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Symmetric powers of the $p+1$-dimensional indecomposable module of a cyclic p-group and the λ-structure of its Green ring ☆

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ABSTRACT

Let K be a field of order p and let G be a cyclic group of order p^k ($k \geqslant 2$). We explicitly decompose the symmetric powers of the indecomposable KG-module of dimension $p+1$ into indecomposable KG-modules. Using this result, for every odd prime p, we give a negative answer to the conjecture posed by Kouwenhoven [F.M. Kouwenhoven, The λ-structure of the Green rings of cyclic p-groups, Proc. Sympos. Pure Math. 47 (1987) 451–466].

1. Introduction

Let K be a field with positive characteristic p and let G be a finite group with order divisible by p. The Green ring $RS_K(G)$ of G over K is a ring formed from isomorphism classes of the finite-dimensional KG-modules, with addition and multiplication coming from direct sums and their tensor products, respectively. If K is algebraically closed and the Sylow p-subgroups of G are cyclic, then the ring structure of $RS_K(G)$ is theoretically approachable [6]. Even in cases of finite cyclic p-groups, however, the λ-structure of $RS_K(G)$ induced from exterior powers is far from being approachable.

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Most obstructions to studies of the λ-structure of $RS_K(G)$ are caused by the fact that $RS_K(G)$ is not a λ-ring. Hence it would be quite natural to search for the smallest λ-ideal I such that the quotient ring $RS_K(G)/I$ is a λ-ring. In 1987, Kouwenhoven settled this problem for the case in which G is a cyclic group of order p for each prime p [9]. Furthermore, he proposed a challenging conjecture for the case in which G is a cyclic group of order p^k $(k \geqslant 2)$. Throughout this paper, the Green ring is simply denoted by $RS(p^k)$ if K is a field of order p and G is a cyclic group of order p^k. It is well known that $RS(p^k)$ is a free $\mathbb{Z}$-module with basis $\{V_n\colon\ 1 \leqslant n \leqslant p^k\}$, where $V_n := K[X]/(X-1)^n$. Against this background, Kouwenhoven's conjecture is stated as follows.

Conjecture. *(See [9].) For every prime p, let $q = p^k$ for $k \geqslant 2$ and let I_q be the ideal generated by $V_{p^i} - V_{p^i-1} - V_1$ for $1 \leqslant i \leqslant k$. Then I_q is closed for the exterior powers and $RS(q)/I_q$ is equipped with the λ-ring structure for the induced exterior powers.*

It was recently shown that the above conjecture is true when $p = 2$, but false when $p = 3$ [10]. We show here that it is false for every odd prime p. In brief, our strategy is to compute $\psi_S^p(V_{p+1}) \pmod{I_q}$ in two different ways. The first is to express $\psi_S^p(V_{p+1})$ as a linear combination of indecomposable modules using Newton's formula (for a precise description, see Theorem 4.3). It should be noted that this method is applicable since the explicit decomposition of $S^n(V_{p+1})$ can be obtained by virtue of the elegant theorem of Shank and Wehlau [11, Theorem 1.3]. The second way is to compute $\psi_S^p(V_{p+1})$ from $\psi_\Lambda^p(V_{p^2-(p+1)})$ modulo $\mathbb{Z}V_{p^2}$ using the result reported by Bryant and Johnson [5, Theorem 6.2]. Under the validity of the conjecture, $\psi_\Lambda^p(V_{p^2-(p+1)})$ is equal to $\psi_\Lambda^p(V_{p^2}) - \psi_\Lambda^p(V_{p+1}) \pmod{I_{p^2}}$. Since $\psi_\Lambda^p(V_{p^2}) = pV_p$ and $\psi_\Lambda^p(V_{p+1})$ is computable by virtue of another result presented by Bryant and Johnson [4, Theorem 4.7], the value of $\psi_S^p(V_{p+1}) \pmod{I_{p^2}}$ is also computable (Lemma 4.1). Finally, by comparing the results thus obtained, we derive a contradiction.

The remainder of the paper is organized as follows. Section 2 recollects the notations and theorems required to develop our arguments. Section 3 describes the explicit decomposition of $S^n(V_{p+1})$ into a direct sum of indecomposable modules for each positive integer n. The final section shows that Kouwenhoven's conjecture does not always hold.

2. Preliminaries

This paper is a sequel to a previous study [10]. To avoid confusion and ambiguity, we adopt all the previous definitions and notation without change. We also introduce further notation and theorems required to develop our argument.

Let K be a field, G a finite group, and $RS_K(G)$ the Green ring of G over K. In the case in which K is a field of order p and G is a cyclic group of order p^k for a positive integer k, we denote $RS_K(G)$ by $RS(p^k)$ throughout the paper. For any KG-module V, let V denote the corresponding element of $RS_K(G)$. It is well known that $RS(p^k)$ is a free $\mathbb{Z}$-module that has a $\mathbb{Z}$-basis consisting of p^k indecomposable modules $V_1, V_2, \ldots, V_{p^k}$ satisfying $\dim V_n = n$ for each $1 \leqslant n \leqslant p^k$. Conventionally, V_0 is defined by the zero element in $RS(p^k)$.

The most significant pre-λ-ring structures of $RS_K(G)$ come from the exterior powers Λ^n and the symmetric powers S^n $(n = 0, 1, 2\ldots)$, respectively. For each positive integer n and for all $x \in RS_K(G)$, we define $\psi_\Lambda^n(x)$ and $\psi_S^n(x)$ as follows.

$$\sum_{n=0}^{\infty}(-1)^n\psi_\Lambda^{n+1}(x)t^n := \frac{d}{dt}\log \Lambda_t(x),$$

$$\sum_{n=0}^{\infty}\psi_S^{n+1}(x)t^n := \frac{d}{dt}\log S_t(x),$$

where

$$\Lambda_t(x) = 1 + \Lambda^1(x)t + \Lambda^2(x)t^2 + \cdots,$$
$$S_t(x) = 1 + S^1(x)t + S^2(x)t^2 + \cdots.$$

We can easily show that $\psi^n_\Lambda(x), \psi^n_S(x) \in RS_K(G)$ and

$$\psi^n_\Lambda(x+y) = \psi^n_\Lambda(x) + \psi^n_\Lambda(y), \qquad \psi^n_S(x+y) = \psi^n_S(x) + \psi^n_S(y)$$

for all positive integers n and all $x, y \in RS_K(G)$. Hence, we have the $\mathbb{Z}$-linear functions

$$\psi^n_\Lambda, \psi^n_S : RS_K(G) \to RS_K(G),$$

which are called the nth Adams operations on $RS_K(G)$. From the definition of Adams operations, we can derive Newton's formula for ψ^n_Λ and ψ^n_S $(n \geqslant 1)$:

$$\sum_{i=0}^{n-1} (-1)^{i+1} \psi^{n-i}_\Lambda \Lambda^i = (-1)^n n \Lambda^n \quad \text{and} \quad \sum_{i=0}^{n-1} \psi^{n-i}_S S^i = nS^n, \quad \forall n \geqslant 1. \tag{2.1}$$

These formulae imply that each ψ^n_Λ and ψ^n_S can be expressed as a polynomial in Λ^i and S^i $(1 \leqslant i \leqslant n)$, respectively, with integer coefficients. As a consequence, if I is invariant under Λ^n (S^n) for all $n \geqslant 1$, then it is also invariant under ψ^n_Λ (ψ^n_S) for all $n \geqslant 1$. In studies of the λ-structure of $RS_K(G)$, Adams operations are useful because they contain much information about λ-operations and are easier to deal with than λ-operations because of their additivity. In particular, Bryant showed that ψ^n_Λ and ψ^n_S behave well provided n is not divisible by the characteristic of K [3].

Theorem 2.1. *(See [3].) For every positive integer n that is not divisible by the characteristic of a field K, we have $\psi^n_\Lambda = \psi^n_S$ and each of these maps is a ring endomorphism of $RS_K(G)$. Furthermore, under the composition of maps we have*

$$\psi^n_\Lambda \circ \psi^m_\Lambda = \psi^{nm}_\Lambda, \qquad \psi^n_S \circ \psi^m_S = \psi^{nm}_S$$

for all positive integers m.

For a KG-module V, let $\mathcal{P}_V$ denote the projective cover of V. It is uniquely determined up to isomorphism and the value of the Heller operation Ω at V is defined as the kernel of the map $\mathcal{P}_V \to V$ so that the short sequence

$$0 \to \Omega(V) \to \mathcal{P}_V \to V \to 0$$

is exact [2]. Then we can have a $\mathbb{Z}$-linear function on $RS_K(G)$ by extending Ω by linearity. We define Ω^0 as the identity map and Ω^n the map obtained by composing Ω n times for each positive integer n. In the case in which $K = \mathbb{F}_p$ and G is a cyclic group of order p^k, V_{p^k} is the only projective indecomposable KG-module and V_{p^k-r} is the Heller translate of V_r in $RS(p^k)$.

For any positive integer n not divisible by p, let $\gamma(n)$ denote the unique integer satisfying the conditions $1 \leqslant \gamma(n) \leqslant p-1$ and $n \equiv \gamma(n)$ (or, $-\gamma(n)$) (mod $2p$) and set $\gamma(0)$ to 0. For $m \in \{0, \dots, k-1\}$ and $i \in \{1, \dots, p-1\}$, we define a $\mathbb{Z}$-linear map $\theta_{ip^m} : RS(p^m) \to RS(p^{m+1})$ so that

$$\theta_{ip^m}(V_r) = V_{ip^m+r} - V_{ip^m-r}$$

for $r = 1, \ldots, p^m$. In particular, θ_0 is defined as the identity map on $RS(p^m)$. The following propositions are useful for computation of the value of Adams operations.

Proposition 2.2. *(See [4,5].) Let k be a positive integer. Then we have the following.*

(a) *Let $m \in \{0, \ldots, k-1\}$ and let n be a positive integer not divisible by p. Let s be a positive integer satisfying $p^m < s \leqslant p^{m+1}$ and write $s = s_0 p^m + s_1$, where $1 \leqslant s_0 \leqslant p-1$ and $1 \leqslant s_1 \leqslant p^m$. Then*

$$\psi_\Lambda^n(V_s) = \sum_{\substack{i \in \{0,\ldots,s_0\} \\ i \equiv s_0 \pmod 2}} \theta_{\gamma(in)p^m}\bigl(\psi_\Lambda^n(V_{s_1})\bigr) + \sum_{\substack{i \in \{0,\ldots,s_0\} \\ i \not\equiv s_0 \pmod 2}} \theta_{\gamma(in)p^m}\bigl(\psi_\Lambda^n(V_{p^m - s_1})\bigr).$$

(b) *Let $q = p^k$. Then for each positive integer n, the nth Adams operations ψ_Λ^n and ψ_S^n on $RS(q)$ satisfy the relation*

$$\psi_S^n(V_s) \equiv (-1)^{n-1}\Omega^n\bigl(\psi_\Lambda^n(V_{q-s})\bigr) + (n,q)V_{n/(n,q)} \pmod{\mathbb{Z}V_q},$$

where $q/p \leqslant s \leqslant q$.

Proposition 2.3. *(See [4,5].) Let k be a positive integer. Then we have the following.*

(a) *For $0 \leqslant i \leqslant k-1$, $\chi_i \psi_\Lambda^p(\chi_i) = 2\psi_\Lambda^{p-1}(\chi_i)$ holds, where χ_i denotes the ith generator of $RS(p^k)$, i.e., $V_{p^i+1} - V_{p^i-1}$.*
(b) *Let q be a power of an odd prime. Then for each positive integer n, $\psi_\Lambda^n(V_q) = (n,q)V_{q/(n,q)}$ holds.*
(c) *For all positive integers n not divisible by p and $1 \leqslant m \leqslant k$,*

$$\psi_\Lambda^n\bigl(V_{p^m-1}\bigr) = \begin{cases} V_{p^m-1} & \text{if } n \text{ is odd}, \\ V_{p^m} - V_1 & \text{if } n \text{ is even}. \end{cases}$$

The following proposition presents the rules necessary for multiplication of $RS(q)$ through decomposition of the tensor product of two indecomposable modules.

Proposition 2.4. *(See [8,9].) Suppose that p is prime and $q = p^k$, where k is a positive integer.*

(a) *For $0 \leqslant n \leqslant q$, we have*

$$V_2 V_n = \begin{cases} V_{n-1} + V_{n+1} & \text{if } p \nmid n, \\ 2V_n & \text{if } p \mid n. \end{cases}$$

(b) *Let m and n be positive integers such that $m \leqslant n \leqslant pq$. We write $n = n_0 q + n_1$ and $m = m_0 q + m_1$ with $0 \leqslant m_1, n_1 \leqslant q-1$. Suppose $V_{n_1}V_{m_1} = \sum_s a_s V_s$. If $m + n \leqslant pq$, then*

$$\begin{aligned} V_n V_m = {}& \sum_{i=1}^{m_0} \sum_{s=1}^{q-1} a_s\bigl(V_{(n_0+m_0+2-2i)q+s} + V_{(n_0+m_0+2-2i)q-s}\bigr) \\ & + \sum_{s=1}^{q-1} a_s V_{(n_0-m_0)q+s} + |n_1 + m_1 - q| \sum_{i=1}^{m_0} V_{(n_0+m_0+1-2i)q} \\ & + |m_1 - n_1| \sum_{i=1}^{m_0} V_{(n_0+m_0+2-2i)q} + \max(0, m_1 + n_1 - q)V_{(n_0+m_0+1)q} \\ & + \max(0, m_1 - n_1)V_{(n_0-m_0)q}. \end{aligned}$$

If $m+n>pq$, then

$$V_n V_m = V_{pq-n} V_{pq-m} + (n+m-pq) V_{pq}.$$

It has been shown that there exists a ring isomorphism, say $\varphi : RS(p)^{\otimes k} \to RS(q)$, defined by [1,7]

$$1 \otimes \cdots \otimes 1 \otimes \overbrace{V_2}^{i\text{th}} \otimes 1 \otimes \cdots \otimes 1 \mapsto V_{2p^{i-1}} - V_{2p^{i-1}-1} + V_1 \quad (1 \leqslant i \leqslant k).$$

Contrary to the simplicity of the ring structure, the λ-structure of $RS(q)$ is extremely complicated. Indeed $RS(p^k)$ $(k \geqslant 1)$ is not a λ-ring for the λ-operations coming from the exterior powers. We define I_p as the ideal of $RS(p)$ generated by $V_p - V_{p-1} - V_1$. In 1987, Kouwenhoven showed that $RS(p)/I_p$ is the largest $\mathbb{Z}$-torsion free quotient of $RS(p)$ that is a λ-ring for the induced exterior powers [9]. He also showed that φ induces a ring isomorphism

$$\bar{\varphi} : \big(RS(p)/I_p\big)^{\otimes k} \to RS(q)/I_q, \qquad \overline{x_1} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_k} \mapsto \overline{\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_k)},$$

where I_q is the ideal generated by $V_{p^i} - V_{p^i-1} - V_1$ for all i with $1 \leqslant i \leqslant k$. Here, bar notation is used to denote a coset. However, if there is no danger of confusion, this will be omitted for simplicity. Motivated by this observation, he proposed the following conjecture on the λ-structure of $RS(q)/I_q$ when $k \geqslant 2$.

Conjecture 2.5. *(See [9].) For every prime p, let $q = p^k$ with $k \geqslant 2$ and let I_q be the ideal generated by $V_{p^i} - V_{p^i-1} - V_1$ for $1 \leqslant i \leqslant k$. The exterior powers induce operations on $RS(q)/I_q$ and the induced ring isomorphism $\bar{\varphi} : (RS(p)/I_p)^{\otimes k} \xrightarrow{\cong} RS(q)/I_q$ commutes with the λ-operations. In particular, $RS(q)/I_q$ is a λ-ring.*

We recently presented a partial result for Conjecture 2.5 [10]: we showed that it is true when $p=2$, but false when $p=3$. The main goal of the present paper is to show that I_{p^k} is not closed for the exterior powers if p is an odd prime and $k \geqslant 2$.

We finish this section by providing a $\mathbb{Z}$-basis of $RS(q)/I_q$, which plays a key role in the final step in disproving Conjecture 2.5.

Let

$$\mathcal{A} := \left\{ (a_1, \ldots, a_{k-1})\colon\ 0 \leqslant a_i \leqslant \frac{p-1}{2} \text{ for all } 1 \leqslant i \leqslant k-1 \right\}.$$

For any $(k-1)$-tuple $(a_1, \ldots, a_{k-1}) \in \mathcal{A}$, we define $\mathbf{sum}(a_1, \ldots, a_{k-1})$ as

$$\sum_{1 \leqslant m \leqslant k-1} a_m p^m.$$

Proposition 2.6. *(See [10].) Let k be a positive integer and let $q = p^k$. Then the set*

$$\big\{ V_{\mathbf{sum}(a_1,\ldots,a_{k-1})+j}\colon\ (a_1, \ldots, a_{k-1}) \in \mathcal{A},\ j = 1, 3, \ldots, p \big\}$$

is a $\mathbb{Z}$-basis of $RS(q)/I_q$.

3. Symmetric powers of V_{p+1}

In this section, we decompose $S^n(V_{p+1})$ into a direct sum of indecomposable modules for each positive integer n. It should be noted that Shank and Wehlau have already succeeded in decomposing $S^n(V_{p+1})$ into a direct sum of indecomposable modules (Theorem 3.4) [11]. Their decomposition, however, is up to induced modules. Here we calculate the multiplicity of each induced module appearing in the decomposition of $S^n(V_{p+1})$ precisely.

We first provide two results on the decomposition of symmetric powers.

Proposition 3.1. *(See [1].)*

(a) $S^r(V_2) \cong V_{r+1}$ *for* $0 \leqslant r \leqslant p-1$.
(b) $S^r(V_{p^m-t}) \cong S^{r/p}(V_{p^{m-1}}) \oplus$ *free for* $r \equiv 0 \pmod p$ *and* t *in the range* $0 \leqslant t \leqslant p-1$.

Lemma 3.2. *For any non-negative integer* s *and* $0 \leqslant r \leqslant p-1$, *we have*

$$S^{sp+r}(V_2) \cong V_{r+1} \oplus sV_p. \tag{3.1}$$

Proof. If $r=0$, then $S^{sp}(V_2) \cong S^s(V_1) \oplus$ *free*, which follows from Proposition 3.1(b), and this holds for all prime p. Comparing the dimension of either side of (3.1) immediately yields the desired result. Thus, r is assumed to be positive hereafter. To accomplish our purpose, we use mathematical induction on $sp+r$. Note that $S^1(V_2) \cong V_2$. Suppose that the decomposition of $S^k(V_2)$ satisfies our assertion for all $k < sp+r$, where $sp+r > 1$. Since $sp+r$ is not divisible by p, in this case we have

$$\sum_{i=0}^{sp+r} (-1)^i \Lambda^i(V_2) S^{sp+r-i}(V_2) = 0 \tag{3.2}$$

in $RS(p)$. Thus,

$$S^{sp+r}(V_2) - V_2 S^{sp+r-1}(V_2) + S^{sp+r-2}(V_2) = 0.$$

If $p \neq 2$, the induction hypothesis implies that

$$\begin{aligned}
0 &= S^{sp+r}(V_2) - V_2(V_r + sV_p) + S^{sp+r-2}(V_2) \\
&= \begin{cases} S^{sp+r}(V_2) - V_2 - 2sV_p + V_p + (s-1)V_p & \text{if } r=1, \\ S^{sp+r}(V_2) - V_{r+1} - V_{r-1} - 2sV_p + V_{r-1} + sV_p & \text{otherwise} \end{cases} \\
&= \begin{cases} S^{sp+r}(V_2) - V_2 - sV_p & \text{if } r=1, \\ S^{sp+r}(V_2) - V_{r+1} - sV_p & \text{otherwise.} \end{cases}
\end{aligned}$$

The second equality follows from Proposition 2.4(a). Conversely, if $p=2$, we have

$$\begin{aligned}
0 &= S^{sp+1}(V_2) - V_2(V_1 + sV_2) + S^{(s-1)p+1}(V_2) \\
&= S^{sp+1}(V_2) - V_2 - 2sV_2 + sV_2 \\
&= S^{sp+1}(V_2) - (s+1)V_2.
\end{aligned}$$

This completes the proof. $\square$

Remark 3.3. For any KG-module V, consider the following exact sequence:

$$0 \to \Lambda^d(V) \to \Lambda^{d-1} \otimes S^1(V) \to \cdots \to \Lambda^1(V) \otimes S^{d-1}(V) \to S^d(V) \to 0.$$

It is well known that the above exact sequence splits whenever d is invertible, and hence Eq. (3.2) follows [1, Theorem 2.3].

Next we introduce a result due to Shank and Wehlau on the decomposition of $S^n(V_{p+1})$ into a direct sum of indecomposable modules [11]. We first review the notation required. Let G be a finite group and let H be a subgroup of G. For any KG-module V, let $Res_{\mathrm{H}}(V)$ denote the KH-module obtained from V by restriction. Since restriction commutes with direct sums and tensor products, we can extend Res_{H} to a ring homomorphism from $RS_K(G)$ to $RS_K(H)$. Furthermore, it is well known that Res_{H} commutes with the exterior powers and symmetric powers. Conversely, for any KH-module W let $Ind^{\mathrm{G}}(W)$ denote the KG-module obtained from W by induction. Since induction commutes with direct sums, Ind^{G} can also be extended to a $\mathbb{Z}$-linear map from $RS_K(H)$ to $RS_K(G)$. A KG-module is said to be induced if it is induced from a KH-module for some subgroup H of G and an element of $RS_K(G)$ is said to be induced if it is a $\mathbb{Z}$-linear combination of induced modules. Now assume that K is a field of characteristic p, G is a cyclic group of order p^k, and H is the unique subgroup of index p in G. Then we have

$$Res_{\mathrm{H}}(V_s) = s_1 V_{s_0+1} + (p - s_1) V_{s_0}, \tag{3.3}$$

where $s = s_0 p + s_1$ with $0 \leqslant s_1 \leqslant p - 1$ and, for $r = 1, \ldots, p^{k-1}$,

$$Ind^{\mathrm{G}}(V_r) = V_{rp}.$$

It is not difficult to show that V_r is induced from a module of a proper subgroup of G if and only if r is divisible by p. Hereafter, we write Res for Res_{H}, the restriction map (3.3), for simplicity. The following theorem is key in the proof of our main result.

Theorem 3.4. *(See [11].) Let K be a field of characteristic p, let G be a cyclic group of order p^2 and let d be any non-negative integer. In the decomposition of $S^n(V_{p+1})$ into a direct sum of indecomposable KG-modules, there is at most one indecomposable summand V_r that is not induced from a representation of a proper subgroup. In particular, writing $n = ap^2 + bp + c$, where $0 \leqslant b, c \leqslant p - 1$, $S^n(V_{p+1})$ is an induced module when $b = p - 1$ and there is exactly one non-induced indecomposable summand when $b \leqslant p - 2$ that is isomorphic to V_{cp+b+1}.*

Note that

$$\begin{aligned} S_t\big(Res(V_{p+1})\big) &= S_t\big((p-1)V_1 + V_2\big) \\ &= \big(1 + t + t^2 + \cdots\big)^{p-1}\big(1 + S^1(V_2)t + S^2(V_2)t^2 + \cdots\big). \end{aligned}$$

Thus, the coefficient of t^n in $S_t(Res(V_{p+1}))$ is given by

$$S^n(V_2) + \left(\!\!\binom{p-1}{1}\!\!\right) S^{n-1}(V_2) + \cdots + \left(\!\!\binom{p-1}{n-1}\!\!\right) S^1(V_2) + \left(\!\!\binom{p-1}{n}\!\!\right) V_1, \tag{3.4}$$

where the notation $\left(\!\binom{n}{k}\!\right)$ denotes the number of k-multicombinations (or k-combinations with repetitions) of an n-element set. Note that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. To apply Lemma 3.2, we write the above summation as

$$\sum_{j=0}^{c+1}\left(\!\!\binom{p-1}{j}\!\!\right)S^{n-j}(V_2)+\sum_{i=0}^{ap+b-2}\sum_{j=0}^{p-1}\left(\!\!\binom{p-1}{c+2+pi+j}\!\!\right)S^{n-(c+2+pi+j)}(V_2)$$
$$+\sum_{j=0}^{p-2}\left(\!\!\binom{p-1}{n+2-p+j}\!\!\right)S^{n-(n+2-p+j)}(V_2), \tag{3.5}$$

where $n=ap^2+bp+c$ with $0\leqslant b,c\leqslant p-1$. Using the convention that $(\!\binom{n}{k}\!)$ is zero when k is a negative integer, in view of Lemma 3.2, we can write the summation in Eq. (3.5) as a linear combination

$$\sum_{k=1}^{p}c(k)V_k$$

of $V_1,\dots,V_p$, where

$$c(k)=\sum_{i=0}^{ap+b}\left(\!\!\binom{p-1}{c+1-k+ip}\!\!\right),\quad 1\leqslant k\leqslant p-1$$

and

$$c(p)=\begin{cases}\sum_{i=0}^{ap+b}(ap+b-i)\sum_{j=0}^{p-1}(\!\binom{p-1}{c+1-j+ip}\!) & \text{if } c\neq p-1,\\ \sum_{i=0}^{ap+b}(ap+b-i)\sum_{j=0}^{p-1}(\!\binom{p-1}{c+1-j+ip}\!)+ap+b+1 & \text{if } c=p-1.\end{cases} \tag{3.6}$$

Moreover, for $0\leqslant c\leqslant p-1$ and $0\leqslant j\leqslant p-1$, it is easy to show that $(\!\binom{p-1}{c+1-j-p}\!)=0$ unless $c=p-1$ and $j=0$. Thus, two cases in Eq. (3.6) can be merged as

$$c(p)=\sum_{i=-1}^{ap+b}(ap+b-i)\sum_{j=0}^{p-1}\left(\!\!\binom{p-1}{c+1-j+ip}\!\!\right).$$

Next, we recall the identity

$$S^n\big(Res(V_{p+1})\big)=Res\big(S^n(V_{p+1})\big),$$

which follows from the fact that the restriction map commutes with symmetric power. Using Theorem 3.4, we can derive

$$S^n(V_{p+1})=\begin{cases}V_{cp+b+1}+c_n(1)V_p+c_n(2)V_{2p}+\dots+c_n(p)V_{p^2} & \text{if } b\leqslant p-2,\\ c_n(1)V_p+c_n(2)V_{2p}+\dots+c_n(p)V_{p^2} & \text{if } b=p-1,\end{cases} \tag{3.7}$$

for some non-negative integers $c_n(i)$ $(1\leqslant i\leqslant p)$. To Eq. (3.7) we apply the formulae

$$Res(V_{cp+b+1})=(p-b-1)V_c+(b+1)V_{c+1}$$

and

$$Res(V_{kp})=pV_k$$

and then compare the coefficient of V_i on either side of $S^n(Res(V_{p+1})) = Res(S^n(V_{p+1}))$. This enables us to derive the explicit decomposition of $S^n(V_{p+1})$ into indecomposables.

Theorem 3.5. *Writing $n = ap^2 + bp + c$ with $0 \leqslant b, c \leqslant p-1$, we have*

$$S^n(V_{p+1}) = (1 - \delta_{b,p-1})V_{cp+b+1} + \sum_{k=1}^{p} c_n(k)V_{kp},$$

where if $b = p-1$, then

$$pc_n(k) == \begin{cases} \sum_{i=0}^{ap+b} \left(\!\binom{p-1}{c+1-k+ip}\!\right) & \text{if } 1 \leqslant k \leqslant p-1, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left(\!\binom{p-1}{c+1-j+ip}\!\right) & \text{if } k = p \end{cases}$$

and if $b \neq p-1$, then

$$pc_n(k) = \begin{cases} \sum_{i=0}^{ap+b} \left(\!\binom{p-1}{c+1-k+ip}\!\right) & \text{if } 1 \leqslant k \leqslant p-1,\ k \neq c, c+1, \\ \sum_{i=0}^{ap+b} \left(\!\binom{p-1}{1+ip}\!\right) - p + b + 1 & \text{if } k = c, \\ \sum_{i=0}^{ap+b} \left(\!\binom{p-1}{ip}\!\right) - b - 1 & \text{if } k = c+1,\ k \neq p, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left(\!\binom{p-1}{(i+1)p-j}\!\right) - b - 1 & \text{if } k = c+1,\ k = p, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left(\!\binom{p-1}{c+1-j+ip}\!\right) & \text{if } k \neq c+1,\ k = p. \end{cases}$$

Example 3.6. Let $p = 5$. According to Theorem 3.5, the first ten $S^n(V_6)$ value ($1 \leqslant n \leqslant 10$) are decomposed into indecomposable modules in the following fashion.

$S^1(V_6)$	V_6
$S^2(V_6)$	$V_{11} \oplus 2V_5$
$S^3(V_6)$	$V_{16} \oplus 4V_5$
$S^4(V_6)$	$V_{21} \oplus 7V_5 \oplus 4V_{10} \oplus 2V_{15}$
$S^5(V_6)$	$V_2 \oplus 11V_5 \oplus 7V_{10} \oplus 4V_{15} \oplus 2V_{20} \oplus V_{25}$
$S^6(V_6)$	$V_7 \oplus 17V_5 \oplus 11V_{10} \oplus 7V_{15} \oplus 4V_{20} \oplus 3V_{25}$
$S^7(V_6)$	$V_{12} \oplus 26V_5 \oplus 17V_{10} \oplus 11V_{15} \oplus 7V_{20} \oplus 7V_{25}$
$S^8(V_6)$	$V_{17} \oplus 37V_5 \oplus 26V_{10} \oplus 17V_{15} \oplus 11V_{20} \oplus 14V_{25}$
$S^9(V_6)$	$V_{22} \oplus 51V_5 \oplus 37V_{10} \oplus 26V_{15} \oplus 17V_{20} \oplus 25V_{25}$
$S^{10}(V_6)$	$V_3 \oplus 68V_5 \oplus 51V_{10} \oplus 37V_{15} \oplus 26V_{20} \oplus 43V_{25}$

4. The value of Adams operations at V_{p+1} and Kouwenhoven's conjecture

The purpose of this section is to disprove Conjecture 2.5 for every odd prime. We first provide a brief outline of our argument. Assume that I_q is closed for the exterior powers. Then it is also closed for all Adams operations associated with the exterior powers, that is, $\psi_\Lambda^n(I_q) \subseteq I_q$ for all positive integers n, because each ψ_Λ^n can be expressed as a polynomial in λ^n ($n \geqslant 1$) with integer coefficients. Consequently, $\psi_\Lambda^p(V_{p^2-(p+1)})$ should be equal to $\psi_\Lambda^p(V_{p^2} - V_{p+1})$ modulo I_{p^2} since $V_{p+1}(V_{p^2} - V_{p^2-1} - V_1) = V_{p^2} - V_{p^2-(p+1)} - V_{p+1} \in I_{p^2}$. However, we show that this phenomenon does not occur if p is odd. Throughout this section, p denotes an odd prime.

First, we express $\psi_\Lambda^p(V_{p^2} - V_{p+1})$ as a linear combination of V_i values modulo I_{p^2}. To use Proposition 2.3(a), we multiply by $V_{p+1} - V_{p-1}$. Since $\psi_\Lambda^p(V_{p^2}) = pV_p$ by Proposition 2.3(b), it follows that

$$\begin{aligned}
&(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p^2}-V_{p+1})\\
&\quad=(V_{p+1}-V_{p-1})\bigl(pV_p-\psi_\Lambda^p(V_{p+1})\bigr)\\
&\quad=pV_p(V_{p+1}-V_{p-1})-(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p+1}-V_{p-1})\\
&\qquad-(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p-1}).
\end{aligned}\tag{4.1}$$

The term $(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p+1}-V_{p-1})$ is equal to $2\psi_\Lambda^{p-1}(V_{p+1}-V_{p-1})$ by Proposition 2.3(a) because $\chi_1=V_{p+1}-V_{p-1}$. Furthermore, from [3, Lemma 3.4] it follows that $\psi_\Lambda^p(V_{p-1})\equiv(p-1)V_1 \pmod{I_{p^2}}$. Substituting these into Eq. (4.1) yields the modulo equivalence

$$\begin{aligned}
&(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p^2}-V_{p+1})\\
&\quad\equiv pV_p(V_{p+1}-V_{p-1})-2\psi_\Lambda^{p-1}(V_{p+1}-V_{p-1})\\
&\qquad-(p-1)(V_{p+1}-V_{p-1}) \pmod{I_{p^2}}.
\end{aligned}\tag{4.2}$$

Moreover, by Propositions 2.3(c) and 2.2(b), we can deduce that $\psi_\Lambda^{p-1}(V_{p-1})=V_p-V_1$, and hence

$$\begin{aligned}
\psi_\Lambda^n(V_{p+1})&=\theta_{\gamma(n)p}\bigl(\psi^n(V_1)\bigr)+\theta_{\gamma(0)p}\bigl(\psi_\Lambda^n(V_{p-1})\bigr)\\
&=V_{np+1}-V_{np-1}+\psi_\Lambda^n(V_{p-1})\\
&=\begin{cases}V_{np+1}-V_{np-1}+V_{p-1} & \text{if } n \text{ is odd},\\ V_{np+1}-V_{np-1}+V_p-V_1 & \text{if } n \text{ is even},\end{cases}
\end{aligned}\tag{4.3}$$

where $1\leqslant n\leqslant p-1$. Applying Eq. (4.3) to Eq. (4.2), we finally have the following identities.

Lemma 4.1.

$$(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p^2}-V_{p+1})\equiv pV_p(V_{p+1}-V_{p-1})-(p+1)(V_{p+1}-V_{p-1}) \pmod{I_{p^2}}$$

and

$$\psi_\Lambda^{p-1}(V_{p+1})\equiv V_{p+1} \pmod{I_{p^2}}.$$

Letting $q=p^2$, $s=p^2-(p+1)$ in Proposition 2.2(b) yields the identity

$$(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p^2-(p+1)})\equiv(V_{p+1}-V_{p-1})\bigl(\Omega\bigl(\psi_S^p(V_{p+1})-pV_p\bigr)\bigr) \pmod{\mathbb{Z}V_{p^2}}.\tag{4.4}$$

Note that $\Omega(V_r)\equiv V_{p^2}-V_r \pmod{I_{p^2}}$ since the Heller operation Ω translates V_r to V_{p^2-r} in $RS(p^2)$. Therefore, we have the following lemma.

Lemma 4.2.

$$\begin{aligned}
&(V_{p+1}-V_{p-1})\psi_\Lambda^p(V_{p^2-(p+1)})\\
&\quad\equiv pV_p(V_{p+1}-V_{p-1})-\psi_S^p(V_{p+1})(V_{p+1}-V_{p-1}) \pmod{I_{p^2}}.
\end{aligned}\tag{4.5}$$

Proof. The desired result can be obtained by comparing the dimension on either side of Eq. (4.5). □

Thus, if Conjecture 2.5 is true, then by Lemmas 4.1 and 4.2,

$$\psi_S^p(V_{p+1})(V_{p+1}-V_{p-1}) \equiv (p+1)(V_{p+1}-V_{p-1}) \pmod{I_{p^2}}.$$

We now show that this modulo equivalence does not hold unless $p=2$ by computing the explicit value of ψ_S^p at V_{p+1}.

We first introduce multiplication formulae necessary for computation of $\psi_S^p(V_{p+1})$, all of which can be derived from Proposition 2.4 by direct calculation.

(P1) If $s>r$, then

$$\begin{aligned}(V_{sp+1}-V_{sp-1}+V_{p-1})V_{rp} &= (V_{sp+1}-V_{sp-1}+V_p-V_1)V_{rp}\\ &= V_{(s+r)p}-V_{(s-r)p}+(p-1)V_{rp}.\end{aligned}$$

(P2) If $s=r$, then

$$\begin{aligned}(V_{sp+1}-V_{sp-1}+V_{p-1})V_{rp} &= (V_{sp+1}-V_{sp-1}+V_p-V_1)V_{rp}\\ &= V_{2rp}+(p-1)V_{rp}.\end{aligned}$$

(P3) If $s<r$, then

$$\begin{aligned}(V_{sp+1}-V_{sp-1}+V_{p-1})V_{rp} &= (V_{sp+1}-V_{sp-1}+V_p-V_1)V_{rp}\\ &= V_{(s+r)p}+V_{(r-s)p}+(p-1)V_{rp}.\end{aligned}$$

(P4) If $1 \leqslant r \leqslant \frac{p-1}{2}$, then

$$\begin{aligned}&(V_{(p-r)p+1}-V_{(p-r)p-1}+V_{p-1})V_{rp+1}\\ &\quad= (p-2)V_{rp}-V_{p^2-1}+2V_{p^2}+V_{(r+1)p-1}-V_{(p-2r)p-1}.\end{aligned}$$

(P5) If $1 \leqslant r \leqslant \frac{p-1}{2}$, then

$$\begin{aligned}&(V_{(p-r)p+1}-V_{(p-r)p-1}+V_p-V_1)V_{rp+1}\\ &\quad= (p-1)V_{rp}-V_{p^2-1}+2V_{p^2}+V_{(r+1)p}-V_{(p-2r)p-1}-V_{rp+1}.\end{aligned}$$

(P6) If $\frac{p+1}{2} \leqslant r \leqslant p-1$, then

$$\begin{aligned}&(V_{(p-r)p+1}-V_{(p-r)p-1}+V_{p-1})V_{rp+1}\\ &\quad= (p-2)V_{rp}-V_{p^2-1}+2V_{p^2}+V_{(r+1)p-1}+V_{(2r-p)p+1}.\end{aligned}$$

(P7) If $\frac{p+1}{2} \leqslant r \leqslant p-1$, then

$$\begin{aligned}&(V_{(p-r)p+1}-V_{(p-r)p-1}+V_p-V_1)V_{rp+1}\\ &\quad= (p-1)V_{rp}-V_{p^2-1}+2V_{p^2}+V_{(r+1)p}+V_{(2r-p)p+1}-V_{rp+1}.\end{aligned}$$

Theorem 4.3. *For every odd prime p, we have*

$$\psi_S^p(V_{p+1}) = pV_2 + V_{p-1} - V_p + (p-2)V_{p^2-1} - (p-2)V_{p^2}.$$

Proof. In view of Theorem 3.5, we obtain

$$S^i(V_{p+1}) = V_{ip+1} + \sum_{r=1}^{i-1} \binom{p-1+i-r}{p-2}_{p^{-1}} V_{rp}, \tag{4.6}$$

where $1 \leqslant i \leqslant p-1$, and

$$S^p(V_{p+1}) = V_2 + \left(\binom{2p-2}{p-2}_{p^{-1}} - 1\right) V_p + \sum_{r=2}^{p-1} \binom{2p-1-r}{p-2}_{p^{-1}} V_{rp} + V_{p^2}. \tag{4.7}$$

In addition, Eq. (2.1) implies that

$$\psi_S^p(V_{p+1}) = pS^p(V_{p+1}) - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) S^i(V_{p+1}).$$

For any V in $RS(p^k)$, let $[V]_{\mathrm{Ind}}$ denote the sum of induced indecomposable summands in the decomposition of V into indecomposables. More precisely, if $V = \sum_{i\geqslant 1} c_i V_i$, then $[V]_{\mathrm{Ind}}$ is defined as $\sum_{i\geqslant 1} c_{pi} V_{pi}$. Using this notation, we can rewrite $\psi_S^p(V_{p+1})$ as

$$pV_2 + p\big[S^p(V_{p+1})\big]_{\mathrm{Ind}} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) V_{ip+1} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) \big[S^i(V_{p+1})\big]_{\mathrm{Ind}}. \tag{4.8}$$

To express Eq. (4.8) as a linear combination of indecomposables, we first focus on the last term,

$$\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) \big[S^i(V_{p+1})\big]_{\mathrm{Ind}}.$$

Utilizing Theorem 2.1 and Eq. (4.3), we can show that $\psi_S^{p-i}(V_{p+1})$ is equal to

$$\begin{cases} V_{sp+1} - V_{sp-1} + V_{p-1} & \text{if } s \text{ is odd}, \\ V_{sp+1} - V_{sp-1} + V_p - V_1 & \text{if } s \text{ is even}, \end{cases}$$

where $s = p - i$. We then multiply $\psi_S^{p-i}(V_{p+1})$ by $[S^i(V_{p+1})]_{\mathrm{Ind}}$ using formulae (P1), (P2) and (P3). For each $1 \leqslant i \leqslant p-1$, the multiplicity of V_{ep} $(1 \leqslant e \leqslant i-1)$ in $S^i(V_{p+1})$ is given by

$$m(i,e) := \binom{p-1+i-e}{p-2}_{p^{-1}}$$

in view of Eq. (4.6). For $e \leqslant i-1$, we set $m(i,e)$ to zero. Therefore, we obtain

$$\psi_S^{p-i}(V_{p+1}) \big[S^i(V_{p+1})\big]_{\mathrm{Ind}} = \sum_e m(i,e) \psi_S^{p-i}(V_{p+1}) V_{ep},$$

where

$$\psi_S^{p-i}(V_{p+1})V_{ep} = \begin{cases} V_{(p-i+e)p} - V_{(p-i-e)p} + (p-1)V_{ep} & \text{if } e < p-i, \\ V_{2ep} + (p-1)V_{ep} & \text{if } e = p-i, \\ V_{(p-i+e)p} + V_{(e+i-p)p} + (p-1)V_{ep} & \text{if } e > p-i. \end{cases} \tag{4.9}$$

It should be noted that $V_{(p-i+e)p}$ in Eq. (4.9) cannot be V_{p^2} because e ranges from 1 to $i-1$ for each $1 \leqslant i \leqslant p-1$. In what follows, for each r with $1 \leqslant r \leqslant p-1$, we compute the multiplicity of V_{rp} in $\sum_i \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\mathrm{Ind}}$, which is equal to

$$\begin{aligned} &\sum_i \sum_{e<p-i} m(i,e)\bigl(V_{(p-i+e)p} - V_{(p-i-e)p} + (p-1)V_{ep}\bigr) \\ &\quad + \sum_i m(i,p-i)\bigl(V_{2p(p-i)} + (p-1)V_{p(p-i)}\bigr) \\ &\quad + \sum_i \sum_{e>p-i} m(i,e)\bigl(V_{(p-i+e)p} + V_{(e+i-p)p} + (p-1)V_{ep}\bigr). \end{aligned}$$

Thus, the coefficient of V_{rp} in $\sum_i \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\mathrm{Ind}}$ is given by

$$\begin{aligned} &\sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e) - \sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i+e=p-r}} m(i,e) + \sum_{\substack{1\leqslant i<p-r\\ r+1\leqslant i}} m(i,r)(p-1) \\ &\quad + m(p-r/2,r/2) + m(p-r,r)(p-1) \\ &\quad + \sum_{\substack{1\leqslant i\leqslant p-1\\ p-i\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e) + \sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i+e=p+r}} m(i,e) + \sum_{\substack{p-r<i\leqslant p-1\\ r+1\leqslant i}} m(i,r)(p-1). \end{aligned}$$

Here, if r is odd, $m(p-r/2,r/2)$ is set to zero. We now simplify the above summation.

First, note that if $i+e=p-r$, then $1 \leqslant i \leqslant p-r-1$ since $e = p-i-r \geqslant 1$. In the same fashion, if $i+e=p+r$, then $r+1 \leqslant i \leqslant p-1$ since $e = p-i+r \leqslant p-1$. As a consequence,

$$\begin{aligned} I &:= -\sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i+e=p-r}} m(i,e) + \sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i+e=p+r}} m(i,e) \\ &= -\sum_{1\leqslant i\leqslant p-r-1} m(i,p-r-i) + \sum_{r+1\leqslant i\leqslant p-1} m(i,p+r-i) \\ &= -\sum_{1\leqslant i\leqslant p-r-1} m(i,p-r-i) + \sum_{1\leqslant i\leqslant p-r-1} m(i+r,p-i) \quad \text{(by replacing } i \text{ by } i+r\text{)} \\ &= 0. \end{aligned}$$

The final equality follows from the identity $m(i,p-r-i) = m(i+r,e)$.

Second, note that

$$\begin{aligned} II &:= \sum_{\substack{1\leqslant i<p-r\\ r+1\leqslant i}} m(i,r)(p-1) + m(p-r,r)(p-1) + \sum_{\substack{p-r<i\leqslant p-1\\ r+1\leqslant i}} m(i,r)(p-1) \\ &= (p-1)\sum_{r+1\leqslant i\leqslant p-1} m(i,r) \\ &= \begin{cases} \frac{p-1}{p}\left[\binom{p}{p-2}+\binom{p+1}{p-2}+\cdots+\binom{2p-2-r}{p-2}\right] & \text{if } 1\leqslant r\leqslant p-2, \\ 0 & \text{if } r=p-1. \end{cases} \end{aligned}$$

Third, note that if $i-e=p-r$, then $m(i,e)=\binom{2p-1-r}{p-2}p^{-1}$. Conversely, if

$$1\leqslant i\leqslant p-1, \qquad 1\leqslant e\leqslant i-1, \qquad i-e=p-r,$$

then $1\leqslant e=i-p+r<p-i$ and thus $p-r+1\leqslant i<p-\frac{r}{2}$. Consequently,

$$\sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e) = \begin{cases} \frac{r-1}{2}\binom{2p-1-r}{p-2}p^{-1} & \text{if } r \text{ is odd,} \\ (\frac{r}{2}-1)\binom{2p-1-r}{p-2}p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

Similarly,

$$\sum_{\substack{1\leqslant i\leqslant p-1\\ p-i\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e) = \begin{cases} \frac{r-1}{2}\binom{2p-1-r}{p-2}p^{-1} & \text{if } r \text{ is odd,} \\ (\frac{r}{2}-1)\binom{2p-1-r}{p-2}p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

We also note that

$$m(p-r/2,r/2) = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \binom{2p-1-r}{p-2}p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

Putting these together, it is evident that

$$III := \sum_{\substack{1\leqslant i\leqslant p-1\\ 1\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e) + m(p-r/2,r/2) + \sum_{\substack{1\leqslant i\leqslant p-1\\ p-i\leqslant e\leqslant i-1\\ i-e=p-r}} m(i,e)$$

equals $(r-1)\binom{2p-1-r}{p-2}p^{-1}$.

Since the coefficient of V_{rp} in $\sum_{i=1}^{p-1}\psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\mathrm{Ind}}$ equals $I+II+III$, we obtain

$$\sum_{i=1}^{p-1}\psi_S^{p-i}(V_{p+1})\left[S^i(V_{p+1})\right]_{\mathrm{Ind}} = \sum_{r=1}^{p-1} x_r p^{-1} V_{rp}, \tag{4.10}$$

where

$$x_r = (r-1)\binom{2p-1-r}{p-2} + \sum_{i=1}^{p-r-1}(p-1)\binom{p-1+i}{p-2}.$$

Note that

$$\begin{aligned}
&p\binom{2p-1-r}{p-2} - (r-1)\binom{2p-1-r}{p-2} - (p-1)\sum_{i=1}^{p-r-1}\binom{p-1+i}{p-2} \\
&= (p-r+1)\binom{2p-1-r}{p-2} - (p-1)\left[-p + \binom{p+1}{p-1} + \binom{p+1}{p-2} + \cdots + \binom{2p-2-r}{p-2}\right] \\
&= (p-r+1)\binom{2p-1-r}{p-2} - (p-1)\binom{2p-1-r}{p-1} + p(p-1) \\
&= p(p-1).
\end{aligned}$$

Here the second and third equalities come from the well-known formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. We now apply this identity to Eqs. (4.7) and (4.10) to obtain

$$p\big[S^p(V_{p+1})\big]_{\mathrm{Ind}} - \sum_{i=1}^{p-1}\psi_S^{p-i}(V_{p+1})\big[S^i(V_{p+1})\big]_{\mathrm{Ind}} = (p-2)V_p + \sum_{r=2}^{p-1}(p-1)V_{rp} + pV_{p^2}.$$

Conversely, by Eq. (4.3) and formulae (P4)–(P7), we derive the identity

$$\begin{aligned}
&\psi_S^{p-i}(V_{p+1})V_{ip+1} \\
&= \begin{cases}
(p-1)V_{ip} + V_{(i+1)p} - V_{p^2-1} + 2V_{p^2} - V_{(p-2i)p-1} - V_{ip+1} & \text{if } 1 \leqslant i \leqslant \frac{p-1}{2} \text{ odd}, \\
(p-2)V_{ip} - V_{p^2-1} + 2V_{p^2} - V_{(p-2i)p-1} + V_{(i+1)p-1} & \text{if } 1 \leqslant i \leqslant \frac{p-1}{2} \text{ even}, \\
(p-1)V_{ip} + V_{(i+1)p} - V_{p^2-1} + 2V_{p^2} + V_{(2i-p)p+1} - V_{ip+1} & \text{if } \frac{p+1}{2} \leqslant i \leqslant p-2 \text{ odd}, \\
(p-2)V_{ip} - V_{p^2-1} + 2V_{p^2} + V_{(2i-p)p+1} + V_{(i+1)p-1} & \text{if } \frac{p+1}{2} \leqslant i \leqslant p-2 \text{ even}, \\
(p-2)V_{ip} + 2V_{p^2} + V_{(2i-p)p+1} & \text{if } i = p-1.
\end{cases}
\end{aligned}$$

This implies that

$$\begin{aligned}
\sum_{i=1}^{p-1}\psi_S^{p-i}(V_{p+1})V_{ip+1} = {}& \sum_{i=1}^{p-1}(p-1)V_{ip} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 \\
& - \sum_{i=1}^{\frac{p-1}{2}} V_{(p-2i)p-1} + \sum_{\substack{1 \leqslant i \leqslant \frac{p-1}{2} \\ \text{even}}} V_{(i+1)p-1} + \sum_{\substack{\frac{p+1}{2} \leqslant i \leqslant p-2 \\ \text{even}}} V_{(i+1)p-1} \qquad (4.11) \\
& + \sum_{i=\frac{p+1}{2}}^{p-1} V_{(2i-p)p+1} - \sum_{\substack{1 \leqslant i \leqslant \frac{p-1}{2} \\ \text{odd}}} V_{ip+1} - \sum_{\substack{\frac{p+1}{2} \leqslant i \leqslant p-2 \\ \text{odd}}} V_{ip+1}. \qquad (4.12)
\end{aligned}$$

Since $\{p-2i\colon 1 \leqslant i \leqslant \frac{p-1}{2}\}$ is the same as $\{i\colon 1 \leqslant i \leqslant p-2, \text{ odd}\}$, (4.11) equals

$$-\sum_{\substack{i=1\\ \text{odd}}}^{p-2} V_{ip-1} + \sum_{\substack{i=2\\ \text{even}}}^{p-2} V_{(i+1)p-1} = -\sum_{\substack{i=1\\ \text{odd}}}^{p-2} V_{ip-1} + \sum_{\substack{i=2\\ \text{odd}}}^{p-2} V_{ip-1} = -V_{p-1}.$$

In the same manner, we can show that (4.12) $= 0$ and hence

$$\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})V_{ip+1} = \sum_{i=1}^{p-1}(p-1)V_{ip} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 - V_{p-1}.$$

As a consequence,

$$\begin{aligned}
\psi_S^p(V_{p+1}) &= pV_2 + p\big[S^p(V_{p+1})\big]_{\mathrm{Ind}} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})V_{ip+1} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})\big[S^i(V_{p+1})\big]_{\mathrm{Ind}} \\
&= pV_2 + (p-2)V_p + \sum_{r=2}^{p-1}(p-1)V_{rp} + pV_{p^2} \\
&\quad - \left(\sum_{r=1}^{p-1}(p-1)V_{rp} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 - V_{p-1}\right) \\
&= pV_2 + V_{p-1} - V_p + (p-2)V_{p^2-1} - (p-2)V_{p^2},
\end{aligned}$$

as required. □

Theorem 4.4. *If $p = 2$, then $\psi_S^p(V_{p+1}) = 2V_2 - V_1$.*

Proof. The desired result is straightforward from Proposition 2.4(b), Theorem 3.5 and Eq. (2.1). □

Recall that we have already shown that if I_{p^2} is closed for the exterior powers, then

$$\psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) \equiv (p+1)(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}.$$

If p is odd, then Theorem 4.3 states

$$\psi_S^p(V_{p+1}) \equiv pV_2 - (p-1)V_1 \pmod{I_{p^2}},$$

and hence

$$\begin{aligned}
&\psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) - (p+1)(V_{p+1} - V_{p-1}) \\
&\quad \equiv p\big(V_2(V_{p+1} - V_{p-1}) - 2(V_{p+1} - V_{p-1})\big) \pmod{I_{p^2}} \\
&\quad \equiv p(V_{p+2} - V_{p-2} - 2V_{p+1} + 2V_{p-1}) \pmod{I_{p^2}} \\
&\quad \equiv p(V_{2p} - V_{2p-2} - 2V_{p+1} + 3V_p - V_{p-2} - 2V_1) \pmod{I_{p^2}}. \qquad (4.13)
\end{aligned}$$

However, (4.13) cannot be zero modulo I_{p^2} because all the indecomposable modules in the last term are contained in $\{V_{ap+j}\colon\ 0 \leqslant a \leqslant \frac{p-1}{2},\ \ j = 1, 3, \ldots, p\}$, which is a $\mathbb{Z}$-basis of $RS(p^2)/I_{p^2}$ (Proposition 2.6).

Corollary 4.5. *For every odd prime p, Conjecture 2.5 is not true.*

Proof. The proof follows from the natural embedding from $RS(q) \to RS(pq)$ mapping V_n to V_n for all $n \leqslant q$. $\square$

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