2\right) = \alpha ax^2 + \beta bx^2$$

for some $\alpha, \beta \in F$. Then

$$0 = ax^2\left(bx^2\right)^3 = \alpha^3 ax^2 + \alpha^2 \beta bx^2$$

implies $\alpha = 0$ and

$$0 = bx^2\left(ax^2\right)^3 = -\beta^3 bx^2$$

gives $\beta = 0$. Thus $ax^2(bx^2) = 0$ and Lx^2 is abelian.

(ii) This follows by (i) and Lemma 5.2(iv), since $Lx \cap C_L(x) = Lx^2$.

(iii) We have

$$0 = -x\{a, x, ax\} = ax\{x, ax\} + ax^2\{a, x\} = ax^2(ax) + ax^2ax$$

and similarly $0 = bx^2(bx) + bx^2bx$.

(iv) By (ii), we know that

$$ax^2(ax) = sax^2 + rbx^2$$

for some $r, s \in F$. Then

$$0 = -x(ax)^3 = ax^2(ax)^2 = s^2ax^2 + srbx^2$$

implies $s = 0$ and hence $ax^2(ax) = rbx^2$.

We get (v) in the same manner. □

Notice that the following result holds with the roles of a and b interchanged.

Lemma 6.3. *If $ax^2(ax) = rbx^2$ for some $r \in F$, then $ax^2(bx) = 0$. Furthermore, $ax^2 \in Z(L)$ when $r = 0$.*

Proof. By (i) of Lemma 5.2, $ax^2a \in Lx$. As $(ax^2a, a) = 0$ and

$$\bigl(ax^2a, ax\bigr) = -\bigl(ax^2(ax), a\bigr) = r,$$

we have

$$ax^2a = \alpha ax + \beta ax^2 - rbx$$

for some $\alpha, \beta \in F$. Then

$$ax^2ax = \alpha ax^2 - rbx^2.$$

But $ax^2ax = -ax^2(ax) = -rbx^2$ by Lemma 6.2(iii), thus $\alpha ax^2 = 0$. It follows that $\alpha = 0$ and

$$ax^2a = \beta ax^2 - rbx,$$

so that ax^2a is orthogonal to bx and thus $ax^2(bx)$ is orthogonal to a. However, $ax^2(bx) \in Lx^2$ by (ii) of Lemma 6.2, hence

$$ax^2(bx) = \gamma ax^2$$

for some $\gamma \in F$. Moreover $0 = ax^2(bx)^3 = \gamma^3 ax^2$, hence $\gamma = 0$ and $ax^2(bx) = 0$.

Now assume $r = 0$. Then

$$ax^2a = \beta ax^2$$

and we have

$$0 = ax^2a^3 = \beta^3 ax^2$$

which gives $\beta = 0$ and

$$ax^2a = 0.$$

We now turn to ax^2u and ax^2t. They both lie in Lx by (ii) of Lemma 5.2 and are orthogonal to a, ax, bx. If $\beta = (ax^2u, b)$ and $\gamma = (ax^2t, b)$, we have

$$ax^2u = \beta ax^2 \quad \text{and} \quad ax^2t = \gamma ax^2.$$

Then, as before, we get $\beta = \gamma = 0$. We have thus seen that ax^2 commutes with $a, ax, ax^2, bx, bx^2, u, t$ and, as the dimension of $C_L(ax^2)$ is even, it follows that $ax^2b = 0$ and $ax^2 \in Z(L)$. □

Corollary 6.4. *Let $y, z \in L$. If $yz^2(yz) = 0$ then $yz^2 \in Z(L)$.*

Proof. If $yz^2 = 0$, this is obvious. Otherwise this follows from Lemma 6.3 with y in the role of a and z in the role of x. □

Remark 6.5. In particular if $yz^2(yz) = 0$ for all $y, z \in L$, then $Lz^2 \subseteq Z(L)$.

Furthermore, we have:

Lemma 6.6. $Z(L) \cap Lx^2 \neq \{0\}$.

Proof. If $ax^2(ax) = 0$, then $ax^2 \in Z(L)$ by the previous lemma. So we may assume $ax^2(ax) \neq 0$. By Lemma 6.2(ii), the multiplication by ax from the right gives us a linear operator on Lx^2 that is a nil-operator and so with a nontrivial kernel. This means that we have

$$(b + \alpha a)x^2(ax) = 0$$

for some $\alpha \in F$. Without loss of generality we can replace b by $b + \alpha a$ and thus assume that

$$bx^2(ax) = 0.$$

By Lemma 6.2(iv) we have $ax^2(ax) = rbx^2$ for some $r \in F \setminus \{0\}$ and hence $ax^2(bx) = 0$ by Lemma 6.3. Then (v) of Lemma 6.2 gives that there exists $s \in F$ such that $bx^2(bx) = sax^2$. This implies

$$0 = bx^2(ax + bx)^3 = rs^2ax^2$$

and we get $s = 0$. It follows $bx^2(bx) = 0$ and $bx^2 \in Z(L)$ again applying Lemma 6.3. □

We now turn to the structure of L. This is determined by the value of all triples $(vz, w) = (zw, v) = (wv, z)$ where v, z, w are pairwise distinct basis vectors. As any such triple has either two vectors from $\{a, ax, ax^2, b, bx, bx^2\}$ or two vectors from $\{u, t\}$, we only need to determine ut and the products of any two elements from $\{a, ax, ax^2, b, bx, bx^2\}$.

According with Lemma 6.6, we will assume

$$bx^2 \in Z(L). \tag{3}$$

Then we also have

$$ax^2(ax) = rbx^2 \quad \text{and} \quad ax^2(bx) = 0 \tag{4}$$

by Lemma 6.2(iv) and Lemma 6.3, respectively.

Step 1. *We can assume that $ax^2b = 0$ and $ax^2a = -rbx$.*

Proof. By Lemma 5.2, (ii) and (i), ax^2b and ax^2a are in Lx. Also ax^2b is orthogonal to ax, b, bx and

$$ax^2b = \alpha bx^2$$

for $\alpha = -(ax^2b, a)$. If $r = 0$, then Lemma 6.3 implies $ax^2 \in Z(L)$ and so $ax^2b = 0$. Let $r \neq 0$, then $ax^2(b - \frac{\alpha}{r}ax) = 0$. Replacing b by $b - \frac{\alpha}{r}ax$, we can assume that $ax^2b = 0$. One can check that (3) and (4) still hold.

Next, we have that ax^2a is orthogonal to a, b, bx and

$$\big(ax^2a, ax\big) = -\big(ax^2(ax), a\big) = -r\big(bx^2, a\big) = r.$$

Thus $ax^2a = -rbx$. □

Suppose now that $x = y + z$ with $y \in \langle a, ax, ax^2, b, bx, bx^2\rangle$ and $z \in \langle u, t\rangle$. Then $0 = yx$ and thus $y \in Lx^2$. Notice that $z \neq 0$ since otherwise $x = y = cx^2$ for some $c \in L$ and $0 = x(-cx)^3 = x$. Without loss of generality, we can suppose that $z = u$. Hence

$$x = u + \alpha ax^2 + \beta bx^2$$

for some $\alpha, \beta \in F$.

Let us calculate the effect of multiplying with

$$u = x - \alpha ax^2 - \beta bx^2.$$

Firstly, we have

$$ut = xt - \alpha ax^2 t.$$

However, $ax^2t \in Lx$ by Lemma 5.2(ii) and is orthogonal to a, ax, b, bx. Thus $ax^2t = 0$ and

$$ut = xt.$$

Recall that $bx^2 \in Z(L)$ and that $ax^2b = ax^2(bx) = 0$, whereas $ax^2a = -rbx$ and $ax^2(ax) = rbx^2$. Using this, we see that

$$au = ax + \alpha ax^2 a = ax - \alpha rbx$$

and

$$\begin{aligned} au^2 &= (ax - \alpha rbx)\big(x - \alpha ax^2 - \beta bx^2\big) \\ &= ax^2 + \alpha ax^2(ax) - \alpha rbx^2 \\ &= ax^2 + \alpha rbx^2 - \alpha rbx^2 \\ &= ax^2. \end{aligned}$$

One also sees that $bu = bx$ and $bu^2 = bx^2$. Replacing x by u and a, ax, ax^2, b, bx, bx^2 by a, au, au^2, b, bu, bu^2, we still have a decomposition into hyperbolic subspaces. One can now check that (3), (4) and Step 1 are still valid with x replaced by u. So without loss of generality we can assume that $u = x$. We thus have a primary decomposition

$$L = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus (\, x \quad t\,)$$

where

$$xt = 0. \tag{5}$$

Step 2. $ax(bx) = 0$.

Proof. From $ax^2b = 0$, we get

$$0 = -x\{a, b, x\} = ax\{b, x\} + bx\{a, x\} = axbx + bxax. \tag{6}$$

Since the values

$$(axb, b), (axb, ax), \big(axb, ax^2\big), \big(axb, bx^2\big)$$

and

$$(bxa, a), (bxa, bx), \big(bxa, ax^2\big), \big(bxa, bx^2\big)$$

are all trivial, we have

$$axb = \alpha ax + y, \quad y \in Fbx^2 + Fx + Ft \tag{7}$$

and

$$bxa = \beta bx + z, \quad z \in Fax^2 + Fx + Ft, \tag{8}$$

respectively. By (6), (7) and (8), it follows that

$$\alpha ax^2 = axbx = -bxax = -\beta bx^2$$

which implies $\alpha = \beta = 0$. Hence $(axb, bx) = (bxa, ax) = 0$ and thus

$$\big(ax(bx), a\big) = \big(ax(bx), b\big) = 0.$$

Clearly, $ax(bx)$ is also orthogonal to ax, bx, ax^2, bx^2, x and thus

$$ax(bx) = \alpha x$$

for some $\alpha \in F$. But we have

$$\begin{aligned} 0 &= -x\{a, ax, bx\} \\ &= ax\{ax, bx\} + ax^2\{a, bx\} + bx^2\{a, ax\} \\ &= ax(bx)(ax) + ax^2 a(bx) \\ &= ax(bx)(ax) - r(bx)^2 \\ &= ax(bx)(ax). \end{aligned}$$

Then

$$0 = ax(bx)(ax) = \alpha x(ax) = -\alpha ax^2$$

and $\alpha = 0$. □

Step 3. *We can assume that $bxb = 0$ and $axa = rb$.*

Proof. Let us first consider bxb. It is orthogonal to ax, ax^2, b, bx, bx^2, x. We then have

$$bxb = \alpha bx^2 + \beta x$$

where $\alpha = -(bxb, a)$ and $\beta = (bxb, t)$. Since

$$0 = xb^3 = -\beta xb,$$

we get $\beta = 0$. It follows that

$$0 = bx(b - \alpha x).$$

Replacing b by $b-\alpha x$ and t by $t-\alpha ax^2$ respectively, (3), (4), (5) and the previous steps still hold. Thus we can assume $bxb=0$.

We turn to axa. It is clear that axa is orthogonal to a, ax, bx, bx^2, x and that

$$\big(axa, ax^2\big)=\big(ax^2, a(ax)\big)=\big(ax^2(ax), a\big)=r\big(bx^2, a\big)=-r.$$

Suppose $(axa, b)=\alpha$ and $(axa, t)=\beta$. Then

$$axa=\alpha ax^2+rb+\beta x. \tag{9}$$

We next show that $axa(bx)\in Lx$ and in order to do this we prove that $a(bx)x=0$. That this is sufficient follows from

$$0=a\{a, x, bx\}=ax\{a, bx\}+a(bx)\{a, x\}=axa(bx)+a(bx)ax+a(bx)xa.$$

As $ax(bx)=0$, by (8) we know that $a(bx)\in Fax^2+Fx+Ft$. But

$$\big(a(bx), b\big)=0 \quad \text{and} \quad \big(a(bx), x\big)=-1,$$

and thus

$$a(bx)=\gamma x+t \quad \text{and} \quad a(bx)x=0. \tag{10}$$

Let $axa(bx)=\alpha_1 ax+\alpha_2 ax^2+\beta_1 bx+\beta_2 bx^2$. Since

$$\big(axa(bx), a\big)=\big(axa(bx), b\big)=\big(axa(bx), ax\big)=\big(axa(bx), bx\big)=0,$$

$axa(bx)$ is trivial and, by (9), we get

$$0=axa(bx)=-\beta bx^2.$$

Thus $\beta=0$ and $ax(a-\alpha x)=rb$. If we replace a by $a-\alpha x$ and t by $t+\alpha bx^2$, then (3), (4), (5) and all the previous steps hold. So we can assume that $axa=rb$. $\square$

Step 4. *$axb=t$ and $bxa=-t$.*

Proof. We first consider axt which is clearly orthogonal to x and t. As the product of ax with a, ax, ax^2, bx, bx^2 is orthogonal to t, axt is also orthogonal to a, ax, ax^2, bx, bx^2. Hence, for some $\alpha\in F$,

$$axt=\alpha ax^2 \quad \text{and} \quad ax(t-\alpha x)=0.$$

Replacing t by $t-\alpha x$ we can assume that

$$axt=0.$$

It follows that $(axb, t)=0$, thus axb is orthogonal to t. As the products of ax with a, ax, bx, ax^2, bx^2 are orthogonal to b, we have that axb is orthogonal to $t, a, ax, bx, ax^2, bx^2, b$. Also $(axb, x)=-1$ and so

$$axb=t.$$

We now turn to bxa. By (10), we know that

$$bxa = -t - \gamma x.$$

Since

$$\begin{aligned} 0 &= -x(a+b)^3 \\ &= (ax+bx)(a+b)^2 \\ &= (axa+axb+bxa)(a+b) \\ &= (rb+t-t-\gamma x)(a+b) \\ &= -rab+\gamma ax+\gamma bx, \end{aligned}$$

we get

$$0 = (-rab+\gamma ax+\gamma bx, bx) = \gamma.$$

Thus $bxa = -t$. $\square$

Step 5. *We can assume that $ab = 0$.*

Proof. Clearly, ab is orthogonal to a, b and, since ax^2, bx, bx^2 commute with b, we have that ab is also orthogonal to ax^2, bx, bx^2. As bx is orthogonal to a we also have ab orthogonal to x. Then

$$(ab, ax) = -(b, axa) = -(b, rb) = 0$$

and the only generator left is t. Hence

$$ab = \alpha x$$

for some $\alpha \in F$.

We consider two cases. Suppose first that $yz^2(yz) = 0$ for all $y, z \in L$. Then $r = 0$ and by Remark 6.5

$$\alpha xb = ab^2 \in Z(L)$$

which is absurd except if $\alpha = 0$. Hence $ab = 0$ in this case.

If the identity $yz^2(yz) = 0$ does not hold for all $y, z \in L$, without loss of generality we can assume $ax^2(ax) = rbx^2$ with $r \neq 0$. Thus

$$0 = ba^3 = \alpha axa = \alpha rb$$

implies $\alpha = 0$ and hence $ab = 0$ also in this case. $\square$

As candidates for our examples we thus have a one parameter family of symplectic alternating algebras

$$L(r) = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus (\,x \quad t\,).$$

Notice that $t \in Z(L(r))$ since vt is orthogonal to x, t and $(vt, w) = -(vw, t) = 0$ for all $v, w \in \{a, ax, ax^2, b, bx, bx^2\}$: the only nontrivial products not involving x are

$$axa = rb,$$
$$ax^2a = -rbx,$$
$$ax^2(ax) = rbx^2,$$
$$axb = t,$$
$$bxa = -t.$$

It remains to check that $L(r)$ is nil-3.

Proposition 6.7. *$L(r)$ is a nil-3 algebra for all $r \in F$.*

Proof. Let $z = \alpha_1 a + \alpha_2 ax + \alpha_3 ax^2 + \beta_1 b + \beta_2 bx + \gamma x$. It suffices to show that $yz^3 = 0$ for the basis elements a, ax, ax^2, b, bx, x. Using the description of $L(r)$, we have $bxz^2 = (-\alpha_1 t + \gamma bx^2)z = 0$ and then:

$$\begin{aligned} az^3 &= (-\alpha_2 rb + \alpha_3 rbx + \beta_2 t + \gamma ax)z^2 \\ &= (-\alpha_2 rb + \gamma ax)z^2 \\ &= \left(\alpha_2^2 rt - \alpha_2\gamma rbx + \gamma\alpha_1 rb - \gamma\alpha_3 rbx^2 + \gamma\beta_1 t + \gamma^2 ax^2\right)z \\ &= \left(-\alpha_2\gamma rbx + \gamma\alpha_1 rb + \gamma^2 ax^2\right)z \\ &= \alpha_2\gamma\alpha_1 rt - \alpha_2\gamma^2 rbx^2 - \gamma\alpha_1\alpha_2 rt \\ &\quad + \gamma^2\alpha_1 rbx - \gamma^2\alpha_1 rbx + \gamma^2\alpha_2 rbx^2 \\ &= 0; \end{aligned}$$

$$\begin{aligned} axz^3 &= \left(\alpha_1 rb - \alpha_3 rbx^2 + \beta_1 t + \gamma ax^2\right)z^2 \\ &= \left(\alpha_1 rb + \gamma ax^2\right)z^2 \\ &= \left(-\alpha_1\alpha_2 rt + \alpha_1\gamma rbx - \gamma\alpha_1 rbx + \gamma\alpha_2 rbx^2\right)z \\ &= 0; \end{aligned}$$

$$ax^2z^3 = \left(-\alpha_1 rbx + \alpha_2 rbx^2\right)z^2 = 0;$$
$$bz^3 = (-\alpha_2 t + \gamma bx)z^2 = 0;$$
$$bxz^3 = \left(-\alpha_1 t + \gamma bx^2\right)z^2 = 0;$$

$$\begin{aligned} xz^3 &= \left(-\alpha_1 ax - \alpha_2 ax^2 - \beta_1 bx - \beta_2 bx^2\right)z^2 \\ &= \left(-\alpha_1 ax - \alpha_2 ax^2\right)z^2 \\ &= \left(-\alpha_1^2 rb + \alpha_1\alpha_3 rbx^2 - \alpha_1\beta_1 t \right. \\ &\quad \left. - \alpha_1\gamma ax^2 + \alpha_2\alpha_1 rbx - \alpha_2^2 rbx^2\right)z \\ &= \left(-\alpha_1^2 rb - \alpha_1\gamma ax^2 + \alpha_2\alpha_1 rbx\right)z \end{aligned}$$

$$\begin{aligned}&= \alpha_1^2\alpha_2 rt - \alpha_1^2\gamma rbx + \alpha_1^2\gamma rbx \\ &\quad - \alpha_1\gamma\alpha_2 rbx^2 - \alpha_2\alpha_1^2 rt + \alpha_2\alpha_1\gamma rbx^2 \\ &= 0. \qquad \square\end{aligned}$$

We finally prove the nilpotency of $L(r)$.

Theorem 6.8. *$L(r)$ is nilpotent of class* 3 *if $r = 0$ and of class* 5 *if $r \neq 0$.*

Proof. Let $r = 0$. Then $Z(L) = Fax^2 + Fbx^2 + Ft$ by Lemma 6.3. Moreover

$$L^2 = Lx + Ft \quad \text{and} \quad L^3 = Lx^2 + Ft = Z(L),$$

so that $L(0)$ is nilpotent of class 3.

Assume $r \neq 0$. Then

$$L^2 = \langle b, ax, bx, ax^2, bx^2, t\rangle, \qquad L^3 = \langle b, bx, ax^2, bx^2, t\rangle$$
$$L^4 = \langle bx, bx^2, t\rangle, \qquad L^5 = \langle bx^2, t\rangle, \qquad L^6 = \{0\}.$$

This proves that $L(r)$ is nilpotent of class 5. $\square$

The parameter $r \in F$ is not unique. Recall that $r = (a, ax^2(ax))$. Now $Z_3(L) = (L^4)^\perp = \langle b, bx, ax^2, bx^2, t\rangle$. Let

$$\bar{a} = \alpha_1 a + \beta_1 ax + \gamma x + u \quad \text{and} \quad \bar{x} = \alpha_2 a + \beta_2 ax + \delta x + v$$

with $u, v \in Z_3(L)$. Tedious but direct calculations show that

$$\big(\bar{a}, \bar{a}\bar{x}^2(\bar{a}\bar{x})\big) = (\alpha_1\delta - \alpha_2\gamma)^3 r.$$

This implies that for $r, s \neq 0$ we have that $L(r) \cong L(s)$ if and only if r and s are in the same coset of the abelian group $F^*/(F^*)^3$ (where $F^* = F \setminus \{0\}$). Adding $L(0)$, we see that there are up to isomorphism exactly $|F^*/(F^*)^3| + 1$ symplectic alternating algebras of dimension 8 that are nil-3 but not nil-2. If F is algebraically closed then this number is 2. As $(\mathbb{R}^*)^3 = \mathbb{R}$, this is also true when the underlying field is the field of real numbers. On the other hand, $\mathbb{Q}^*/(\mathbb{Q}^*)^3$ is infinite so over the rational field we have an infinite number of examples. If F is finite then F^* is cyclic and thus $|F^*/(F^*)^3|$ is 1 or 3 depending on whether 3 divides $|F| - 1$ or not.

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